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The Group of Local Biholomorphisms of $\mathbb{C}^1$ and Conformal Field Theory in the Operator Formalism

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Abstract. Motivated by the operator formulation of conformal field theory on Riemann surfaces, we study the properties of the infinite dimensional group of local biholomorphic transformations (conformal reparametrizations) of $\mathbb{C}^1$ and develop elements of its representation theory.

1. Introduction

String theory [1] has provided and continues to provide motivation for much interesting mathematics. It suffices to recall the vertex operator construction of representations of Kac-Moody algebras [2], the impulse it gave to the study of the global geometry of the moduli space of Riemann surfaces, and the surge of interest in two dimensional conformal field theory (CFT), inspired mainly by the role of CFT in string theory. Indeed, recent developments, to mention only [3, 4], have raised CFT to the role of the main technical tool in the study of quantum string dynamics, at least within the first quantized approach. It is also worth adding that CFT has considerable interest in itself, for it has a quite intricate mathematical structure, which is nonetheless much more tractable than that of more general quantum field theories. CFT has moreover found important applications within the theory of two dimensional critical phenomena.

The present work was motivated mainly by recent developments concerning the operator formulation of CFT on higher genus Riemann surfaces [5, 6]. One of the characteristics of these approaches is the use they make of a space $\mathscr{M}$ of geometrical data, consisting in the simplest case of triples (M, P, z), with M a (closed) Riemann surface, P a distinguished point on M, and z a local uniformizer at P, i.e. a holomorphic function in a neighborhood of P with $dz(P) \neq 0$. The freedom of choosing any z with the above properties corresponds to the action on $\mathscr{M}$ of an infinite dimensional complex continuous group, which we refer to in the following as the group $\mathscr{G}$ of local biholomorphisms; it would seem natural to view $\mathscr{M}$ as a principal fiber bundle with structure group $\mathscr{G}$. Moreover, considering a

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CFT on M leads one naturally to the idea of associating to each (M, P, z) a "space of states," isomorphic to the Hilbert space $\mathscr{H}$ of the given CFT. Naively, the result of this should be an infinite dimensional vector bundle over M, associated to the above mentioned principal bundle. This is essentially the "bundle of observables" introduced by Witten [6].

However, to make this precise we need to know more about the properties of $\mathscr{G}$ and its representation on $\mathscr{H}$. The subject of the present paper is to fill in some gaps in our understanding of this problem. We think that such a study is also of interest from the mathematical point of view. $\mathscr{G}$ is an example of a non-Banach infinite dimensional Lie group, and to our knowledge no general theory of such groups exists. On the other hand, non-Banach groups occur frequently both in mathematical physics and pure mathematics, e.g. as various diffeomorphism groups and loop groups [7, 8]. A systematic investigation of their properties and representation theory thus seems to be worthwhile.

Our strategy, in outline, is the following: on $\mathscr{H}$ we are given the action of the Virasoro algebra [9], a certain completion of which gives a central extension of the algebra of meromorphic vector fields on M. The Virasoro algebra contains a subalgebra, generated (in standard notation) by L_n, $n \geqq 0$, which may be completed to the algebra $\mathscr{g}$ of holomorphic germs of vector fields at P, vanishing at P, and this subalgebra is "anomaly free." This leaves the problem of whether $\mathscr{G}$ is in fact generated by $\mathscr{g}$, via an exponential map with suitable continuity properties, and of the existence and properties of the representation of $\mathscr{G}$ on $\mathscr{H}$ obtained by exponentiating the corresponding representation of $\mathscr{g}$.

It should be noted that one cannot expect such a representation to be defined on all of $\mathscr{H}$: for example, $\exp(L_0)$ is an unbounded self-adjoint operator. An essential part of the problem is therefore to find the maximal invariant subspace of $\mathscr{H}$ on which this representation exists.

We find that most of the properties of ordinary Lie groups indeed hold also for $\mathscr{G}$, with an important modification: the exponential map is not locally invertible in any neighborhood of 0. We have however found a way to deal with this problem, and the net result is that $\mathscr{G}$ is spanned by one parameter subgroups, as any Lie group.

As to the second question, we describe the maximal invariant subspace of a Verma module (more precisely: of the space obtained by taking the quotient of a Verma module by null vectors), on which the representation of $\mathscr{g}$ obtained by restricting the given representation of the Virasoro algebra integrates to a representation of the universal covering group of $\mathscr{G}$. The latter is also explicitly described. For non-negative integer values of the highest weight of the Verma module this representation can be projected to one of $\mathscr{G}$. In the case of a holomorphic CFT this solves the problem of representing $\mathscr{G}$ on $\mathscr{H}$, since $\mathscr{H}$ is then a direct sum of Verma modules. In the general case, one must consider the tensor product of a representation corresponding to highest weight h with the complex conjugate of the one corresponding to $\bar{h}$, and sum over all $(h, \bar{h})$ occurring in the given model. This necessary generalization does not seem to pose problems.

This paper is organized as follows: in Sect. 2 we describe the group of local biholomorphisms and its topology, the latter by making use of the concept of inductive limit of Fréchet spaces. In Sect. 3 we do the same for the Lie algebra $\mathscr{g}$ and introduce the exponential map. The properties of the exponential map are studied in detail in Sect. 4. In Sect. 5 we describe the universal covering group of $\mathcal{G}$ and the properties of the corresponding exponential map. Section 6 is devoted to developing the representation theory. A few final remarks are gathered in Sect. 7.

2. The Group of Local Biholomorphisms: General Overview

We consider the space of germs of holomorphic functions at zero in $\mathbb{C}^1$. It is a fiber of the sheaf of holomorphic germs on $\mathbb{C}^1$. Within this space we can distinguish the subspace of germs of functions with a fixed point at zero and locally invertible at zero, i.e. $f(0)=0$ and $f'(0)\neq 0$. We are therefore dealing with the space of germs of holomorphic transformations of some neighborhood of zero, which send zero to zero in $\mathbb{C}^1$.

It is easy to see that this space can be given the following explicit description:

Definition.

$$\mathcal{G}=\left\{f(z):f(z)=\sum_{n=1}^{\infty} f_n z^n,\ f_1\neq 0,\ \limsup_{n\to\infty}|f_n|^{1/n}<\infty\right\}. \tag{2.1}$$

$\mathcal{G}$ is naturally equipped with a group structure $(\mathcal{G},\circ)$, where "$\circ$" is composition of mappings: $(f\circ g)(z)=f(g(z))$ for $f,g\in\mathcal{G}$. It is obvious that the group axioms are satisfied, moreover we may introduce the following topological structure: let

$$\mathcal{G}_R=\left\{\sum_{n=1}^{\infty} f_n z^n,\ \limsup_{n\to\infty}|f_n|^{1/n}<R^{-1}\right\}.$$

For every $R>0$, $\mathcal{G}_R$ forms a Fréchet space, with a countable family of seminorms given by

$$\|f\|_r=\sup_{|z|=r}|f(z)|,\qquad r=R\left(1-\frac{1}{n}\right),\qquad n=2,3,\ldots. \tag{2.2}$$

Now take $\bigcup_{R>0}\mathcal{G}_R$ and endow this set with the topology of an inductive limit of Fréchet spaces (ILF topology). Thus $\mathcal{G}$ is a topological space, as an open subset in an ILF topological space. It is easy to see that $\mathcal{G}$ enjoys the following properties: (a) it is Hausdorff, (b) separable, (c) arcwise connected, and therefore connected. $\mathcal{G}$ is also non-metrizable as a strict inductive limit of Fréchet spaces [10].

