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Existence, Uniqueness and Cohomology of the Classical BRST Charge with Ghosts of Ghosts

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Abstract. A complete canonical formulation of the BRST theory of systems with redundant gauge symmetries is presented. These systems include p-form gauge fields, the superparticle, and the superstring. We first define the Koszul–Tate differential and explicitly show how the introduction of the momenta conjugate to the ghosts of ghosts makes it acyclic. The global existence of the BRST generator is then demonstrated, and the BRST charge is proved to be unique up to canonical transformations in the extended phase space, which includes the ghosts. Finally, the BRST cohomology in classical mechanics is investigated and shown to be equal to the cohomology of the exterior derivative along the gauge orbits, as in the irreducible case. This is done by re-expressing the exterior algebra along the gauge orbits as a free differential algebra containing generators of higher degree, which are identified with the ghosts of ghosts. The quantum cohomology is not dealt with.

I. Introduction

The most transparent and useful formulation of a field theory appears to be one in which locality is manifest. This also makes relativistic invariance manifest, since locality ensures that signals do not propagate at infinite speed.

It appears in practice that, for systems of physical interest, manifest locality can be maintained only at the price of formulating the theory in terms of more variables than what would naively seem to be necessary. One then obtains what is called a gauge theory.

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Since in a gauge theory there are redundant variables, one finds that in the Hamiltonian formulation, there are relations among the canonical variables, called constraint equations. This is a familiar occurrence in, for example, Yang–Mills theory and Einstein's theory of gravitation.

Moreover, it has been realized that the most natural way to describe a gauge theory is to further enlarge the space on which the theory is defined by bringing in new variables, called "ghosts," that do not appear in the original action principle. Actually, locality is again a key point here, since the ghosts make it possible to maintain that property in the quantum theory.

The formulation of gauge theories in terms of ghosts constitutes the subject of BRST theory.

The notion that "gauge invariance" is ultimately tied with "redundancy" is taken even further in a more general level of gauge theory, called "reducible," which has played an increasingly important role. These are gauge theories in which not only the variables are mutually dependent, but also the relations among them are themselves mutually dependent. Again, the reason for this new type of redundancy is that it leads to the most transparent form of the theory, namely, that one which makes locality and relativistic invariance manifest. Systems of this kind include p-form gauge fields, the superparticle, the Green Schwarz superstring, and string field theories.

We discuss in this article the BRST formulation of reducible gauge theories. Its main novel feature—as compared with the irreducible case—is the necessity of enlarging now the ghost spectrum, by introducing ghosts of ghosts.

One thus faces yet another example of what appears to be a general principle: theories take their simplest form when one enlarges sufficiently the number of functions used to describe them.

The general method for handling arbitrary reducible theories, with closed or "open" algebras, has been given by Batalin and Fradkin [1, 2] and is reviewed in [3]. This method is based on the canonical formalism and guarantees the existence of an off-shell nilpotent BRST symmetry in a sufficiently small region of phase space. It also shows that the BRST generator involves, in general, ghost powers of order higher than three. Alternative approaches based on the Lagrangian have also been developed [4]. However, the global structure of the equations determining the BRST generator, and its uniqueness, have not been studied as thoroughly as in the irreducible case. In particular, the cohomology has not been dealt with.

This article is devoted to settle those two issues. The first one is to show that the BRST charge exists globally and is unique up to canonical transformations. The second is to establish that the BRST cohomology in classical mechanics is isomorphic with that of the exterior derivative along the gauge orbits on the constraint surface.

The key tool which enables us to derive these results is the Koszul–Tate differential operator δ[5, 6], which generalizes the operator δ_2 of refs. [3, 7]. In the irreducible case, δ_2 is acyclic [5, 3, 7], but this is no longer so in the reducible case. However, following a method of Tate's [6], acyclicity can be recovered by adding extra variables, which are, in the present context, the momenta conjugate to the extra ghosts. The extra ghost momenta can thus be understood from a cohomological point of view; they are the weapon that kills unwanted nontrivial cycles of the operator δ_2.

The study of the cohomology also sheds some light on the geometrical meaning of the ghosts of ghosts. Indeed, the analysis of the irreducible case [8] already showed that the ghosts can be considered as 1-forms along the gauge orbits. In the reducible case one must bring, in addition, new forms of higher rank. They are necessary to express the exterior derivative along the gauge orbits in a form that is free from conditions on the generators of the algebra. One arrives in this way to a realization of the concept of differential free algebra or, as one also says, "free differential algebra (FDA)". The extra forms in the free exterior differential algebra along the gauge orbits are the ghosts of ghosts.

The paper is organized as follows. In Sect. II, which is next, we briefly review the canonical description of reducible theories. We then construct in Sect. III the Koszul–Tate operator and explicitly establish that the extra ghosts make it acyclic. This result is used in Sect. IV to prove both the global existence of the BRST generator and its uniqueness up to canonical transformations.

Section V deals with the classical BRST cohomology and establishes its equivalence with the cohomology of the exterior derivative along the gauge orbits. Lastly, Sect. VI, is devoted to concluding remarks.

The present article may be thought of as an extension of the treatment given for the irreducible case in ref. [8].

II. Canonical Description of Reducible Theories

1. Reducibility and Order of Reducibility. In the Hamiltonian formalism, the gauge transformations are canonical transformations whose generators G_{a_0}, which will be assumed to be real and of definite Grassmann parity, are phase space functions that are constrained to vanish [9],

$$G_{a_0}=0, \quad a_0=1,\ldots,m_0. \tag{2.1}$$

These constraints are first class in Dirac's terminology,

$$[G_{a_0}, G_{b_0}]=C^{c_0}_{a_0 b_0} G_{c_0}. \tag{2.2}$$

The gauge transformations read

$$\delta F=[F, \varepsilon^{a_0} G_{a_0}], \tag{2.3}$$

and close on the constraint surface (2.1). Here, ε^{a_0} are arbitrary phase space functions, whereas [,] stands for the graded Poisson bracket. By closure of the gauge transformations on the surface (2.1), one means that the Hamiltonian vector fields X_{a_0} associated with the constraint-generators G_{a_0},

$$X_{a_0} F \equiv [F, G_{a_0}], \tag{2.4a}$$

close with respect to the Lie bracket on the constraint surface (2.1),

$$[X_{a_0}, X_{b_0}] \approx X_{c_0} C^{c_0}_{a_0 b_0} (-)^{\varepsilon_{c_0}+\varepsilon_{c_0}(\varepsilon_{a_0}+\varepsilon_{b_0})} \tag{2.4b}$$

(see Appendix for conventions on differential calculus on a supermanifold).

Now in the reducible case, the gauge transformations (2.3) are not all independent. In other words, the vector fields X_{a_0} are not linearly independent on the constraint surface; there exist functions $Z^{a_0}_{a_1} \not\approx 0$ such that

$$(-)^{\varepsilon_{a_0}+\varepsilon_{a_1}} Z^{a_0}_{a_1} X_{a_0} \approx 0, \quad a_1 = 1, \ldots, m_1, \tag{2.5}$$

where $\approx$ means "equality up to the constraints." Here and in the sequel, independence or dependence of vector fields will always be with respect to functions.

By appropriately extending $Z^{a_0}_{a_1}$ off $G_{a_0} = 0$, one can replace (2.5) by the equivalent equations[1]

$$Z^{a_0}_{a_1} G_{a_0} = 0, \quad a_1 = 1, \ldots, m_1. \tag{2.6}$$

Thus, one sees that in the reducible case the constraint surface is not of dimension $n - m_0$, where n is the dimension of phase space. Rather, it is of dimension $n - m_0'$, where $m_0' < m_0$ is the number of independent constraints G_{a_0}. At the same time, the dimension of the gauge orbits generated by the Hamiltonian vector fields X_{a_0} associated with G_{a_0} on $G_{a_0} = 0$ is not of dimension m_0 but of dimension $m_0' < m_0$.

We will assume throughout this paper that the constraint functions G_{a_0} can be locally separated into "independent constraint functions" G_{A_0}, $A_0 = 1, \ldots, m_0'$, and "dependent constraint functions" G_{α_0}, $\alpha_0 = m_0' + 1, \ldots, m_0$, which are expressible in terms of G_{A_0} by (2.6) as

$$G_{\alpha_0} = N^{A_0}_{\alpha_0} G_{A_0},$$

with coefficients $N^{A_0}_{\alpha_0}$ that are regular in the vicinity of $G_{a_0} = 0$. The matrix of the gradients dG_{A_0} of the independent functions should be of maximum rank on $G_{a_0} = 0$, i.e., one should be able to take locally G_{A_0} as the first m_0' coordinates of a regular (possibly noncanonical) coordinate system in phase space. For the physical applications that have been met so far, this assumption always holds, or can be made to hold by redefinition of the constraints.

Although we postulate that the constraints can be separated into two groups as above, we by no means assume that this can be done globally with a single set of independent constraints G_{A_0}. It may instead be necessary to cover phase space with more than one open set, with a different set of independent constraints for each open set. This is because there may be global obstructions to the existence of a single set G_{A_0}. These obstructions are related to the topology of the normal bundle over the constraint surface.

Furthermore, even when the splitting $G_{a_0} \to (G_{A_0}, G_{\alpha_0})$ has global validity, it spoils in practice manifest Lorentz invariance or locality in physical space. For these reasons, it is appropriate to construct the BRST formalism with the full set of constraints G_{a_0} without assuming that any definite splitting has been performed.

The coefficients $Z^{a_0}_{a_1}$ in (2.5) and (2.6) are required to exhaust all existing relations among the vector fields X_{a_0}, i.e.,

$$\lambda^{a_0} G_{a_0} = 0 \Rightarrow \lambda^{a_0} = \lambda^{a_1} Z^{a_0}_{a_1} + \nu^{a_0 b_0} G_{b_0} (-)^{\varepsilon_{b_0}}, \tag{2.7}$$

[1] For an arbitrary extension, the Taylor expansion of $Z^{a_0}_{a_1} G_{a_0}$ around the constraint surface must start at second order in order to reproduce (2.5); that is, one must have $Z^{a_0}_{a_1} G_{a_0} = S^{b_0 a_0}_{a_1}(q, p) G_{b_0} G_{a_0}$. Then, by redefining $Z^{a_0}_{a_1} \to Z^{a_0}_{a_1} - S^{b_0 a_0}_{a_1} G_{b_0}$, one arrives at (2.6)

where $\nu^{a_0 b_0}$ can be assumed to be "antisymmetric" in the sense of [1–3],

$$\nu^{a_0 b_0} = (-)^{(\varepsilon_{a_0}+1)(\varepsilon_{b_0}+1)} \nu^{b_0 a_0}. \tag{2.8}$$

It may turn out that the relations (2.6) are furthermore independent, namely, $\lambda^{a_1} Z^{a_0}_{a_1} \approx 0 \Rightarrow \lambda^{a_1} \approx 0$. In that case, the number m'_0 of independent constraints is equal to $m_0 - m_1$. But it may very well be that the system $Z^{a_0}_{a_1}$ is overcomplete, so that there exist functions λ^{a_1} that do not vanish even weakly and have the property $\lambda^{a_1} Z^{a_0}_{a_1} \approx 0$. Then, the number of independent gauge generators is equal to $m_0 - m'_1$, where m'_1 is the number of independent relations (2.6) and is greater than $m_0 - m_1$. The consideration of an overcomplete set of $Z^{a_0}_{a_1}$ may again be desirable on the grounds of preserving manifest Lorentz invariance or locality in physical space, or may be mandated by topological obstructions.

Taking a complete set $Z^{a_1}_{a_2}$ of functions which (weakly) annihilate $Z^{a_0}_{a_1}$, one finds

$$Z^{a_1}_{a_2} Z^{a_0}_{a_1} = C^{a_0 b_0}_{a_2} G_{b_0} (-)^{\varepsilon_{a_0}}, \quad a_2 = 1, \ldots, m_2, \tag{2.9}$$

$$\lambda^{a_1} Z^{a_0}_{a_1} \approx 0 \Rightarrow \lambda^{a_1} = \lambda^{a_2} Z^{a_1}_{a_2} + \nu^{a_1 a_0} G_{a_0}, \tag{2.10}$$

where, without loss of generality, one can assume

$$C^{a_0 b_0}_{a_2} = (-)^{(\varepsilon_{a_0}+1)(\varepsilon_{b_0}+1)} C^{b_0 a_0}_{a_2}, \tag{2.11}$$

since only the "symmetric part" of $C^{b_0 a_0}_{a_2}$ contributes to $Z^{a_1}_{a_2} Z^{a_0}_{a_1} G_{a_0}$.

