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On the Invariant Mass Conjecture in General Relativity

Piotr T. Chruściel*,**
Physics Department, Yale University, New Haven, CT 06511, USA

Abstract. An asymptotic symmetries theorem is proved under certain hypotheses on the behaviour of the metric at spatial infinity. This implies that the Einstein–von Freud–ADM mass can be invariantly assigned to an asymptotically flat four dimensional end of an asymptotically empty solution of Einstein equations if the metric is a no-radiation metric or if the end is defined in terms of a collection of boost-type domains.

1. Introduction

One of the still unsolved classical problems in general relativity is to establish well-posedness of at least one of the existing definitions of energy-momentum at spatial infinity of an asymptotically flat space-time. Whatever the framework used to define energy-momentum [Ei, We, ADM, Ge, AH, Som, AD] the problems arising are closely related to the one which appears when one tries to define it via the so-called von Freud superpotential [vF]:

$$p_\mu = \lim_{R\to\infty} \frac{3}{16\pi} \int_{\substack{r(x)=R \\ x^0=\text{const}}} \delta^{\alpha\beta\gamma}_{\lambda\nu\mu}\eta^{\lambda\rho}\eta_{\gamma\sigma}g^{\nu\sigma}_{,\rho}\,dS_{\alpha\beta},\ r(x)=\{\textstyle\sum(x^i)^2\}^{1/2}. \tag{1.1}$$

To make sense of (1.1) one selects some asymptotically Minkowskian coordinates $\{x^\mu\}$ in which the metric $\mathfrak{g}$ takes the form[1]

$$\mathfrak{g}(\partial/\partial x^\mu, \partial/\partial x^\nu) = g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ h_{\mu\nu}(x) = \mathcal{O}_1(r(x)^{-\alpha}), \tag{1.2}$$

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[1] The signature is $+2$, greek indices run from 0 to 4, latin ones from 1 to 3, $\eta_{\mu\nu}$ is the flat Minkowski metric, $dS_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}dx^\alpha \wedge dx^\beta/2$, $\varepsilon_{0123} = 1$. We shall write $f = \mathcal{O}_n(r^\alpha)$, $\alpha\in\mathbb{R}$, if f satisfies $|f| \leq C\sigma^\alpha$, $|\nabla_\mu f| \leq C\sigma^{\alpha-1}$, $|\nabla_{\mu_1}\cdots\nabla_{\mu_n} f| \leq C\sigma^{\alpha-n}$, with $\sigma = (1+r^2)^{1/2}$, for some constant C, $O(r^\alpha) = \mathcal{O}_0(r^\alpha)$, $f = o(r^\alpha)$ if $\lim_{r\to\infty} r^{-\alpha} f = 0$, r^0 is always understood as $\ln r$. $B(R)$ and $S(R)$ denote a coordinate ball and sphere of radius R respectively (if ambiguities are likely to occur the coordinate sphere in e.g. coordinates y will be denoted by $S_y(R)$, etc.). Letters C, C', etc. are used throughout to denote strictly positive constants which may vary from line to line

with some $1/2 < \alpha \leqq 1$ (cf. [Ba, Ch1, Ch2, OM, Sol]), and the integrals (1.1) are performed in this coordinate system. If another system of coordinates $\{y^\mu\}$ in which analogous inequalities are satisfied is chosen

$$\mathrm{g}(\partial/\partial y^\mu, \partial/\partial y^\nu) = g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}, h'_{\mu\nu}(y) = \mathcal{O}_1(r(y)^{-\alpha}),$$
$$r(y) = \{\textstyle\sum (y^i)^2\}^{1/2}, \tag{1.3}$$

one will obtain

$$p'_\mu = \lim_{R\to\infty} \frac{3}{16\pi} \int_{\substack{r(y)=R \\ y^0=\mathrm{const}}} \delta^{\alpha\beta\gamma}_{\lambda\nu\mu} \eta^{\lambda\rho} \eta_{\gamma\sigma} g'^{\nu\sigma}_{,\rho} dS_{\alpha\beta}. \tag{1.4}$$

Is there any relationship between p_μ and p'_μ? If there is the slightest physical meaning in the above prescription p and p' should differ at most by a Lorentz transformation. It is well known that this will be the case if the coordinates y^μ and x^μ differ by a boost together with, eventually, a "supertranslation" ζ^μ:

$$y^\mu = \Lambda^\mu_\nu x^\nu + \zeta^\mu, \quad \Lambda - \text{a Lorentz matrix}, \quad \zeta = \mathcal{O}_2(r^{1-\alpha}) \tag{1.5}$$

with $\alpha > 1/2$ (cf. [Ch2, OM], and also e.g. [Tr, We]), if Einstein equations are satisfied with $T_{\mu\nu} = O(r^{-3-\varepsilon})$. For more general coordinate transformations there may be no simple relation between p and p', which is well illustrated by the following "mass generating coordinate transformations": starting from a Minkowski metric $-d\tau^2 + \Sigma(dy^i)^2$ (for which $p' = 0$) and performing the coordinate transformation

$$\tau = t + ar(x)^{1/2}, \quad y^i = (1 + br(x)^{-1/2})x^i, \quad a, b \in \mathbb{R},$$

one obtains a non-vanishing p unless $a = \pm b$. The expected legitimacy of the Einstein–von Freud–Arnowitt–Deser–Misner–Geroch–Ashtekar–Hansen prescription for defining energy relies upon the following conjecture, or some variation thereof:

Asymptotic Symmetries Conjecture. All twice differentiable coordinate transformations preserving the boundary conditions (1.2) are of the form (1.5).

A slightly different formulation of the mass problem in which various difficulties become separated is the following: p_μ can be defined purely in terms of Cauchy data on an asymptotically flat three dimensional region N of a spacelike hypersurface Σ—a "three dimensional end" or, shortly, a "three-end" (a set diffeomorphic to the complement of a ball in $\mathbb{R}^3$ on which the gravitational Cauchy data satisfy fall off conditions analogous to (1.2), cf. e.g. [ADM, Ba, Ch 1])—and the invariant mass $m(N) = (-p^\mu p_\mu)^{1/2}$ can be calculated. The asymptotic symmetries conjecture, if true, would imply the following:

Invariant Mass Conjecture. Let M be an asymptotically flat end of a Lorentzian manifold $\mathcal{M}$, with metric g satisfying vacuum Einstein equations, let N_1 and N_2 be two asymptotically flat three-ends included in M, with invariant masses $m(N_1)$ and $m(N_2)$. We have

$$m(N_1) = m(N_2).$$

The first obstacle one meets when trying to prove such a statement is our lack of knowledge of long time behaviour of solutions of Einstein equations: we want to speak about a four dimensional asymptotic region—a "four dimensional end", or, shortly, a "four-end"—which is large enough to include every spacelike asymptotically flat hypersurface lying in the manifold under consideration[2], and we need to impose some asymptotic conditions on the behaviour of the metric in this four dimensional region. What should these conditions be to have compatibility with a large class of solutions of vacuum Einstein equations? There is at least one class of space-times for which no doubts about sensible boundary conditions arise—the stationary asymptotically flat metrics. The asymptotic symmetries theorem we show implies that the invariant mass conjecture holds for these metrics.