Since we can write $f\in\mathcal{G}$ as $re^{i\varphi}z+f_2z^2+f_3z^3+\ldots$ for $r>0$, $\varphi\in\mathbb{R}$, and $f_i\in\mathbb{C}$, $i=2,3,\ldots$, there exists a one to one continuous map representing f as $(e^{i\varphi},r,f_2,f_3,\ldots)\in S^1\times\{\text{contractible set}\}$. This shows that $\mathcal{G}$ is homotopically equivalent to the circle S^1.

Proposition 2.1. *$(\mathcal{G},\circ)$ is a topological group.*

Proof. Use of elementary properties of holomorphic functions yields the estimate

$$\begin{aligned}\sup_{|z|=r}|f\circ g^{-1}(z)-f_n\circ g_n^{-1}(z)|\leqq C_1\sup_{|z|=r_1}&|f(z)-f_n(z)|\\ &+C_2\sup_{|z|=r_2}|g(z)-g_n(z)|,\end{aligned}$$

where the constants C_1, C_2, r_1, r_2 have finite limits as $f_n\to f$, $g_n\to g$. The continuity of $(f,g)\to f\circ g^{-1}$ follows. $\square$

The ILF structure of $\mathscr{G}$ seems to be natural for our purposes. One could attempt to equip $\mathscr{G}$ with the structure of a Lie group, but that would require at least a differentiable structure in the Fréchet space (cf. [11]), extendible to the ILF in a way which would make the map $(f, g) \to f \circ g^{-1}$ smooth.

3. The Lie Algebra of $\mathscr{G}$

Although we have not shown explicitly that $\mathscr{G}$ may be considered a Lie group, there is however a natural associated Lie algebra which may be easily described.

Definition.

$$\mathscr{g} = \left\{ v : v = \sum_{n=1}^{\infty} v_n z^n (d/dz),\ v_n \in \mathbb{C},\ \limsup_{n \to \infty} |v_n|^{1/n} < \infty \right\}. \tag{3.1}$$

Let us compute explicitly $[v, w]$ for $v, w \in \mathscr{g}$.

$$\begin{aligned} [v, w] &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} v_n w_m [z^n (d/dz), z^m (d/dz)] \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} v_n w_m (m-n) z^{m+n-1} (d/dz) \in \mathscr{g}. \end{aligned} \tag{3.2}$$

One may also think of $\mathscr{g}$ as the Lie algebra of germs of holomorphic vector fields vanishing at zero in $\mathbb{C}^1$. Now, proceeding in the same way as in the case of the group $\mathscr{G}$, we can introduce the ILF topological structure on $\mathscr{g}$. In addition we show

Proposition 3.1. *$\mathscr{g}$ is a topological Lie algebra.*

Proof. The only nontrivial point is the continuity of $(v, w) \to [v, w]$. Let $v = v(z)(d/dz)$. Using the Cauchy inequality we find

$$|v_n| \leqq r^{-n} \|v\|_r. \tag{3.3}$$

Thus

$$\begin{aligned} \|[v, w]\|_R &= \sup_{|z|=R} \left| \sum_{k=1}^{\infty} \sum_{n=1}^{k} (k+1-2n) v_n w_{k+1-n} z^k \right| \\ &\leqq \sup_{|z|=R} \|v\|_r \|w\|_r \sum_{k=1}^{\infty} \sum_{n=1}^{k} |k+1-2n| r^{-(k+1)} |z|^k \\ &= C(R, r) \|v\|_r \|w\|_r, \end{aligned}$$

where $C(R, r)$ is a constant which depends only on R and r, and $R < r$. The continuity of the Lie bracket follows. □

We now explore the relation between $\mathscr{g}$ and $\mathscr{G}$. By analogy with the case of ordinary Lie algebras and groups, it should be given by the exponential map.

Definition. By the exponential map $\exp : \mathscr{g} \to \mathscr{G}$ we mean

$$\mathscr{g} \ni v \to \sum_{k=0}^{\infty} \frac{1}{k!} (v(z) d/dz)^k z. \tag{3.4}$$

Proposition 3.2. *The map* exp *is well defined and continuous.*

Proof. First of all observe that, by direct calculation, if $f=\exp(v)$ then $f(0)=0$ and for the first coefficient we have $f_1=\exp(v_1)\neq 0$. Now it should be shown that for every $v\in\mathcal{g}$, $\exp(v)\in\mathcal{G}$.

Following the same procedure as in the case of Proposition 3.1 we obtain by induction the following inequality for the n^{th} coefficient of the k^{th} term of $\exp(v)$:

$$|a_n(k)|=|[(v(z)d/dz)^k z]_n|\leqq \frac{\|v\|_r^k}{r^{n+k-1}}\sigma_{k-1}(n).$$

where

$$\sigma_k(n)=\sum_{n_k=1}^{n} n_k \ldots \sum_{n_2=1}^{n_3} n_2 \sum_{n_1=1}^{n_2} n_1 .$$

It is seen that $\sigma_k(n)\leqq \frac{1}{k!}\left(\frac{n^2}{2}\right)^k$. Thus

$$|a_n(k)|\leqq \frac{1}{(k-1)!}\frac{\|v\|_r^k}{r^{n+k-1}}\left(\frac{n^2}{2}\right)^{k-1}.$$

Now let $b_n=\sum_{k=0}^{\infty}\frac{a_n(k)}{k!}$. Then for $n>1$ we have

$$|b_n|\leqq \frac{\|v\|_r}{r^n}\sum_{k=0}^{\infty}\frac{\|v\|_r^k}{r^k}\left(\frac{n^2}{2}\right)^k\frac{1}{(k!)^2(k+1)}.$$

Because of $\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2}\leqq \exp(2\sqrt{x})$, we obtain

$$|b_n|\leqq [r^{-1}\exp(2(\|v\|_r/2r)^{1/2})]^n\|v\|_r .$$

Therefore

$$\limsup |b_n|^{1/n}\leqq r^{-1}\exp(2(\|v\|_r/2r)^{1/2}). \tag{3.5}$$

We have thus obtained an estimate for the radius of convergence of the analytic mapping $\exp(v)$, which shows that $\exp(v)\in\mathcal{G}$.

