Commun. Math. Phys. 120, 121–148 (1988)

Communications in
Mathematical
Physics

Renormalizability and Infrared Finiteness of Non-Linear σ-Models: A Regularization-Independent Analysis for Compact Coset Spaces

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Abstract. The non-linear σ models in two space-time dimensions corresponding to compact homogeneous coset spaces G/H are studied with particular attention to three problems: first, independence of coordinate choice and regularization, second, the physical content of the theory, and finally the regularity of the "physics" in the infrared limit. Concerning in particular the physical content of the theory, we construct a set of local observables whose correlation functions depend on a finite number of parameters identified among those defining the metric tensor of the coset space. For these models, we give a general proof of renormalizability based on the introduction of a nilpotent BRS operator which describes the non-linear isometries and a classical action which contains a mass term for all quantized fields. The mass term belongs to a finite dimensional representation of the group G, which allows us to prove the conjecture that the correlation functions of local observables, i.e., the local operators invariant under G, are regular in the infrared limit.

1. Introduction

The non-linear σ-models were introduced more than 15 years ago [1, 2] to describe the infrared properties in $d>2$ space-time dimensions of systems with a symmetry spontaneously broken according to the Goldstone-Nambu mechanism. In 2 space-time dimensions – where the theory is power counting renormalizable – they appear as an interesting testing ground of theoretical ideas due to their asymptotic freedom property [3] and, more recently, because of their connection with the ground state of the string theories [4]. Consequently many efforts have been devoted to their investigation, both in perturbative and non-perturbative quantum field theory [5].

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In spite of these efforts, it is not very clear in what sense the non-linear σ-model is a complete field theory, i.e. a theory of observables in Hilbert space. Indeed, for a generic non-linear σ-model built on a compact manifold whose coordinates are the fields [6], one immediately encounters two problems:

i) the physical content of the theory should be given in terms of a *system of observables* whose Green functions should depend on a finite number of parameters to be considered as physical;

ii) in a perturbative framework, the theory is affected by *infrared singularities* which are due to the masslessness of the coordinate fields, which, in 2 space-time dimensions, excludes any physical interpretation for these fields.

These are, in our opinion, the two issues to be clarified. The first is connected with the coordinate choice on the manifold and the other with the infrared properties of the model. In the standard approach, one introduces a mass term as an infrared regulator, but then a generic Green function will not a priori possess a finite zero-mass limit. It is commonly believed that this infrared problem and the first one are linked in the sense that *observables – which correspond to coordinate frame-independent operators – and hence to intrinsic geometrical properties of the manifold* (such as the geodesic distance between two points) *do have infrared regular correlation functions:* the so-called Elitzur conjecture [7]. The idea is that the infrared singularities are connected with large coordinate fluctuations which should not affect the intrinsic geometrical properties of a compact manifold [8]. A parallel problem is that of the stability of the manifold, i.e. the problem of its parametrization. Indeed, the definition of a metric on a generic Riemannian manifold a priori requires an infinite number of parameters on which the geometry is going to depend. Consequently, from the point of view of perturbative renormalization, one immediately encounters the difficulty of treating an infinite number of normalization conditions, which makes the whole treatment ill defined.

For these reasons, and to investigate the above-mentioned problems in a well-defined context, *we restrict ourselves to the particular class of non-linear σ-models built on compact homogeneous* (coset G/H) *spaces* [9]. Then, up to a field redefinition, the metric depends only on a finite number of parameters and, as we shall show, the construction of these models as an operator theory of local observables in Hilbert space is possible within a perturbative approach.

This analysis generalizes the results obtained for the $O(N+1)/O(N)$ model in the orthogonal projection by Brézin, Le Guillou, and Zinn-Justin in 1976 for the ultraviolet problem [3] and by David in 1981 for the infrared limit (with a special choice of the mass term, i.e. one with a particular covariance under the isometries) [10a]. The choice of the orthogonal projection is very convenient since in these coordinates the renormalization is strictly multiplicative, which results in a considerable simplification of the analysis (see also [11] for other examples of σ-models in special coordinates). More recently, the generalization to other theories has brought about the necessity of studying the same model in other projections [12] or without reference to any special coordinate frame in order to rely only on the algebraic properties of the isometry group G [13]. Furthermore, the late interest in *supersymmetric* non-linear σ-models – which are involved in superstring theories – needs, lacking a consistent regularization which respects both the supersymmetry and the geometric properties [14], a regulator free treatment. This justifies the necessity of considering the *usual bosonic coset space non-linear σ-models within an approach which does not depend on the regularization and which is compatible with a suitably general choice of coordinates.* As pointed out above, and as seems to be necessary for the treatment of the ultraviolet problem, a model in this class has an action that depends on a finite number of parameters; on the other hand, Elitzur's conjecture is also expected to hold, provided one is able to construct an infrared regulating mass term depending on a finite number of parameters and hence belonging to a finite dimensional representation of G. This point has been overlooked in [10b].

To our knowledge, there are, up to now, no other classes of σ-models for which the "physics" is defined by a finite number of normalization conditions.

The paper is so organized: Sect. 2 contains the strategy of our approach to the ultraviolet renormalizability and the infrared limit. In this section, we give a non-technical description of our method and of the results. In Sect. 3, we introduce the notation for the non-linear σ-models built on homogeneous (coset) compact spaces and describe their classical action, including the mass term. The B.R.S. operator associated to the non-linear symmetry is also identified at the classical level and translated in functional form in view of quantization which is analyzed in Sect. 4. The renormalizability of the linear symmetry associated with any compact group, not necessarily semi-simple, is discussed in Appendix A and the tools needed to analyze the cohomology of the B.R.S. operator are given in Appendix B. In Sect. 5 we prove the validity of Elitzur's conjecture by means of a recursive relation derived from the Ward identities. Some final remarks are contained in the concluding section.

2. The Strategy

Let us now describe the strategy adopted in this paper to analyze the perturbative definition of a bosonic non-linear σ-model built on a homogeneous compact space, within an algebraic approach which depends neither on a coordinate choice nor on a regularization.

In the geometric approach to general non-linear σ-models, à la Friedan [6], a coordinate frame is attached to each point of the manifold and the first characterization of the theory is through its independence with respect to any particular coordinate choice. This is ensured by a system of identities which guarantee that changes of the reference point and of the coordinate system do not affect the action [6, 15]. For coset spaces G/H *the transitive action of the isometry group* G, connecting the different points of the manifold, ensures these independence properties. Therefore the theory is completely defined by the usual set of Ward identities corresponding to the generators of G. In particular, the generators of the isotropy subgroup H of the reference point, for a suitable but quite general class of coordinates *act linearly*, while the remaining symmetry G/H is non-linearly realized [2]. This in fact will be the only restriction on the coordinate choice. As a consequence, the classical action contains a "kinetic" part defined through the invariant metric g_{ij} (the coordinates being the fields ϕ^i $i=1, \ldots, N$) and a "mass" term which has H as its isotropy group and belongs to a finite dimensional representation of G. Due to *the compactness of* G, such a representation does exist as ensured by a mathematical theorem on G-spaces [16].

From the point of view of renormalization, the set of Ward identities separates naturally into two classes which are treated differently. Indeed, the renormalization of the linear symmetry can be discussed through a purely algebraic method, straightforward in the semi-simple case [17 a)]. Here, Abelian factors frequently appear in H, even when G is semi-simple [18]: therefore, in Appendix A, we generalize this method to the case of any *compact* group H.

The renormalization of the non-linear symmetry can then be analyzed on the restricted class of fully quantized actions which are invariant under H. Following the approach of [13], to each non-linear generator we associate an anticommuting c-number. These Faddeev-Popov parameters are used to transform the set of non-linear Ward identities into an equation à la B.R.S.: the resulting anticommuting external differential operator d becomes nilpotent when acting on H-invariant effective actions. The renormalizability proof is therefore reduced to the characterization of the first two cohomology classes of the d operator: the Faddeev-Popov neutral sector is expected to contain only the classical action while the class with a unit charge, to which possible anomalies belong, ensures, if it is empty, the renormalizability of the Ward identities. In this paper, the cohomology of the d operator is studied through a purely algebraic method, sufficient to obtain the desired result.

The whole analysis is based on the transitive action of G on the manifold (homogeneity). This guarantees that *the only scalar invariant functions are the constants*, a necessary condition to have an action defined by a finite number of physical parameters. Homogeneity also implies that the mass term is identified by a finite number of parameters, once the finite linear representation of G to whom it belongs is chosen: therefore, as an operator it will renormalize multiplicatively with a field-independent renormalization matrix [1]. Furthermore, homogeneity also implies that *the d operator is a "perturbation" of the external differential operator on the tangent space to the manifold.* The exact meaning of "perturbation" is defined by introducing in the space of formal power series in the coordinate fields a filtration [19] with respect to the number of fields: this isolates in d the flat space operator d_0 from the rest. Now *the cohomology of d_0 turns out to be trivial in the Faddeev-Popov charged sectors*, which, as explained in Appendix B, is sufficient to exclude the presence of anomalies and to obtain an isomorphism between the neutral sector of the cohomology of d and that of d_0. The latter is then shown to contain the same number of independent parameters as the classical theory, thereby ensuring the stability of these models and proving the ultraviolet renormalizability.

Finally we discuss the validity of Elitzur's conjecture for a generic compact homogeneous space: the first step is the identification of local observables *with the class of invariant operators under G*. The nonexistence of invariant scalar functions implies that non-trivial local invariant operators must depend upon the space-time derivatives of the fields, so the simplest ones are given by the invariant quadratic forms in these field derivatives. Hence they correspond to derivatives of

[1] For the $O(N+1)/O(N)$ case in the orthogonal projection discussed in [10a]', the natural choice of the mass term is the coordinate σ along the axis of projection and the finite linear representation is carried by π^i, σ alone: there is then only one mass parameter which renormalizes multiplicatively

the Lagrangian density with respect to the parameters defining the metric tensor. Thus the theory admits only a finite number of dimension-two local observables which, under renormalization, mix with constant (field-independent) coefficients [20]. In the case of irreducible symmetric spaces where the metric depends on only one parameter, the corresponding local observable, i.e. the Lagrangian density, renormalizes multiplicatively. Multilocal observables, for instance the finite geodesic distance on the manifold, have been considered in the first checks of Elitzur's conjecture [21] and provide dimensionless invariant quantities which are discussed by David in the $O(N+1)/O(N)$ case [10a]. In the concluding section, we indicate how our analysis could be extended to discuss the infrared behaviour of their Green functions too.