Even though complete, the set of coefficient functions $Z^{a_1}_{a_2}$ may itself be overcomplete, and so on. This leads to a tower of reducibility equations

$$Z^{a_{k-1}}_{a_k} Z^{a_{k-2}}_{a_{k-1}} = (-)^{\varepsilon_{a_{k-2}}} C^{a_{k-2} a_0}_{a_k} G_{a_0}, \quad k = 1, \ldots, L, \quad a_k = 1, \ldots, m_k, \tag{2.12}$$

where we have set $Z^{a_{-1}}_{a_0} = G_{a_0}$. The order of reducibility L is defined as the last k for which $Z^{a_{k-1}}_{a_k} \neq 0$, so an irreducible theory has $L = 0$.

We will allow L to be arbitrarily large or even infinite (this happens in some treatments of the superparticle [10]). At each stage, the $Z^{a_{k-1}}_{a_k}$, $k \leqq L$ provide a (over-)complete set of "reducibility functions," i.e.,

$$\lambda^{a_{k-1}} Z^{a_{k-2}}_{a_{k-1}} \approx 0 \Rightarrow \lambda^{a_{k-1}} = \lambda^{a_k} Z^{a_{k-1}}_{a_k} + \nu^{a_{k-1} a_0} G_{a_0}, \tag{2.13}$$

for finite order of reducibility L, this equation implies

$$\lambda^{a_L} Z^{a_{L-1}}_{a_L} \approx 0 \Rightarrow \lambda^{a_L} = \nu^{a_L a_0} G_{a_0} \tag{2.14}$$

for the last coefficients $Z^{a_{L-1}}_{a_L}$.

The number of independent gauge generators is given by $m_0 - m_1 + m_2 - m_3 + \cdots + (-)^k m'_k$, where m'_k is the number of independent $Z^{a_{k-1}}_{a_k}$. For theories of finite order, this sum is equal to $\sum_{i=0}^{L} (-)^i m_i$.

Since the G_a are of definite Grassmann parity, the Z's may be assumed to have the same property. We recursively define ε_{a_k} by

$$\varepsilon(Z^{a_{k-1}}_{a_k}) = \varepsilon_{a_k} + \varepsilon_{a_{k-1}} \tag{2.15a}$$

with $\varepsilon_{a_0} = \varepsilon(G_{a_0})$. Similarly, since the constraints are real, we may demand

$$(Z^{a_{k-1}}_{a_k})^* = (-)^{k + k(\varepsilon_{a_k} + \varepsilon_{a_{k-1}}) + \varepsilon_{a_k} \varepsilon_{a_{k-1}}} Z^{a_{k-1}}_{a_k}. \tag{2.15b}$$

2. *Ambiguity in the Reducibility Functions.* Given the constraint surface, the functions G_{a_0} are only determined up to

$$G_{a_0} \to \bar{G}_{a_0} = M^{b_0}_{a_0} G_{b_0}. \tag{2.16a}$$

For given $\bar{G}_{a_0}, G_{b_0}$, the matrix $M^{b_0}_{a_0}$ in (2.16a) is defined by that equation up to linear combinations $k^{a_1}_{a_0} Z^{b_0}_{a_1} + n^{c_0 b_0}_{a_0} G_{c_0}(-)^{c_0}$. In order for G_{a_0} and $\bar{G}_{a_0}$ to be equivalent one must be able to choose the linear combination so that

$$M^{a_0}_{b_0} \text{ is invertible.} \tag{2.16b}$$

Given G_{a_0}, the reducibility functions of the first stage $Z^{a_0}_{a_1}$ are, in turn, only determined up to

$$Z^{a_0}_{a_1} \to \bar{Z}^{a_0}_{a_1} = M^{b_1}_{a_1} Z^{a_0}_{b_1} + v^{a_0 b_0}_{a_1} G_{b_0}(-)^{\varepsilon_{b_0}}, \tag{2.17a}$$

with

$$v^{a_0 b_0}_{a_1} = (-)^{(\varepsilon_{a_0}+1)(\varepsilon_{b_0}+1)} v^{b_0 a_0}_{a_1}, \quad M^{b_1}_{a_1} \text{ invertible.} \tag{2.17b}$$

Similarly, if the reducibility functions of order $Z^{a_{s-1}}_{a_s}$, $s \leqq k-1$ are given, the $Z^{a_{k-1}}_{a_k}$ are determined up to

$$Z^{a_{k-1}}_{a_k} \to \bar{Z}^{a_{k-1}}_{a_k} = M^{b_k}_{a_k} Z^{a_{k-1}}_{b_k} + v^{a_{k-1} b_0}_{a_k} G_{b_0}, \tag{2.18}$$

with

$$M^{b_k}_{a_k} \text{ invertible.}$$

Under the redefinitions (2.16)–(2.18), the functions $C^{a_{k-1} a_0}_{a_k}$ transform in a manner that can be straightforwardly computed. That computation will not be given here, since it is rather messy and not particularly useful. Furthermore, even for fixed $Z^{a_{k-1}}_{a_k}$, the C's carry some ambiguity, which will be analyzed later on.

3. *Local Canonical Form.* In a sufficiently small region of phase space, one can take advantage of the freedom (2.16)–(2.18) to bring the constraints and the functions $Z^{a_{k-1}}_{a_k}$ to an extremely simple form. Indeed, we have assumed that the constraints G_{a_0} can be split into independent and dependent ones. The dependent constraints G_{α_0} can be expressed in terms of the independent ones G_{A_0} as $G_{\alpha_0} = N^{A_0}_{\alpha_0} G_{A_0}$. Redefining the dependent constraints as $\bar{G}_{\alpha_0} = G_{\alpha_0} - N^{A_0}_{\alpha_0} G_{A_0}$, one gets $\bar{G}_{\alpha_0} = 0$ (identically). Hence, one can take

$$G_{a_0} = (G_{A_0}, G_{\alpha_0} = 0). \tag{2.19}$$

An independent set of reducibility functions of first order is then given by

$$Z^{a_0}_{A_1} = (0, \delta^{\alpha_0}_{A_1}), \tag{2.20}$$

where A_1 ranges over the same values as α_0 does. The remaining (dependent) reducibility functions of first order can be taken to vanish,

$$Z^{a_0}_{\alpha_1} = 0. \tag{2.21}$$

Going on in the same fashion, one sees that, at each stage, the $Z^{a_{k-1}}_{a_k}$ can be split into independent ($Z^{a_{k-1}}_{A_k}$) and dependent ($Z^{a_{k-1}}_{\alpha_k}$) functions. The range of the index A_k of the independent $Z^{a_{k-1}}_{A_k}$ is equal to the range of the index α_{k-1}

of the dependent functions $Z^{a_{k-2}}_{\alpha_{k-1}}$ of the previous stage, and one has

$$Z^{a_{k-1}}_{A_k} = (0, \delta^{\alpha_{k-1}}_{A_k}), \tag{2.22a}$$

$$Z^{a_{k-1}}_{\alpha_k} = 0. \tag{2.22b}$$

III. The Koszul–Tate Operator δ

1. Review of Results in the Irreducible Case. An important ingredient of the BRST construction in the irreducible case is the Koszul differential operator δ [5]. This operator arises just because the constraints define a surface in phase space. Its definition does not even rely on the first class property and is as follows. For each constraint G_{a_0}, one introduces a variable $\mathscr{P}_{a_0}$ of opposite Grassmann parity and of ghost number -1,

$$\varepsilon(\mathscr{P}_{a_0}) = \varepsilon_{a_0} + 1 \tag{3.1a}$$

$$\operatorname{antigh}(\mathscr{P}_{a_0}) = +1 = -\operatorname{gh}(\mathscr{P}_{a_0}). \tag{3.1b}$$

This variable will turn out to be the momentum conjugate to the ghost η^{a_0}.

The operator δ acts as a derivation on polynomials in $\mathscr{P}_{a_0}$, with coefficients that are phase space functions, and decreases the antighost number and Grassmann parity by one unit. Its defining properties are

$$\delta\mathscr{P}_{a_0} = G_{a_0}, \tag{3.2a}$$

$$\delta(\text{canonical variables}) = 0, \tag{3.2b}$$

$$\delta(AB) = A\delta B + (-)^{\varepsilon_B}\delta A B. \tag{3.2c}$$

Furthermore, when the ghosts η are introduced below, we will set

$$\delta\eta = 0. \tag{3.2d}$$

In (3.2c), we have taken δ to act as a right derivation in order to conform to the standard Hamiltonian conventions where the BRST generator Ω acts also from the right in the Poisson bracket.

A crucial feature of the Koszul operator δ is its nilpotency, $\delta^2 = 0$. This follows from the fact that δ changes the Grassmann parity. Indeed, one has, for any polynomial of arbitrary order p,

$$\delta(M^{a_0 b_0 \cdots c_0}\mathscr{P}_{c_0}\cdots\mathscr{P}_{b_0}\mathscr{P}_{a_0}) = pM^{a_0 b_0 \cdots c_0}\mathscr{P}_{c_0}\cdots\mathscr{P}_{b_0}G_{a_0}, \tag{3.3a}$$

and thus

$$\delta^2(M^{a_0 b_0 \cdots c_0}\mathscr{P}_{c_0}\cdots\mathscr{P}_{b_0}\mathscr{P}_{a_0}) = p(p-1)M^{a_0 b_0 \cdots c_0}\mathscr{P}_{c_0}\cdots G_{b_0}G_{a_0}(-)^{\varepsilon_{a_0}} = 0. \tag{3.3b}$$

The "components" $M^{a_0 \cdots c_0}$ can be assumed to possess the same symmetry properties as the product $\mathscr{P}_{c_0}\cdots\mathscr{P}_{a_0}$. The rule (3.2c) automatically brings in the appropriate phase $(-)^{\varepsilon_{a_0}}$ in (3.3) so as to have $\delta^2 = 0$. Note that the components of δM are directly related to the operators δ_2 and δ'_2 of ref. [3], which are thus unified with the present notations in a single operator.

It has been shown in [5, 3] that δ is acyclic in the irreducible case except for ghost number 0 (see also [7]) for the pure bosonic case). This means that if $\delta M = 0$, then $M = \delta N$ with antigh $N = \text{antigh}\, M + 1$, provided antigh $M \geqq 1$. If gh $M = 0$, then M is a phase space function and $\delta M = 0$. But M can be written as δN only if it vanishes on the constraint surface, the space $(\text{Ker}\,\delta/\text{Im}\,\delta)^\circ$ is thus isomorphic to the quotient space of the ring of phase space functions modulo the ideal of phase space functions vanishing on $G_a = 0$. Elements of $(\text{Ker}\,\delta/\text{Im}\,\delta)^\circ$ are functions defined on the constraint surface.[2]

2. Nontrivial Cycles in the Reducible Case and How to Kill Them. When the constraints are reducible, the cohomology of δ at antighost number zero is the same as in the irreducible case. But new features emerge at antighost number 1 with the appearance of nontrivial cycles. For instance, the combinations $Z^{a_0}_{a_1}\mathscr{P}_{a_0}$ are closed,

$$\delta(Z^{a_0}_{a_1}\mathscr{P}_{a_0}) = Z^{a_0}_{a_1}G_{a_0} = 0, \tag{3.5a}$$

but they cannot be exact

$$Z^{a_0}_{a_1}\mathscr{P}_{a_0} \neq \delta N, \tag{3.5b}$$

for this would imply $Z^{a_0}_{a_1}\mathscr{P}_{a_0} \approx 0$, which, in turn, requires $Z^{a_0}_{a_1} \approx 0$; but this would mean that $Z^{a_0}_{a_1}$ is of the trivial form $C^{a_0 b_0}_{a_1} G_{b_0}$ and could thus be taken to vanish identically.