The continuity of exp can be shown by using the integral du Hamel formula

$$\exp(v_1 d/dz)z-\exp(v_2 d/dz)z=\int_0^1 \exp(tv_1 d/dz)(v_1-v_2)\frac{d}{dz}\exp[(1-t)v_2 d/dz]z\,dt,$$

and formula (4.1) below. $\square$

Example 3.3. Let $\ell_n\in\mathcal{g}$ be of the form

$$\ell_n=-z^{n+1}d/dz,\qquad n=0,1,2,3,\ldots. \tag{3.6}$$

We immediately obtain

$$[\ell_n,\ell_m]=(n-m)\ell_{n+m}. \tag{3.7}$$

Then for $\alpha\in\mathbb{C}$, $\exp(\alpha\ell_n)$ can be computed as

$$\exp(\alpha\ell_n)=\begin{cases} z(1+\alpha n z^n)^{-1/n}, & n>0\\ ze^{-\alpha}, & n=0.\end{cases} \tag{3.8}$$

4. Properties of the Exponential Map

By analogy with the situation for finite dimensional Lie algebras and groups we expect exp to display a series of convenient algebraic properties. In the following we present the most elementary among these properties

Proposition 4.1. *Take $v(z)d/dz \in \mathfrak{g}$ and $f \in \mathcal{G}$. Then:*

1. $$\exp[v(z)d/dz]f(z) = f(\exp[v(z)d/dz]z)\,; \tag{4.1}$$

2. $$v(z)df/dz = v(f(z)) \quad \text{if} \quad f = \exp(v)\,; \tag{4.2}$$

3. *Let $f_\lambda(z) = \exp[\lambda v(z)d/dz]z$. Then*
$$f_{\lambda_1+\lambda_2} = f_{\lambda_1} \circ f_{\lambda_2}\,. \tag{4.3}$$

4. $$\exp(v_f) = f^{-1} \circ \exp(v) \circ f\,, \tag{4.4}$$

where $v_f = (df/dz)^{-1} v(f(z))$.

Proof. To show 1. take two maps

$$t \to A_1(t) = \exp[tv(z)d/dz]f(z) \quad \text{and} \quad t \to A_2(t) = f(\exp[tv(z)d/dz]z)\,.$$

Observe that $A_1(0) = A_2(0)$ and they satisfy the same differential equation:

$$\frac{d}{dt}A_i(t) = v(z)\frac{d}{dz}A_i(t)\,, \qquad i = 1, 2\,.$$

The above may be viewed as a system of ordinary differential equations (in t) for the coefficients of the series expansion of $A_i(t)$ in powers of z. Therefore A_1 and A_2 must coincide.

To show 2. let us replace v with f in property 1. Then

$$v(f(z)) = \exp[v(z)d/dz]v(z) = v(z)df/dz\,.$$

3. is a direct consequence of 1. To show 4. we can take two mappings

$$t \to \exp(tv_f) \quad \text{and} \quad t \to f^{-1} \circ \exp(tv) \circ f\,,$$

and repeat the same reasoning as in 1. □

As is well known, not all the properties that the exponential map displays in finite dimensional cases carry over in infinite dimensions. To illustrate this we shall show with two counterexamples that exp is not an homeomorphism (even locally in the neighborhood of zero) and is not surjective on $\mathcal{G}$.

Example 4.2. We demonstrate below that exp is not injective, even locally in the neighborhood of zero. Let us take a one-parameter subgroup in $\mathcal{G}$, i.e. $f_0 = \mathrm{id}$, $f_{t+s} = f_t \circ f_s$, of the form

$$f_t(z) = e^{it}z(1 + (e^{int} - 1)z^n)^{-1/n}\,. \tag{4.5}$$

For $t = 2\pi/n$ we have $f_{2\pi/n}(z) = e^{2\pi i/n}z$. There exists a $v \in \mathfrak{g}$ such that

$$f_t(z) = \exp[tv(z)d/dz]\,.$$

In fact, $v(z)$ is given by

$$\frac{d}{dt} f_t(z)\bigg|_{t=0} = v(z).$$

Performing the computation we obtain

$$\frac{d}{dt} f_t(z)\bigg|_{t=0} = i(z - z^{n+1}),$$

hence a single element $e^{2\pi i/n} z \in \mathscr{G}$ may be obtained in two different ways:

$$\exp[(2\pi i/n)z d/dz]z = e^{2\pi i/n} z = \exp[(2\pi i/n)z(1-z^n)d/dz]z.$$

This is the counterexample, as claimed.

Example 4.3. Here we present a family $f_n \in \mathscr{G}$, the elements of which are not contained in the image of exp:

$$f_n(z) = e^{2\pi i/n} z + \frac{1}{n} z^2. \tag{4.6}$$

Below we will explain how this counterexample works, and thus show that exp is not a surjective map onto an arbitrarily small neighborhood of the identity in $\mathscr{G}$.

We will need the following lemma, due to H. Poincaré (cf. [12]), which concerns the linearization problem for germs of complex analytic diffeomorphisms. This lemma provides the possibility of showing the existence of a large subset of $\mathscr{g}$ on which exp is an homeomorphism.

Lemma 4.4. *If $f \in \mathscr{G}$ and the first coefficient of its power series expansion satisfies $|f_1| \neq 1$, there exists a $g \in \mathscr{G}$ such that*

$$g \circ f \circ g^{-1} = f_1 z. \tag{4.7}$$

The proof is a standard exercise and will not be given here.

Now we are ready to state the following

Theorem 4.5. a) *Let $\mathscr{g}_{(1)} = \{v \in \mathscr{g} : \mathrm{Re}\, v_1 \neq 0\}$, and $\mathscr{G}_{(1)} = \{f \in \mathscr{G} : |f_1| \neq 1\}$.*

Then $\exp: \mathscr{g}_{(1)} \to \mathscr{G}_{(1)}$ is a covering map, i.e. a local homeomorphism with each fiber being a discrete space (in fact, isomorphic to $\mathbb{Z}$).

b) *For each $f \in \mathscr{G}$ there exist v_1, v_2 such that*

$$f = \exp v_1 \circ \exp v_2. \tag{4.8}$$

Proof. From Proposition 3.2 we know that exp is well defined and continuous. By Lemma 4.4 and Eq. (4.4), $\exp^{-1}$ exists locally:

$$\exp^{-1}(f) = \log f_1 \frac{g(z)}{dg/dz} \frac{d}{dz}, \tag{4.9}$$

where g is the solution to the linearization problem of Lemma 4.4. It is straightforward to show that $\exp^{-1}$ is also locally continuous; we leave this to the reader as an exercise.