Having so defined the local observables, we analyze the zero-mass limit of Green functions built only with invariant operators at non-exceptional momenta in the sense of Symanzik [22]. The proof relies on known results concerning the infrared behaviour of Feynman graphs [23], in particular the fact that the dominant infrared singular part of a generic graph decomposes into a finite sum of contributions, each of which appears as a product of an I.R. regular subgraph (called a link) of the original diagram times the graph which is obtained contracting the link to a point. It is this last graph which carries the possible infrared singularities. The link contains all the external vertices, and the legs connecting it to the rest of the diagram are amputated and carry zero momentum. Furthermore, being a subgraph, it belongs to a lower perturbative order compared to the original diagram. Of course, the link with no external legs gives the regular part of the diagram.

The iterative argument of the proof is based on *the use of a Ward identity: the symmetry implies*, recursively with respect to their number of legs, *that the sum of all links with a given, non-zero, number of amputated legs at zero momentum vanishes with a power law in the zero-mass limit.* This is sufficient to guarantee the infrared regularity of the complete graph *since the infrared singular factors in its decomposition may diverge at most as powers of* $Ln\, m^2$ due to the fact that the action, in the zero-mass limit, contains only dimension-2 couplings. This scheme also parallels the one employed in [10a] to analyze the same problem for the sphere in the orthogonal projection.

3. The Classical Theory

3.1. The Algebra

It is well known that any compact homogeneous space, i.e. a space with a transitive action of a compact Lie group of isometries G, is isomorphic to a coset space G/H, where H is a closed subgroup [9]. Denoting with $\mathscr{G}$, $\mathscr{H}$ the corresponding Lie algebras, the commutation relations are:

$$[h_a, h_b] = f_{ab}{}^c h_c, \qquad h_a \in H; \qquad a,b,c=1,2,\ldots,L, \tag{3.1 a}$$

$$[h_a, W_i] = f_{ai}{}^k W_k, \qquad W_i \in G-H; \qquad i,j,k=1,2,\ldots,N, \tag{3.1 b}$$

$$[W_i, W_j] = f_{ij}{}^k W_k + f_{ij}{}^c h_c. \tag{3.1 c}$$

All structure constants are chosen to be real.

Notice that if a subset of the generators W_i commutes with the elements of $\mathscr{H}$, this subset generates a subgroup X of G[2]. Then we decompose $\mathscr{G}-\mathscr{H}$ into the generators of X, denoted by x_u, and a system of elements w_n such that

$$[h_a, x_u]=0, \tag{3.2a}$$

$$[h_a, w_n]=f_{an}{}^m w_m, \tag{3.2b}$$

$$[x_u, w_n]=f_{un}{}^m w_m, \tag{3.2c}$$

$$[x_u, x_v]=f_{uv}{}^w x_w, \tag{3.2d}$$

$$[w_n, w_m]=f_{nm}{}^k w_k+f_{nm}{}^u x_u+f_{nm}{}^c h_c. \tag{3.2e}$$

In order to have a description independent of any particular choice of coordinates, we adopt the parametrization of Coleman, Wess, and Zumino [2], where the coordinates on the coset space are given by a real scalar field $\phi^i(x)$ $(i=1,\ldots,N)$ such that for each value of ϕ^i there is an associated group element $L[\phi]$. If the coordinates ϕ^i and ϕ'^i are different, the corresponding group elements $L[\phi]$ and $L[\phi']$ belong to different equivalence classes in G/H. A group element g of G acts on ϕ^i in the following way:

$$L[\phi]\to L[\phi']=gL[\phi]\,h^{-1}[\phi,g], \tag{3.3}$$

where $h[\phi,g]$ is an element of the subgroup H, depending on g and ϕ, uniquely determined by the choice of the element $L[\phi]$.

The infinitesimal generators h_a, W_i of G may be realized as differential operators acting on functionals

$$h_a=-f_{ai}{}^j\int d^2x\phi^i(x)\frac{\delta}{\delta\phi^j(x)}, \tag{3.4a}$$

$$W_i=\int d^2x W_i^j[\phi]\frac{\delta}{\delta\phi^j(x)}, \tag{3.4b}$$

where, due to the homogeneity property of the coset space G/H,

$$W_i^j[\phi]=\delta_i^j+\mathcal{O}(\phi). \tag{3.5}$$

Notice that the generators (3.4b) provide a non-linear realization of the algebra of X. However, there are two ways to realize the subgroup X in G/H. The standard left action has already been described in formula (3.3). As X commutes with the subgroup H, one can also define a right action of X on G/H. A combination of the left and right actions may be realized linearly on ϕ^i through the formula

$$L[\phi]\to L[\phi']=x^{-1}L[\phi]x, \tag{3.6}$$

where x is an element of X. The algebra corresponding to this group action is spanned by the generators

$$\hat{x}_u=-f_{ui}{}^j\int d^2x\phi^i(x)\frac{\delta}{\delta\phi^j(x)},\qquad u=1,\ldots,N_x. \tag{3.7}$$

[2] This does occur in most non-symmetric compact coset spaces: one notable exception are the Kählerian manifolds where X contains only the identity

These linear generators satisfy, with the non-linear generators W_i and with themselves, the following commutation relations:

$$[\hat{x}_u, x_v] = f_{uv}{}^w x_w\,, \tag{3.8a}$$

$$[\hat{x}_u, w_m] = f_{um}{}^n w_n\,, \tag{3.8b}$$

$$[\hat{x}_u, \hat{x}_v] = f_{uv}{}^w \hat{x}_w\,, \tag{3.8c}$$

and

$$[\hat{x}_u, h_a] = 0\,. \tag{3.8d}$$

3.2. *The Classical Action*

3.2.a. The Invariants. The transitive action of the group G on the manifold expressed in Eq. (3.5) implies the following (essential) property:

on an homogeneous manifold G/H, the only scalar G-invariant functions of the coordinates ϕ^i are the constants.

Therefore, the classical invariant action must contain space-time derivatives of the fields, and we write it as [6],

$$A^{cl}{}_{(0)} = \int d^2x g_{ij}[\phi]\,\partial_\mu\phi^i(x)\,\partial^\mu\phi^j(x)\,, \tag{3.9}$$

where $g_{ij}[\phi]$ is the metric function on the manifold. The easiest way [2c)] to characterize $A^{cl}{}_{(0)}$ is to use the group elements $L[\phi]$ and the following elements of the Lie algebra:

$$L^{-1}[\phi]\,\partial_\mu L[\phi] = e_\mu{}^i[\phi]\,W_i + e_\mu{}^a[\phi]\,h_a\,.$$

An element g of G acts on $e_\mu{}^i$, $e_\mu{}^a$ according to Eq. (3.3) as

$$e_\mu{}^i[\phi]\,W_i \to e_\mu{}^i[\phi']\,W_i = h[\phi, g]\,e_\mu{}^i[\phi]\,W_i h^{-1}[\phi, g]\,, \tag{3.10a}$$

$$e_\mu{}^a[\phi]\,h_a \to e_\mu{}^a[\phi']\,h_a = h[\phi, g]\,e_\mu{}^a[\phi]\,h_a h^{-1}[\phi, g] + h[\phi, g]\,\partial_\mu h^{-1}[\phi, g]\,, \tag{3.10b}$$

and the classical action may then be written as

$$A^{cl}{}_{(0)} = \int d^2x \lambda_{ij} e_\mu{}^i[\phi]\,e_\mu{}^j[\phi]\,, \tag{3.11}$$

where the λ_{ij} are constant parameters. The conditions for the invariance of (3.11) under the full group G reduce to the invariance under the linear subgroup H, which reads:

$$f_{ai}{}^k\lambda_{kj} + f_{aj}{}^k\lambda_{ik} = 0\,, \tag{3.12}$$

i.e. λ_{ij} *is an invariant 2-tensor under H.* The number of linearly independent solutions of (3.12) is thus equal to the number of independent quadratic invariants under the action of H. The stability analysis of Sect. 4 will also show that, modulo a coordinate redefinition, this is the general solution. Therefore the classical action (3.9) depends on the finite number of parameters λ_{ij} determined through Eq. (3.12).

Notice that if there is a non-trivial subgroup X commuting with H, its generic infinitesimal element $\hat{x}$:

$$\hat{x} = 1 + \tau^u \hat{x}_u \tag{3.13a}$$

transforms, by Eqs. (3.6, 7), an invariant action with parameters λ_{ij} into a new invariant action with parameters

$$\lambda'_{ij}=\lambda_{ij}-\tau^u(f_{ui}{}^k\delta_j{}^l+f_{uj}{}^k\delta_i{}^l)\lambda_{kl}\,. \tag{3.13b}$$

This points out that, for a given choice of coordinates, independent invariant Lagrangians are connected by the action of the subgroup X. Therefore, it is a signal that among the parameters λ_{ij} there are some, whose number is equal to the dimension of the subgroup X, which are related to a coordinate redefinition and which we shall call "non-physical" in the zero mass limit. This redefinition is not a symmetry of the classical action, but it corresponds to a set of parametric differential generators; indeed from Eq. (3.13b) we have

$$\hat{x}_u A^{cl}{}_{(0)}=-(f_{ui}{}^k\delta_j{}^l+f_{uj}{}^k\delta_i{}^l)\lambda_{kl}\frac{\partial}{\partial\lambda_{ij}}A^{cl}{}_{(0)}\,. \tag{3.14}$$

We shall also see that this situation is altered by the introduction of an infrared regulating mass, but, as discussed in Sect. 5, it is recovered in the zero mass limit.

3.2.b. Infrared Regulator. As remarked in the Introduction, the classical action (3.9), where all fields ϕ^i are massless, is not suitable to discuss the possible quantum extensions of the theory and an I.R. regulator is needed.