Another well understood description of spatial infinity is obtained within the Ashtekar–Hansen conformal approach [AH]. While there is certainly some interest in proving the invariant mass theorem in such a framework, it is well known that the Ashtekar–Hansen asymptotic conditions are much more restrictive than what is compatible with Einstein equations (cf. [COM]). One can try to relax some of the spi conditions, retaining from it the light cone structure at spatial infinity together with some Hoelder continuity requirements on the rescaled metric at i^0. It is however not known whether even such weakened conditions are generically compatible with Einstein equations. To avoid this drawback we shall assume only what is rigorously known from the boost theorem [COM]: in this picture the basic blocks from which spatial infinity can be constructed are boost-type domains[3]:

$$\Omega_{\theta,R,T} = \{x^\mu : r \geqq R, |t| \leqq \theta r + T\} \quad \text{for some} \quad \theta, R, \geqq 0, \quad T \in \mathbb{R}.$$

Since the constant T plays little role in our considerations we shall often write $\Omega_{\theta,R}$ instead of $\Omega_{\theta,R,T}$. Whenever useful for the clarity of the presentation we shall write $\Omega^x_{\theta,R}$ and $\Omega^y_{\theta,R}$ to stress that we are speaking about x or y coordinates boost-type domains. Given asymptotically flat Cauchy data for vacuum Einstein equations (cf. [COM] for the appropriate definition of "asymptotically flat" in terms of weighted Sobolev spaces) for every $0 \leqq \theta < 1$ there exists R, $T \in \mathbb{R}$ and a metric $g_{\mu\nu}$ defined in $\Omega_{\theta,R,T}$, solution of vacuum Einstein equations, such that

$$|g_{\mu\nu}(x) - \eta_{\mu\nu}| \leqq C(\theta, R, T, g_{\mu\nu})\sigma(x)^{-\alpha},$$
$$|\partial g_{\mu\nu}(x)/\partial x^\sigma| \leqq C(\theta, R, T, g_{\mu\nu})\sigma(x)^{-\alpha-1}, \quad \sigma = (1 + r^2)^{1/2},$$

with an $\alpha > 0$. We shall say that two three-ends N_1 and N_2 lie in the same four-end if there exists a boost-type domain $\Omega_{\theta,R}$ such that $N_1 \subset \Omega_{\theta,R}$, $N_2 \subset \Omega_{\theta,R}$. We show that if the asymptotically flat three dimensional coordinate systems on N_i are restrictions to N_i of some regular four dimensional coordinate systems then the coordinate transformations relating these coordinate systems to the reference coordinates in $\Omega_{\theta,R}$ are of the form (1.5). In the vacuum case this implies that the

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[2] More precisely, if many "spatial infinities" are present as in the case of e.g. Reissner-Nordrstroem solutions, we want the end to include the "large r regions" of all spacelike hypersurfaces which "extend to the spatial infinity under consideration"

[3] Strictly speaking $\Omega_{\theta,R}$ deserves the name of a boost-type domain for $\theta < 1$ only. We shall however use this name for $\Omega_{\theta,R}$ even if $\theta \geqq 1$. It should also be noted that the term "domain" does not have its familiar mathematical meaning here since our $\Omega_{\theta,R}$ is closed

invariant mass conjecture holds under some mild supplementary conditions on the induced metric and extrinsic curvature (cf. [COM]; it is not too difficult to show from the results of Christodoulou and O'Murchadha that every appropriately regular coordinate system in a three-end can be extended to a four-dimensional one in at least a small boost-type domain).

For notational simplicity $\dim \mathcal{M} = 4$ will always be assumed, although the asymptotic symmetries theorem holds whatever $\dim \mathcal{M} \geqq 2$.

2. Asymptotic Conditions

In this section we shall present in detail the regularity and asymptotic conditions on the metric and the coordinate systems.

Definition 1. A coordinate system $\Phi_x = \{x^\mu\}$ defined on a subset M_x of a Lorentzian manifold $\mathcal{M}$ with be called α-admissible or, shortly, admissible, if there exist θ_x, $R_x \in \mathbb{R}^+$, $T_x \in \mathbb{R}$ such that

1) $\Phi_x(M_x) \supset \Omega_{\theta_x, R_x, T_x}$ with $\Omega_{\theta_x, R_x, T_x} = \{x^\mu : r \geqq R_x, |t| \leqq \theta_x r + T_x\}$,

2) in $\mathcal{O}_x = \Phi_x(M_x)$ we have, with $g_{\mu\nu} = \mathrm{g}(\partial/\partial x^\mu, \partial/\partial x^\nu) \in C_2(\mathcal{O}_x)$,

$$0 < -g_{00}, \quad \forall X^i \in \mathbb{R}^{n-1}, \quad 0 < g_{ij} X^i X^j, \tag{2.1}$$

3) for $0 < \Psi < \theta_x$, for $x \in \Omega_{\Psi, R_x, T_x}$, with $\sigma(x) = \{1 + \sum (x^i)^{1/2}$:

$$|g_{\mu\nu}(x) - \eta_{\mu\nu}| \leqq C_x(\Psi)\sigma(x)^{-\alpha}, \quad |\partial g_{\mu\nu}(x)/\partial x^\sigma| \leqq C_x(\Psi)\sigma(x)^{-\alpha-1}, \tag{2.2}$$

with some function $C_x(\Psi) < \infty$.

Let us note that (2.1)–(2.2) imply the existence of strictly positive functions $a_x(\Psi)$, $c_x(\Psi)$ such that, for $x \in \Omega_{\Psi, R_x, T_x}$,

$$a_x(\Psi) \leqq -g_{00}, \quad \forall X^i \in \mathbb{R}^{n-1} \, c_x(\Psi) \sum (X^i)^2 \leqq g_{ij} X^i X^j. \tag{2.3}$$

As discussed in the introduction, these conditions are compatible with a large class of solutions of Einstein equations [COM]. The first condition requires the coordinate system to cover a large, boost-type region. The second demands that t be a regular time coordinate, and that the slices $t = \mathrm{const}$ be riemannian manifolds. The last condition requires the metric to tend uniformly to the flat one in space-like directions, allowing however for gravitational radiation: $C_x(\Psi)$ may blow up to ∞ when Ψ tends to θ_x, which one expects to occur if $\theta_x = 1$ and gravitational radiation is present.

Definition 2. Let x^μ and y^μ be admissible coordinate systems, with appropriate constants θ_x, R_x, θ_y, R_y, etc. The three end $N_R^x = \{x^\mu : x^0 = 0, r(x) \geqq R\}$, $R \geqq R_x$, will be said to lie within the asymptotic four end defined by the coordinates y^μ and the three end $N_{R_y}^y = \{y^\mu : y^0 = 0, r(y) \geqq R_y\}$ if $\Phi_x^{-1}(N_R^x) \subset M_y$ (M_y being the domain of definition of the coordinates y) and if

$$\Phi_y \circ \Phi_x^{-1}(N_R^x) \subset \Omega_{\theta, R_y}^y \quad \text{for some} \quad \theta < \theta_y.$$

If $\theta_y > 1$ we shall say that the metric g is a no-radiation metric.

Let us note that our main theorem will still hold with the condition $\theta < \theta_y$ replaced by $\theta \leqq \theta_y$ provided $C_y(\theta_y) < \infty$.

It must be emphasized that in this work an asymptotic four-end is defined by a single reference coordinate system which covers at least some boost-type domain. Thus Minkowski space-time has one asymptotic four-end while Minkowski space-time from which one has removed the set of points $\{t=0, r \geqq R\}$ has none (unless $\dim \mathscr{M}=2$ in which case it has four asymptotic ends). It is worth realizing that in $\mathbb{R}^4 \backslash \{t=0, r \geqq R\}$ an asymptotically flat metric for which the invariant mass conjecture does not hold can be constructed: for $t<0$ and $r>2R>4m_1$ let the metric be the Schwarzschild metric with mass m_1, for $t>0$ and $r>2R>4m_2$ let the metric be the Schwarzschild metric with mass m_2, and interpolate smoothly between these metrics in any way; vacuum Einstein equations are satisfied everywhere for $r>2R$, the metric tends to the flat one as r goes to infinity, a single coordinate system covering the whole space-time exists; still the slices $t=\text{const}$ do not all have the same mass if $m_1 \neq m_2$.