As to point b): let $f=\lambda(\lambda^{-1}f)$, $\lambda \neq 0$ and chosen in such a way that $|\lambda^{-1}f_1| \neq 1$. By Theorem 4.5 a), there exists a $v \in \mathfrak{g}$ which satisfies

$$\lambda^{-1}f=\exp(v).$$

Hence $g=\exp(z\log\lambda d/dz)\circ\exp(v)$. □

Corollary 4.6. *From Theorem 4.5* b) *it follows that $\mathcal{G}$ is spanned by a family of one-parameter subgroups.*

Now we wish to make a few comments concerning Example 4.3. The counterexamples are provided by the fact that the linearization problem has different solutions in $\mathfrak{g}$ and in $\mathcal{G}$. Were exp a surjective map, there would be a correspondence between the respective solution spaces. We observed that, if $|f_1| \neq 1$, there exists a $g \in \mathcal{G}$ such that $g\circ f\circ g^{-1}=f_1 z$, given by a solution to the linearization problem in $\mathcal{G}$ (cf. Lemma 4.4). Now let $v \in \mathfrak{g}$, and $v_1 \neq 0$. We ask: does there exist a $g \in \mathcal{G}$ such that

$$v_1\left(\frac{dg}{dz}\right)^{-1} g(z)=v(z), \tag{4.10}$$

where $v(z)=\sum\limits_{n=1}^{\infty} v_n z^n$? This is the linearization problem in $\mathfrak{g}: v_{g^{-1}}=v_1 z$, cf. (4.4). To solve this problem we will apply the method of Lemma 4.4. Write down both sides of (4.12):

$$v_1 g(z)=\sum_{n=1}^{\infty} v_1 g_n z^n,$$

$$v(z)\frac{dg}{dz}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} v_{n-k+1}g_k k\right)z^n.$$

We therefore obtain the recursive equations (with g_1 arbitrary):

$$g_n=\frac{1}{v_1(1-n)}\sum_{k=1}^{n-1} v_{n-k+1}g_k k, \tag{4.11}$$

which give a formal solution g, as required. We now verify the convergence of $\sum\limits_{k=1}^{\infty} g_k z^k$, i.e. $g \in \mathcal{G}$. To this end we introduce $N=R|v_1|$, with R satisfying

$$|v_n| \leqq \frac{N}{R^n}.$$

Then

$$|g_n| \leqq \frac{R}{N(n-1)}\sum_{k=1}^{n-1}\frac{N}{R^{n-k+1}}|g_k|k.$$

Now we put

$$x_n=\frac{R}{N(n-1)}\sum_{k=1}^{n-1}\frac{N}{R^{n-k+1}}x_k k, \quad \text{with } x_1 \text{ arbitrary},$$

in order to obtain inductively the inequality

$$|g_n| \leqq x_n .$$

But x_n can be computed explicitly, by setting up a differential equation. Indeed, denote

$$X(z) = \sum_{n=1}^{\infty} x_n z^n ,$$

and

$$V(z) = \sum_{n=1}^{\infty} \frac{N}{R^n} z^n = \frac{Nz}{R-z} .$$

We obtain the equation

$$\frac{Rz}{R-z} \frac{dX}{dz} = 2z \frac{dX}{dz} - X ,$$

which can be explicitly solved, yielding

$$X(z) = \text{const}\, z(R-2z)^{-1/2} .$$

Since this function is holomorphic in a neighborhood of zero, so must be g.

At this point we are ready to prove that Example 4.3 is indeed correct. For simplicity, take $f = -z + z^2$. Suppose that there exists a $v \in \mathscr{g}$ such that $f = \exp(v)$. Then $v_1 = \log(-1) = i\pi \neq 0$, hence for some $g \in \mathscr{G}$, $v_g = (dg/dz)^{-1} v(g(z)) = i\pi z d/dz$. It follows that $f = g \circ (-z) \circ g^{-1}$, i.e. $f \circ f = z$. But obviously $f \circ f = z - 2z^3 + z^4 \neq z$, so we obtain a contradiction.

5. Covering Group

The properties of $\mathscr{G}$ make it obvious that $\mathscr{G}$ posesses a universal covering group $(\hat{\mathscr{G}}, \mathscr{G}, \pi)$. In this case we are able to present an explicit form of $(\hat{\mathscr{G}}, \mathscr{G}, \pi)$.

Definition.

$$\hat{\mathscr{G}} = \left\{ (\lambda, f) \in \mathbb{C} \times \mathscr{G} : f_1 = e^{\lambda}, \text{ where } f = \sum_{n=1}^{\infty} f_n z^n \right\} . \tag{5.1}$$

If $\hat{\mathscr{G}}$ is endowed with the product topology, as a subset of $\mathbb{C} \times \mathscr{G}$, then it is easily observed that the operation $\hat{\circ}$:

$$(\lambda, f) \,\hat{\circ}\, (\mu, g) = (\lambda + \mu, f \circ g) \tag{5.2}$$

gives $(\hat{\mathscr{G}}, \hat{\circ})$ the structure of a topological group. Observe that the first coefficient of $f \circ g$ indeed satisfies $f_1 g_1 = e^{\lambda + \mu}$.

It is clear that $\hat{\mathscr{G}}$ is contractible: to see this, it is enough to consider the homeomorphism $(\lambda, f) \to (\lambda, f_2, f_3, \ldots)$; the first coefficient f_1 is then determined by λ, and $\hat{\mathscr{G}}$ is mapped into a contractible space. Therefore, in particular, $\hat{\mathscr{G}}$ is simply connected.

Definition: Let us define

$$\pi : \hat{\mathcal{G}} \to \mathcal{G} \quad \text{as} \quad \pi(\lambda, f) = f. \tag{5.3}$$

We see that π is a group homomorphism, and a local homeomorphism with discrete fiber:

$$\pi^{-1}(f) = \bigcup_{n \in \mathbb{Z}} (\log|f_1| + 2\pi n i, f) \simeq \mathbb{Z}. \tag{5.4}$$

Thus we have a covering $(\hat{\mathcal{G}}, \mathcal{G}, \pi)$ of the topological group $\mathcal{G}$, and since $\hat{\mathcal{G}}$ is simply connected, $(\hat{\mathcal{G}}, \mathcal{G}, \pi)$ is the universal covering.

Let us check the properties of the exponential mapping to $\hat{\mathcal{G}}$.

Definition. $\widehat{\exp} : \mathscr{g} \to \hat{\mathcal{G}}$ is given by

$$\widehat{\exp}(v) = (v_1, \exp(v)). \tag{5.5}$$

The map $\widehat{\exp}$ is obviously not injective nor surjective for any neighborhood of zero in $\mathscr{g}$.

Definition.

$$\hat{\mathcal{G}}_{(1)} = \{(\lambda, f) \in \hat{\mathcal{G}} : \operatorname{Re} \lambda \neq 0\}. \tag{5.6}$$

Note that $\widehat{\exp} : \mathscr{g}_{(1)} \to \hat{\mathcal{G}}_{(1)}$ is a homeomorphism (where $\mathscr{g}_{(1)}$ was defined in Theorem 4.5). Indeed, take $g \in \mathcal{G}$ such that $g \circ f \circ g^{-1}(z) = e^{\lambda} z$ (cf. Lemma 4.4); then

$$\widehat{\exp}^{-1}(\lambda, f) = g(dg/dz)^{-1} \lambda d/dz. \tag{5.7}$$

Arguing as in Theorem 4.5 b) we can show that for each element $(\lambda, f) \in \hat{\mathcal{G}}$ there exist $v, w \in \mathscr{g}$ such that

$$(\lambda, f) = \widehat{\exp}(v) \mathbin{\hat{\circ}} \widehat{\exp}(w). \tag{5.8}$$

Hence $\hat{\mathcal{G}}$ is spanned by a family of one-parameter subgroups.