The idea is to include in the action a term $v^0(\phi)$ which provides a mass to all fields, together with a set of partners $v^A(\phi)$ in such a way that $\{v^0, v^A\}$ carry a finite dimensional linear representation of G. In order to give mass to all fields, $v^0(\phi)$ must contain a term quadratic in ϕ^i, i.e.

$$v^0(\phi)=v^0(0)+a_{ij}\phi^i\phi^j+\mathcal{O}(\phi^3)\,, \tag{3.15}$$

where a_{ij} is a symmetric, positive definite matrix which can be reduced to the identity; now, recalling Eq. (3.5), we have

$$W_i v^0(\phi)=v^i=a_{ij}\phi^j+\mathcal{O}(\phi^2)\,, \tag{3.16}$$

so that the set $\{v^A\}$ must contain at least the elements v^i, which, according to Eq. (3.16), can be chosen as interpolating fields for $\phi^i(x)$[3].

Next, the general analysis of Coleman, Wess, and Zumino shows that the linear representation carried by $\{v^0, v^A\}$ cannot be arbitrary but it must contain the identity representation when restricted to the subgroup H. It is therefore natural to ask that v^0 be invariant under H (i.e. $h_a v^0=0$ for the algebra $\mathscr{H}$), and Eq. (3.16) then implies that H is also the isotropy group of v^0.

The existence of a set of functions $\{v^0, v^A\}$ carrying the linear representation of G with the desired properties is ensured by the following:

For any compact coset space G/H, one can always find a finite dimensional linear orthogonal representation of G – hereafter called the mass representation – containing at least one vector which has H as isotropy group.

This assertion follows directly from a mathematical theorem proved in [16].

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[3] The minimal situation where $\{v^A\}=\{v^i\}$ is realized in the $O(N+1)/O(N)$ symmetric model where the functions $\{v^0, v^i\}$ are explicitly given in [24]

Let now $D^{\alpha}{}_{\beta}(g)$ be the representative of the group element g in the mass representation and η label a basis for the H invariant subspace. Then the most general system of functions transforming according to the mass representation is

$$v^{\alpha}[\phi]=\sum_{\eta} D^{\alpha}{}_{\eta}(L[\phi])\mu^{\eta} \tag{3.17}$$

for an arbitrary choice of the μ^{η} parameters.

However, we wish to introduce v^0 as a mass term in the action: hence we have to be sure that *it does not contain linear contributions* in the fields ϕ^i. This condition is automatically satisfied if the subgroup X is trivial, otherwise it induces as many constraints on the μ^{η} parameters as there are infinitesimal generators of X, i.e. as there are fields invariant under H. Notice that one can neglect these constraints and reabsorb the linear terms through a right action of the subgroup X. The action of X on the μ^{η} parameters restores the constraints when the linear contributions are eliminated. After this we can identify the mass term in the Lagrangian with the vector v^0; the number of new parameters is equal to the difference between the H invariant independent directions in the mass representation and those in the representation carried by the fields ϕ^i.

In order to characterize the *covariance under the isometries of the mass term* in the action, we couple the v^{α}'s to dimension-2 external fields $K_{\alpha}(x)$ with vacuum value $q_{\alpha}=m^2\delta_{\alpha}{}^{0}$, and define the new classical action

$$A^{cl}=\int d^2x g_{ij}[\phi]\,\partial_{\mu}\phi^i(x)\,\partial^{\mu}\phi^j(x)+\int d^2x(K_{\alpha}(x)+q_{\alpha})\,v^{\alpha}[\phi]\,. \tag{3.18}$$

From the definition (3.17) we have the infinitesimal representations

$$[D(h_a)]^{\alpha}{}_{\beta}v^{\beta}=(T_a)^{\alpha}{}_{\beta}v^{\beta}\,, \tag{3.19a}$$

$$[D(W_i)]^{\alpha}{}_{\beta}v^{\beta}=(T_i)^{\alpha}{}_{\beta}v^{\beta}\,, \tag{3.19b}$$

and this new action is invariant under the transformations generated by the modified operators

$$\mathbf{h}_a=h_a+\int d^2xK_{\alpha}(x)\,(T_a)^{\alpha}{}_{\beta}\frac{\delta}{\delta K_{\beta}(x)}, \tag{3.20a}$$

$$\mathbf{W}_i=W_i+\int d^2x(K_{\alpha}(x)+q_{\alpha})\,(T_i)^{\alpha}{}_{\beta}\frac{\delta}{\delta K_{\beta}(x)}. \tag{3.20b}$$

Notice the absence of q_{α} in Eq. (3.20a) due to the invariance of v^0 under h_a. Clearly $\mathbf{h}_a$ and $\mathbf{W}_i$ in Eqs. (3.20) obey the algebra (3.1).

If the subgroup X commuting with H is non-trivial, its generic element x acts on the mass terms in Eq. (3.18) as:

$$\begin{aligned}&\sum(K_{\alpha}+q_{\alpha})\,D(L[\phi])^{\alpha}{}_{\eta}\mu^{\eta}\to\sum(K_{\alpha}+q_{\alpha})\,D(x^{-1}L[\phi]x)^{\alpha}{}_{\eta}\mu^{\eta}\\&=\sum(K_{\alpha}+q_{\alpha})\,D(x^{-1})^{\alpha}{}_{\beta}\,D(L[\phi])^{\beta}{}_{\gamma}\,D(x)^{\gamma}{}_{\eta}\mu^{\eta}\,,\end{aligned} \tag{3.21}$$

which amounts to a linear transformation of the sources K_{α} and a linear mixing of the parameters μ^{η}. In particular, recalling that x acts non-trivially on v^0 (since it does not belong to its stability group), the action on the parameters μ^{η} provides a faithful representation of the group X. According to this discussion we can extend the parametric equations in (3.14) to the complete classical action

$$\left(\hat{x}_u + \int d^2x [K_\alpha(x) + q_\alpha] (T_u)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)}\right) A_{cl}$$
$$= \left(\mu^\eta (T_u)^\zeta{}_\eta \frac{\partial}{\partial \mu^\zeta} - \lambda_{ij}(f_{uk}{}^i \delta_l{}^j + f_{ul}{}^i \delta_k{}^j) \frac{\partial}{\partial \lambda_{kl}}\right) A_{cl}. \tag{3.22}$$

Notice that these parametric equations refer not only to a change of parameters defining the action, but also to a variation of the covariance of the infrared regulator and hence of the symmetry properties of the model. Moreover, these equations are compatible with the constraint that the classical action does not contain linear terms since the violation induced by the μ^η dependent term on the right-hand side is compensated by the action on q_α on the left-hand side.

3.3. B.R.S. Symmetry

The symmetry (3.20a) corresponding to the H subgroup is linearly realized; therefore its quantum implementability can be discussed at the level of Ward identities. On the other hand, a non-linear symmetry such as (3.20b) is more easily analyzed by means of a B.R.S. operator, as proposed for the $O(N+1)/O(N)$ case in [13]. Following these lines we introduce anticommuting, positively charged parameters C^i, C^a and the anticommuting, negatively charged, sources $\gamma_i(x)$, which are assigned canonical dimension equal to two. The new classical action is

$$\Gamma_{cl} = A_{cl} + \int d^2x [C^i W_i^j[\phi] - C^a f_{ai}{}^j \phi^i(x)] \gamma_j(x). \tag{3.23}$$

The tree approximation connected functional generator $Z_c^{cl}[J, C, \gamma] = \Gamma_{cl} + \int d^2x J_i(x) \phi^i(x)$ satisfies

$$\int d^2x \left(-J_i(x) \frac{\delta}{\delta \gamma_i(x)} - (K_\alpha(x) + q_\alpha)(C^i T_i + C^a T_a)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)}\right) Z_c^{cl}(J, C, \gamma)$$
$$+ \left(\frac{1}{2} C^i C^j f_{ij}{}^k \frac{\partial}{\partial C^k} + \frac{1}{2} C^a C^b f_{ab}{}^c \frac{\partial}{\partial C^c} + \frac{1}{2} C^i C^j f_{ij}{}^a \frac{\partial}{\partial C^a} + C^a C^i f_{ai}{}^j \frac{\partial}{\partial C^j}\right)$$
$$\times Z_c^{cl}(J, C, \gamma) \equiv S Z_c^{cl}(J, C, \gamma) = 0. \tag{3.24}$$

Since the C^a parameters appear only in couplings linear in the fields, we also have

$$\frac{\partial}{\partial C^a} Z_c^{cl}(J, C, \gamma) + \int d^2x f_{ai}{}^j \gamma_j(x) \frac{\delta}{\delta J_i(x)} Z_c^{cl}(J, C, \gamma) \equiv \mathcal{R}_a Z_c^{cl}(J, C, \gamma) = 0. \tag{3.25}$$

Anticommuting the two operators in (3.24) and (3.25), we obtain:

$$\{\mathcal{R}_a, S\} = \int d^2x \left(-J_j(x) f_{ai}{}^j \frac{\delta}{\delta J_i(x)} - \gamma_j(x) f_{ai}{}^j \frac{\delta}{\delta \gamma_i(x)} - K_\alpha(x) (T_a)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)}\right)$$
$$+ \left(C^i f_{ai}{}^j \frac{\partial}{\partial C^j} + C^b f_{ab}{}^c \frac{\partial}{\partial C^c}\right) \equiv -\mathcal{H}_a + f_{ab}{}^c C^b \frac{\partial}{\partial C^c}, \tag{3.26a}$$

and the equation

$$\left(\mathscr{H}_a - f_{ab}{}^c C^b \frac{\partial}{\partial C^c}\right) Z_c^{cl}(J, C, \gamma) = 0 \tag{3.26b}$$

expresses the invariance of the classical theory under the linear subgroup H.

A notable simplification occurs if we now suppress the C^a parameters, thus decoupling the analysis of the linear symmetry from that of the non-linear one. Computing Eq. (3.24) for $C^a = 0$ and taking (3.25) into account, we get the final B.R.S. identity corresponding to the non-linear invariance

$$\begin{aligned} &SZ_c(J, K, C, \gamma) \\ &\equiv \int d^2x \left(-J_i(x)\frac{\delta}{\delta\gamma_i(x)} - (K_\alpha(x) + q_\alpha)\, C^i (T_i)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)}\right) Z_c(J, K, C, \gamma) \\ &\quad + \frac{1}{2} C^i C^j \left(f_{ij}{}^k \frac{\partial}{\partial C^k} - f_{ij}{}^a f_{al}{}^k \int d^2x \gamma_k(x) \frac{\delta}{\delta J_l(x)}\right) Z_c(J, C, K, \gamma) = 0\,, \end{aligned} \tag{3.27}$$

while the Ward operators $\mathscr{H}_a$ for the linear symmetry are defined in (3.26a).