To prove the asymptotic symmetries theorem we have to assume that the hypersurface $x^0=0$ lies within the region where the coordinates y^μ are well behaved in the sense of Definition 1. On the other hand one would be tempted to expect that asymptotic flatness of N_R^x forces it in fact to extend to this region, if e.g. space-time can be covered by a single coordinate chart. That this need not be the case is shown by the following example, due to R. Geroch: let $g_{\mu\nu}$ be the Minkowski metric outside the future light cone of the point $y_0=(-1,0,0,0)$, let the coordinates y^μ be the standard Minkowskian coordinates on $\mathbb{R}^4$. Whatever the metric in the future light cone of y_0 the asymptotic conditions of Definition 1 will be satisfied with $\theta_y=1$, because in every $\Omega_{\theta,R}$ with $\theta<1$ for sufficiently large r one will be in the region where $g_{\mu\nu}$ is flat. Let $\varphi(s)$ be any smooth function defined on the positive real axis which tends to minus infinity as s goes to zero and tends to infinity as s goes to infinity. Let $\bar{y}^\mu = y^\mu - y_0^\mu$. For $\bar{y}^\mu \bar{y}_\mu < 0$ let $t(y)=x^0(y) \equiv \varphi(-\bar{y}^\mu \bar{y}_\mu)$, for definiteness let $x^i = y^i$[4], finally let $\mathscr{S}$ be the hypersurface $t(x) = -(1+r(x)^2)^{1/2}$. Let the metric be equal to $-dt^2 + \sum (dx^i)^2$ for y^μ such that $y^0 > s(\vec{y})$, where $s(\vec{y})$ is the height function of $\mathscr{S}$ considered as a graph over the hypersurface $y^0=0$; interpolate between this metric and the flat one outside the future light cone of y_0 in any way. In this example the asymptotic symmetries conjecture fails to be satisfied which is easy to understand because the four-ends defined by the hypersurfaces $x^0=0$ and $y^0=0$ can be considered as physically distinct (in the conformal picture this space-time has two i^0's). It should be stressed that vacuum Einstein equations are satisfied in every boost-type domain with $\theta<1$ for sufficiently large $r(x)$ or $r(y)$.

3. Asymptotic Symmetries

Throughout this section we shall assume that the coordinate systems x^μ and y^μ are admissible, that the coordinates y^μ are the defining coordinates for the four-end under consideration, and that $\underline{N}_R^x = \Phi_y \circ \Phi_x^{-1}(N_R^x) \subset \Omega^y_{\theta,R_y,T_y}$ with $\theta < \theta_y$.

Lemma 1. *Let* $\Phi = \Phi_y \circ \Phi_x^{-1}$, *let* $\Phi_t(x^i) = \{\Phi^j(x^0=t, x^i)\}$, *let* $\mathscr{N}_R^x = \Phi_0(N_R^x)$.

[4] The actual definition of the coordinates x^i is irrelevant in this example

a) Φ_0 *is a diffeomorphism between* N_R^x *and* $\mathcal{N}_R^x$ *(in particular* $\underline{N}_R^x = \Phi(N_R^x)$ *is a graph over* $\{y^\mu : y^0 = 0, \vec{y} \in \mathcal{N}_R^x\}$).

b) *Shifting the coordinates* y^μ *by a constant vector if necessary we may assume that* $0 \notin \mathcal{N}_R^x$. Let $\chi(R) = \min_{x \in S(R)} r(\Phi_0(x))$. χ *is strictly monotonous.*

Proof. a) Suppose that the matrix $\partial y^i/\partial x^j$ is degenerate, i.e. there exists a vector $A^i \neq 0$ such that $(\partial y^i/\partial x^j)A^j = 0$. Since $\partial y^\mu/\partial x^\alpha$ is non-degenerate we have $(\partial y^\mu/\partial x^j)A^j \neq 0$, which implies that the image by Φ^* of the vector $A^j\partial/\partial x^j$, which is spacelike by virtue of our hypotheses on g, is the vector $(\partial y^0/\partial x^j)A^j\partial/\partial y^0$, which is timelike in virtue of our hypotheses on g'—a contradiction.

b) At every point x_R for which $\chi(R)$ is attained $y^i(\partial y^i/\partial r)$ must be different from zero, simultaneously positive or simultaneously negative for all such points, because $y^i(\partial y^i/\partial x^A) = 0$ at these points, where x^A are the angular variables θ and ϕ. If $y^i(\partial y^i/\partial r)$ were equal to zero y^i would be a zero eigenvalue eigenvector of $\partial y^i/\partial x^j$, contradicting point a) above. The sign of $(\partial r(y(x))/\partial x^i)x^i/r(x)$ is x_R-independent because the image by Φ_* of the outwards pointing vector field $\partial/\partial r$ is either everywhere outwards pointing or everywhere inwards pointing.

Remark. In what follows we shall often identify $\underline{N}_R^x = \Phi(N_R^x)$ with N_R^x.

The "transformation laws" of the metric and of the connection coefficients

$$g_{\alpha\beta}(x) = g'_{\mu\nu}(y(x))\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}, \tag{3.1}$$

$$\Gamma^\alpha_{\beta\gamma}(x) = \left\{\Gamma'^\mu_{\nu\rho}(y(x))\frac{\partial y^\nu}{\partial x^\beta}\frac{\partial y^\rho}{\partial x^\gamma} + \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\gamma}\right\}\frac{\partial x^\alpha}{\partial y^\mu} \tag{3.2}$$

can be considered as a set of differential equations satisfied by the transformation functions $y^\mu(x)$. With the notation

$$a = -g_{00}, \quad a' = -g'_{00}, \quad b_i = g_{0i}, \quad b'_i = g'_{0i}, \quad \tau = y^0, \quad t = x^0, \tag{3.3}$$

(3.1) written out in detail leads to the following set of equations:

$$a'\left(\frac{\partial\tau}{\partial t}\right)^2 = a + g'_{ij}\frac{\partial y^i}{\partial t}\frac{\partial y^j}{\partial t} + 2b'_i\frac{\partial y^i}{\partial t}\frac{\partial\tau}{\partial t}, \tag{3.4}$$

$$a_{ij} \equiv g'_{kl}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} = g_{ij} + a'\tau_{,i}\tau_{,j} - b'_k\left(\tau_{,i}\frac{\partial y^k}{\partial x^j} + \tau_{,j}\frac{\partial y^k}{\partial x^i}\right), \tag{3.5}$$

$$g'_{kl}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial t} = a'\frac{\partial\tau}{\partial t}\tau_{,i} + b_i - b'_j\left(\frac{\partial\tau}{\partial t}\frac{\partial y^j}{\partial x^i} + \frac{\partial\tau}{\partial x^i}\frac{\partial y^j}{\partial t}\right). \tag{3.6}$$

It is also useful to write (3.2) in the form

$$\frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta} = \Gamma^\sigma_{\alpha\beta}(x)\frac{\partial y^\mu}{\partial x^\sigma} - \Gamma'^\mu_{\nu\rho}(y(x))\frac{\partial y^\nu}{\partial x^\alpha}\frac{\partial y^\rho}{\partial x^\beta}. \tag{3.7}$$