6. Representations

One of the main goals of this paper is to construct a large class of representations of $\hat{\mathcal{G}}$ and of $\mathcal{G}$. They will be obtained as restrictions of the unitary highest weight representations (UHWR) of the Virasoro algebra [13]. Let us recall here an abstract definition of this algebra.

Definition. Let us take a family of formal elements $\{C, L_n\}$, where $n \in \mathbb{Z}$. By the Virasoro algebra we mean the complex vector space

$$\mathrm{Vir} = \operatorname{span}\{C, L_n\}, \tag{6.1}$$

together with the Lie bracket defined by

$$\begin{gathered} [L_n, L_m] = (n-m) L_{n+m} + \frac{C}{12} n(n^2 - 1) \delta_{n,-m}, \\ [L_n, C] = 0 \quad \text{for all } n, m \in \mathbb{Z}. \end{gathered} \tag{6.2}$$

The unitary highest weight representation is given by

Definition. There exists a Hilbert space $V(c,h)$, $c,h\in\mathbb{R}$, with a distinguished vector $|h\rangle\in V(c,h)$, $\langle h|h\rangle=1$, and a dense subspace D, described below. $V(c,h)$ carries a representation of Vir by unbounded operators, with D as a common invariant domain. The following conditions hold:

a) C is represented as multiplication by $c\in\mathbb{R}$.

b) $L_0|h\rangle=h|h\rangle$.

c) $L_n|h\rangle=0$ for $n>0$.

d) $|h\rangle$ is a cyclic vector for the representation of Vir.

e) $L_n^+=L_{-n}$ for all $n\in\mathbb{Z}$.

We have used here the same notation for the abstract elements of Vir and for the operators that represent them on $V(c,h)$.

By virtue of c) and d) we can introduce the notation

$$|\{n_k\}_{k=1}^N\rangle=(L_{-N})^{n_N}\dots(L_{-2})^{n_2}(L_{-1})^{n_1}|h\rangle\,. \tag{6.3}$$

We take D to be the subspace of $V(c,h)$ spanned by vectors of the form (6.3).

The UHWR exists for special values of c and h, in particular it is necessary that $c>0$, $h\geqq 0$, and then it is unique [14, 15]. From now on c and h will be fixed.

Moreover, L_0 is an essentially self-adjoint operator on D with the discrete spectrum $\{h+N\}$ given by b) and

$$L_0|\{n_k\}_{k=1}^N\rangle=\sum_{k=1}^N(h+kn_k)|\{n_k\}_{k=1}^N\rangle\,. \tag{6.4}$$

The basic observation is that $\ell_n=-z^{n+1}d/dz\in\mathcal{g}$, $n\geqq 0$, have the same commutation relations as $L_n\in\mathrm{Vir}$ [see (3.7)]. Thus any representation of Vir is simultaneously a representation of the generators of $\mathcal{g}$. In particular, this holds for the UHWR described above.

Since L_0 is an unbounded operator, we cannot expect to have here a representation of $\mathcal{g}$, and so of $\hat{\mathcal{G}}$, into a space of bounded operators acting on the Hilbert space $V(c,h)$. We will thus take an approach which may be summarized in the following points:

1. Choose a dense domain $S\subset V(c,h)$, invariant under the representation of Vir;

2. Introduce a Fréchet topology on S (stronger than the induced one);

3. Show that there exists a continuous representation of $\mathcal{g}$ in the space of bounded operators on S;

4. This representation may be now integrated to a representation of $\hat{\mathcal{G}}$.

Below we assume that $h>0$. If not, replace L_0 by L_0+1 in all formulas.

Definition.

$$S=\{x\in V(c,h):\|\exp(\lambda L_0)x\|<\infty \text{ for every } \lambda>0\}\,. \tag{6.5}$$

Example. Observe that $D\subset S$, and moreover, $x=\sum_{n=1}^{\infty}e^{-n^2}L_{-n}|h\rangle\in S$. More generally, S consists of all combinations $\sum_{\{n_k\}}\alpha_{\{n_k\}}|\{n_k\}_{k=1}^N\rangle$ with coefficients $\alpha_{\{n_k\}}$ which decay faster than exponentially.

As is natural, we take $\|x\|_n=\|\exp(nL_0)x\|$, $n=0,1,2,\dots$ as the family of seminorms defining the topology on S. In this way S becomes a Fréchet space.

In addition, there obviously hold the following inequalities:

$$\|x\|_m = \|\exp(mL_0)x\| = \|\exp[-(n-m)L_0]\exp(nL_0)x\| \leqq \|x\|_n \quad \text{for} \quad n>m. \tag{6.6}$$

Proposition 6.1. *The map $\ell_n \to L_n$, $n \geqq 0$, uniquely extends to a continuous representation $d\varrho$ of $\mathscr{g}$ in the space of bounded operators acting on S.*

Proof. Obviously for each $v \in \mathscr{g}$, given in the form

$$v = \sum_{n=0}^{\infty} v_{n+1}\ell_n \quad \text{with} \quad \limsup_{n\to\infty} |v_n|^{1/n} < \infty,$$

we have a unique corresponding operator $\sum_{n=0}^{\infty} v_{n+1}L_n$ acting on S. It is now necessary to show that this operator is bounded, i.e.

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} v_{n+1}L_n x\right\|_k &= \left\|\exp(kL_0)\sum_{n=0}^{\infty} v_{n+1}L_n x\right\| \\ &\leqq \left(\sum_{n=0}^{\infty} |v_{n+1}|\, \|\exp(kL_0)L_n\exp(-sL_0)\|\right)\|x\|_s.\end{aligned}$$

Note that

$$\exp(kL_0)L_n\exp(-sL_0) = e^{-sn}\exp[-(s-k)L_0]\,L_n, \qquad s>k,$$

which implies that

$$\left\|\sum_{n=0}^{\infty} v_{n+1}L_n x\right\|_k \leqq \left(\sum_{n=0}^{\infty} |v_{n+1}|e^{-sn}\|\exp[-(s-k)L_0]L_n\|\right)\|x\|_s. \tag{6.7}$$

We now define $A_n = \exp(-tL_0)L_n$ and study $\|A_n\|$. Using $A_n^+ = L_{-n}\exp(-tL_0)$, and the following formula:

$$A_nA_n^+ = e^{-2tn}A_n^+A_n + \exp(-2tL_0)\left(2nL_0 + \frac{c}{12}(n^3-n)\right),$$

we obtain

$$\|A_n\|^2 \leqq (1-e^{-2tn})^{-1}\left\|\exp(-2tL_0)\left(2nL_0 + \frac{c}{12}(n^3-n)\right)\right\|,$$

and

$$\|A_n\|^2 \leqq (1-e^{-2tn})^{-1}\left(\frac{n}{et} + \frac{c}{12}(n^3-n)e^{-2th}\right). \tag{6.8}$$

We see that for s sufficiently large the series

$$\sum_{n=0}^{\infty} |v_{n+1}|e^{-sn}\|\exp[-(s-k)L_0]L_n\|$$

is convergent. Hence the operator $\sum_{n=0}^{\infty} v_{n+1}L_n$ is bounded, as claimed. Using the Cauchy inequality and Eqs. (6.7), (6.8), we obtain

$$\left\| \sum_{n=0}^{\infty} v_{n+1}L_n x \right\|_k \leqq \mathrm{const}(k,s,r)\,\|v\|_r \|x\|_s . \tag{6.9}$$

The continuity of the representation follows. □

Now we are ready to integrate the representation of $\mathscr{g}$ to one of $\hat{\mathscr{G}}$. At first, however, we restrict our attention to the simpler case of a one-parameter subgroup of $\hat{\mathscr{G}}$. As was shown above, the group $\hat{\mathscr{G}}$ is spanned by such subgroups.