Notice also that after the insertion of the sources $\gamma_i(x)$, the parametric equation (3.22) should be modified to

$$\begin{aligned} &\int d^2x \left[f_{ui}{}^j \left(J_j(x) \frac{\delta}{\delta J_i(x)} + \gamma_j(x) \frac{\delta}{\delta \gamma_i(x)}\right) + (K_\alpha(x) + q_\alpha)\,(T_u)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)}\right] Z_c(J, K, C, \gamma) \\ &\quad - C^i f_{ui}{}^j \frac{\partial}{\partial C^j} Z_c(J, K, C, \gamma) \\ &= \left(\mu^\eta (T_u)^\zeta{}_\eta \frac{\partial}{\partial \mu^\zeta} - \lambda_{ij} (f_{uk}{}^i \delta_l{}^j + f_{ul}{}^i \delta_k{}^j) \frac{\partial}{\partial \lambda_{kl}}\right) Z_c(J, K, C, \gamma)\,, \end{aligned} \tag{3.28}$$

which is equivalent to

$$\begin{aligned} &\left\{ S, -f_{ui}{}^j \int d^2x \gamma_j(x) \frac{\delta}{\delta J_i(x)} - \frac{\partial}{\partial C^u}\right\} Z_c(J, K, C, \gamma) \\ &= \left(\mu^\eta (T_u)^\zeta{}_\eta \frac{\partial}{\partial \mu^\zeta} - \lambda_{ij} (f_{uk}{}^i \delta_l{}^j + f_{ul}{}^i \delta_k{}^j) \frac{\partial}{\partial \lambda_{kl}}\right) Z_c(J, K, C, \gamma)\,. \end{aligned} \tag{3.29}$$

The algebraic relations obeyed by the operators S, $\mathscr{H}_a$ in (3.26a), (3.27) are

$$[\mathscr{H}_a, \mathscr{H}_b] = f_{ab}{}^c \mathscr{H}_c\,, \tag{3.30a}$$

$$[S, \mathscr{H}_a] = 0\,, \tag{3.30b}$$

$$S^2 = \tfrac{1}{2} C^i C^j f_{ij}{}^a \mathscr{H}_a\,, \tag{3.30c}$$

which suggest the strategy we shall follow in the analysis of the quantum extensions of the model. Indeed due to (3.30b) the linear $\mathscr{H}_a$ symmetry and the non-linear S symmetry can be discussed independently, and by (3.30c) we also see that if we succeed in implementing first the linear Ward identities, then the S operator becomes nilpotent in the restricted subspace of the $\mathscr{H}_a$-invariant functionals.

It is more convenient, in the study of the quantum extension of our identities, to rewrite them for the vertex functional Γ whose classical limit is given by Eq. (3.23) at $C^a=0$, i.e.

$$\Gamma^{cl}(\phi,\gamma,C)=\int d^2x g_{ij}[\phi]\,\partial_\mu\phi^i(x)\,\partial^\mu\phi^j(x)+\int d^2xC^iW_i^j[\phi]\,\gamma_j(x) + \int d^2x(K_\alpha+q_\alpha)\,v^\alpha[\phi]\,. \tag{3.31}$$

The linear Ward identities become

$$\mathscr{H}_a\Gamma\equiv -f_{ai}{}^j\left[\int d^2x\left(\phi^i(x)\frac{\delta\Gamma}{\delta\phi^j(x)}-\gamma_j(x)\frac{\delta\Gamma}{\delta\gamma_i(x)}\right)+C^i\frac{\partial\Gamma}{\partial C^j}\right] + \int d^2xK_\alpha(x)\,(T_a)^\alpha{}_\beta\frac{\delta\Gamma}{\delta K_\beta(x)}=0\,, \tag{3.32a}$$

while the B.R.S. identity is

$$D\Gamma\equiv\int d^2x\frac{\delta\Gamma}{\delta\phi^i(x)}\frac{\delta\Gamma}{\delta\gamma_i(x)}-C^i\int d^2x(K_\alpha(x)+q_\alpha)\,(T_i)^\alpha{}_\beta\frac{\delta\Gamma}{\delta K_\beta(x)} +\frac{1}{2}C^iC^jf_{ij}{}^k\frac{\partial\Gamma}{\partial C^k}=\frac{1}{2}C^iC^jf_{ij}{}^af_{ak}{}^l\int d^2x\gamma_l(x)\,\phi^k(x)\,. \tag{3.32b}$$

The possibility of extending to all perturbative orders the identities in (3.32) will be discussed in the next section.

4. Ultraviolet Renormalizability

Whenever one wishes to renormalize a theory characterized by a system of Ward identities without referring to any special invariant regularization procedure, one meets two main problems. These are:

i) all possible breakings which might affect the Ward identities, order by order in the radiative corrections, should be reabsorbed by a suitable choice of counterterms,

ii) these counterterms should be uniquely identified, up to a field redefinition, by the parameters characterizing the classical action.

To discuss the first point, one looks for the general solution of a linear system of consistency equations [25] for the breakings and compares it with the possible corrections which are introduced in the classical Ward identities by an arbitrary choice of counterterms. The second problem is solved by showing, at the classical level, that the general solution of the Ward identities in a neighbourhood of the classical action can be obtained by a variation of the parameters in the action itself.

We shall refer to i) and ii) as the *anomaly and stability problems* respectively.

The first step in our study is the analysis of the *linear symmetry*. The possibility of implementing to all orders the linear symmetry in Eq. (3.32a) is not trivial due to the presence of possible Abelian factors in H (for example, homogeneous Kählerian non-linear σ-models always contain at least one $U(1)$ [18]). For this reason, in Appendix A we prove that *in a generalized 2-dimensional σ model, any linearly realized symmetry corresponding to a compact group H can be implemented to all orders in perturbation theory.*

Consequently, from now on we shall restrict ourselves to the class of functionals Γ whose classical approximation is given in (3.31) and which obey to all orders

$$\mathscr{H}_a \Gamma = 0. \tag{4.1}$$

Concerning the non-linear symmetry (3.32b), both its anomalies and stability can be discussed in terms of the linearized operator

$$D_L = \int d^2x \left[\frac{\delta \Gamma^{cl}}{\delta \phi^i(x)} \frac{\delta}{\delta \gamma_i(x)} + \frac{\delta \Gamma^{cl}}{\delta \gamma_i(x)} \frac{\delta}{\delta \phi^i(x)} \right] + \frac{1}{2} C^i C^j f_{ij}{}^k \frac{\partial}{\partial C^k}$$
$$- C^i \int d^2x (K_\alpha(x) + q_\alpha)(T_i)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)}, \tag{4.2}$$

satisfying

$$[D_L, \mathscr{H}_a] = 0, \tag{4.3a}$$

$$D_L{}^2 = \tfrac{1}{2} C^i C^j f_{ij}{}^a \mathscr{H}_a. \tag{4.3b}$$

Indeed the Quantum Action Principle [26] ensures that at the first non-trivial order

$$D\Gamma = \Delta^{(+)}, \tag{4.4}$$

where $\Delta^{(+)}$ is a local functional in the field variables ϕ^i, K_α and the Grassmann variables γ_i and C^k, which has at most dimension-2, *carries a positive unit of Faddeev-Popov charge*, and is, due to Eqs. (4.1, 3), constrained by [26e)]

$$D_L \Delta^{(+)} = 0. \tag{4.5}$$

Notice that the absence of anomalies, i.e. the compensability condition for $\Delta^{(+)}$, is $\Delta^{(+)} = D_L \Delta^0$. Concerning the stability problem, let $\Gamma^{(1)}$ be a perturbation of the classical action where $\Gamma^{(1)}$ *is a Faddeev-Popov neutral local* functional with canonical dimension at most equal to two. The stability equation

$$D(\Gamma^{cl} + \varepsilon \Gamma^{(1)}) = \tfrac{1}{2} C^i C^j f_{ij}{}^a f_{ak}{}^l \int d^2x \gamma_l(x) \phi^k(x) \tag{4.6}$$

reduces to

$$D_L \Gamma^{(1)} = 0. \tag{4.7}$$

Let us remark that the contributions to $\Gamma^{(1)}$ which can be written as $D_L \Delta^{(-)}$ correspond to field and source redefinitions.

Thus the stability and anomaly analyses are reduced to a standard cohomology computation in the space of *local, $\mathscr{H}_a$-invariant functionals* which have at most dimension two and are limited to the sector with zero Faddeev-Popov charge (stability) or with a positive unit of Faddeev-Popov charge (anomaly).

4.1. From Functionals to Functions

According to the previous discussion, it is sufficient to study the action of D_L on the space of integrated local functions of dimension ≤ 2. The first step in the analysis of D_L consists in its reduction to an ordinary differential operator, together with the reduction of the space of functionals upon which D_L acts to a space of functions. The generic element of the functional space is

$$\int d^2x[\gamma_i P^i(\phi, C) + h_{ij}(\phi, C)\,\partial_\mu \phi^i(x)\,\partial_\mu \phi^j(x) + K_\alpha(x)\,V^\alpha(\phi, C) + U(\phi, C, q_\alpha)], \tag{4.8}$$

where the integrand is identified up to an integration by parts. (Let us recall that the canonical dimension of ϕ^i is zero while γ_i and K_α have dimension two and the mass scale is given by $q_\alpha = m^2 \delta_\alpha{}^0$.)