Lemma 2. *There exist positive constants* c, c' *such that*

$$\forall x \in N_R^x \; cr(x) - c' \leqq r(\Phi(x)). \tag{3.8}$$

Proof. It is sufficient to establish (3.8) for sufficiently large $r(x)$. Suppose first that $r(y(x))$ goes to infinity as $r(x)$ does; therefore one may choose $\rho(\varepsilon)$ such that $|b'_i| \leqq \varepsilon$ on $\mathcal{N}^x_\rho = \Phi_0(N^x_\rho)$. On $\mathcal{N}^x_\rho$ parametrized by x^i we can define the riemannian metric $\mathbf{a} = a_{ij}dx^i dx^j$ (a_{ij} defined in (3.5)) (this is *not* the natural metric induced on $\mathcal{N}^x_\rho$ by g). Let us show that

$$\forall X^i \in \mathbb{R}^{n-1} c_1 \sum (X^i)^2 \leqq a_{ij} X^i X^j \tag{3.9}$$

for some constant $c_1 > 0$. Contracting (3.5) with g'^{ij}—the 3 dimensional inverse of g'_{ij}—and using the inequalities

$$g'^{ij} a_{ij} \geqq C^{-1} |\partial y/\partial x|^2,$$

$$\left| b'_k \tau_{,i} g'_{ij} \frac{\partial y^k}{\partial x^j} \right| \leqq \varepsilon C |\nabla\tau| |\partial y/\partial x| \leqq \varepsilon C(|\nabla\tau|^2 + |\partial y/\partial x|^2)/2,$$

$$C^{-1} \leqq g'^{ij} g_{ij} \leqq C,$$

$$C^{-1} |\nabla\tau|^2 \leqq a' \tau_{,i} \tau_{,j} g'^{ij} \leqq C |\nabla\tau|^2,$$

which follow from (2.2)–(2.3) one obtains, for ε small enough

$$C^{-1}(1 + |\nabla\tau|^2) \leqq |\partial y/\partial x|^2 \leqq C(1 + |\nabla\tau|^2), \tag{3.10}$$

where

$$|\nabla\tau|^2 = \sum \tau_{,i}\tau_{,i}, \quad |\partial y/\partial x|^2 = \sum \left(\frac{\partial y^i}{\partial x^j} \right)^2.$$

Contracting (3.5) with $X^i X^j$ we have

$$a_{ij} X^i X^j = g_{ij} X^i X^j + a'(\tau_{,i} X^i)^2 - 2b'_i \frac{\partial y^i}{\partial x^j} X^j \tau_{,k} X^k,$$

therefore the estimation, which makes use of (3.10),

$$b'_i \frac{\partial y^i}{\partial x^j} X^j \tau_{,k} X^k \leqq \varepsilon C |\nabla\tau| (1 + |\nabla\tau|^2)^{1/2} \{\sum (X^i)^2\}^{1/2} \leqq \varepsilon C(1 + |\nabla\tau|^2) \{\sum (X^i)^2\}^{1/2},$$

together with (2.2)–(2.3) lead indeed to (3.9) for all $r(x) > \rho(\varepsilon_0)$, for some ε_0. Let us show that (3.9) implies (3.8). Let $R = \inf_{y \in \Phi_0(S(\rho(\varepsilon_0)))} r(y)$, $R_0 = \sup_{y \in \Phi_0(S(\rho(\varepsilon_0)))} r(y)$. For $r(y(p)) > R$ let $\sigma(p)$ be the geodesic distance with respect to the metric $\mathbf{a}$ from p to $S_y(R)$. Parametrizing $\mathcal{N}^x_\rho$ with the coordinates y^i (Lemma 1) we have by definition

$$\sigma(y) = \inf_\Gamma \int_\Gamma \left(g'_{ij} \frac{dy^i}{dt} \frac{dy^j}{dt} \right)^{1/2} dt,$$

and inf is taken over piecewise differentiable curves joining y with $S_y(R)$. Let $\Gamma_0 = \{ty + (1-t)Ry/r(y), t \in [0, 1]\}$. We have clearly

$$\sigma(y) \leqq \int_{\Gamma_0} \left(g'_{ij} \frac{dy^i}{dt} \frac{dy^j}{dt} \right)^{1/2} dt \leqq C(r(y) - R). \tag{3.11}$$

Let $\sigma'(x)$ be the geodesic distance (still with respect to **a**) from $p=x$ to $S_x(\rho(\varepsilon_0))$. Since for $x_1 \in S_x(\rho(\varepsilon_0))$ $R \leqq r(y(x_1)) \leqq R_0$ we have, for $r(y(x)) > R_0$,

$$\sigma'(p) \leqq \sigma(p). \tag{3.12}$$

In coordinates x^i the metric **a** is represented by the matrix a_{ij}. By virtue of (3.9) we have

$$\int_\Gamma \left(a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt \geqq c_1^{1/2} \int_\Gamma \left(\sum \frac{dx^i}{dt} \frac{dx^i}{dt} \right)^{1/2} dt$$

for $r(x) > \rho(\varepsilon_0)$, so that

$$\sigma'(x) = \inf_{\Gamma'} \int_{\Gamma'} \left(a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt \geqq c_1^{1/2} \inf_{\Gamma'} \int_{\Gamma'} \left(\sum \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt = c_1^{1/2}(r(x) - \rho(\varepsilon_0)) \tag{3.13}$$

(the last equality being the consequence of the well known variational inequality for the geodesic distance with respect to the flat metric). Equation (3.11) together with (3.12) and (3.13) give (3.8) if $\lim_{R\to\infty} \chi(R) = \infty$, χ being defined in the statement of Lemma 1.

Suppose that $\lim_{R\to\infty} \chi(R) = r_0 < \infty$. Let x_N be any sequence of points x_R, with $R = N \in \mathbb{N}$, for which the infinimum $\chi(N)$ is attained. By hypothesis the sequence $y_N = \Phi(x_N)$ is bounded in space, therefore by $\Phi_y \circ \Phi_x^{-1}(N_R^x) \subset \Omega_{\theta,R_y}$ it is also bounded in time so that we may choose a subsequence y_i converging to a point y_∞. Let A^α_β be the matrix which transforms $g'_{\mu\nu}(y_\infty)$ to the Minkowski metric: $g'_{\mu\nu}(y_\infty) A^\mu_\alpha A^\mu_\beta = \eta_{\alpha\beta}$, and perform the coordinate transformation $y^\mu \to A^\mu_\nu y^\nu - y^\mu_\infty$, the new coordinates and the pulled-back metric still being denoted by y^μ and $g'_{\mu\nu}$. By continuity of $g'_{\mu\nu}$ there exists a ball of radius ρ_1 such that $|b'_i| < \varepsilon$ in the hypercube $C(\rho_1) = \{|y^i| \leqq \rho_1, |y^0| \leqq \rho_1\}$, with ε small enough so that the algebraic manipulations leading to (3.9) (with some other constant c_2) can be performed. Let $N \in \mathbb{N}$ be large enough so that $\forall i > N$ $y(x_i) \in C(\rho_1/2)$. Let $S^i_x(\varepsilon)$, $B^i_x(\varepsilon)$ denote the coordinate sphere and ball of radius ε around x_i. Let $\chi_i(\varepsilon) = \inf_{x \in S^i_x(\varepsilon)} r(y(x) - y(x_i))$. Since Φ_0 is a diffeomorphism the image by Φ_0 of $B^i_x(\varepsilon)$ contains the coordinate ball $B^i_y(\chi_i(\varepsilon))$ of radius $\chi_i(\varepsilon)$ centered at $y_i = y(x_i)$. Let us fix i and let $x(\varepsilon)$ be any point on $S^i_x(\varepsilon)$ such that $r(y(x(\varepsilon)) - y(x_i)) = \chi_i(\varepsilon)$. Let σ_i denote the geodesic distance from $y(x(\varepsilon))$ to y_i with respect to the metric **a**. Decreasing ρ_1 if necessary one derives, by an argument very similar to the one presented above, the inequalities

$$c_2^{1/2} r(x - x_i) \leqq \sigma_i(y(x)) \leqq 2(r(y(x) - y_i))$$

which hold for $r(y(x) - y_i) \leqq \rho_1/2$. If $\chi_i(1/4) \geqq \rho_1/2$ then $\Phi_0(B^i_x(1/4))$ contains the ball $B^i_y(\rho_1/2)$. If $\chi_i(1/4) \leqq \rho_1/2$ we have

$$\chi_i(1/4) = r(y(x(\varepsilon = 1/4)) - y_i) \geqq \sigma_i(y(x(1/4)))/2 \geqq c_2^{1/2}/8,$$

so that in any case $\Phi_0(B^i_x(1/4))$ contains the ball $B^i_y \equiv B^i_y(c_4)$, with $c_4 = \min(c_2^{1/2}/8, \rho_1/2)$. Consider now the sequence of space balls B^i_x of radius 1/4 centered around x_i. We have $B^i_x \cap B^j_x = \varnothing$ for $i \neq j$. Their images $\Phi_0(B^i_x)$ contain balls B^i_y of radius c_4, $B^i_y \subset C(\rho_1)$, and clearly $B^j_y \cap B^i_y = \emptyset$ for $i \neq j$. This leads to a contradiction since the infinite sequence of disjoint balls of constant radius B^i_y cannot be contained in the compact cube $|y^i| \leqq \rho_1$.