Let us consider an operator $\exp\left(\sum_{n=0}^{\infty} v_{n+1}L_n\right)$ acting on S. This operator is well defined at least on D, since for every $x \in D$ a sufficiently high power of L_n annihilates x. We find the following inequality:

$$\left\|\exp\left(\sum_{n=0}^{\infty} v_{n+1}L_n\right)x\right\|_k \leqq \left\|\exp(kL_0)\exp\left(\sum_{n=0}^{\infty} v_{n+1}L_n\right)\exp(-sL_0)\right\| \|x\|_s . \tag{6.10}$$

By virtue of $[L_n, L_0] = nL_0$ we have

$$\exp\left(\sum_{n=0}^{\infty} v_{n+1}L_n\right)\exp(-sL_0) = \exp(-sL_0)\exp\left(\sum_{n=0}^{\infty} v_{n+1}e^{-sn}L_n\right).$$

We now introduce $\varphi(t)$, $t \geqq 0$, by the formula

$$\varphi(t) = \exp[-t(s-k)L_0]\exp\left(t\sum_{n=0}^{\infty} v_{n+1}e^{-sn}L_n\right). \tag{6.11}$$

It satisfies the following linear differential equation:

$$\frac{d\varphi}{dt} = -A(t)\varphi(t), \quad \text{where for } p = s-k \geqq 0,$$
$$A(t) = pL_0 - \sum_{n=0}^{\infty} v_{n+1}e^{-np-nk+tnp}L_n . \tag{6.12}$$

From (6.10) we see that the boundedness of the operator $\exp\left(\sum_{n=0}^{\infty} v_{n+1}L_n\right)$ on the Fréchet space S depends crucially on the existence of the operator norm $\|\varphi(1)\|$, therefore it is necessary to study Eq. (6.12) more closely. To this end we will make use of the following two lemmas:

Lemma 6.2. *For s and k sufficiently large, and for all $t \in [0,1]$, the operator $A(t)$ is the generator of a semigroup of contractions.*

Proof. We shall verify the assumptions of Phillips' theorem (Theorem X.48, [10]), i.e.

a) $\frac{1}{2}(A(t) + A^*(t))$ is positive.

b) $\mathrm{Im}(\lambda + A(t)) = V(h,c)$ for a certain $\lambda > 0$.

It was proved in [16] that $L_nL_0^{-1}$ is a bounded operator with $\|L_nL_0^{-1}\|$ bounded by a certain polynomial in $|n|$.

To prove a) we decompose

$$\tfrac{1}{2}(A(t)+A^*(t))=(p-\operatorname{Re}v_1)L_0+B,$$

where

$$B=\tfrac{1}{2}\left(-\sum_{n=1}^{\infty} v_{n+1}e^{-np-nk+tnp}L_n+\text{herm. conj.}\right).$$

Now observe that

$$\|Bx\| \leqq \|B((p-\operatorname{Re}v_1)L_0)^{-1}\| \, \|(p-\operatorname{Re}v_1)L_0x\|,$$

where

$$\|B((p-\operatorname{Re}v_1)L_0)^{-1}\| \leqq 1$$

for p, k sufficiently large, by [16]. By virtue of the Kato-Rellich theorem (Theorem X.12, [10]) and of the positivity of L_0, $A(t)+A^*(t)$ is positive.

To prove b) we observe that for λ positive and sufficiently large the operator $\lambda+A(t)$ is invertible. In fact, decomposing as above and using [16] we can define $(\lambda+A(t))^{-1}$ by a suitable Neumann series. Thus, by virtue of Phillips' theorem, $A(t)$ is a generator of a semigroup of contractions. □

Lemma 6.3. *Let X be a Hilbert space, and $A(t)$ for $t\in[0,1]$ – the generator of a semigroup of contractions. Suppose that there exists a dense domain $S\subset X$, common to all $A(t)$ (which allows $A(t)A(s)^{-1}$ to be bounded). Let $C(t,s)=A(t)A(s)^{-1}-I$. Assume that $(t-s)^{-1}C(t,s)$ is uniformly continuous and uniformly bounded as a function of t and s (for $t\neq s$). Furthermore, assume also that the limit*

$$C(t)=\lim_{s\to t}(t-s)^{-1}C(t,s)$$

exists uniformly in t, and $C(t)$ thus defined is bounded and uniformly continuous.

Then there exists a unique solution to the equation

$$\frac{d\varphi}{dt}=-A(t)\varphi(t), \qquad \varphi(0)=\varphi_0.$$

Moreover, for every t, $\|\varphi(t)\|\leqq\|\varphi_0\|$.

The above lemma is a direct consequence of Kato's theorem (see e.g. [10]).

We are now ready to state the following

Theorem 6.4. *The representation $d\varrho$ of $\mathscr{g}$ in $V(c,h)$ is integrable along one-parameter subgroups in $\hat{\mathscr{G}}$, i.e.*

a) $\exp\left(\lambda\sum_{n=0}^{\infty}v_{n+1}L_n\right)$ *is a one-parameter group of bounded operators on S;*

b) *Let $v\in\mathscr{g}$ and let $\{\widehat{\exp}(\lambda v)\}$ be a one-parameter subgroup in $\hat{\mathscr{G}}$, generated by $v=\sum_{n=0}^{\infty}v_{n+1}\ell_n$. Then*

$$\{\widehat{\exp}(\lambda v)\}\ni f_\lambda\to\varrho(f_\lambda)=\exp\left(\lambda\sum_{n=0}^{\infty}v_{n+1}L_n\right) \tag{6.13}$$

is a continuous representation.

c) *The differential of the representation ϱ is $d\varrho$.*

Proof. We only need to prove a), since b) and c) then follow easily. To do this we return to Eq. (6.12) and check whether the assumptions of Lemma 6.3 are satisfied:

$$C(t,s)=A(t)A(s)^{-1}-I=(I+B(t))(I+B(s))^{-1}-I,$$

where

$$B(t)=\frac{-1}{p-v_1}\sum_{n=1}^{\infty} v_{n+1}e^{-np-nk+tnp}L_nL_0^{-1}. \tag{6.14}$$