Thus, wishing to translate our functional differential equation into an ordinary differential one, we have to take into account the above-mentioned freedom of integration by parts and use it to eliminate the second derivatives of ϕ^i. To be more explicit, we introduce a set of independent fields $u_\mu{}^i(x) = \partial_\mu \phi^i(x)$ – and hence assigned canonical dimension equal to one – and we study the action of D_L in the space of functions:

$$\Xi \equiv \{\gamma_i P^i(\phi, C) + h_{ij}(\phi, C) u_\mu{}^i u_\mu{}^j + K_\alpha(x)\,V^\alpha(\phi, C) + U(\phi, C, q_\alpha)\}, \tag{4.9}$$

where the coefficients P^i, h_{ij}, V^α, and U are formal power series. The action of D_L on (4.8) induces on (4.9) a differential operator $\mathscr{D}_L$ given by

$$\begin{aligned}\mathscr{D}_L = &-C^k\left(W_k{}^i \frac{\partial}{\partial \phi^i} + \frac{\partial W_k{}^i}{\partial \phi^j}\left(u_\mu{}^j \frac{\partial}{\partial u_\mu{}^i} - \gamma_i \frac{\partial}{\partial \gamma_j}\right)\right.\\ &\left. + (K_\alpha + q_\alpha)(T_k)^\alpha{}_\beta \frac{\partial}{\partial K_\beta} - \frac{1}{2} C^l f_{kl}{}^i \frac{\partial}{\partial C^i}\right)\\ &+ \left(u_\mu{}^k u_\mu{}^l \left(\frac{\partial g_{kl}}{\partial \phi^i} + 2 g_{ki} \frac{\partial}{\partial \phi^l}\right) + (K_\alpha + q_\alpha) \frac{\partial v^\alpha}{\partial \phi^i}\right) \frac{\partial}{\partial \gamma_i}.\end{aligned} \tag{4.10}$$

The introduction of the $u_\mu{}^i(x)$ as independent fields requires a modification of the linear operators $\mathscr{H}_a$ in Eq. (3.32a) which become

$$\tilde{\mathscr{H}}_a = -f_{ai}{}^j\left(\phi^i \frac{\partial}{\partial \phi^j} + u_\mu{}^i \frac{\partial}{\partial u_\mu{}^j} - \gamma_j \frac{\partial}{\partial \gamma_i} + C^i \frac{\partial}{\partial C^j}\right) + K_\alpha (T_a)^\alpha{}_\beta \frac{\partial}{\partial K_\beta}. \tag{4.11}$$

The operators $\mathscr{D}_L$ and $\tilde{\mathscr{H}}_a$ still obey the relations (4.3) and we have reduced our problem to the analysis of the cohomology of $\mathscr{D}_L$ in the space Ξ.

4.2. The Cohomology Space of $\mathscr{D}_L$

The space Ξ decomposes into the *finite* sum of eigenspaces of the Faddeev-Popov charge

$$Q = -\gamma_i \frac{\partial}{\partial \gamma_i} + C^i \frac{\partial}{\partial C^i}. \tag{4.12}$$

We also introduce a second counting operator

$$N=\phi^i\frac{\partial}{\partial\phi^i}+u_\mu{}^i\frac{\partial}{\partial u_\mu{}^i}+C^i\frac{\partial}{\partial C^i}, \tag{4.13}$$

commuting with Q, and find that

i) Ξ decomposes into eigenspaces $\Xi^{(v)}$ corresponding to the integer non-negative eigenvalues $v=0,1,\dots$ of N.

ii) $\mathscr{D}_L$ can be written as: $\mathscr{D}_L=\Sigma_v\mathscr{D}_L{}^{(v)}$ with $[N,\mathscr{D}_L{}^{(v)}]=v\mathscr{D}_L{}^{(v)}$ and in particular $\mathscr{D}_L{}^{(0)}$ is a nilpotent operator.

Now, as proved in a general context in Appendix B, i) and ii) imply, without other restrictions on the operator $\mathscr{D}_L$ and the space Ξ, that:

if the cohomology of $\mathscr{D}_L{}^{(0)}$ is contained in the Faddeev-Popov neutral sector, then that of $\mathscr{D}_L$ belongs to the same sector and the two cohomology spaces are isomorphic.

From Eq. (4.10) we find

$$\mathscr{D}_L{}^{(0)}=-C^i\frac{\partial}{\partial\phi^i}+K_i'\frac{\partial}{\partial\gamma_i}, \tag{4.14}$$

where K_i', a linear combination of K_α, is defined by

$$K_i'=K_\alpha\left.\frac{\partial v^\alpha[\phi]}{\partial\phi^i}\right|_{\phi^i=0}. \tag{4.15}$$

The introduction of the new sources K_i' corresponds to a change of basis for the mass representation and, according to (4.15), the K_i' transform with the representation contragradient to that of the ϕ^i which coincides with the latter since it is orthogonal. On the other hand, this representation was carried, in the old basis, by the sources coupled to the operator v^i in (3.16); hence it is possible to complete the new basis for the mass representation in such a way as to leave unaltered the action of H. In particular there are as many independent K_i as field components ϕ^i and as many independent K_η' as there were independent $K_\alpha \mp K_i$ invariant under H in the old basis. Notice also, for future reference, that the number of K_i' invariant under H coincides with the number of generators of the subgroup X.

Clearly the cohomology of $\mathscr{D}_L{}^{(0)}$ can be discussed separately in every eigenspace $\Xi^{(v)}$. In each $\Xi^{(v)}$ the coefficients P^i, h_{ij}, V^α, and U in (4.9) are *polynomials* in the fields ϕ^i, so that this eigenspace can be embedded into a Fock space were the action of a creation operator is identified with the multiplication by the corresponding variable and the annihilation operator is given by the derivative with the respect to the same variable. In this framework the adjoint of $\mathscr{D}_L{}^{(0)}$ is

$$\mathscr{D}_L{}^{(0)\dagger}=-\phi^i\frac{\partial}{\partial C^i}+\gamma_i\frac{\partial}{\partial K'_i}, \tag{4.16}$$

and the cohomology space H_0 of $\mathscr{D}_L{}^{(0)}$ coincides with the kernel of the Laplace-Beltrami operator

$$\nabla^2=\{\mathscr{D}_L{}^{(0)},\mathscr{D}_L{}^{(0)\dagger}\}=C^i\frac{\partial}{\partial C^i}+\phi^i\frac{\partial}{\partial\phi^i}+\gamma_i\frac{\partial}{\partial\gamma_i}+K'_i\frac{\partial}{\partial K'_i}. \tag{4.17}$$

It is evident that H_0 cannot contain contributions from the C^i, ϕ^i, K_i', and γ_i. Consequently, all subspaces $H^{(q)}{}_0$ of H_0 with a non-vanishing Faddeev-Popov charge q are empty, which is sufficient to ensure the isomorphism of the cohomology spaces of $\mathscr{D}_L{}^{(0)}$ and $\mathscr{D}_L$. The only non-empty subspace is $H^{(0)}{}_0$ which can be parametrized as

$$H^{(0)}{}_0 = \{f + \eta_{ij} u^i{}_\mu u^j{}_\mu + K'_\eta f^\eta\}, \tag{4.18}$$

where f, η_{ij}, and f^η are constants (field independent). The invariance under $\tilde{\mathscr{H}}_a$ in (4.11) has been taken into account by limiting the K'_α dependence to the sole K'_η components; the same invariance also implies that η_{ij} is a H invariant 2-tensor, so it can be identified with λ_{ij} in (3.11, 12). Therefore, the cohomology space of $\mathscr{D}_L$, which is isomorphic to that of $\mathscr{D}_L{}^{(0)}$, depends on the same number of parameters appearing in the classical action since the η_{ij} correspond to the λ_{ij} and the f^η to the non-vanishing parameters μ^η in the mass representation [see (3.17) and the following discussion].

This result is sufficient to establish *the full renormalizability of the theory.* Indeed there are no anomalies since all Faddeev-Popov charged sectors of the cohomology of $\mathscr{D}_L$ are empty and the model is also stable since the classical action is identified, modulo a field redefinition, as the general solution of the B.R.S. identity, i.e. it is isomorphic to the neutral sector of the cohomology of $\mathscr{D}_L$.

We include here a discussion of the behaviour under renormalization of the parametric equations (3.28), since they will play a rôle in identifying the physical parameters of the theory, although they have no direct relevance to the effects of the ultraviolet renormalizability of the model. Now the left-hand side of Eq. (3.28) is well defined at the fully-quantized level being a functional differential operator which commutes with the B.R.S. identity by construction [see Eq. (3.29)]. According to the Quantum Action Principle [26], this operator is equivalent to the insertion of an integrated vertex of dimension 2 which, being compatible with the B.R.S. identity, is itself equivalent to a partial derivative with respect to the parameters of the theory. Since these partial derivatives act non-trivially on the parameters appearing in the kinetic part of the Lagrangian at the tree level, they will also do so at the fully-quantized level.

5. Local Observables and the Infrared Limit

5.1. Green Functions of Local Invariant Operators

According to the argument outlined in the Introduction, we identify the local observables of the theory with the set of G-invariant non-trivial local operators $\mathscr{L}_A(x)$ built with the fields $\phi^i(x)$ and their space-time derivatives. As a consequence of the homogeneity of the manifold, the lowest dimensionality $\mathscr{L}_A$'s are provided by the independent terms in the classical invariant action (3.11). For this set of local observables we shall prove Elitzur's conjecture [7] that the connected Green functions with external vertices given by local observables at non-exceptional momenta do possess a finite zero mass limit. We shall call them "*invariant Green functions.*" The proof parallels the one given in [10a] for the $O(N+1)/O(N)$ model.

In order to keep track of the renormalization of the invariant operators $\mathscr{L}_A(x)$, we introduce them in the effective action (3.31), coupled to a set of external fields $\omega_A(x)$. As the $\mathscr{L}_A$ are G-invariant, the B.R.S. and linear Ward identities hold unchanged for $Z_c(J_i, C^i, \gamma_i, K_\alpha, \omega_A)$; in particular

$$SZ_c(J_i, C^i, \gamma_i, K_\alpha, \omega_A) = 0\,,$$

$$S \equiv \int d^2x \left(-J_i(x) \frac{\delta}{\delta\gamma_i(x)} - (K_\alpha(x) + q_\alpha)\, C^i (T_i)^\alpha{}_\beta \frac{\delta}{\delta K_\beta(x)} \right) + \frac{1}{2} C^i C^j \left(f_{ij}{}^k \frac{\partial}{\partial C^k} - f_{ij}{}^a f_{al}{}^k \int d^2x \gamma_k(x) \frac{\delta}{\delta J_l(x)} \right). \tag{5.1}$$

Under renormalization the $\mathscr{L}_A(x)$ operators will mix among themselves with a constant (field-independent) matrix [20].