Lemma 3. *Suppose that the functions $\partial y^\mu/\partial x^\nu$ are uniformly bounded on N^x_R. There exists a Lorentz matrix Λ such that, for $x \in N^x_R$,*

$$y^\mu = \Lambda^\mu_\nu x^\nu + \zeta^\mu, \quad \partial\zeta = \mathcal{O}_1(r^{-\alpha}). \tag{3.14}$$

Proof. Equation (3.7) with (3.8) show that

$$\frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta} = O(r(x)^{-1-\alpha}),$$

and an easy argument making use of the lemma of Appendix A establishes (3.14).

Lemma 4. *Either one of the quantities $A = \sum(\partial x^0/\partial y^\nu)^2$ or $B = \sum(\partial y^0/\partial x^\nu)^2$ is bounded, in which case (3.14) holds, or $\lim_{r\to\infty} A = \infty = \lim_{r\to\infty} B$, and there exists a constant C such that, for $r \geqq \max(1, R_x)$,*

$$A \geqq Cr^\alpha, \quad B \geqq Cr^\alpha.$$

Proof. The hypotheses on $g_{\mu\nu}$ and $g'_{\mu\nu}$ imply, by a simple algebraic argument which makes e.g. use of the ADM formulae relating $g_{\mu\nu}$ and $g^{\mu\nu}$, that the matrices $g^{\mu\nu}$ and $g'^{\mu\nu}$ have uniformly bounded coefficients. The equations

$$\frac{\partial y^\mu}{\partial x^\alpha} = g'^{\sigma\mu} g_{\alpha\beta} \frac{\partial x^\beta}{\partial y^\sigma}, \quad \frac{\partial x^\beta}{\partial y^\sigma} = g^{\alpha\beta} g'_{\mu\sigma} \frac{\partial y^\mu}{\partial x^\alpha}$$

show that $\sum((\partial y^\mu/\partial x^\sigma))^2 < \infty$ is equivalent to $\sum((\partial x^\mu/\partial y^\sigma))^2 < \infty$. Manipulations similar to the proof of (3.10) starting from (3.4) imply

$$C^{-1}\left\{1 + \left(\frac{\partial \vec{y}}{\partial t}\right)^2\right\} \leqq \left(\frac{\partial \tau}{\partial t}\right)^2 \leqq C\left\{1 + \left(\frac{\partial \vec{y}}{\partial t}\right)^2\right\}. \tag{3.15}$$

Equations (3.10) and (3.15) imply that boundedness of A is equivalent to boundedness of $\sum((\partial x^\mu/\partial y^\sigma))^2$.

Let $D = (\sum(\partial y^\mu/\partial x^\sigma)^2)^{1/2}$. Equation (3.15) shows in particular that D is strictly separated from zero. Let $E = D^{-1}$. From (3.7) and (3.8) we have

$$\frac{\partial E}{\partial x^\alpha} = -\sum \frac{\partial y^\mu}{\partial x^\beta} \frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta} / D^3 = O(r(x)^{-1-\alpha}) \tag{3.16}$$

and Appendix A shows that E has a finite direction independent limit E_∞ as r tends to infinity. If E_∞ is different from zero D is bounded, therefore all the derivatives $\partial y^\mu/\partial x^\sigma$ are bounded and (3.14) follows by Lemma 3. If $E_\infty = 0$, (3.16) implies $E \leqq C\sigma^{-\alpha}$, and from what has been said the estimate on A and B follows.

Proposition 1. *Suppose there exists a spacelike geodesic $\Gamma \subset \Omega^x_{\theta,R_x}$, $\theta < \theta_x$, $\Gamma: [0,\infty) \ni s \to x(s)$ such that $\lim_{s\to\infty} r(x(s)) = \infty$ and such that for $s > s_0$ there exists a family of timelike differentiable curves $\Gamma_s: [0,1] \ni u \to y_s(u) \in \Omega^y_{\theta',R_y}$, $\theta' < \theta_y$, $x_s(u) \equiv$*

$x(y_s(u)) \in \Omega^x_{\theta,R_x}$, satisfying $x_s(0) = (0, \vec{x}(s))$, $x_s(1) = (x^0(s), \vec{x}(s))$, $r(x_s(u)) \geqq cr(x_s(0))$, $r(y_s(u)) \geqq cr(y_s(0))$ for some constant c. Then (3.14) holds.

Proof. Let $F = \ln D$, $D = (\sum (\partial y^\mu / \partial x^\sigma)^2)^{1/2}$. From (3.7) we have

$$dF = D^{-2} \sum \frac{\partial y^\mu}{\partial x^\beta} \frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta} dx^\alpha = f_\alpha dx^a + f'_a dy^a, \tag{3.17}$$

where

$$f_a = D^{-2} \sum \frac{\partial y^\mu}{\partial x^\beta} \Gamma^\sigma_{\alpha\beta} \frac{\partial y^\mu}{\partial x^\sigma} \leqq C\sigma(x)^{-\alpha-1},$$

$$f'_a = D^{-2} \sum \frac{\partial y^\mu}{\partial x^\beta} \Gamma'^\mu_{\alpha\rho} \frac{\partial y^\rho}{\partial x^\beta} \leqq C\sigma(y)^{-\alpha-1}. \tag{3.18}$$

Consider the Eq. (3.17) along Γ:

$$dF/ds = f_\alpha dx^\alpha/ds + f'_\alpha dy^\alpha/ds. \tag{3.19}$$

By Proposition B1 (Appendix B) we have for large s $x^\mu(s) = s(X^\mu_\infty + o(1))$, $y^\mu(s) = s(Y^\mu_\infty + o(1))$ for some constant vectors X^μ_∞ and Y^μ_∞, and also $r(x(s)) \geqq C^{-1}s - C'$, $r(y(s)) \geqq c^{-1}s - c'$ for some positive constants c, C, c', C', the right-hand side of (3.19) is therefore integrable in s and F remains bounded along Γ. The differentiable timelike curve Γ_s can be parametrized by t: $[0, x^0(s)] \ni t \to x_s(t) = (t, \vec{x}_s(t)) \equiv (x^0(y_s(u(t))), \vec{x}(y_s(u(t))))$, timelikeness of $x_s(t)$ and our hypotheses on the coordinate system imply $|(d\vec{x}_s/dt)| \leqq C$. Let E be as in Lemma 4. By our hypotheses (3.16) holds again and we have

$$|E(x^\mu(s)) - E(0, \vec{x}(s)| = \left| \int_0^{x^0(s)} dE/dt dt \right| \leqq C' r^{-\alpha}$$

($|x^0(s)| \leqq \theta r(s) + T_x$), therefore $E_\infty = \lim_{s \to \infty} E(x^\mu(s)) \neq 0$ (E_∞ as in the proof of Lemma 4) and the result follows by Lemma 4.

As a corollary of Proposition 1 one obtains immediately the asymptotic symmetries conjecture under a somewhat stronger assumption than $N^x_R \subset \Omega^y_{\theta,R_y}$:

Corollary 1. *Suppose that there exist $0 < \varepsilon < \theta_x$, $R_1 \geqq R_x$, such that $\Omega^x_{\varepsilon,R_1} \subset \Omega^y_{\theta,R_y}$, $\theta < \theta_y$. Then (3.14) holds.*

Proof. By Proposition B2 we can construct a complete spacelike geodesic $\Gamma \subset \Omega^x_{\varepsilon,R_1}$. The family of differentiable curves Γ_s is given by Γ_s: $[0, 1] \ni u \to x_s(u) = (ux^0(s), \vec{x}(s)) \in \Omega^x_{\varepsilon,R_1} \subset \Omega^y_{\theta,R_y}$.