Now, the acyclicity of δ at antighost number $\geqq 1$ is crucial for the existence of the BRST generator. This was already encountered in [3] for the irreducible case and will also appear hereafter for reducible theories. Therefore, it is necessary to find a way to recover the acyclicity of δ in the reducible case.

The method by which this can be done was devised by Tate [6]. One adds extra variables, one for each element of a basis of nontrivial cycles, and extends δ so that the previous nontrivial cycles become trivial. We will follow this method here, with the slight modification that we will introduce more new variables than independent cycles when the $Z^{a_0}_{a_1}$ are overcomplete. To be specific, we introduce as many new $\mathscr{P}_{a_1}$ as there are $Z^{a_0}_{a_1}$.

The way to kill the cycles $Z^{a_0}_{a_1}\mathscr{P}_{a_0}$ is simply to set

$$\delta\mathscr{P}_{a_1} = Z^{a_0}_{a_1}\mathscr{P}_{a_0}. \tag{3.6}$$

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[2] It is possible to define an operator ε acting on phase space functions and yielding the corresponding elements of $(\text{Ker}\,\delta/\text{Im}\,\delta)^\circ$. Element of $(\text{Ker}\,\delta/\text{Im}\,\delta)^\circ$ are then mapped by ε on zero. One thus has

$$\begin{array}{ccccc} \xrightarrow{\delta} \overset{(1)}{E} & \xrightarrow{\delta} & \overset{(0)}{E} & \xrightarrow{\delta} & 0 \\ & & \downarrow{\scriptstyle\varepsilon} & & \\ & & \left(\dfrac{\text{Ker}\,\delta}{\text{Im}\,\delta}\right)^\circ & \xrightarrow{\varepsilon} & 0, \end{array} \tag{3.4}$$

and it is easy to check that $\varepsilon\delta = \varepsilon^2 = 0$.

The replacement of δ by ε in the last step of (3.4) yields an acyclic operator, even at order zero, which was actually the operator δ_2 considered in [3]. However, the introduction of this operator requires new objects (functions on the constraint surface) belonging to a new complex. For that reason, it is more economical to use the unmodified δ, as will be done in the sequel

By construction, $\delta^2\mathscr{P}_{a_1}=0$. In order to preserve the grading properties of δ, one also demands

$$\varepsilon(\mathscr{P}_{a_1})=\varepsilon_{a_1}=\varepsilon(Z^{a_0}_{a_1}\mathscr{P}_{a_0})+1, \tag{3.7a}$$

$$\operatorname{antigh}(\mathscr{P}_{a_1})=2=-\operatorname{gh}(\mathscr{P}_{a_1}), \tag{3.7b}$$

where ε_{a_1} is given by (2.15a). One then extends δ as a derivation and one easily checks that the parity assignment (3.7a) ensures that $\delta^2=0$ on any polynomial in $\mathscr{P}_{a_0}, \mathscr{P}_{a_1}$.

With the extension (3.6) of δ to $\mathscr{P}_{a_1}$, it is easy to see that all the homology has been killed at antighost number $+1$: if $\delta M=0$, antigh $M=+1$, then $M=\delta N$ with antigh $N=+2$. This simply follows from the relation (2.7) expressing the completeness of the first order reducibility functions $Z^{a_0}_{a_1}$.

Turn now to the homology at antighost number $+2$. Here again, one finds nontrivial cycles. These are given by

$$Z^{a_1}_{a_2}\mathscr{P}_{a_1}+\tfrac{1}{2}C^{a_0b_0}_{a_2}\mathscr{P}_{b_0}\mathscr{P}_{a_0}. \tag{3.8}$$

The polynomial (3.8) is clearly annihilated by δ as a result of (2.9) but may not be written as δN, for the coefficient of $\mathscr{P}_{a_1}$ in (3.8) should then vanish on the constraint surface—which would again mean that the $Z^{a_1}_{a_2}$ can be taken to be strongly zero by an appropriate redefinition.

One thus introduces further new variables $\mathscr{P}_{a_2}$, with the properties

$$\delta\mathscr{P}_{a_2}=Z^{a_1}_{a_2}\mathscr{P}_{a_1}+\tfrac{1}{2}C^{a_0b_0}_{a_2}\mathscr{P}_{b_0}\mathscr{P}_{a_0}, \tag{3.9a}$$

$$\varepsilon(\mathscr{P}_{a_2})=\varepsilon_{a_2}+1=\varepsilon(\delta\mathscr{P}_{a_2})+1, \tag{3.9b}$$

$$\operatorname{antigh}(\mathscr{P}_{a_2})=3=-\operatorname{gh}(\mathscr{P}_{a_2}), \tag{3.9c}$$

where ε_{a_2} is given by (2.15a). The extension of δ as a graded derivation is again nilpotent. We will show in the next subsection that the introduction of $\mathscr{P}_{a_2}$ is enough to completely kill the homology at antighost number 2 (other than (3.8), no nontrivial cycles exist). This results from (2.13).

The process continues along the same lines at higher antighost numbers. One finds nontrivial cycles of the form

$$Z^{a_{k-1}}_{a_k}\mathscr{P}_{a_{k-1}}+M_{a_k}, \tag{3.10}$$

where M_{a_k} only involves $\mathscr{P}_{a_u}$, $u\leqq k-2$. The explicit expression of M_{a_k} is tedious, except when $C^{a_{s-2}a_0}_{a_s}=0$ for all s, in which case $M_{a_k}=0$. We will give a recursive proof of the existence of M_{a_k} below.

In order to kill these cycles, one introduces m_k variables $\mathscr{P}_{a_k}$, such that

$$\delta\mathscr{P}_{a_k}=Z^{a_{k-1}}_{a_k}\mathscr{P}_{a_{k-1}}+M_{a_k}, \tag{3.11a}$$

$$\operatorname{antigh}(\mathscr{P}_{a_k})=k+1, \tag{3.11b}$$

$$\varepsilon(\mathscr{P}_{a_k})=\varepsilon(\delta\mathscr{P}_{a_k})+1=\varepsilon_{a_k}+k+1 \tag{3.11c}$$

with ε_{a_k} defined by (2.15a). These definitions of the derivation δ preserve nilpotency because δ again changes the Grassmann parity. As we now show, they also make δ acyclic: there are no nontrivial cycles other than (3.10).

3. Explicit Proofs. It has been shown in [6] that the process of adding extra variables ultimately leads to an acyclic operator, provided one identifies correctly all the nontrivial cycles. To implement the procedure here, we will proceed recursively. First, we will assume that the action of δ has been constructed up to the k^{th} ghost momentum ("order k"), and we will demonstrate that it is acyclic in the space of polynomials in the momenta $\mathscr{P}_{a_i}$ $(i < k)$ with coefficients that are phase space functions (Theorems 1 and 2 below). We will then show that δ can be extended to order $k+1$ (Theorem 3), thus completing the proof of acyclicity. We will also observe that the extension of δ is unique up to a δ-exact polynomial (Theorem 4). The techniques used in the demonstration of theorems 1–4 are close to standard methods in homological algebra.

3.a. δ Taken as Given to Order k. Suppose that we have built the nilpotent operator δ up to order k,

$$\begin{aligned} \delta\mathscr{P}_{a_0} &= G_{a_0} \\ &\vdots \\ \delta\mathscr{P}_{a_k} &= Z^{a_{k-1}}_{a_k}\mathscr{P}_{a_{k-1}} + M_{a_k}, \end{aligned} \tag{3.12}$$

where M_{a_s} only involves $\mathscr{P}_{a_i}$, $i \leqq s-2$. The space of all polynomials in $\mathscr{P}_{a_0}, \mathscr{P}_{a_1}, \ldots$ up to $\mathscr{P}_{a_k}$, upon which δ acts as a graded derivation, will be called the k^{th} level complex and denoted by K_k. The action of δ on K_1 and K_2 is determined by (3.6) and (3.9a), respectively.

We first show that any δ-closed polynomial in K_i is δ-exact in K_{i+1} for each $i < k$. This property will be proved in two steps. First, we will demonstrate it in an open set where one can split the constraints G_{a_0} and the reducibility functions $Z^{a_{i-1}}_{a_i}$ into independent and dependent ones. (Any split will do; one does not need here the canonical form of Sect. 2.3.) Second, we will show that there is no global subtlety in patching together the various open sets on which the property has been proved.

So, start with an open set where one can make the split and consider the ghost momenta $\mathscr{P}_{A_0}$ associated with the independent constraints G_{A_0}. Because the subset G_{A_0} is irreducible, we know that δ is acyclic on polynomials in $\mathscr{P}_{A_0}$ of order greater than zero.

Next, introduce the remaining ghosts $\mathscr{P}_{\alpha_0}$ associated with the dependent constraints G_{α_0} and the ghosts $\mathscr{P}_{A_1}$ associated with the independent $Z^{\alpha_0}_{A_1}$. One has

$$\delta\mathscr{P}_{A_0} = G_{A_0}, \tag{3.13a}$$

$$\delta\mathscr{P}_{\alpha_0} = G_{\alpha_0}, \tag{3.13b}$$

$$\delta\mathscr{P}_{A_1} = Z^{A_0}_{A_1}\mathscr{P}_{A_0} + Z^{\alpha_0}_{A_1}\mathscr{P}_{\alpha_0}, \tag{3.13c}$$

and one wants to prove the following:

Lemma 1. *One can get rid of the newly introduced ghosts $(\mathscr{P}_{\alpha_0}, \mathscr{P}_{A_1})$ in a δ-closed polynomial by adding an appropriate δ-exact term. This implies, in particular, that a δ-closed polynomial in K_0 is δ-exact in K_1.*

Proof. The square matrix $Z^{\alpha_0}_{A_1}$ has an inverse Z^{-1}, since the relations $Z^{\alpha_0}_{A_1} G_{\alpha_0} +$

$Z^{A_0}_{A_1} G_{A_0} = 0$ enable one to express G_{α_0} in terms of G_{A_0}. By replacing the $\mathscr{P}_{A_1}$ by linear combinations $(Z^{-1}\mathscr{P})_{A_1}$ of themselves, one can bring (3.13) to the simple form

$$\delta \mathscr{P}_{A_0} \doteq 0, \tag{3.14a}$$

$$\delta \mathscr{P}_{\alpha_0} \doteq 0, \tag{3.14b}$$

$$\delta \tilde{\mathscr{P}}_{A_1} \doteq \delta^{\alpha_0}_{A_1} \mathscr{P}_{\alpha_0}, \tag{3.14c}$$

where $\delta^{\alpha_0}_{A_1}$ is the unit $m'_1 \times m'_1$ matrix and where we have systematically dropped terms which do not involve the ghosts $\mathscr{P}_{\alpha_0}, \tilde{\mathscr{P}}_{A_1}$ from the righthand sides of (3.14) (this is the meaning of $\doteq$). These terms play no role in the proof that $\mathscr{P}_{\alpha_0}$ and $\tilde{\mathscr{P}}_{A_1}$ (and thus $\mathscr{P}_{\alpha_0}, \mathscr{P}_{A_1}$) can be eliminated from δ-closed polynomials.