The generic connected invariant Green function $\langle T(\mathscr{L}_{A_1}(x_1) \dots \mathscr{L}_{A_n}(x_n))\rangle_{\text{conn}}$ can be obtained by applying to Z_c a multilocal functional derivative operator, denoted as $T_n(\delta/\delta\omega_A)$, which commutes with the S operator in Eq. (5.1). We have

$$\langle T(\mathscr{L}_{A_1}(x_1) \dots \mathscr{L}_{A_n}(x_n))\rangle_{\text{conn}} = T_n \left[\frac{\delta}{\delta\omega_A} \right] Z_c|_{J=C=\gamma=K=\omega=0} \tag{5.2a}$$

and

$$ST_n \left[\frac{\delta}{\delta\omega_A} \right] Z_c = 0\,. \tag{5.2b}$$

Elitzur's conjecture asserts that these invariant amplitudes *at non-exceptional momenta* have a finite I.R. limit. To simplify the notation, we shall omit in the formulae the $T_n(\delta/\delta\omega_A)$ operator and refer the discussion to the connected functional generator $Z_c(J_i, C^i, \gamma_i, K_\alpha, \omega_A)$.

5.2. *Infrared Finiteness*

As anticipated in the strategy (Sect. 2), the proof relies on known results concerning the infrared behaviour of Feynman graphs [23], in particular the fact that the dominant infrared singular part of a generic graph is given by a finite sum of contributions, each of which appears as a product of an I.R. regular subgraph (called a link in [23b] of a dominant subgraph in [23a]) of the original diagram times the graph which is obtained contracting the link to a point, and whose leading I.R. behaviour is determined, in a minimal renormalization scheme[4], up to logarithms of m^2, by standard power counting. It is this last graph which carries the possible infrared singularities. Such amplitude *with no external vertices* can be computed, without loss of generality, in the framework of dimensional renormalization. It reduces to the sum of a finite number of contributions of the type:

$$\mu^{n\varepsilon} \int d^{n(2-\varepsilon)}k \frac{Q^{2r}(m)\, R^{2s}(k)}{\prod_{i=1}^{n+r+s} (k_i^2 + m^2)}\,, \tag{5.3}$$

[4] By "minimal renormalization" scheme we mean a scheme where the counterterms do not depend on the mass appearing in the propagators: minimal dimensional renormalization is a possible one

where μ is the renormalization mass of the theory, $r+s+1$ the number of vertices of the diagram, Q^{2r} and R^{2s} are monomials of degree $2r$ and $2s$ respectively in m and in the components of the internal momenta k_i (where k_i is the i^{th} line momentum). Indeed, in the absence of external vertices, m is the only dimensional parameter of the theory, except for the renormalization mass μ. The renormalized contribution is computed following the general procedure described by Breitenlohner and Maison [26d] and in the limit $\varepsilon \to 0$ produces only logarithmic singularities in the mass since this amplitude is dimensionless and develops only pole singularities in the ε parameter.

Given a graph G, its *links or dominant subgraphs* are those which contain all external vertices of G (that is those carrying non-exceptional momenta) and which are minimal in the sense that the removal of any piece[5] destroys this property. Moreover, the legs connecting it to the rest of the diagram are amputated and carry zero momentum. Of course, among the possible links, there can be the whole graph which does not contain any amputated zero momentum field leg and to which does not correspond any singular factor. This is the regular part of our amplitude. In the following we shall disregard it and concentrate our attention on the singular parts. Hence we shall call links those with at least one field leg.

The crucial point will be that, as a consequence of the symmetry, *the finite sum of all links with a given, non-zero, number of amputated legs at zero momentum, and with only invariant external vertices, vanishes with a power law in the zero mass limit.* We call these combinations "*invariant links*" and shall prove this property at the tree approximation, and then recursively in the loop expansion.

5.2.a) *Vanishing of the "Invariant Tree Links."* Taking the derivative of the B.R.S. identity (5.1) with respect to the parameter C^i and setting $J_i = \gamma_i = C^i = 0$, we have

$$\mathscr{W}_i Z_c \equiv \int d^2x (K_\alpha(x) + m^2 \delta_\alpha{}^0)(T_i)^\alpha{}_\beta \left.\frac{\delta Z_c}{\delta K_\beta(x)}\right|_{J=C=\gamma=0} = 0\,. \tag{5.4}$$

At $K_\alpha = 0$, denoting the Fourier transform with a $\tilde{}$, we get

$$m^2 (T_i)^0{}_\beta \tilde{\delta} Z_c / \delta K_\beta(0)|_{J=C=\gamma=K=0} = 0\,. \tag{5.5}$$

Now the operator generated by the functional derivative with respect to K_α will in general contain a linear part in the fields and a non-linear part. In the tree approximation, the contributions to Eq. (5.5) coming from the non-linear part vanish proportionally to m^2, since in this case the graphs contributing to $\tilde{\delta} Z_c/\delta K_\beta(0)$ are regular in the zero mass limit. Now, for every choice of the index (i), the insertions appearing in Eq. (5.5) contain an independent term linear in the fields, hence Eq. (5.5) implies the vanishing, in the tree approximation, of the invariant amplitudes containing *one amputated*[6] field leg carrying zero momentum, i.e. of "invariant tree links" with one leg.

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[5] Each graph decomposes naturally into pieces consisting of maximal 1 PI subgraphs and single lines

[6] The amputation at zero momentum corresponds to a multiplication by a factor m^2

Then, consider the identity

$$\mathscr{W}_{i_1}\mathscr{W}_{i_2}\ldots\mathscr{W}_{i_n}Z_c=0\,. \tag{5.6}$$

At $K_\alpha=0$, it implies

$$\left\{\prod_{k=1}^{n}\left[(T_{i_k})^0{}_{\beta_k}\frac{m^2\tilde{\delta}}{\delta K_{\beta_k}(0)}\right]+P^{(n-1)}\left[\frac{m^2\tilde{\delta}}{\delta K_\gamma(0)}\right]\right\}Z_c[J,C,\gamma,K,\omega]|_{J=C=\gamma=K=0}=0\,, \tag{5.7}$$

where $P^{(n-1)}(x)$ is a polynomial of degree $(n-1)$ in the variable x without constant term. At the tree approximation, in both terms of Eq. (5.7), the contributions of the non-linear couplings of the K_α fields vanish again as positive integer powers of m^2. Hence Eq. (5.7) shows that in the zero mass limit, the invariant amplitudes with n amputated field legs are linearly related to amplitudes with less than n amputated field legs, and hence one recursively sees that they all vanish since they do for one field leg.

5.2.b) *Vanishing of the "Invariant Links."* We shall now show that the vanishing at $m^2=0$ of the "invariant links," proved at the tree approximation in the previous subsection, extends to all orders. As a recursive hypothesis, we assume that they vanish up to the loop order v.

Let us introduce an arbitrary system of operators of zero canonical dimension coupled to external sources τ_A. We shall first show that, at the order $v+1$, we have for a generic invariant amplitude with N such zero dimensional insertions,

$$\left\{\prod_{a=1}^{N}\frac{m^2\tilde{\delta}}{\delta\tau_{A_a}(0)}\right\}Z_c[J,C,\gamma,\tau,\omega]|^{v+1}_{J=C=\gamma=\tau=0}$$
$$=\left\{\prod_{a=1}^{N}\left(Z_{A_a,i_a}\frac{m^2\tilde{\delta}}{\delta J_{i_a}(0)}\right)\right\}Z_c[J,C,\gamma,\tau,\omega]|^{v+1}_{J=C=\gamma=\tau=0}+\mathcal{O}[(m^2)^\alpha]\,, \tag{5.8}$$

where the first term on the right-hand side represents a linear combination of *invariant amplitudes with amputated field legs carrying zero momentum* and α is a positive number.

To prove Eq. (5.8), let us analyze the infrared properties of its left-hand side and remark that every operator insertion can contain a linear part and a non-linear one. The linear part automatically contributes to the first term on the right-hand side.

Considering amplitudes where *at least one* operator is non-linear, we have to distinguish contributions coming from links containing the non-linear operators from the others (notice that these integrated operators have to be considered as *internal* vertices of the diagram). In the case in which at least one non-linear operator belongs to the link, the corresponding m^2 factor makes it vanish since the link is by construction regular in the limit $m^2\to 0$. The singular part corresponding to this situation diverges at most logarithmically since every quadratic divergence introduced by a zero dimensional operator is compensated by an m^2 factor. We remain with contributions where the links do not contain any non-linear operators. These are the "invariant links" *of loop order less than* $v+1$, hence, by the recursion hypothesis, they vanish as a positive power of m^2 which cannot be overcome by the logarithmically divergent part. This proves Eq. (5.8).

We now complete the recursive proof and show that the "invariant links" vanish up to order $v+1$ by going back to the Ward identities (5.5) and (5.7) which have already been discussed in the tree approximation. Consider these identities at $v+1$ loops and assume as before that the "invariant links" vanish up to v loops. Since Eq. (5.8) is valid at the order $v+1$ as shown before, we can use it by identifying the sources τ_A with the K_α, and write Eq. (5.5) as

$$\left\{Z_{ij}\frac{m^2\tilde{\delta}}{\delta J_j(0)}\right\} Z_c[J, C, \gamma, K, \omega]|_{J=C=\gamma=K=0}^{v+1} = \mathcal{O}[(m^2)^\alpha], \tag{5.9}$$

and Eq. (5.7) as

$$\left\{\prod_{a=1}^{n}\left[Z_{i_a j_a}\frac{m^2\tilde{\delta}}{\delta J_{j_a}(0)}\right] + P^{(n-1)}\left[\frac{m^2\tilde{\delta}}{\delta J_k(0)}\right]\right\} Z_c[J, C, \gamma, K, \omega]|_{J=C=\gamma=K=0}^{v+1}$$
$$= \mathcal{O}[(m^2)^\beta], \tag{5.10}$$

with α and β positive, and the matrix Z_{ij} is non-singular due to Eq. (4.15). These two equations, in much the same way as in the tree approximation, prove the vanishing of the "invariant links" up to the order $v+1$.

5.2.c) *Infrared Regularity*. Having thus shown by recursion in the loop order that the "invariant links" vanish in the zero mass limit as positive powers of the mass, the regularity of the invariant amplitudes in this limit follows from the fact that the singular parts diverge at most logarithmically.