The rest of the proof will consist in showing that there exists a geodesic satisfying the hypotheses of Proposition 1. Although this will not be necessary for further considerations, it is of some interest to note that boundedness of $\partial\tau/\partial x^\sigma$ is controlled by $|(\partial\tau/\partial t)|$, as is the case for a Lorentz transformation. A simple algebra exercise leads to the following relationships between the components of a Lorentz matrix Λ^μ_ν:

$$\Lambda^i_0 = \pm \Lambda^0_i, \quad \sum |\Lambda^0_i|^2 = (\Lambda^0_0)^2 - 1, \quad \sum |\Lambda^i_j|^2 = 2 + (\Lambda^0_0)^2. \tag{3.20}$$

Proposition 2. *Either the derivatives $\partial y^\mu/\partial x^\nu$ are uniformly bounded on N^x_R or there exist constants r_0, c_1 and c_2 such that, for $r \geqq r_0$,*

$$\left|\frac{\partial \tau}{\partial t}\right| \geqq c_1 r^\alpha, \quad \sum \left|\frac{\partial \tau}{\partial x^i}\right| \geqq c_2 r^a.$$

Proof. Let e^a_μ, e'^a_μ be tetrads, defined for sufficiently large r, satisfying

$$e^a_\mu(x) = \delta^a_\mu + h^a_\mu(x), \quad h^a_\mu = O(r(x)^{-\alpha}),$$
$$e'^a_\mu(y) = \delta^a_\mu + h'^a_\mu(y), \quad h'^a_\mu = O(r(y)^{-\alpha}),$$

(existence of such tetrads can be established by methods similar to e.g. the proof of Proposition 4.1 of [ChK]). Let $\Lambda^a_b = e'^a_\mu(\partial y^\mu/\partial x^\sigma)e^\sigma_b$, where e^σ_b is the inverse matrix to e^a_μ. Equation (3.1) shows that Λ^a_b is a Lorentz matrix, Proposition 2 follows from (3.8), (3.10), (3.15), (3.20) and from the fact that both h^a_μ and h'^a_μ can be both made sufficiently small for sufficiently large $r(x)$.

By inspection of the slopes of the light cones one shows the following:

Lemma 7. *For all $0 < \varepsilon < \min(\theta_x, 1)$ there exists $R'_x \geqq R_x$ such that for all $x \in \Omega^x_{\varepsilon, R'_x, -\varepsilon R'_x}$ satisfying $x^0 > 0$ ($x^0 < 0$) and for every past (future) directed differentiable timelike curve $[0,1] \ni u \to p(u) \in \mathcal{M}$ satisfying $p(0) = x$, $p(1) \in N^x_{R_x}$, for which $p(u) \in N^x_{R_x} \Rightarrow u = 1$, we have $\forall u \in [0,1]$ $p(u) \in M_x$ (the domain of definition of the coordinates x) and $x(u) \equiv x(p(u)) \in \Omega^x_{\varepsilon, R'_x, -\varepsilon R'_x}$. Moreover there exists a constant c such that $c^{-1} r(x(1)) \leqq r(x(u)) \leqq c r(x(0))$.*

Theorem 1. (Asymptotic Symmetries at Spatial Infinity). *Let the coordinates $\{x^\mu\}$ and $\{y^\mu\}$ be α-admissible, $0 < \alpha \leqq 1$, let $y(x)$ be twice differentiable and suppose that there exist constants $R \geqq R_x$ and $\theta < \theta_y$ such that*

$$N^x_R = \{x^\mu : x^0 = 0, r(x) \geqq R\} \subset \Omega^y_{\theta, R_y, T_y} = \{y^\mu : r(y) \geqq R_y, |y^0| \leqq \theta r(y) + T_y\}.$$

There exists a Lorentz matrix Λ and a constant C such that $\forall x \in N^x_R$,

$$\begin{gathered} y^\mu = \Lambda^\mu_\nu x^\nu + \zeta^\mu, \quad |\zeta^\mu_{,\nu}| \leqq C(1+r)^{-\alpha}, \quad |\zeta^\mu_{,\sigma\nu}| \leqq C(1+r)^{-\alpha-1} \\ |\zeta^\mu| \leqq C(1+r)^{1-\alpha} \quad \text{for} \quad 0 < \alpha < 1, \quad |\zeta^\mu| \leqq C(1 + \ln(1+r)) \quad \text{if} \quad \alpha = 1. \end{gathered} \tag{3.21}$$

Proof. Multiplying τ by -1 if necessary we may assume that the local time orientations of coordinates x and y coincide (cf. (3.15)). Let $R_2 \geqq R$ be large enough so that, for an appropriate $R_3 \geqq R_y$, $\Phi_0(N^x_{R_2}) \subset \mathbb{R}^3 \backslash B^y(R_3) \subset \Phi_0(N^x_R)$ (cf. Lemma 1, point a)). Fix some $0 < \theta_0 < \min(1, \theta_x)$, let $R_4 \geqq \max(R'_x, R_2)$, $R'_x \equiv R'_x(\theta_0)$ given by Lemma 7. Increasing R_4 if necessary we may assume (cf. Proposition B2) that a) all the geodesics satisfying $x(0) \in \Omega_{\theta_0, R_4}$, $r(x(0)) = R_4$, $x^0(0) = 0$, $d\vec{x}/ds_{|s=0} = \vec{x}/R_4$, $|dt/ds|_{|s=0} \leqq \theta_0/2$ remain in Ω_{θ_0, R_4} for all $s > 0$, b) $dt/ds_{|s=0} \geqq \theta_0/4 \Rightarrow t(s) > 0$ for all $s > 0$, c) $dt/ds_{|s=0} \leqq -\theta_0/4 \Rightarrow t(s) < 0$ for all $s > 0$. Let $r_0 = \sup_{r(x) = R_4, x^0 = 0} r(y(x))$. Increasing R_4 if necessary we may assume that all the geodesics satisfying $y(0) \in \Omega^y_{\theta, r_0}$, $r(y) = r_0$, $|dy^0/ds|_{|s=0} \leqq \theta/2$, $d\vec{y}/ds_{|s=0} = \vec{y}/r_0$ remain in Ω^y_{θ, r_0} for all $s \geqq 0$. Let $x_0 \in \partial N^x_{R_4}$ be any of the points satisfying $r(y(x_0)) = r_0$. Consider the family of geodesics Γ_ε, $\varepsilon \in [-\theta_0/2, \theta_0/2]$, where Γ_ε is a geodesic starting radially outwards at x_0 ($d\vec{x}/ds_{|s=0} = \vec{x}/R_4$) with $dt/ds_{|s=0} = \varepsilon$. For $\varepsilon > \theta_0/4$ Γ_ε lies to the local future of N^x_R. Suppose that $y(x(s)) \in \Omega^y_{\theta, r_0}$, consider the timelike differentiable curve $\Gamma_{s,\varepsilon}: u \to y_s(u) =$