The ghosts $\mathscr{P}_{\alpha_0}$ and $\tilde{\mathscr{P}}_{A_1}$ are in equal number and of opposite parity. Consider the operator s (similar to a contracting homotopy)

$$s\mathscr{P}_{\alpha_0} = \delta^{A_1}_{\alpha_0} \tilde{\mathscr{P}}_{A_1}, \tag{3.15a}$$

$$s\tilde{\mathscr{P}}_{A_1} = 0, \tag{3.15b}$$

$$s\mathscr{P}_{A_0} = s(\text{original canonical variables}) = 0. \tag{3.15c}$$

One extends s to polynomials by requiring that it acts as a right derivative of degree $+1$. A straightforward computation shows that, for an arbitrary polynomial in $\mathscr{P}_{\alpha_0}, \tilde{\mathscr{P}}_{A_1}$ of order n, one has

$$(s\delta + \delta s) P_n = nP_n + P_{n-1}, \tag{3.16}$$

where P_{n-1} is a polynomial of order $n-1$. Therefore, if P_n is δ-closed ($\delta P_n = 0$) and of order $n > 0$, then it is also δ-exact modulo a polynomial of lower order,

$$P_n = \frac{1}{n}\delta(sP_n) - P_{n-1} \quad \text{if} \quad \delta P_n = 0. \tag{3.17}$$

It follows from (3.17) that P_{n-1} itself must be δ-closed, and by going in the same fashion, one arrives at the conclusion that $\mathscr{P}_{A_1}$ and $\mathscr{P}_{\alpha_0}$ can indeed be completely eliminated from δ-closed polynomials in $\mathscr{P}_{\alpha_0}, \mathscr{P}_{A_1}, \mathscr{P}_{A_0}$. □

Similarly, one can successively introduce the remaining pairs $(\mathscr{P}_{\alpha_1}, \mathscr{P}_{A_2})$, $(\mathscr{P}_{\alpha_2}, \mathscr{P}_{A_3}), \ldots$, maintaining at each stage the acyclicity property, until one reaches $\mathscr{P}_{A_k}$. We have thus proven the following.

Theorem 1. *In appropriate open sets U_s, any δ-closed polynomial belonging to K_i is δ-exact in K_{i+1} for each $i < k$. Furthermore, if there is no $\mathscr{P}_{\alpha_k}$ variable*, i.e., *if $\mathscr{P}_{A_k}$ exhaust all the ghost momenta of the k^{th} generation, δ is even acyclic in K_k.*

The next step is to establish.

Theorem 2. *Theorem 1 holds not only locally, but throughout phase space.*

Proof. One relies on the crucial property that δ is algebraic in phase space in the sense that it does not take derivatives of phase space functions. It is, therefore, very easy to patch together the open sets U_s.

Consider a partition of unity $\{f_s\}$ associated with the U_s's. If $\delta M = 0$, $M \in K_i$, then one has, on each U_s, $M = \delta N_s$, $N_s \in K_{i+1}$. Define, on U_s, $N'_s = N_s + \sum_t (N_t - N_s) f_t$. Clearly, one has $\delta N'_s = \delta N_s$[since $\delta((N_t - N_s)f_t) = (\delta(N_t - N_s))f_t = (M - M)f_t = 0$] and furthermore, $N'_s = N'_t$ on $U_s \cap U_t$, so that N'_s can be viewed as the restriction to U_s of a single polynomial $N = \Sigma_t f_t N_t \in K_{i+1}$ well defined throughout phase space, with $M = \delta N$. □

3.b. Passing from Order k to Order $k+1$. To finish with the explicit proofs that δ can be extended so as to become acyclic, we need to show that one can pass to the next complex K_{k+1} of level $k+1$. We thus establish Theorem 3.

Theorem 3. *δ can be extended to $\mathscr{P}_{a_{k+1}}$ while preserving the nilpotency condition.*

Proof. From the discussion of Sect. 3.2, we take

$$\delta \mathscr{P}_{a_{k+1}} = Z^{a_k}_{a_{k+1}} \mathscr{P}_{a_k} + M_{a_{k+1}}, \tag{3.18}$$

where $M_{a_{k+1}}$ should be of antighost number $k+1$ (as $\mathscr{P}_{a_k}$) and should be such that $\delta^2 \mathscr{P}_{a_{k+1}} = 0$. By Taking $\varepsilon(\mathscr{P}_{a_{k+1}}) = \varepsilon(\delta \mathscr{P}_{a_{k+1}}) + 1$, we will ensure that $\delta^2 = 0$ on any element of K_{k+1}.

Now, one has

$$\begin{aligned} \delta(Z^{a_k}_{a_{k+1}} \mathscr{P}_{a_k}) &= (-)^{\varepsilon_{a_{k+1}}} C^{a_{k-1} b_0}_{a_{k+1}} G_{b_0} \mathscr{P}_{a_{k-1}} + Z^{a_k}_{a_{k+1}} M_{a_k} \\ &= (-)\delta(C^{a_{k-1} b_0}_{a_{k+1}} \mathscr{P}_{a_0} \mathscr{P}_{a_{k-1}}) + S_{a_{k+1}}, \end{aligned} \tag{3.19a}$$

where

$$S_{a_{k+1}} = (-)^{k+1} C^{a_{k-1} b_0}_{a_{k+1}} \mathscr{P}_{b_0} \delta \mathscr{P}_{a_{k-1}} + Z^{a_k}_{a_{k+1}} M_{a_k}. \tag{3.19b}$$

In order to derive (3.19a), we used (2.12) and (3.12). The function $S_{a_{k+1}}$ in (3.19) is independent of $\mathscr{P}_{a_{k-1}}$, i.e., belongs to K_{k-2}. Furthermore, it is annihilated by δ (as can be seen from (3.19a). Hence, there exists a polynomial $M_{a_{k+1}}$ of antighost number $k+1$ in K_{k-1} such that

$$\delta M_{a_{k+1}} = -S_{a_{k+1}}. \tag{3.20}$$

Since $M_{a_{k+1}}$ belongs to K_{k-1}, it does not involve $\mathscr{P}_{a_k}$ and it is clearly the searched-for last term in the definition (3.18) of $\delta \mathscr{P}_{a_{k+1}}$. □

We have thus established that a nilpotent δ exists. Is this δ unique?

Theorem 4. *The polynomials $M_{a_{k+1}}$ appearing in $\delta \mathscr{P}_{a_{k+1}}$ are unique up to a δ-exact polynomial.*

Proof. The difference between two solutions $M_{a_{k+1}}$, $\tilde{M}_{a_{k+1}}$ of (3.20) is a polynomial in K_{k-1} which is δ-closed, thus, by Theorem 2, it is δ-exact in K_k, i.e.,

$$M_{a_{k+1}} - \tilde{M}_{a_{k+1}} = \delta N_{a_{k+1}}. \quad \square \tag{3.21}$$

Note that the ansatz (3.18) for $\delta \mathscr{P}_{a_{k+1}}$ assumes that $Z^{a_k}_{a_{k+1}}$ is given. Then Theorem 4 establishes that $M_{a_{k+1}}$ is unique up to (3.21), where $N_{a_{k+1}}$ is such that $\delta N_{a_{k+1}}$ does not contain $\mathscr{P}_{a_k}$. What would happen if $\delta N_{a_{k+1}}$ is allowed to contain $\mathscr{P}_{a_k}$, namely, if

$$N_{a_{k+1}} = D^{a_k a_0}_{a_{k+1}} \mathscr{P}_{a_0} \mathscr{P}_{a_k} + \text{“more,”} \tag{3.22}$$

with no restriction on $D^{a_k a_0}_{a_{k+1}}$?

Answer: The term containing $\mathscr{P}_{a_k}$ in $\delta N_{a_{k+1}}$ can be absorbed in an allowed redefinition of $Z^{a_k}_{a_{k+1}}$:

$$Z^{a_k}_{a_{k+1}} \to Z^{a_k}_{a_{k+1}} + D^{a_k a_0}_{a_{k+1}} G_{a_0}. \tag{3.23}$$

One thus sees that once the $\delta\mathscr{P}_{a_s}$, $s \leqq k$ have been defined and the $Z^{a_k}_{a_{k+1}}$ have been given only on the constraint surface, the ambiguity in the definition of $\delta\mathscr{P}_{a_{k+1}}$ is just of the form δ (something).

We have completed at this stage the proof of the acyclicity of δ in the space of polynomials in the ghost momenta $\mathscr{P}_{a_i}$. Indeed, we have constructed a sequence of complexes $\ldots K_k, K_{k+1}, \ldots$ (which is finite if the order of reducibility is finite) such that $\delta M = 0$, $M \in K_i \to M = \delta N$, $N \in K_{i+1}$. Given an arbitrary polynomial in the ghost momenta, one can decompose it as a sum of terms of definite antighost number,

$$M = \sum_{k>0} \overset{(k)}{M}.$$

If $\delta M = 0$, then $\delta \overset{(k)}{M} = 0$ for each k, and thus $\overset{(k)}{M} = \delta \overset{(k+1)}{N}$ because $\overset{(k)}{M}$ belongs to K_k. This implies $M = \delta N$, $N = \sum \overset{(k+1)}{N}$, as stated, i.e., δ is acyclic for $k > 0$.

IV. Existence and Uniqueness of the BRST Generator

1. Ghost Spectrum. Up to this point, we have only needed the ghost momenta $\mathscr{P}_{a_k}$. We complete the ghost spectrum by introducing their canonically conjugate variables η^{a_k} and assign the standard reality condition [1, 3],

$$[\mathscr{P}_{a_k}, \eta^{b_k}] = -\delta^{b_k}_{a_k}, \tag{4.1}$$

$$\varepsilon(\eta^{b_k}) = \varepsilon(\mathscr{P}_{a_k}) = \varepsilon_{b_k} + k + 1 \ (\mathrm{mod}\, 2), \tag{4.2a}$$

$$\mathrm{gh}\,(\eta^{b_k}) = k + 1 = -\mathrm{gh}\,(\mathscr{P}_{b_k}), \tag{4.2b}$$

$$(\eta^{b_k})^* = \eta^{b_k}, \quad (\mathscr{P}_{a_k})^* = -(-)^{\varepsilon_{b_k}+k}\mathscr{P}_{a_k}. \tag{4.2c}$$

All other brackets involving the new variables other than (4.1) vanish.

That (4.1) and (4.2) yield the correct number of ghosts η^{a_k} will be seen in a different way when we analyze the BRST cohomology.

The antighost number of the η^{a_k}'s is zero. Their pure ghost number is equal to $k+1$. We define the pure ghost number of the $\mathscr{P}$'s to vanish, so that in all cases the (total) ghost number is equal to the difference of the pure ghost number minus the antighost number.

2. Conditions Which Implement the Nilpotency of the BRST *Generator.* With a nilpotent, homology-free Koszul–Tate differential at hand, the construction of the BRST generator goes much along the same lines as in the irreducible case.

The BRST function must fulfill the following requirements:

$$\text{(i)}\ \mathrm{gh}\,\Omega = 1, \quad \varepsilon(\Omega) = 1, \quad \Omega^* = \Omega, \tag{4.3a}$$

$$\text{(ii)}\ \ \Omega = \eta^{a_0} G_{a_0} + \text{“more,”} \tag{4.3b}$$

$$\text{(iii)}\ \ [\Omega, \Omega] = 0\ \text{(“nilpotency”)}. \tag{4.3c}$$

It turns out that in the reducible case, these conditions do not completely determine Ω, even up to a canonical transformation. One must add the condition

$$\text{(iv)}\ \ \frac{\partial^2 \Omega}{\partial^r \mathscr{P}_{a_{k-1}} \partial^l \eta^{a_k}} = Z^{a_{k-1}}_{a_k}, \tag{4.3d}$$

which amounts to replacing (4.3b) by

$$\text{(ii)}'\ \ \Omega = \eta^{a_0} G_{a_0} + \eta^{a_k} Z^{a_{k-1}}_{a_k} \mathscr{P}_{a_{k-1}} + \text{“more,”} \tag{4.3b$'$}$$

where “more” now refers to terms containing at least two η's and one $\mathscr{P}$ or two $\mathscr{P}$'s and one η.

It is furthermore only if one imposes (4.3b′) that Ω reproduces the Koszul–Tate operator through the Poisson bracket (see below) and yields the appropriate cohomology.