5.3. Physical Parameters in the Infrared Limit

Let us finally discuss the case in which there exists a subgroup X commuting with H, and hence the I.R. regularized theory satisfies the parametric equation (3.28). The right transformations corresponding to X act non-trivially on the space of local invariant operators $\mathcal{L}_A(x)$ in much the same way as they do for the kinetic part of the Lagrangian. Thus, in the presence of the sources ω_A, the parametric equations should be modified in order to take into account this action. In the zero-mass limit, considering only $Z_c(J_i=0,\ C^i=0,\ \gamma_i=0,\ K_\alpha=0,\ \omega_A)$ they become

$$\left(\int d^2x\left[\omega_A(x)(T_u)^A{}_B\frac{\delta}{\delta\omega_B(x)}\right] + \Lambda_{ij}(\lambda_{kl})\frac{\partial}{\partial\lambda_{ij}}\right) Z_c(\omega) = 0, \tag{5.11}$$

where Λ_{ij} is given, at the tree approximation, by $\Lambda_{ij} = \lambda_{kl}(f_{ui}{}^k\delta_j{}^l + f_{uj}{}^k\delta_i{}^l)$. This exhibits the degeneracy of the theory with respect to as many parameters appearing in the metric as there are generators in the subgroup X.

6. Conclusion

We have proved in a regularization independent way the renormalizability of all non-linear σ-models built on compact coset spaces, and we have shown the existence of a finite zero-mass limit for the Green functions of a suitable class of invariant local operators. We have thus exhibited the parameters upon which these Green functions depend, noticing in particular that the theory could have less free parameters than the metrics defined on the coset space have. Indeed, two theories corresponding to different metrics could lead to the same invariant Green functions. We have identified the origin of this degeneracy with the presence of a subgroup X of the isometry group G, commuting with the isotropy group H.

The infrared properties of multilocal invariant operators, such as the geodesic distance between two points on the manifold, could be investigated following the lines we have adopted for the local observables, provided one is able to control their behaviour under renormalization.

The analysis presented here for the bosonic case can be generalized to $N=1$ supersymmetric σ-models built on homogeneous compact spaces. Indeed our proof does not rely on any regularization procedure, and a theorem proved by Piguet et al. [27] ensures that there are no new anomalies due to a global supersymmetric extension.

We have a final remark, concerning the generalization to other models of the ultraviolet renormalizability analyzed for homogeneous coset spaces in Sect. 4. The same technique can be employed to study the coordinate dependence of any renormalized version of non-linear σ-models; the argument we have used to show the vanishing of the second cohomology class would guarantee the independence with respect to field parametrization in a quite general context [6, 15], since in this case too, one encounters differential operators which are a perturbation of the one corresponding to the same dimensionality flat manifold.

Appendix A. Renormalization of a Linear Symmetry in a 2-Dimensional Compact σ-Model

The theory is classically assigned through the *H-invariant* action

$$\int d^2x[g_{ij}[\phi]\,\partial_\mu\phi^i(x)\,\partial^\mu\phi^j(x)+m(\phi)] \tag{A.1}$$

in terms of the dimensionless coordinate field with components ϕ^i $(i=1,\dots,N)$. We want to show that it is always possible to build a quantum extension of the theory preserving a *linearly realized symmetry* corresponding to a *compact group H* with infinitesimal transformations

$$\delta\phi^i(x)=w^\alpha(T_\alpha)^i{}_j\,\phi^j(x), \tag{A.2}$$

in other words that *in a generalized bosonic 2-dimensional σ-model, any linearly realized symmetry corresponding to a compact group H can be implemented to all orders in perturbation theory.*

This means that the theory (A.1) admits a quantum extension whose vertex functional $\Gamma[\phi]$ satisfies the perturbative Ward identity

$$H_\alpha\Gamma\equiv\int d^2x(T_\alpha)^i{}_j\,\phi^j(x)\frac{\delta\Gamma}{\delta\phi^i(x)}=0. \tag{A.3}$$

The proof proceeds via a standard recursive method based on the assumption that Eq. (A.3) holds true up to the n^{th} order in the loop parameter $\hbar$ in power series of which Γ is formally developed,

$$\Gamma=\sum_\nu\Gamma^{(\nu)}\hbar^\nu,\qquad H_\alpha\Gamma^{(\nu)}=0,\qquad \nu\leqq n. \tag{A.4}$$

According to the Quantum Action Principle [26], for a generic choice of the $(n+1)^{\text{th}}$ order counterterms we shall have

$$H_\alpha \Gamma^{(n+1)} = \Delta_\alpha{}^{n+1}, \tag{A.5}$$

where $\Delta_\alpha{}^{n+1}$ stands for an integrated local functional of dimension $\leqq 2$, which, in the absence of skew-symmetric tensors, can be written as

$$\Delta_\alpha{}^{n+1} = \int d^2x [\gamma_{\alpha,ij}(\phi)\,\partial_\mu \phi^i\,\partial^\mu \phi^j + \varepsilon_\alpha(\phi)], \tag{A.6}$$

where $\gamma_{\alpha,ij}$ and ε_α are formal power series in the field components. We want to prove that by a suitable choice of supplementary finite counterterms at the $(n+1)^{\text{th}}$ order,

$$\Theta = \int d^2x [\sigma_{ij}(\phi)\,\partial_\mu \phi^i\,\partial^\mu \phi^j + \tau(\phi)], \tag{A.7}$$

we can implement Eq. (A.4) to this order (as $\gamma_{\alpha,ij}$ and ε_α are, σ_{ij} and τ will also be formal power series in ϕ). Of course this result would imply the renormalizability of Eq. (A.3).

The first step of the proof consists in applying the Ward operator H_β to both sides of Eq. (A.5) and selecting the skew-symmetric contribution in the indices α and β, to obtain, as a consequence of the algebra of H,

$$[H_\alpha, H_\beta] = f_{\alpha\beta}{}^\gamma H_\gamma, \tag{A.8a}$$

the Wess-Zumino [25] consistency conditions

$$f_{\alpha\beta}{}^\gamma \Delta_\gamma{}^{n+1} = H_\alpha \Delta_\beta{}^{n+1} - H_\beta \Delta_\alpha{}^{n+1} \tag{A.8b}$$

(the coefficients $f_{\alpha\beta}{}^\gamma$ represent the structure constants of the group H). Let us introduce a non-degenerate H-invariant rank two symmetric tensor $K^{\alpha\beta}$, whose existence is guaranteed by the *compactness of H:*

$$K^{\alpha\beta} = K^{\beta\alpha}, \qquad f_{\eta\gamma}{}^\alpha K^{\eta\beta} + f_{\eta\gamma}{}^\beta K^{\alpha\eta} = 0. \tag{A.9}$$

From Eq. (A.8b) we obtain:

$$K^{\alpha\beta} H_\alpha (H_\beta \Delta_\gamma{}^{n+1} - H_\gamma \Delta_\beta{}^{n+1}) = K^{\alpha\beta} H_\alpha f_{\beta\gamma}{}^\eta \Delta_\eta{}^{n+1} \Rightarrow H^2 \Delta_\gamma{}^{n+1} = H_\gamma K^{\alpha\beta} H_\alpha \Delta_\beta{}^{n+1}, \tag{A.10}$$

where $H^2 = K^{\alpha\beta} H_\alpha H_\beta$ is the quadratic Casimir operator of H which commutes with H_α. Due to the compactness of H, the linear space Φ whose elements are the local functionals appearing in Eqs. (A.6, 7) decomposes into an infinite sequence of *finite dimensional* subspaces Φ_N (in which the coefficient functions σ_{ij}, $\gamma_{\alpha,ij}$, τ, and ε_α are *polynomials* in ϕ^i) each carrying an irreducible representation of H. For the moment, we distinguish in this functional space two subspaces $\Phi^\#$, carrying the identity representation of H, and Φ^b carrying the rest; $\Phi^\#$ is the Kernel of H^2. Decomposing Eq. (A.10) according to the # and b components, we find for the b component:

$$H^2 \Delta^b{}_\gamma{}^{n+1} = H_\gamma K^{\alpha\beta} H_\alpha \Delta^b{}_\beta{}^{n+1} \equiv H_\gamma \hat{\Delta}'^{b,n+1} \Rightarrow \Delta^b{}_\beta{}^{n+1} = H_\beta \hat{\Delta}^{b,n+1}, \tag{A.11}$$

where $\hat{\Delta}'^{b,n+1}$ and $\hat{\Delta}^{b,n+1}$ belong to the same space Φ^b of local functionals. On the other hand, the # component of Eq. (A.10) gives no information. However it is possible to find a constraint on $\Delta^\#{}_\gamma{}^{n+1}$ expressing its invariance under H:

Eq. (A.8b) gives

$$f_{\alpha\beta}{}^{\gamma} \Delta^{\#}{}_{\gamma}{}^{n+1} = 0. \tag{A.12}$$

Hence the # component of $\Delta_{\gamma}{}^{n+1}$, if any, is restricted to the Abelian invariant factors of H.

Up to this point we have followed the standard treatment used to analyze the perturbative renormalizability of a linear symmetry in *the semi-simple case* [17a)]. In our case, as remarked in Sect. 2, H generally contains $U(1)$ factors which contribute # Abelian components of $\Delta_{\gamma}{}^{n+1}$: we shall now exclude the existence of these components. The idea of the proof is that, due to *the complete reducibility*, it is contradictory for a local functional to be at the same time invariant under H and a variation under H (of a non-local functional) [17b)].

It is possible to define the *adjoint space* $\tilde{\Phi}$ to Φ by introducing its formal basis whose elements are the functional differential operators:

$$\tilde{\Omega}_N = -\frac{1}{8} F_{N_{ij}} \left[\int d^2y \frac{\delta}{\delta\phi(y)} \right] \int d^2x \left(x^2 \frac{\delta^2}{\delta\phi^i(x)\,\delta\phi^j(0)} \right), \tag{A.13}$$

and

$$\tilde{\Xi}_N = H_{N_i} \left[\int d^2y \frac{\delta}{\delta\phi(y)} \right] \frac{\delta}{\delta\phi^i(0)}, \tag{A.14}$$

where the coefficient functions $F_{N_{ij}}[x]$ and $H_{N_i}[x]$ vary over all the possible monomials of degree N. $\tilde{\Xi}_N$ and $\tilde{\Omega}_N$ define linear functionals on Φ and it is easy to verify that the conditions on $\Theta \in \Phi$,

$$\langle \tilde{\Omega}_N, \Theta \rangle = \langle \tilde{\Xi}_N, \Theta \rangle = 0, \quad \forall N \tag{A.15}$$

implies $\Theta = 0$ (of course in the sense of formal power series).