We already observed in the proof of Lemma 6.2 that $B(t)$ is uniformly bounded for $t\in[0,1]$, and furthermore $\|B(t)\|<1$ for p and k sufficiently large. Since

$$\frac{C(t,s)}{t-s}=\frac{B(t)-B(s)}{t-s}(I+B(s))^{-1},$$

it is enough to show that $B(t)$ is uniformly continuous. Indeed,

$$\|B(t)-B(s)\|\leqq\frac{1}{|p-v_1|}\sum_{n=1}^{\infty}|v_{n+1}|e^{-nk}|t-s|\,\|L_nL_0^{-1}\|=\mathrm{const}\,|t-s|.$$

Note that the following inequality holds:

$$\left\|\frac{C(t,s)}{t-s}\right\|\leqq\left\|\frac{B(t)-B(s)}{t-s}\right\|\frac{1}{1-\|B(s)\|}\leqq\mathrm{const}.$$

Now let

$$C(t)=\frac{1}{p-v_1}\sum_{n=1}^{\infty} v_{n+1}pne^{-np-nk+tnp}L_nL_0^{-1}.$$

Then

$$\left\|\frac{C(t,s)}{t-s}-C(t)\right\|\leqq\frac{1}{|p-v_1|}\sum_{n=1}^{\infty}|v_{n+1}|\frac{p^2n^2}{2}e^{-nk}|t-s|\,\|L_nL_0^{-1}\|=\mathrm{const}\,|t-s|.$$

The existence of $C(t)$ as a limit as $s\to t$, uniform in t, follows. It is not difficult to check that $C(t)$ is bounded and uniformly continuous. Thus all assumptions of Lemma 6.3 hold and the solution of (6.12) obeys $\|\varphi(t)\|\leqq 1$ for $t\in[0,1]$. Hence by (6.10), $\exp\left(\sum_{n=0}^{\infty} v_{n+1}L_n\right)$ is a continuous operator on S. □

Two essential facts hold: first, that $d\varrho$ is integrable along one-parameter subgroups in $\hat{\mathscr{G}}$ (Theorem 6.4), and second, that $\hat{\mathscr{G}}$ is spanned by a family of one-parameter subgroups (Corollary 4.6); these allow us to define the representation ϱ of $\hat{\mathscr{G}}$. We will need some additional notation: denote the partial sum of a series $\Omega=\sum_{k=1}^{\infty}\omega_k z^k$ by $S_n(\Omega)\equiv\sum_{k=1}^{n}\omega_k z^k$. Now, let

$$\hat{\mathscr{G}}_n=\left\{(\lambda,f):f=\sum_{k=1}^{n} f_k z^k, f_1=e^{\lambda}\right\}.$$

The group structure in $\hat{\mathscr{G}}_n$ is given by

$$(\lambda,f)\circ(\mu,g)=(\lambda+\mu,S_n(f\circ g)).$$

It is clear that $(\hat{\mathcal{G}}_n, \circ)$ forms an n-dimensional, simply connected complex Lie group. Denote by $\hat{S}_n$ the following projection:

$$\hat{S}_n : \hat{\mathcal{G}} \to \hat{\mathcal{G}}_n , \qquad \hat{S}_n((\lambda, f)) = (\lambda, S_n(f)) .$$

It may be verified that $\hat{S}_n$ is a group homomorphism.

We clearly have the following commuting diagram [this may be checked by using recursive formulas for the coefficients of $\widehat{\exp}(v)$]:

$$\begin{tikzcd} \mathcal{g} \arrow[r, "S_n"] \arrow[d, "\widehat{\exp}"'] & \mathcal{g}_n \arrow[d, "\widehat{\exp}_n"] \\ \hat{\mathcal{G}} \arrow[r, "\hat{S}_n"] & \hat{\mathcal{G}}_n \end{tikzcd} \tag{6.15}$$

where $\mathcal{g}_n = \left\{ v : \sum_{k=1}^{n} v_k z^k (d/dz) \right\}$, and $\mathcal{g} \supset \mathcal{g}_n \ni v \mapsto \widehat{\exp}_n(v) = \hat{S}_n(\widehat{\exp}(v))$.

To see that ϱ is a representation, the following lemma is needed:

Lemma 6.5. *Let u, v, w be elements of $\mathcal{g}$ such that $\widehat{\exp}(u) \hat{\circ} \widehat{\exp}(v) = \widehat{\exp}(w)$. Then* $\exp(d\varrho(u)) \circ \exp(d\varrho(v)) = \exp(d\varrho(w))$.

Proof. Take $|\lambda\rangle$ to be an eigenvector of L_0 with eigenvalue λ. Then we can form

$$V_\lambda = \mathrm{span}(L_N^{n_N} \dots L_1^{n_1} |\lambda\rangle) .$$

Observe that:

– V_λ is a finite dimensional subspace in $S \subset V(h, c)$. Indeed, we have

$$L_0 L_N^{n_N} \dots L_1^{n_1} |\lambda\rangle = \left(\lambda - \sum_{k=1}^{N} k n_k \right) (L_N^{n_N} \dots L_1^{n_1} |\lambda\rangle) ,$$

and

$$L_N^{n_N} \dots L_1^{n_1} |\lambda\rangle = 0 \quad \text{if} \quad \sum_{k=1}^{N} k n_k > \lambda - h ,$$

since L_0 is bounded from below. Thus the number of nonzero vectors of the above type is finite.

– V_λ is an invariant space for $d\varrho$.

– $L_n|_{V_\lambda} = 0$ for sufficiently large n ($n \geqq \lambda - h + 1$).

The above observations imply that

$$d\varrho(v)|_{V_\lambda} = d\varrho(S_m(v))|_{V_\lambda} ,$$

for all $v \in \mathcal{g}$ and sufficiently large m.

Similarly, we have

$$\exp(d\varrho(v))|_{V_\lambda} = \exp[d\varrho(S_m(v))]|_{V_\lambda} .$$

The meaning of the above formulas may be expressed by saying that both ϱ and $d\varrho$ reduce on V_λ to representations of $\hat{\mathcal{G}}_m, \mathcal{g}_m$. We thus have a finite dimensional representation $d\varrho$ of the finite dimensional Lie algebra $\mathcal{g}_m$, which by a classical theorem can be integrated to the unique representation ϱ of the simply connected, finite dimensional Lie group $\hat{\mathcal{G}}_m$. Making use of this fact and of the diagram (6.15) we obtain

$$\exp(d\varrho(u))|_{V_\lambda} \circ \exp(d\varrho(v))|_{V_\lambda} = \exp(d\varrho(w))|_{V_\lambda} .$$

Moreover, for any $x \in D$,

$$\exp(d\varrho(u)) \circ \exp(d\varrho(v))x = \exp(d\varrho(w))x,$$

since D is spanned by all V_λ. But D is dense in S and $\exp(d\varrho(v))$ is continuous on S for each $v \in \mathscr{g}$, hence the lemma follows. $\square$

Corollary 6.6. *Let u_i, $i=1,\ldots,n$ be elements of $\mathscr{g}$ such that $\prod_{i=1}^{n} \widehat{\exp}(u_i) = \widehat{\exp}(w)$. Then*

$$\prod_{i=1}^{n} \exp(d\varrho(u_i)) = \exp(d\varrho(w)).$$

Now we are ready to present the definition of the representation ϱ.