In order to investigate the nilpotency condition (4.3c), it is convenient to expand Ω according to the antighost number,

$$\Omega = \sum_{p \geqq 0} \overset{(p)}{\Omega}, \tag{4.4a}$$

$$\operatorname{antigh}(\overset{(p)}{\Omega}) = p. \tag{4.4b}$$

Let us compute $[\Omega, \Omega]$. One finds, after elementary calculations,

$$[\Omega, \Omega] = \sum_{p \geqq 0} \overset{(p)}{B}, \tag{4.5a}$$

$$\operatorname{antigh}(\overset{(p)}{B}) = p, \tag{4.5b}$$

$$\overset{(p)}{B} = -2(\delta \overset{(p+1)}{\Omega} - \overset{(p)}{D}), \tag{4.5c}$$

where we have set

$$\overset{(p)}{D} = \frac{1}{2}\left\{ \sum_{k=0}^{p} [\overset{(p-k)}{\Omega}, \overset{(k)}{\Omega}]_{\text{orig}} + \sum_{k=1}^{p} \sum_{s=0}^{k-1} [\overset{(p-k+s+1)}{\Omega}, \overset{(k)}{\Omega}]_{(\text{gh})_s} \right\}, \tag{4.6}$$

and where the operator δ is defined, when acting on a function A of antighost number $\leqq p+1$, as

$$\delta A = -\sum_{s=0}^{p} [A, \overset{(s)}{\Omega}]_{(\text{gh})_s} = \sum_{s=0}^{p} \frac{\partial^r A}{\partial \mathscr{P}_{a_s}} \frac{\partial^l \overset{(s)}{\Omega}}{\partial \eta^{a_s}}, \quad \operatorname{antigh}(A) \leqq p+1. \tag{4.7}$$

We will see below that δ defined by (4.7) is indeed the Koszul–Tate operator.

In (4.6), $[\ ,\ ]_{\text{orig}}$ refers to the original Poisson bracket not involving the ghosts—which preserves the antighost number—while $[\ ,\ ]_{(\text{gh})_s}$ denotes the Poisson bracket with respect to the ghost pairs $\eta^{a_s}, \mathscr{P}_{a_s}$. This latter bracket destroys one $\mathscr{P}_{a_s}$, and hence δ in (4.7) diminishes the antighost number of A by one unit, since $\partial^l \overset{(s)}{\Omega} / \partial \eta^{a_s}$ possesses antighost number s.

The function $\overset{(p)}{D}$ only involves the structure functions $\overset{(i)}{\Omega}$ with $i \leqq p$. So, the idea is to solve for Ω step by step and to regard the equation $[\Omega, \Omega] = 0$, which is equivalent to the equations $\overset{(p)}{B} = 0$,

$$\delta \overset{(p+1)}{\Omega} = \overset{(p)}{D}, \tag{4.8}$$

as determining $\overset{(p+1)}{\Omega}$ once the $\overset{(i)}{\Omega}$, $i \leqq p$ are known.

Suppose, then, that we have solved (4.8) up to order p,

$$\delta \overset{(1)}{\Omega} = \overset{(0)}{D}, \ldots, \delta \overset{(p)}{\Omega} = \overset{(p-1)}{D}, \tag{4.9}$$

so that Ω is nilpotent up to order $p-1$ (i.e., $\overset{(i)}{B} = 0, i \leqq p-1$), and we want to investigate whether we can solve (4.8) for $\overset{(p+1)}{\Omega}$.

If Ω is nilpotent up to order $p-1$, the operator δ defined by (4.7) must indeed be the Koszul–Tate operator when acting on functions of antighost number $\leqq p+1$. This is because (4.7) yields

$$\begin{aligned} &\delta \mathscr{P}_{a_0} = 0, \quad \delta \mathscr{P}_{a_k} = Z^{a_{k-1}}_{a_k} \mathscr{P}_{a_{k-1}} + \text{“extra terms,”} \quad k \leqq p, \\ &\delta \eta^{a_k} = 0, \qquad \delta q = \delta p = 0, \end{aligned} \tag{4.10}$$

where the extra terms contain at least two $\mathscr{P}$'s of order $< k-1$. Furthermore, one easily finds that (4.7) is nilpotent when $\operatorname{antigh}(A) \leqq p+1$ if Ω is nilpotent up to order $p-1$. Thus, for $\operatorname{antigh} A \leqq p+1$, (4.7) coincides with the Koszul–Tate operator modulo the irrelevant ambiguities in the definition of δ discussed at the end of Sect. 3 and analyzed in more detail in Sect. 4.3 below.

It follows from this analysis that the equations determining the structure functions $\overset{(p+1)}{\Omega}$ of order $p+1$ from those of order $k \leqq p$ read

$$\delta \overset{(p+1)}{\Omega} = \overset{(p)}{D}, \tag{4.11}$$

just as in the irreducible case.

3. The BRST *Generator is Unique up to Canonical Transformations.* Before proving the existence of the BRST generator, we show its uniqueness modulo canonical transformations in the extended phase space.

Start by considering two different, nilpotent BRST charges Ω, Ω' associated with the same constraint hypersurface but with possibly different $G_{a_0}, Z^{a_0}_{a_1}, Z^{a_1}_{a_2}$, etc....

By exactly the same argument as in the irreducible case, one can make the coefficients of η^{a_0} in Ω and Ω' coincide by means of canonical transformation in the extended phase space [2, 3, 8]. Assume therefore that Ω and Ω' coincide up to order p, but that start differing at order $p+1$,

$$\overset{(i)}{\Omega} = \overset{(i)}{\Omega}{}', \quad i \leqq p, \tag{4.12a}$$

$$\overset{(p+1)}{\Omega} \neq \overset{(p+1)}{\Omega}{}'. \tag{4.12b}$$

The equations determining $\overset{(p+1)}{\Omega}$ and $\overset{(p+1)}{\Omega'}$ read

$$\delta \overset{(p+1)}{\Omega} = \overset{(p)}{D}, \tag{4.13a}$$

$$\delta \overset{(p+1)}{\Omega'} = \overset{(p)}{D}, \tag{4.13b}$$

with the same right-hand sides and with the same δ, since, as it follows from (4.7), δ acting on polynomials of antighost number $\leqq p+1$ is entirely determined by $\overset{(i)}{\Omega}$, $i \leqq p$. This implies

$$\overset{(p+1)}{\Omega'} = \overset{(p+1)}{\Omega} + \delta_{\Omega} \overset{(p+2)}{M}, \tag{4.14}$$

where we have added the index Ω to δ in the right-hand side of (4.14) to emphasize that δ_{Ω} and $\delta_{\Omega'}$ may differ at order $p+2$, since $\overset{(p+1)}{\Omega}$ may contribute to the definition of δ at that order.

Next, consider the canonical transformation generated by $\overset{(p+2)}{M}$. It clearly induces no change in Ω at antighost number $\leqq p$, while the change in $\overset{(p+1)}{\Omega}$ is given by

$$\overset{(p+1)}{\Omega} \rightarrow \overset{(p+1)}{\Omega} + \delta_{\Omega} \overset{(p+2)}{M}. \tag{4.15}$$

This is the required result, since (4.15) coincides with (4.14). It enables us to conclude that the BRST function is unique up to a canonical transformation. Note that, in particular, the change $Z^{a_0}_{a_1} \rightarrow M^{b_1}_{a_1} Z^{a_0}_{b_1}$ is generated by $\eta^{a_1} m^{b_1}_{a_1} \mathscr{P}_{b_1}$ when $M^{b_1}_{a_1} = (\exp m)^{b_1}_{a_1}$. The extra ambiguity in $Z^{a_0}_{a_1}$ is generated by $\eta^{a_1} v^{a_0 b_0}_{a_1} \mathscr{P}_{b_0} \mathscr{P}_{a_0}$, and its exponentiation is trivial, since the transformation $Z^{a_0}_{a_1} \rightarrow Z^{a_0}_{a_1} + v^{a_0 b_0}_{a_1} G_{b_0} (-)^{\varepsilon_{a_0}}$ is additively abelian.

4. Existence of the BRST *Generator*

Theorem 5. *The* BRST *generator exists.*

Proof. The question of proving the existence of the BRST function is completely equivalent to the question of proving the existence of the structure functions $\overset{(p+1)}{\Omega}$, which should obey equation (4.11). This, in turn, amounts to establishing the identity

$$\delta \overset{(p)}{D} = 0 \tag{4.16}$$

for the structure functions $\overset{(i)}{\Omega}$ of order $\leqq p$, since δ is both nilpotent and acyclic (except at order zero, where (4.16) should be replaced by $\varepsilon \overset{(0)}{D} = 0$; this case is analyzed in more detail below).

To demonstrate (4.16), one can again proceed as in the irreducible case. The condition (4.16) can be verified pointwise. So, we first check it in any open set in which the constraints and the irreducibility functions take the canonical form given in Sect. 2.3. If it holds in any such set, it will hold everywhere, and since $\overset{(p)}{D}$ is globally defined (the $\overset{(i)}{\Omega}, i \leqq p$ are globally defined), the acyclicity of δ guarantees the existence of a global $\overset{(p+1)}{\Omega}$ solution of $\delta \overset{(p+1)}{\Omega} = \overset{(p)}{D}$.

Now, in any open set where the canonical form of Sect. 2.3. is attained, it is easy to see that a (local) BRST generator Ω' exists. It is just given by

$$\Omega' = \Omega(G_{A_0}) + \sum_{k=0}^{L-1} \mathscr{P}_{\alpha_k} \delta^{\alpha_k}_{A_{k+1}} \eta^{A_{k+1}}, \tag{4.17}$$

where (i) $\Omega(G_{A_0})$ is the BRST charge for the irreducible constraints G_{A_0} and only contains the ghost pairs $\eta^{A_0}, \mathscr{P}_{A_0}$, and (ii) the remaining piece in Ω clearly anticommutes with $\Omega(G_{A_0})$ and is nilpotent by itself.

By a canonical transformation valid on the open set on which Ω' is defined, one can then turn Ω' into Ω'', which is, of course, still nilpotent and which, furthermore, can be chosen to coincide up to order p with the searched-for global Ω,

$$\overset{(i)}{\Omega''} = \overset{(i)}{\Omega}, \quad i \leqq p. \tag{4.18}$$

This follows from the results of Sect. IV.3. and from the fact that Ω is nilpotent up to order $p-1$, i.e., the $\overset{(i)}{\Omega}$'s that have been constructed so far obey (4.9).

One has, from (4.18),

$$\overset{(p)}{D}(\Omega'') = \overset{(p)}{D}(\Omega),$$

since $\overset{(p)}{D}$ only involves $\overset{(i)}{\Omega}$, $i \leqq p$. Therefore, from the nilpotency of Ω'', which holds in the given open set under consideration, one has

$$\delta \overset{(p)}{D}(\Omega'') = \delta \overset{(p)}{D}(\Omega) = 0.$$

This demonstrates (4.16). □

It is worth noticing that while the existence of the Koszul–Tate differential does not require the first class property, this feature is essential for the existence of the BRST symmetry. Indeed, in order for Eq. (4.11) to possess a solution at $p=0$, it is necessary that $\overset{(0)}{D}$ vanishes on the constraint surface. This requirement, in turn, is equivalent to the first class condition (2.2) among the constraints.

The first class property also plays a key role in the discussion of the BRST cohomology and in the interpretation of the ghosts η^{a_1} as exterior forms, as we now pause to discuss.

V. Classical BRST Cohomology

1. Exterior Derivative Operator along the Gauge Orbits

1.a. Intrinsic Definition. Through each point of the constraint surface, there is a gauge orbit generated by the gauge transformations. These gauge orbits define what may be called "vertical directions."

A vertical form field α on the constraint surface can be defined as a form field that only acts on vector fields defined over all the constraint surface and tangent to the orbits ("vertical vectors"). Because the Lie bracket of two vertical vector fields is again a vertical vector field, one can define the exterior derivative along the gauge orbits of a vertical p-form as the vertical $p+1$-form given by

$$d\alpha(X_0,\ldots,X_p)=(-)^{\sum_{i=0}^{p}\varepsilon_{X_i}}\Bigg[\sum_{j=0}^{p}(-)^{\rho_j+\varepsilon_{X_j}}\alpha(X_0,\ldots,\hat{X}_j,\ldots,X_p)\overleftarrow{\mathscr{L}}_{X_j}$$
$$+\sum_{0\leq i<j\leq p}(-)^{\rho_{ij}+\varepsilon_{X_j}+1}\alpha(X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p,[X_i,X_j])\Bigg]. \tag{5.1}$$

(Cartan–Chevalley–Eilenberg coboundary formula for the vertical Lie algebra cohomology. See appendix for the unexplained symbols; compare [11, 12, 13]).