$(uy^0(s), \vec{y}(s))$. Since $\Phi_0(N^x_{R_4}) \subset \mathbb{R}^3 \backslash B^y(R_3) \subset \Phi_0(N^x_R)$, there exists $u_0(s)$ such that $y_s(u_0(s)) \in \Phi_0(N^x_R)$. For small s, u_0 must be smaller than 1 because Φ is a diffeomorphism and the local time orientations agree. This must also be true for all $s > 0$ such that $y(x(s)) \in \Omega^y_{\theta,r_0}$ because, if not, continuity would imply the existence of a point on Γ_ε which belongs to N^x_R; this does however not occur for $\varepsilon > \theta_0/4$. This shows that, for $\varepsilon > \theta_0/4$, either $\Gamma_\varepsilon \subset \Omega^y_{\theta,r_0}$ and then also $\Gamma_{s,\varepsilon} \subset \Omega^y_{\theta,r_0}$, for $u \in [u_0(s), 1]$, or there exists $s_0(\varepsilon)$ for which Γ_ε crosses $B^+ = \{y \in \Omega_{\theta,r_0,T_y}, y^0 = \theta r(y) + T_y\}$. Similarly for $\varepsilon < -\theta_0/4$ either $\Gamma_\varepsilon \subset \Omega^y_{\theta,r_0}$ or there exists $s_0(\varepsilon)$ for which Γ_ε crosses $B^- = \{y \in \Omega^y_{\theta,r_0,T_y}, y^0 = -\theta r(y) - T_y\}$. Let $\eta(\varepsilon) = dy^0(x(s))/ds_{|s=0}$. If $\Gamma_{\theta_0/2}$ crosses B^+ then we must have $\eta(\theta_0/2) \geqq \theta/2$, similarly if $\Gamma_{-\theta_0/2}$ crosses B^- then we must have $\eta(-\theta_0/2) \leqq -\theta/2$. If $\Gamma_{\theta_0/2} \subset \Omega^y_{\theta,r_0}$ set $\varepsilon_0 = \theta_0/2$, if $\Gamma_{-\theta_0/2} \subset \Omega^y_{\theta,r_0}$ set $\varepsilon_0 = -\theta_0/2$, if neither of the previous cases occurs, then by continuity of $\eta(\varepsilon)$ there exists ε_0 such that $-\theta/2 < \eta(\varepsilon_0) < \theta/2$, and so in all cases $\Gamma_{\varepsilon_0} \subset \Omega^y_{\theta,r_0}$. The family of curves Γ_{s,ε_0} parametrized by u is included in Ω^y_{θ,r_0} for appropriate ranges of u, $u \in [u_1(s,\varepsilon_0), u_2(s,\varepsilon_0)]$, by Lemma 7 it is also included in $\Omega^x_{\theta_0,R_4}$, and the theorem follows by Proposition 1.

For the applications of the asymptotic symmetries theorem it is useful to have more information about the geometry of the domain of overlap of coordinates x and y. We have the following:

Corollary 2. *Under the hypotheses of Theorem 1:*

1) *For every θ', $\theta < \theta' < \theta_y$ there exists $R_1 \geqq R$ and $\varepsilon > 0$ such that $\Omega^x_{\varepsilon,R_1} \subset \Omega^y_{\theta',R_y}$.*

2) *For all $\theta_1 < \theta_x$, $\theta_2 < \theta_y$, $R \geqq R_x$, if $\Omega^x_{\theta_1,R} \subset \Omega^y_{\theta_z,R_y}$, then there exists a constant C such that (3.21) holds for all $x \in \Omega^x_{\theta_1,R}$.*

Outline of proof. Point 1) can be established by considering timelike geodesics orthogonal to N^x_R. Methods of Appendix B show that such geodesics remain both in $\Omega^x_{\varepsilon,R_1}$ and Ω^y_{θ',R_y} for an appropriate range of affine parameter, which can be used to establish the result. Point 2) is essentially Corollary 1 (cf. also Appendix A).

In some applications one would like to be able to claim that (3.21) holds throughout the overlap region. This is clealy false in general when the overlap has disconnected components. The following corollary shows that (3.21) holds indeed in the "largest x^0-connected component" of the overlap:

Corollary 3. *Under the hypotheses of Theorem 1, suppose that either*

a) $\theta_1 < \theta_x$, $\theta_2 < \theta_y$, *or*

b) $\theta_1 \leqq \theta_x$, $\theta_2 \leqq \theta_y$ *and* $C_x(\theta_x) < \infty$, $C_y(\theta_y) < \infty$.

For $\vec{x} \in N_{R_x}$ and $(s, \vec{x}) \in \Omega^x_{\theta_1,R_x}$ define

$$t^+(\vec{x}) = \begin{cases} \text{the least positive value of } s \text{ for which the curve} \\ s \to y(s) = y(x^0 = s, \vec{x}) \text{ meets } \partial\Omega^y_{\theta_2,R_y}, \\ \theta_1 r(x) + T_x \text{ if } y(s) \text{ is included in } \Omega^y_{\theta_2,R_y}, \end{cases}$$

$$t^-(\vec{x}) = \begin{cases} \text{the largest negative value of } s \text{ for which the curve} \\ s \to y(s) = y(x^0 = s, \vec{x}) \text{ meets } \partial\Omega^y_{\theta_2,R_y}, \\ -\theta_1 r(x) - T_x \text{ if } y(s) \text{ is included in } \Omega^y_{\theta_2,R_y}. \end{cases}$$

Let $V_{\theta_1,\theta_2} = (x : r(x) \geqq R_x, t^- \leqq x^0 \leqq t^+\}$. There exists a constant C such that (3.21) holds for all $x \in V_{\theta_1,\theta_2}$.

Outline of Proof. One considers timelike geodesics starting from $N^x_{R^0}$ along the x^0 axis, for sufficiently large R^0. Methods of Appendix B give uniform estimates on the coordinate components of the velocity vector of the geodesics both in the x and y charts, arguments similar to the proof of Proposition 2 establish (3.21) for r larger than some constant and the claim follows.

Let us finally mention that in the case of a stationary metric we can set $\theta_y = \infty$, and we can also assume $\forall \Psi \; C_y(\Psi) < C_y^\infty < \infty$, $a_y(\Psi) > a_y^\infty > 0$, $c_y(\Psi) > c_y^\infty > 0$. Under such hypotheses the asymptotic symmetries theorem remains true (the only modifications of the proof occur in the proof of Lemma 2 where time-boundedness of appropriate sequences follows from spacelikeness of N^x_R). As mentioned in the introduction this implies the invariant mass theorem without having to assume a priori that the three-end N^x_R lies within a finite boost domain of y-coordinates—it suffices that N^x_R lies within the domain of definition and regularity of the y coordinates.

Conclusions

We have shown that the asymptotic symmetries conjecture holds provided the hypersurfaces $x^0 = \text{const}$ and $y^0 = \text{const}$ lie within a finite boost of each other ($\theta_y \leqq 1$) or if the metric is a no-radiation metric ($\theta_y > 1$). The example by R. Geroch discussed in Sect. 2 shows that our results cannot be improved without some further hypotheses on the metric. One can write down various supplementary conditions which will ensure *a priori* that any two spacelike three-ends will lie within some finite boost of each other, it seems however that any physically significant progress will only be made possible after we will have reached a better understanding of the infinite time behaviour of solutions of vacuum or asymptotically vacuum Einstein equations.