Definition.

$$\varrho(\lambda, f) = \exp(d\varrho(v)) \circ \exp(d\varrho(u)),$$

where u, v are chosen such that $(\lambda, f) = \widehat{\exp}(v) \mathbin{\hat{\circ}} \widehat{\exp}(u)$.

To see that this definition is correct we must verify that $\varrho(\lambda, f)$ does not depend on the choice of u, v. Indeed, let

$$(\lambda, f) = \widehat{\exp}(v_1) \mathbin{\hat{\circ}} \widehat{\exp}(u_1) = \widehat{\exp}(v_2) \mathbin{\hat{\circ}} \widehat{\exp}(u_2),$$

i.e. $\widehat{\exp}(v_1) = \widehat{\exp}(v_2) \mathbin{\hat{\circ}} \widehat{\exp}(u_2) \mathbin{\hat{\circ}} \widehat{\exp}(-u_1)$. Then, by Corollary 6.6,

$$\exp(d\varrho(v_1)) = \exp(d\varrho(v_2)) \circ \exp(d\varrho(u_2)) \circ \exp(-d\varrho(u_1)),$$

and thus ϱ is well defined.

Now we present the main theorem of this section:

Theorem 6.7. *ϱ is the unique continuous representation of $\hat{\mathscr{G}}$ in the space of bounded operators on S with differential equal to $d\varrho$.*

Proof. The uniqueness of ϱ follows from Theorem 6.4. Corollary 6.6 implies that ϱ is indeed a representation, i.e.

$$\varrho((\lambda, f) \mathbin{\hat{\circ}} (\mu, g)) = \varrho((\lambda, f)) \circ \varrho((\mu, g)).$$

The continuity of ϱ on $\hat{\mathscr{G}}_{(1)}$ follows easily: by Theorem 4.5 and Proposition 6.1, ϱ is the composition of three continuous maps: $\varrho = \exp \circ d\varrho \circ \widehat{\exp}^{-1}$ (see also remarks following formula 5.6). However, as ϱ is a homomorphism of topological groups, it suffices that it be continuous at one point for continuity on all of $\hat{\mathscr{G}}$ to follow. $\square$

Now the following question may be posed: for what values of h, c, the parameters of $V(c, h)$, the representation ϱ of $\hat{\mathscr{G}}$ can be projected to $\mathscr{G}$, i.e.

$$\begin{tikzcd} \hat{\mathscr{G}} \arrow[r, "\varrho"] \arrow[d, "\pi"'] & B(S) \\ \mathscr{G} \arrow[ur, "?"'] & \end{tikzcd}.$$

Fortunately there exists a simple answer to this question, which is given by the theorem below.

Theorem 6.8. *ϱ can be projected to $\mathcal{G}$ if and only if h is a natural number.*

Proof. We need to show that ϱ is constant on the fibers of π, i.e. $\varrho(\lambda, f) = \varrho(\lambda + 2\pi i n, f)$, $n \in \mathbb{Z}$. There are two cases to consider:

a) $(\lambda, f) \in \hat{\mathcal{G}}_{(1)}$. Then

$$(\lambda, f) = (\mu, g) \,\hat{\circ}\, (\lambda, e^{\lambda} z) \,\hat{\circ}\, (-\mu, g^{-1}),$$

by Lemma 4.4. As we have the freedom to choose μ, we can take $\mathrm{Re}\,\mu \neq 0$. Then $(\mu, g) = \widehat{\exp}(v)$ for some $v \in \mathfrak{g}$, and

$$(\lambda, f) = \widehat{\exp}(\exp(\mathrm{Ad}_v)\lambda \ell_0).$$

Likewise,

$$(\lambda + 2\pi i n, f) = \widehat{\exp}(\exp(\mathrm{Ad}_v)(\lambda + 2\pi i n)\ell_0),$$

with the same $v \in \mathfrak{g}$.

These assertions imply that

$$\begin{aligned} \varrho(\lambda, f) &= \exp\{d\varrho[\exp(\mathrm{Ad}_v)\lambda \ell_0]\} = \exp\{\exp[\mathrm{Ad}_{d\varrho(v)}]\lambda L_0\} \\ &= \exp(d\varrho(v)) \circ \exp(\lambda L_0) \circ \exp(-d\varrho(v)). \end{aligned} \tag{6.16}$$

Similarly,

$$\varrho(\lambda + 2\pi i n, f) = \exp(d\varrho(v)) \circ \exp((\lambda + 2\pi i n)L_0) \circ \exp(-d\varrho(v)). \tag{6.17}$$

Now, (6.16) is equal to (6.17) if and only if

$$\exp(2\pi i n L_0) = \mathrm{id} \quad \text{for all } n \in \mathbb{Z}.$$

In other words, the spectrum of L_0 may consist of natural numbers only. This is the case if and only if h is natural.

b) $(\lambda, f) \notin \hat{\mathcal{G}}_{(1)}$, then (λ, f) is the composition of two elements of $\hat{\mathcal{G}}_{(1)}$. The above argument may now be applied to these elements, and hence to (λ, f) itself. □

7. Additional Comments

We would like to conclude this paper with remarks on some open questions which we have not touched upon:

a) Observe that $\mathcal{G}$ has many normal subgroups – these are the kernels of the homomorphisms $S_n : \mathcal{G} \to \mathcal{G}_n$. It seems likely that these are, up to conjugation, its only nontrivial normal subgroups. We expect a direct correspondence to hold between the (closed) ideals of $\mathfrak{g}$ and the normal subgroups of $\mathcal{G}$, via the exponential map (cf. [7], where the analogous problem is considered for groups of diffeomorphisms);

b) As is known, in complex groups the Borel (maximal solvable) subgroups take over the role that maximal tori play in real groups. We expect that the Borel subgroups of $\mathcal{G}$ coincide with one-parameter subgroups. In particular, a natural conjecture is that $\mathcal{G}$ is semisimple.

c) The representation of $\mathcal{G}$ in $V(c, h)$ is reducible (a fact which was exploited in our construction), but it is not likely to be fully reducible in S, in the sense of the existence of a direct sum decomposition into invariant closed subspaces.

d) We leave it to the reader to verify the formula

$$\varrho(f)T(z)\varrho^{-1}(f)=(df/dz)^2T(f(z))+(c/12)\{f,z\},$$

(see [9]), which was *not* used in the present paper.

e) It is usual to call a topological group a Lie group if it is spanned by its one-parameter subgroups (this may be called the algebraic point of view), or if it is equipped with a smooth structure compatible with the group structure (when the geometrical aspect is stressed). In the former sense, we have shown $\mathscr{G}$ to be in fact a Lie group. It seems possible to satisfy also the second definition (see [7, 11]).

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