One easily checks that $d^2=0$. Since only tangent vector fields appear in (5.1), the exterior derivative operator d only involves (appropriately antisymmetrized) derivatives along the gauge orbits. It does not measure how α changes as one goes from one orbit to the next.

When one works in terms of a basis of tangent vectors and its dual, the definition (5.1) reduces to the usual expression of the exterior derivative along the orbits in terms of antisymmetrized derivatives of the components. However, (5.1) as it stands has the advantage that it is manifestly basis-independent and can thus be applied directly to vectors that are expanded in terms of an overcomplete set.

1.b. Overcomplete Representation. There is a useful representation of vertical forms in terms of "components" along the vectors X_{a_0} associated with the constraints. Because these vectors completely span the tangent space to the orbits, a p-form is completely determined by its action on $X_{a_0},\ldots,X_{c_0}$,

$$\alpha(X_{a_0},X_{b_0},\ldots,X_{c_0}). \tag{5.2}$$

These "components" are, however, redundant and subject to algebraic relations following from the overcompleteness of the X_{a_0}'s. One finds from (2.5)

$$\alpha(X_{a_0},\ldots,X_{c_0})Z^{c_0}_{a_1}(-)^{\varepsilon_{c_0}(\varepsilon_{c_0}+\varepsilon_{a_1})}=0, \tag{5.3}$$

and similarly for the other arguments in α.

Conversely, any set of functions $\alpha_{a_0\cdots c_0}$ with the appropriate symmetry in $a_0,\ldots,c_0$,

$$\alpha_{a_0\cdots b_0b'_0\cdots c_0}=(-)^{(\varepsilon_{b_0}+1)(\varepsilon_{b'_0}+1)}\alpha_{a_0\cdots b'_0b_0\cdots c_0}, \tag{5.4}$$

and obeying the algebraic conditions (5.3), can be viewed as the components $\alpha(X_{a_0},\ldots,X_{c_0})$ of a vertical p-form in the overcomplete set of tangent vectors X_{a_0},

$$\alpha_{a_0\cdots c_0}=(-)^{(\varepsilon_\alpha+\varepsilon_{a_0}+\cdots+\varepsilon_{c_0}+p)(\varepsilon_{a_0}+\cdots+\varepsilon_{c_0}+p)}\alpha(X_{a_0},\ldots,X_{c_0}), \tag{5.5}$$

where ε_α is the Grassmann parity of α. We have included a phase in (5.5) for later convenience. With this phase, the functions $\alpha_{a_0\ldots c_0}$ can be viewed as the components of the vertical form α, as in formula (5.6) below, and this will match the standard BRST notations when we relate the vertical forms to functions in the extended phase space.

Since the X_{a_0} are not linearly independent, one cannot define their dual basis. However, it is useful to introduce formal 1-forms ω^{a_0} which play a similar role because, as we shall see right now, one can then give simple rules for the intrinsic exterior derivative (5.1) expressed in terms of the components (5.5) obeying (5.3).

The ω^{a_0} are simply used to saturate the indices in $\alpha_{a_0 \cdots c_0}$ by writing

$$\alpha = \frac{1}{p!} \omega^{a_0} \omega^{b_0} \cdots \omega^{c_0} \alpha_{c_0 \cdots b_0 a_0} \tag{5.6}$$

with the symmetry property

$$\omega^{a_0} \omega^{b_0} = (-)^{(\varepsilon_{b_0}+1)(\varepsilon_{a_0}+1)} \omega^{b_0} \omega^{a_0} \Leftrightarrow \varepsilon(\omega^{a_0}) = \varepsilon_{a_0} + 1. \tag{5.7}$$

We will assume for simplicity that there are no relations among the ω^{a_0}, i.e., we formally set $\omega^{a_0}(X_{b_0}) = \delta^{a_0}_{b_0}$. This demand is largely motivated by the desire to identify ω^{a_0} later with the ghosts η^{a_0}, which are independent. As we shall see, the independence of the ω^{a_0} will lead to another reason for introducing ghosts of ghosts.

On account of the remarks made above, the representation (5.6) enables one to view vertical forms as polynomials in ω^{a_0} whose coefficients are constraint surface functions obeying the algebraic condition (5.3).

In terms of the representation (5.6), the exterior derivative operator along the gauge orbits can be computed according to the following rules:

$$d\omega^{a_0} = \tfrac{1}{2}(-)^{\varepsilon_{a_0}+\varepsilon_{b_0}} \omega^{b_0} \omega^{c_0} C^{a_0}_{c_0 b_0} \tag{5.8a}$$

for ω^{a_0},

$$dF = \partial_{a_0} F \omega^{a_0} \quad (\partial_{a_0} F \equiv [F, G_{a_0}]) \tag{5.8b}$$

for a function F, and

$$d(AB) = AdB + (-)^{\varepsilon_B}(dA)B \tag{5.8c}$$

for arbitrary A, B with B of definite parity ε_B.

It should be stressed that in order for α to be a vertical form, the coefficients $\alpha_{c_0 \cdots a_0}$ in (5.6) must obey the condition (5.3). This means that the ω^{a_0} themselves are not vertical 1-forms, since we take them to be independent, and as a matter of fact they do not appear to have a direct geometrical meaning. This is why we have called them formal 1-forms. Equation (5.3) for a vertical form can be rewritten as

$$Z^{a_0}_{a_1} \omega^{b_0} \cdots \omega^{c_0} \alpha_{c_0 \cdots b_0 a_0} = 0. \tag{5.9}$$

The functions $\alpha_{c_0 \cdots a_0}$ have so far been defined only over the constrained surface. They may be arbitrarily extended off that surface to yield phase space functions. When that has been done, one writes (5.9) as a weak equality

$$Z^{a_0}_{a_1} \omega^{b_0} \cdots \omega^{c_0} \alpha_{c_0 \cdots b_0 a_0} \approx 0 \tag{5.10a}$$

and identifies two forms that coincide on $G_{a_0} = 0$,

$$\alpha \sim \alpha' \quad \text{if} \quad \alpha' - \alpha \approx 0. \tag{5.10b}$$

One defines d on the vertical forms defined throughout phase space by the same rules (5.8). This makes sense because the functions $C^{a_0}_{c_0 b_0}$ and $Z^{a_0}_{a_1}$, as well as the vector fields X_{a_0} along which one takes derivatives, are also defined throughout phase space.

The operator d defined in this manner on polynomials in ω^{a_0} with coefficients that are phase space functions is easily checked to be weakly nilpotent when acting on polynomials obeying the algebraic conditions (5.10a). These polynomials will still be named "vertical forms." The cohomological classes of the exterior derivative operator along the orbits is given by the classes of weakly closed, vertical p-forms modulo weakly exact, vertical p-forms.

The (weak) nilpotency of the vertical exterior derivative is guaranteed only for vertical forms. It does not hold for arbitrary polynomials in ω^{a_0} and, for example, one finds that $d^2\omega^{a_0}$ is in general different from zero, even on the constrained surface. This is because one allows in (5.8a) for an arbitrary set of $C^{a_0}_{c_0b_0}$ compatible with (2.2); i.e., one can freely add to $C^{a_0}_{c_0b_0}$ a term of the form $M^{a_0d_0}_{c_0b_0}(-)^{\varepsilon_{d_0}}G_{d_0}+N^{a_1}_{c_0b_0}Z^{a_0}_{a_1}$. One may define (as shown in Subsect. V.2. below) an extended exterior derivative that is nilpotent. This is done by introducing additional formal objects $\omega^{a_{p-1}}$, which are p-forms.

The cohomology of the extended derivative turns out to be identical with that of the original, vertical d defined by (5.8). However, it is the cohomology of the extended d that may be related straightforwardly with the cohomology of the BRST generator (the additional p-forms correspond to the "ghosts of ghosts").

2. *Extended Exterior Derivative along Gauge Orbits*

2.a. Motivation. The definition of the extended nilpotent exterior derivative suggests itself after one analyzes how the original d fails to be nilpotent. Direct calculations using (5.8) yield

$$d^2\omega^{a_0}=-(-)^{\varepsilon_{a_1}}\omega^{b_0}\omega^{c_0}\omega^{d_0}U^{a_1}_{d_0c_0b_0}Z^{a_0}_{a_1}. \tag{5.11a}$$

Here, the functions $U^{a_1}_{b_0c_0d_0}$ are the coefficients that appear in the identities

$$(-)^{(\varepsilon_{d_0}+1)(\varepsilon_{a_0}+\varepsilon_{b_0})+\varepsilon_{d_0}\varepsilon_{c_0}}\partial_{d_0}C^{a_0}_{c_0b_0}+(-)^{\varepsilon_{c_0}}C^{a_0}_{e_0b_0}C^{e_0}_{d_0c_0}+\text{cyclic}\approx-(-)^{\varepsilon_{a_1}}U^{a_1}_{d_0c_0b_0}Z^{a_0}_{a_1}, \tag{5.11b}$$

which are direct consequences of the Jacobi identity $[X_{[a_0},[X_{b_0},X_{c_0]}]]=0$ for the Lie bracket.

In order to restore nilpotency at the level of ω^{a_0}, we bring in a new object ω^{a_1} that will be, this time, a formal 2-form, through the equations

$$d\omega^{a_0}=\tfrac{1}{2}(-)^{\varepsilon_{a_0}+\varepsilon_{b_0}}\omega^{b_0}\omega^{c_0}C^{a_0}_{c_0b_0}+(-)^{\varepsilon_{a_0}}\omega^{a_1}Z^{a_0}_{a_1}, \tag{5.12a}$$

$$d\omega^{a_1}=\omega^{a_0}\omega^{b_0}\omega^{c_0}U^{a_1}_{c_0b_0a_0}+\text{"more,"} \tag{5.12b}$$

where "more" stands for terms of the form $\omega^{a_0}\omega^{b_1}$ to be determined later on. The new term that has been added to $d\omega^{a_0}$ does not modify the exterior derivative of a true vertical form α, since this term is annihilated in that case by the components $\alpha_{a_0\cdots c_0}$ of α because of (5.9).

In order to preserve the grading properties of d, ω^{a_1} should not only be a form of degree 2, but it should also possess Grassmann parity $\varepsilon(\omega^{a_1})=\varepsilon_{a_1}$, as follows from $\varepsilon(d)=1$, $\varepsilon(\omega^{a_0})=\varepsilon(G_{a_0})+1=\varepsilon_{a_0}+1$, and $\varepsilon(Z^{a_0}_{a_1})=\varepsilon_{a_0}+\varepsilon_{a_1}$.

With (5.11), the terms with three ω^{a_0} disappear from $d^2\omega^{a_0}$, so one is left only with terms of the form $\omega^{a_0}\omega^{b_1}$. The idea then is to adjust "more" in (5.12b) so as to cancel these terms as well. This turns out to be possible and can be seen either directly or indirectly. The direct way of tackling the issue is through examination of the new identities obtained by taking further antisymmetrized Lie derivatives of both (5.11b) and the reducibility equation $(-)^{\varepsilon_{a_0}} Z^{a_0}_{a_1} X_{a_0} \approx 0$. We shall not pursue that route. Instead, we will rely immediately below on the existence of the BRST generator, which has already been demonstrated.

Now, even after having established $d^2\omega^{a_0} = 0$, one needs to show also that $d^2\omega^{a_1} = 0$. To that end it becomes necessary in general to introduce yet another object ω^{a_2}, a formal 3-form, and to add to the right side of (5.12b) a term that brings in $Z^{a_1}_{a_2}$, namely, $\omega^{a_2} Z^{a_1}_{a_2}$. Such a term does not spoil $d^2\omega^{a_0} \approx 0$ because $Z^{a_1}_{a_2} Z^{a_0}_{a_1} \approx 0$. The procedure goes on by introducing new forms ω^{a_k} for each $Z^{a_{k-1}}_{a_k}$. Again, the feasibility of the whole construction will be established indirectly through the BRST generator.

It is noteworthy that the spirit underlying the extension of d is quite similar to the construction of the Koszul–Tate operator δ. Indeed, also here one incorporates a desired property by introducing extra variables. In the case of δ, one had nilpotency to begin with and incorporated the acyclicity, or, what is the same, the lack of cohomology. For d, one consistently incorporates the requirement that the generators ω^{a_0} be treated as independent, while maintaining nilpotency. After this is done, one finds that the cohomology is unchanged.