Appendix A

Lemma. *Let a differentiable function $f: \Omega_{\theta,R} \to \mathbb{R}$ satisfy*

$$\sup_{r(x)=R} |f(0,x)| \leqq C, \quad |\partial_\mu f| \leqq C\sigma^{-1-\alpha}, \quad \alpha > 0.$$

Then f has a direction-independent limit f_∞ as r goes to infinity and

$$f - f_\infty = \mathcal{O}_1(r^{-\alpha}) \quad \text{in} \quad \Omega_{\theta,R}.$$

Proof. We have

$$|f(0,x^i) - f(0,\rho x^i/r(x))| = \left| \int_\rho^r \partial f/\partial r dr \right| \leqq C\rho^{-\alpha}, \quad \text{for} \quad \rho < r, \tag{A.1}$$

so that

$$\forall x \in \mathbb{R}^{n-1} \setminus B(R) \; |f(0,x)| \leqq \max_{y \in S(R)} |f(0,y)| + CR^{-\alpha}$$

which shows that $f(0,x)$ is uniformly bounded. Let r_i be any sequence of numbers tending to ∞—(A.1) implies that the sequence $f(0, r_i \vec{n})$ is a Cauchy sequence and therefore $f_\infty(\vec{n}) = \lim_{r_i \to \infty} f(0, r_i\vec{n})$ exists, (A.1) also implies that $f_\infty(\vec{n})$ is sequence-independent. Consider two unit vectors n_1 and n_2. We have

$$f_\infty(n_2) - f_\infty(n_1) = \int_\rho^\infty \partial f/\partial r(0, rn_1)dr - \int_\rho^\infty \partial f/\partial r(0, rn_2)dr + \int_\Gamma df, \tag{A.2}$$

where Γ is an arc lying on the sphere $S(\rho)$ joining ρn_1 with ρn_2. Equation (A.2) implies $|f_\infty(n_2) - f_\infty(n_1)| \leqq C\rho^{-\alpha}$ for any ρ, therefore f_∞ is direction independent. From (A.1) one finds that $|f(0,x) - f_\infty| \leqq Cr^{-\alpha}$. Finally

$$f(t,x) = f(0,x) + \int_0^t \partial f/\partial t dt,$$

which yields

$$|f(t,x) - f_\infty| \leqq Cr^{-\alpha} + C|t|r^{-\alpha-1},$$

which proves our assertion because $|t| \leqq \theta r + T$ in $\Omega_{\theta,R}$.

Appendix B

Proposition B1. *Let g be a C_2 metric in $\Omega_{\theta,R,T}$ satisfying*

$$|g_{\mu\nu}(x) - \eta_{\mu\nu}| \leqq C\sigma(x)^{-\alpha}, \quad |\partial g_{\mu\nu}(x)/\partial x^\sigma| \leqq C\sigma(x)^{-\alpha-1}, \quad 0 < \alpha \leqq 1.$$

For every complete non-timelike geodesic $\Gamma \subset \Omega_{\theta,R,T}$ such that $\lim_{s\to\infty} r(x(s)) = \infty$, where s is an affine parameter, there exists a vector η^μ_∞ such that

$$x^\mu(s) = \eta^\mu_\infty s + \zeta^\mu + x^\mu(0),$$

ζ^μ satisfying for $s \geqq 0$, with $r_0 = r(x(0))$,

$$|d\zeta^\mu(s)/ds| \leqq C(r_0 + s)^{-\alpha}, \quad |d^2\zeta^\mu(s)/ds^2| \leqq C(r_0 + s)^{-\alpha-1},$$
$$|\zeta^\mu(s)| \leqq Cs(r_0 + s)^{-\alpha} \quad \text{for} \quad 0 < \alpha < 1, \quad |\zeta^\mu(s)| \leqq C\ln(1 + s/r_0) \quad \text{if} \quad \alpha = 1.$$

Proof. Since $\lim_{s\to\infty} r(x(s)) = \infty$, for every s_1 there exists $s_0 > s_1$ such that $dr/ds_{|s=s_0} \geqq 0$. Let us parametrize the geodesic by an affine parameter such that $s_0 = 0$, $(dx^i/ds)(dx^i/ds)_{|s=0} = 1$. For $r_0 > r_1$, r_1 sufficiently large and for all s such that $3/4 < (dx^i/ds)(dx^i/ds) < 5/4$ non-timelikeness of dx/ds implies $|dt/ds| < 2$. A simple calculation gives

$$\frac{d^2r^2}{ds^2} = 2\left(\frac{dx^i}{ds}\frac{dx^i}{ds} - x^i\Gamma^i_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right),$$

which again for $3/4 < (dx^i/ds)(dx^i/ds) < 5/4$ and for $r(s) > r_1$ gives, increasing r_1 if necessary,

$$\frac{d^2r^2}{ds^2} > 1 \Rightarrow r^2(s) \geqq r_0^2 + s^2/2,$$

since $dr/ds_{|s=0} \geqq 0$. Let s_2 be such that for all $0 \leqq s < s_2$, $(3/4)^{1/2} \leqq |dx^i/ds| \leqq (5/4)^{1/2}$ ($s_2 > 0$ because $(dx^i/ds)(dx^i/ds)_{|s=0} = 1$ and dx/ds depends continuously on

s). For $s < s_2$ we have $|\Gamma^{\mu}_{\nu\rho}(dx^{\mu}/ds)(dx^{\nu}/ds)| \leqq C(r_0 + s)^{-1-\alpha}$ so that

$$\left|\frac{dx^{\mu}}{ds}(s) - \frac{dx^{\mu}}{ds}(0)\right| \leqq \int_0^s C(r_0 + u)^{-1-\alpha} du \leqq C r_0^{-\alpha}/\alpha,$$

therefore for sufficiently large r_0 we have $s_2 = \infty$ and the propostion follows.

Proposition B2. *Under the hypotheses of Proposition* **B1**, *for every* $\rho > r_1$ *there exist* $\theta_{\pm}(\rho)$ *satisfying* $-\theta \leqq \theta_- < 0 < \theta_+ \leqq \theta$ *such that every geodesic satisfying*

$$x(0) \in \Omega_{\theta,\rho,T}, \frac{dx^i}{ds}\Big|_{s=0} = x^i/r, \quad \theta_- \leqq \frac{dt}{ds}\Big|_{s=0} \leqq \theta_+ \tag{B.1}$$

remains in $\Omega_{\theta,\rho,T}$ *for all* $s > 0$, *and satisfies* $r(x(s)) \geqq (r_0 + s)/2$ (*in particular the estimates of* **B1** *hold*). *We have* $\lim_{\rho \to \infty} -\theta_-(\rho) = \lim_{\rho \to \infty} \theta_+(\rho) = \theta$. *There also exist* $\varepsilon_{\pm}(\rho)$ *satisfying* $\theta_-(\rho) < \varepsilon_-(\rho) \leqq 0 \leqq \varepsilon_+(\rho) < \theta_+(\rho)$ *such that if, for the above geodesics,* $t(0) \geqq 0$, $dt/ds_{|s=0} > \varepsilon_+(\rho)$, *then for all* s $t(s) \geqq 0$ (*if* $t(0) \leqq 0$, $dt/ds_{|s=0} < \varepsilon_-(\rho)$, *then for all* s $t(s) \leqq 0$). *We have* $\lim_{\rho \to \infty} \varepsilon_-(\rho) = \lim_{\rho \to \infty} \varepsilon_+(\rho) = 0$.

Proof. Let $\rho(\varepsilon)$ be large enough so that for any curve $x(s) \in \Omega_{\theta,\rho(\varepsilon),T}$ satisfying $|(dx^{\mu}/ds)| \leqq 2$ and $r(s) \geqq (r_0 + s)/2$ we have $\int_0^{\infty} |\Gamma^{\mu}_{\nu\rho}(dx^{\mu}/ds)(dx^{\nu}/ds)| \leqq \varepsilon$. By methods similar to the proof of Proposition B1 one shows that every radially outgoing geodesic with $x(0) \in \Omega_{\theta,\rho(\varepsilon),T}$ will satisfy

$$(1 - \delta)s + r_0 \leqq r(s) \leqq (1 + \delta)s + r_0,$$
$$(\eta - \delta)s + t_0 \leqq t(s) \leqq (\eta + \delta)s + t_0, \quad \text{where} \quad \eta = dt/ds_{|s=0},$$

with some $\delta(\varepsilon)$, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, provided $x(s)$ remains in $\Omega_{\theta,\rho(\varepsilon),T}$, and the result readily follows.

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Note added in proof. It should be stressed that throughout this paper the addition of a sub- or superscript x or y to a constant, e.g. c_x, C_y, $a_x(\psi)$ etc., does not indicate a pointwise dependence of the constants c, C, $a(\psi)$, but is meant to stress that the constant in question depends upon the coordinate system $\{x^\mu\}$ or $\{y^\mu\}$.