2.b. Definition. We define the extended exterior derivative operator d along the gauge orbits by

$$dF = [F, \Omega]\big|_{\substack{\eta = \omega \\ \mathscr{P} = 0}}, \tag{5.13a}$$

and, in particular,

$$d\omega^{a_k} = [\eta^{a_k}, \Omega]\big|_{\substack{\eta = \omega \\ \mathscr{P} = 0}}. \tag{5.13b}$$

The meaning of (5.13) is as follows. In (5.13a), F is a polynomial in ω^{a_i}, with coefficients that are arbitrary phase space functions. By replacing ω^{a_i} with η^{a_i}, F becomes a polynomial in the ghost variables, so that $[F, \Omega]$ is a well-defined function in the extended phase space. Once $\mathscr{P}_{a^i}$ is set equal to zero, $[F, \Omega]$ reduces to a polynomial in η^{a_i} only. The further replacement $\eta^{a_i} \to \omega^{a_i}$ in $[F, \Omega]$ yields a polynomial in ω^{a_i}, which defines dF. It is clear from this definition that $dF \approx 0$ if $F \approx 0$.

In order for the replacement of η by ω in (5.13) to be reasonable, one needs these objects to possess the same algebraic properties when the multiplication in the Grassmann algebra of the q's, the p's, and the ghosts are identified with the exterior product of forms.

With the conventions adopted here that involve a single grading for the exterior product (see appendix), the question amounts to whether the Grassmann parities of η^{a_k} and ω^{a_k} coincide. Because $\varepsilon(\Omega) = 1$ and because the Grassmann parity of

ω^{a_k} is recursively determined from $\varepsilon(\omega^{a_0})$ by repeated use of (5.13b) so as to have $\varepsilon(d) = 1$, this will be the case if $\varepsilon(\omega^{a_0}) = \varepsilon(\eta^{a_0})$, i.e., if the Grassmann parities coincide at level zero. Mere comparison of formulas (5.7) and (4.2a) shows that this is indeed the case, so that the identification of ω^{a_k} with η^{a_k} does indeed make sense.

The definition (5.13) of d yields an operator that manifestly obeys the Leibnitz rule and clearly reproduces (5.8b) or (5.12a) when F is a 0-form or is equal to ω^{a_0}. Furthermore, because Ω is nilpotent, one finds $d^2 \approx 0$,

$$d^2 A = [dA, \Omega]\big|_{\substack{\eta=\omega \\ \mathscr{P}=0}} \approx [[A, \Omega], \Omega]\big|_{\substack{\eta=\omega \\ \mathscr{P}=0}} = 0,$$

since the brackets $[\mathscr{P}_{a_i}, \Omega]$ vanish on the constraint surface when the momenta $\mathscr{P}_{a_i}$ are set equal to zero. This shows that (5.13) is the appropriate definition of the extended exterior derivative operator along the gauge orbits.

Under the identification of the ghosts η^{a_k} with the forms ω^{a_k}, the pure ghost number becomes equal to the form degree. In that sense, the pure ghost number is the grading associated with the exterior derivative operator.

The algebra generated by the ω^{a_k}, together with the exterior derivative d acting on it, possesses the structure of a free differential algebra [14]. The independent generators of this algebra are just the forms $\omega^{a_0}, \omega^{a_1}, \ldots$ of increasing degree. The coefficient ring is the space of constraint surface functions, or, equivalently, the space of phase space functions with the identification (5.10b).

3. Cohomology of the Extended Exterior Derivative. The extended exterior derivative operator acts on polynomials in the ghosts $\eta^{a_0}, \ldots, \eta^{a_k}$. Because it is nilpotent, one can define and study its cohomology. A natural question to ask is: what is the relationship between its cohomology and that of the original vertical derivative acting on polynomials in η^{a_0} with coefficients obeying (5.9)? We show in this section that they are the same.

First, we establish,

Theorem 6. *The cohomological classes of the original exterior derivative along the gauge orbits (acting on true vertical forms) define cohomological classes of the extended derivative.*

Proof. True vertical forms along the gauge orbits are polynomials in η^{a_0} obeying the algebraic condition (5.9). For such polynomials, one has

$$d^{\text{orig}}\alpha = d\alpha. \tag{5.14}$$

Hence, if $d^{\text{orig}}\alpha \approx 0$, then also $d\alpha \approx 0$. Moreover, if $\alpha \approx d^{\text{orig}}\beta$, with β obeying the algebraic condition (5.9), then also $\alpha \approx d\beta$. $\square$

Hence, we can define a map which sends any cohomological class of d^{orig} onto the corresponding cohomological class of d. For the cohomologies of d^{orig} and of d to be equal, this map should possess an everywhere defined inverse.

To prove the invertibility, we introduce an algebraic operator Δ, which acts again as a right differential, by

$$\Delta q = \Delta p = 0, \tag{5.15a}$$

$$\Delta \eta^{a_i} = \eta^{a_{i+1}} Z^{a_i}_{a_{i+1}} (-)^{\varepsilon_{a_i}}. \tag{5.15b}$$

This operator is (weakly) nilpotent,

$$\Delta^2 \approx 0, \tag{5.15c}$$

as a consequence of the reducibility equations.

The interest of the operator Δ is that it is one of the building blocks of the extended exterior derivative along the gauge orbits. Indeed, because Ω contains the terms $\eta^{a_{i+1}} Z^{a_i}_{a_{i+1}} \mathscr{P}_{a_i}$, the exterior derivative defined by (5.13) clearly reads

$$d\alpha = \Delta\alpha + \bar{d}\alpha, \tag{5.16a}$$

where $\bar{d}\alpha$ is obtained by taking the bracket of α with $\Omega - \sum \eta^{a_{i+1}} Z^{a_i}_{a_{i+1}} \mathscr{P}_{a_i}$.

Now, Δ is the only piece of d that replaces η^{a_i} by a combination of the ghosts of higher order $\eta^{a_{i+1}}$, and this is why it is useful to isolate it. Indeed, by the definition of Ω, $\bar{d}$ will only transform η^{a_i} into a combination of products $\eta^{a_j} \eta^{a'_{j'}} \cdots \eta^{a''_{j''}}$, where $j, j', \ldots \leqq i$. No higher order ghost appears in $\bar{d}\eta^{a_i}$. Furthermore, $\bar{d}\eta^{a_i}$ contains η^{a_i} at most linearly when $i \geqq 1$, since in that case the product of two η^{a_i} possesses pure ghost number strictly greater than the pure ghost number of η^{a_i} plus one unit.

Therefore, one has

Lemma 2. *Let η^{a_s} $(s \geqq 1)$ be the ghosts of highest degree occurring in the p-form α (because α is of finite degree p, there is such an s, and $s \leqq p-1$). Let k be the order of the polynomial α in the highest degree ghosts η^{a_s},* i.e., *each term in α contains at most k powers of the η^{a_s}'s. Then:*

(i) *$\bar{d}\alpha$ is a polynomial in η^{a_i} $(i \leqq s)$ of order at most k in η^{a_s}, and*
(ii) *$\Delta\alpha$ involves $\eta^{a_{s+1}}$ at most linearly,*

$$\Delta\alpha = P_{a_{s+1}} \eta^{a_{s+1}} + \beta. \tag{5.16b}$$

Here the coefficients $P_{a_{s+1}}$ are polynomials in η^{a_i} $(i \leqq s)$ of order at most $k-1$ in η^{a_s} and β is a polynomial in η^{a_i} $(i \leqq s)$ of order at most $k+1$ in η^{a_s},

$$\beta = \overset{(k+1)}{\beta} + \beta' \tag{5.16c}$$

with $\overset{(k+1)}{\beta}$ of order $k+1$ in η^{a_s}, and β' of order $\leqq k$.

Proof. Point (i) was already established in the comments immediately following (5.16a). Point (ii) results from the definition of Δ, the first term in $\Delta\alpha$ arises when η^{a_s} is replaced by a linear combination of the $\eta^{a_{s+1}}$'s and the term of order $k+1$ in β comes in when $\eta^{a_{s-1}}$ is replaced by a linear combination of the η^{a_s}'s. □

In general, both $P_{a_{s+1}}$ and $\overset{(k+1)}{\beta}$ will be nonvanishing. However, one has the following

Lemma 3. *If $\Delta\alpha$ is weakly equal to a polynomial in η^{a_i} $(i \leqq s)$ of order at most k in η^{a_s} $(k, s \geqq 1)$,* i.e., *if in (5.16b)–(5.16c) both $P_{a_{s+1}}$ and $\overset{(k+1)}{\beta}$ weakly vanish so that*

$$\Delta\alpha \approx \beta', \tag{5.17a}$$

then, α can be written as

$$\alpha \approx \Delta\gamma + \varepsilon, \tag{5.17b}$$

where γ and ε are both polynomials in η^{α_i} ($i \leqq s$) of order at most $k-1$ in η^{α_s}.

What this lemma says is that if $\Delta\alpha$ does not involve $\eta^{\alpha_{s+1}}$ and is still of order k in η^{α_s}, then one can remove one power of η^{α_s} from α by the addition of a Δ-exact term. By repeated use of (5.17b), one can then show that the cohomology of Δ is determined by the polynomials in η^{a_0} that are Δ-closed, but we will need below the stronger result (5.17b). In the theory of free differential algebras, Δ would generate the contractible part of the algebra [14], since it replaces a generator of degree p by a generator of degree $p+1$.

Proof. To establish (5.17b) one proceeds as in the proof of the acyclicity of δ. One first checks (5.17b) in an open region where the constraints and the reducibility functions can be split into dependent and independent sets, so that Δ can be written in the canonical form $\Delta\eta^{\alpha_i} = \delta^{\alpha_i}_{A_{i+1}}\eta^{A_{i+1}}$, $\Delta\eta^{A_i} = 0$. One then extends the proof to the whole of phase space.

The explicit local check of (5.17b) proceeds as follows. (a) One first observes that if α obeys (5.17a), then it cannot depend on η^{α_s}. (b) A map σ is defined by $\sigma\eta^{A_s} = \delta^{A_s}_{\alpha_{s-1}}\eta^{\alpha_{s-1}}$, $\sigma\eta^{\alpha_{s-1}} = 0$ and σ (other variables in α) $= 0$. For each component of α of definite order k in η^{A_s} and l in $\eta^{\alpha_{s-1}}$, this map obeys $(\Delta\sigma + \sigma\Delta)\alpha = (k+l)\alpha$. It follows that $\alpha \approx \Delta\gamma + \varepsilon$ with $\gamma = (k+l)^{-1}\sigma\alpha$, $\varepsilon = (k+l)^{-1}\sigma\Delta\alpha$. The polynomials γ and ε obey the required properties because σ decreases the number of η^{a_s} and does not introduce any $\eta^{a_{s+1}}$.

From (a) and (b) one infers that (5.17b) holds in appropriate open sets. To complete the proof, one observes that there is no global subtlety in patching these sets together because Δ is algebraic in phase space. □

The use of Lemma 3 above is to establish the following important theorem,

Theorem 7. *If α is an extended p-form such that $d\alpha$ only involves η^{a_0}, then it can be written as*

$$\alpha \approx \alpha' + d\beta,$$

where α' only involves η^{a_0}. In other words, the dependence of α on the extra ghosts $\eta^{a_1}, \eta^{a_2}, \ldots$ can be removed by the addition of an exact form.

Proof. One has $d\alpha = \Delta\alpha + \bar{d}\alpha$. If $d\alpha$ only involves η^{a_0}, then $\Delta\alpha$ obeys the assumptions of Lemma 3; therefore, $\alpha \approx \Delta\gamma + \varepsilon = d\gamma - \bar{d}\gamma + \varepsilon$. The form $\varepsilon - \bar{d}\gamma$ involves one less power of η^{a_s}, and, since $d^2 \approx 0$, it must be such that $d(\varepsilon - \bar{d}\gamma)$ only involves η^{a_0}. By proceeding recursively in the same way, one completely eliminates η^{a_s}, and then $\eta^{a_{s-1}}, \ldots$, until one reaches a p-form α' that only contains η^{a_0}. □

Theorem 7 is important because it tells us that it is sufficient to analyze the cohomology of the extended exterior derivative d in the space of polynomials in η^{a_0}. Indeed, the theorem indicates that any cohomological class possesses a representative that does not involve $\eta^{a_1}, \eta^{a_2}, \ldots$. Furthermore, if γ is exact and does not contain the extra ghosts, one has $\gamma \approx d\alpha'$, where α' is also independent of $\eta^{a_1}, \eta^{a_2}, \ldots$.

The last step in showing that the cohomologies of the original and the extended d coincide is given by

Theorem 8. *If both α and $d\alpha$ do not contain the ghosts of ghosts, then α is a true vertical form along the gauge orbits,* i.e., *it obeys the algebraic condition* (*5.9*).

Proof. One works out explicitly the term containing η^{a_1} in $d\alpha$ and finds out that it is just the left-hand side of (5.9). □

Hence, one sees that in the extended formalism with the extra ghost forms, one does not need to impose by hand the algebraic condition (5.9), and one can treat the forms as "free." The algebraic condition (5.9) is automatically enforced by the formalism when one imposes the closedness condition.

We have thus arrived at

Theorem 9. *The cohomologies of the original exterior derivative operator along the gauge orbits and of the extended exterior derivative d coincide.*

Proof. Any cohomological class of d contains a closed form involving only η^{a_0} (Theorem 7), which is vertical (Theorem 8) and thus also closed for d^{orig}. Moreover, if a vertical form γ is exact for d, $\gamma \approx d\alpha$, then α can be taken to depend only on η^{a_0} (Theorem 7) and must then be vertical (Theorem 8), so that γ is also exact for d^{orig}. It follows that the map that takes the cohomological classes of d^{orig} into those of d is an isomorphism. □

The conclusion of this section is that the extended operator d is an extension of the original vertical exterior derivative along the gauge orbits, which does not change its cohomology. The theorem that has been established is the analog of a theorem of free differential algebra theory, which states that the most general free differential algebra is the tensor product of a contractible and a minimal one [14]. If the constraints and the reducibility functions are split into independent and dependent ones, the ghosts η^{A_0} associated with the independent constraints belong to the minimal part carrying all the cohomology, whereas the extra ghosts belong to the cohomologically trivial contractible subalgebra.

4. BRST *Cohomology.* We have at this point everything needed to relate the classical BRST cohomology to the cohomology of the exterior derivative operator along the gauge orbits.

Indeed, we have shown that Ω incorporates both the Koszul–Tate operator δ and the exterior derivative operator d in the sense that

$$[F(q,p), \Omega] = (\delta + d)F + (\text{more}), \tag{5.18a}$$

$$[\eta^{a_1}, \Omega] = (\delta + d)\eta^{a_1} + (\text{more}), \tag{5.18b}$$

$$[\mathscr{P}_{a_1}, \Omega] = (\delta + d)\mathscr{P}_{a_1} + (\text{more}). \tag{5.18c}$$

Here, we have extended d to act on $\mathscr{P}_{a_1}$ as

$$d\mathscr{P}_{a_1} = 0. \tag{5.18d}$$

The abbreviation "more" in (5.18a) and (5.18b) refers to terms of "higher rank," coming from contributions that vanish if one sets $\mathcal{P}_{a_1}=0$ (after the bracket is computed) but that are nonzero otherwise. Similarly, "more" in (5.18c) refers to terms that vanish if one sets $\eta^{a_1}=0$. These extra terms mix the ghosts with their momenta in the BRST transformation. They are absent in the abelian group case (vanishing C's) with constant Z's but are present in general.

Now the Koszul–Tate operator δ is acyclic, and the cohomology of d is the cohomology of the exterior derivative along the gauge orbits. We have thus been able, by introducing extra ghosts and ghost momenta, to reach a point where the reducible case possesses exactly the same features as the irreducible one. Thus, by repeating exactly the same steps as in the irreducible case, one can prove

Theorem 10.

$$\left(\frac{\operatorname{Ker}\Omega}{\operatorname{Im}\Omega}\right)^{p}_{\text{classical}}=\begin{cases}0 & p<0\\ \left(\dfrac{\operatorname{Ker}d}{\operatorname{Im}d}\right) & p\geqq 0\end{cases}. \tag{5.19}$$

We refer the reader to [8] for a detailed proof, carried out on irreducible theories.

VI. Conclusion

The main conclusion of this paper is that the BRST analysis of reducible theories goes through in much the same way as in the irreducible case, once it is realized that the formalism can be expressed in terms of the Koszul–Tate differential, which is acyclic thanks to the presence of the extra ghosts.

In particular, the BRST generator always exists and is unique modulo a canonical transformation in the extended phase space. Furthermore, BRST invariance appears again as synonymous of gauge invariance at zero ghost number, since one has

$$\left(\frac{\operatorname{Ker}\Omega}{\operatorname{Im}\Omega}\right)^{\circ}_{\text{classical}}=\left(\frac{\operatorname{Ker}d}{\operatorname{Im}d}\right)^{\circ}, \tag{6.1}$$

and $(\operatorname{Ker}d/\operatorname{Im}d)^{\circ}$ just contains the gauge invariant functions.

This property, as well as the general cohomological analysis of the previous section, holds even if the reducibility of the theory is infinite. This is because when the ghost number is fixed, the tower of ghosts at antighost number zero is effectively cut at a finite order (of course, where the ghost tower is cut depends on the actual value of the ghost number under consideration).

Another interesting feature that arises from the analysis is the fact that the ghosts and the ghost momenta serve different purposes to start with. While the former can be viewed as forms along the gauge orbits and are thus related to the gauge generator aspect of the constraints, the latter are linked with the other role played by the constraints, namely, that of restricting the dynamics to the surface $G_{a_0}=0$. These two different aspects join forces in the BRST formalism through the extended phase space Poisson bracket structure, which is defined in such a way that the generators of the Kozsul–Tate complex (ghost momenta) become canonically conjugate to the generators of the exterior algebra along the gauge orbits (ghosts).

The proofs of the properties of the BRST formalism given here are largely based on the invariance of the BRST theory under canonical transformations. It would be of interest to compare the present analysis with other approaches that are of a more algebraic topological nature [15, 16]. Let us also note that the direct but lengthy pedestrian checking of the identity (4.16) worked out in [3] for the irreducible case can be extended to the irreducible case as well, provided one develops Ω in powers of the ghosts η instead of the ghost momenta $\mathscr{P}$. The details of this alternative method will not be given here.

Appendix: Conventions

In the main text, we consider three different nilpotent differential operators: the Koszul–Tate operator, the exterior derivative operator along the gauge orbits, and the BRST generator.

The gradings associated with these operators are, respectively, the antighost number, the pure ghost number, and the (total) ghost number. Hence, the Koszul–Tate operator is a differential operator of degree -1 in the antighost number, the exterior derivative d is a differential operator of degree $+1$ in the pure ghost number, whereas the BRST generator defines a differential operator of degree $+1$ in the ghost number.

The only variables with nonvanishing antighost number are the momenta $\mathscr{P}_{a_s}$. The only variables with nonvanishing pure ghost number are the ghosts η^{a_s}. The (total) ghost number is the difference between the pure ghost number and the antighost number. Therefore, one finds

$$\operatorname{antigh}(\mathscr{P}_{a_s}) = -\operatorname{gh}(\mathscr{P}_{a_s}); \quad \operatorname{pure\,gh}(\mathscr{P}_{a_s}) = 0, \tag{A.1}$$

$$\operatorname{pure\,gh}(\eta^{a_s}) = \operatorname{gh}(\eta^{a_s}); \quad \operatorname{antigh}(\eta^{a_s}) = 0, \tag{A.2}$$

and the Koszul–Tate operator is of degree $+1$ in the ghost number.

Concerning the exterior calculus on a supermanifold, we adopt the conventions of Berezin and Leites [11] and those in differential graded algebras [14], characterized by the fact that the exterior product is governed by a single grading, called here the Grassmann parity. Hence, if A and B are, respectively, p-forms and q-forms of Grassmann parity ε_A and ε_B, one has

$$AB = (-)^{\varepsilon_B \varepsilon_A} BA \tag{A.3}$$

with no explicit reference to the form degree of A or B. The Grassmann parity of the p-form A is equal to the sum of its degree as a form plus its "intrinsic grading." The only difference is that the differentials here are all derivatives from the right.

One has

$$\varepsilon(AB) = \varepsilon_A + \varepsilon_B, \tag{A.4a}$$

$$\varepsilon(dA) = \varepsilon_A + 1, \tag{A.4b}$$

so that d possesses Grassmann parity $+1$,

$$\varepsilon(d) = 1. \tag{A.5}$$

It results that the Lie derivative operator also carries Grassmann parity $+1$. Thus, if the constraints G_{a_0} are of Grassmann parity ε_{a_0}, the associated vector fields X_{a_0},

$$X_{a_0} F = [F, G_{a_0}] \tag{A.6a}$$

possess Grassmann parity $\varepsilon_{a_0} + 1$,

$$\varepsilon(Z_{a_0}) = \varepsilon_{a_0} + 1. \tag{A.6b}$$

Berezin and Leites's conventions are particularly convenient because the algebraic manipulation of functions, forms, and vectors are the same as in an ordinary Grassmann algebra.

Another convention adopted here is that our derivative operators (vector fields, d) act from the right. Hence, one has, for instance,

$$d(AB) = A\, dB + (-)^{\varepsilon_B}(dA)B. \tag{A.7}$$

This is done in order to comply with standard Hamiltonian conventions where, as in (A.6a), canonical generators are usually written to the right in the Poisson brackets.

With these conventions, the explicit expression for the exterior derivative $d\alpha$ of the p-form α is given by

$$\begin{aligned}(d\alpha)(X_0, \ldots, X_p) = (-)^{\sum_{i=0}^{p} \varepsilon_{X_i}} \Bigg[&\sum_{j=0}^{p} (-)^{\rho_j + \varepsilon_{X_j}} \alpha(X_0, \ldots, \hat{X}_j, \ldots, X_p) \overleftarrow{\mathscr{L}}_{X_j} \\ &+ \sum_{0 \leq i < j \leq p} (-)^{\rho_{ij} + \varepsilon_{X_j} + 1} \alpha(X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p, [X_i, X_j]) \Bigg],\end{aligned} \tag{A.8}$$

where ^ denotes omission, $f \overleftarrow{\mathscr{L}}_{X_j}$ is the right derivative of f along the vector field X_j, and $[X_i, X_j]$ stands for the (graded) Lie bracket of X_i and X_j. Also, the phases ρ_i and ρ_{ij} in (A.8) are just the phases resulting from moving X_i (and X_j) past the other X_k and are given by

$$\rho_j = \sum_{k=j+1}^{p} \varepsilon_{X_j} \varepsilon_{X_k}, \tag{A.9a}$$

$$\rho_{ij} = \rho_i + \rho_j + \varepsilon_{X_i} \varepsilon_{X_j}. \tag{A.9b}$$

Here, ε_{X_j} is the Grassmann parity $\varepsilon(X_j)$ of the vector X_j.

Finally, we note that, given a basis ω^{A_1} of 1-forms, one can expand any p-form as

$$\alpha = \frac{1}{p!} \omega^{A_1} \cdots \omega^{A_p} \alpha_{A_1 \cdots A_p} \tag{A.10a}$$

$$= \frac{1}{p!} \tilde{\alpha}_{A_1 \cdots A_p} \omega^{A_p} \cdots \omega^{A_1}, \tag{A.10b}$$

the functions $\alpha_{A_a \cdots A_p}$ and $\tilde{\alpha}_{A_1 \cdots A_p}$ are the "left–left" and "right–right" components of α and differ by an elementary phase. They possess the appropriate symmetry properties

$$\alpha_{A_1 \cdots A_i A_{i+1} \cdots A_p} = (-)^{\varepsilon(\omega_{A_i})\varepsilon(\omega_{A_{i+1}})} \alpha_{A_1 \cdots A_{i+1} A_i \cdots A_p}, \tag{A.11}$$

etc. . . .

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