We also notice that, even if the functional $\Gamma^{(n+1)}$ does not belong to Φ since it is a formal power series in the fields ϕ^i but with *non-local coefficients*, the action of $\tilde{\Omega}_N$ and $\tilde{\Xi}_N$ on $\Gamma^{(n+1)}$ is well defined for every N since our theory contains an I.R. regulator (mass term). A second important remark is that in much the same way as Φ, its adjoint space $\tilde{\Phi}$ decomposes into an infinite sequence of *finite dimensional* subspaces, whose elements

$$\begin{aligned} \tilde{\Sigma}_N^{\alpha} = A_{N_{ij}}^{\alpha} &\left[\int d^2y \frac{\delta}{\delta\phi(y)} \right] \int d^2x \left(x^2 \frac{\delta^2}{\delta\phi^i(x)\,\delta\phi^j(0)} \right) \\ &+ B_{N_i}^{\alpha} \left[\int d^2y \frac{\delta}{\delta\phi(y)} \right] \frac{\delta}{\delta\phi^i(0)} \end{aligned} \tag{A.16}$$

have homogeneous polynomial coefficients $A_{N_{ij}}^{\alpha}[x]$ and $B_{N_i}^{\alpha}[x]$ and transform, under the induced action of H, according to its irreducible representations. In particular we shall label by $\tilde{\Sigma}^{\#}{}_N$ the invariant element of $\tilde{\Phi}$. It is apparent that the condition

$$\langle \tilde{\Sigma}^{\#}{}_N, \Theta^{\#} \rangle = 0, \quad \text{for} \quad \Theta^{\#} \in \Phi^{\#} \quad \text{and} \quad \forall N \tag{A.17}$$

implies $\Theta^{\#} = 0$. Now, applying $\tilde{\Sigma}^{\#}{}_N$ to both sides of Eq. (A.5) we get, taking Eq. (A.11) into account,

$$\langle \tilde{\Sigma}^{\#}{}_N, H_{\alpha} \Gamma^{(n+1)} \rangle = 0 \equiv \langle \tilde{\Sigma}^{\#}{}_N, \Delta_{\alpha}{}^{n+1} \rangle = \langle \tilde{\Sigma}^{\#}{}_N, \Delta^{\#}{}_{\alpha}{}^{n+1} \rangle \tag{A.18}$$

for any N. It follows that $\Delta^{\#}{}_{\alpha}^{n+1}=0$ and then

$$\Delta_{\alpha}^{n+1}=\Delta^{b}{}_{\alpha}^{n+1}\equiv H_{\alpha}\hat{\Delta}^{b,n+1}. \qquad \text{(A.19)}$$

If we now introduce into the effective action the finite supplementary counterterms: $-[\hbar^{n+1}]\hat{\Delta}^{b,n+1}$, we have that $\Gamma^{(\nu)}$ remains unchanged for $\nu\leqq n$ while

$$[\hbar^{n+1}]\Gamma^{(n+1)}\rightarrow[\hbar^{n+1}]\Gamma^{(n+1)}-[\hbar^{n+1}]\hat{\Delta}^{b,n+1}\equiv[\hbar^{n+1}]\Gamma'^{(n+1)} \qquad \text{(A.20)}$$

and, for the modified theory, we have

$$H_{\alpha}\Gamma'^{(n+1)}=0. \qquad \text{(A.21)}$$

As promised, we have shown that the Ward identity can be implemented to the next $(n+1)^{\text{th}}$ order, and thus the proof of the renormalizability of the linear symmetry holds to all orders.

Appendix B. Isomorphism of the Cohomology Spaces of $\mathscr{D}_L$ and of $\mathscr{D}_L^{(0)}$

In this Appendix we prove the assertion of Sect. 4 about the isomorphism of the cohomology spaces of $\mathscr{D}_L$ and $\mathscr{D}_L{}^{(0)}$ when *the last one has a non-trivial cohomology only in the neutral Faddeev-Popov sector*. Since the results we shall derive do not depend on the explicit form of $\mathscr{D}_L$, we refer the analysis to *any linear nilpotent operator D of Faddeev-Popov charge* $+1$, acting on a linear space V. Let us emphasize that *we do not suppose V to be a Hilbert space*: this, in the absence of an adjoint operator for D and a scalar product in V, makes the study more delicate.

In the space V we have two compatible gradings; one due to the Faddeev-Popov charge Q whose eigenvalues in V are the integers q with associated eigenspaces $V_{(q)}$. This grading corresponds to a separation of V according to the ghost number, and recall that the D operator is homogeneous with ghost number $+1$. *The second, essential grading in our analysis is due to the existence of a linear (counting) operator N acting on V with the following properties:*

i) *N has integer, non-negative eigenvalues* $\nu=0,1,2,\dots$ in V with corresponding eigenspaces $V^{(\nu)}$.

ii) *The operator D decomposes* as

$$D=\sum_{\nu=0}^{\infty}D^{(\nu)} \quad \text{such that} \quad [N,D^{(\nu)}]=\nu D^{(\nu)}, \qquad \text{(B.1)}$$

hence $D^{(\nu)}V^{(\mu)}$ is a subspace of $V^{(\mu+\nu)}$. The special role played by $D^{(0)}$ comes from the fact that it leaves each $V^{(\mu)}$ invariant. Moreover, the nilpotency of D induces on the $D^{(\nu)}$ operators the relations

$$\sum_{\mu=0}^{\nu}D^{(\mu)}D^{(\nu-\mu)}=0, \qquad \nu=0,1,2,\dots, \qquad \text{(B.2)}$$

hence *$D^{(0)}$ is still a nilpotent operator.*

iii) *N commutes with the Faddeev-Popov charge operator Q*, so each eigenspace $V^{(\nu)}$ can be further analyzed in ghost content and each $D^{(\nu)}$ has a Faddeev-Popov charge $+1$.

The cohomology space H_D of D is the subspace of vectors $h\in V$ such that $Dh=0$ and $h\neq Dy$ for $y\in V$, i.e. the vectors h are cocycles but not coboundaries.

Since $D^{(0)}$ commutes with N, *we can separately analyze its cohomology* $H_0{}^{(\nu)}$ *in each subspace* $V^{(\nu)}$. In this general framework we first show that, *if the cohomology of* $D^{(0)}$ *is contained in the zero Faddeev-Popov sector*, then, given a basis for the cocycles of $D^{(0)}$, one can construct a set of cocycles for D. This follows from:

Lemma. *If the cohomology subspaces* $H_0{}^{(\nu)}$ *of* $D^{(0)}$ *are empty in all non-zero Faddeev-Popov sectors for all* $\nu=0,1,2,\ldots$, *then for each non-trivial* $h^{(\nu)}\in H_0{}^{(\nu)}$ *there exist Faddeev-Popov neutral elements* $K_\nu{}^{(\nu+\tau)}\in V^{(\nu+\tau)}$ $(\tau=0,1,2,\ldots)$, *with* $K_\nu{}^{(\nu)}=0$, *identified by the equation:*

$$D^{(\mu)}h^{(\nu)}+\sum_{\tau=0}^{\mu}D^{(\mu-\tau)}K_\nu{}^{(\nu+\tau)}=0 \qquad \mu=0,1,2,\ldots. \tag{B.3}$$

The proof of (B.3) will be by induction on the index μ. The value $\mu=0$ simply yields $D^{(0)}h^{(\nu)}=0$ and, since $h^{(\nu)}\in H_0{}^{(\nu)}$, the vanishing of $K_\nu{}^{(\nu)}$ in Eq. (B.3) is consistent. At the next step $\mu=1$ we have to analyze $D^{(1)}h^{(\nu)}$; now from (B.2) we find

$$D^{(0)}D^{(1)}h^{(\nu)}=-D^{(1)}D^{(0)}h^{(\nu)}=0\,. \tag{B.4}$$

The cohomology of $D^{(0)}$ *being trivial in the Faddeev-Popov charged sectors by hypothesis* and since $D^{(1)}h^{(\nu)}$ has a Faddeev-Popov charge $+1$, we obtain as the solution of (B.4)

$$D^{(1)}h^{(\nu)}=-D^{(0)}K_\nu{}^{(\nu+1)}, \tag{B.5}$$

which fixes $K_\nu{}^{(\nu+1)}$ up to a $D^{(0)}$-cocycle.

This simple case illustrates the idea of the procedure; suppose that Eq. (B.3) holds up to the value $\mu=\lambda$. At the next step the nilpotency of D expressed through Eq. (B.2) is written as

$$\left[D^{(0)}D^{(\lambda+1)}+\sum_{\sigma=0}^{\lambda}D^{(\lambda+1-\sigma)}D^{(\sigma)}\right]h^{(\nu)}=0\,. \tag{B.6}$$

Substituting the induction hypothesis for $D^{(\sigma)}h^{(\nu)}$, one obtains after simple algebraic manipulations on summations and use of the nilpotency of D,

$$D^{(0)}D^{(\lambda+1)}h^{(\nu)}=-\sum_{\tau=0}^{\lambda}D^{(0)}D^{(\lambda+1-\tau)}K_\nu{}^{(\nu+\tau)} \tag{B.7a}$$

$$\Leftrightarrow D^{(0)}\left[D^{(\lambda+1)}h^{(\nu)}+\sum_{\tau=0}^{\lambda}D^{(\lambda+1-\tau)}K_\nu{}^{(\nu+\tau)}\right]=0\,. \tag{B.7b}$$

The parenthesis being Faddeev-Popov charged, the corresponding triviality of the cohomology of $D^{(0)}$ allows the identification of a neutral $K_\nu{}^{(\nu+\lambda+1)}$ such that

$$D^{(\lambda+1)}h^{(\nu)}+\sum_{\tau=0}^{\lambda}D^{(\lambda+1-\tau)}K_\nu{}^{(\nu+\tau)}=-D^{(0)}K_\nu{}^{(\nu+\lambda+1)}, \tag{B.8}$$

thus completing the recursive proof of Eq. (B.3).

This lemma can now be used to construct a linear relation, denoted as τ, between the cocycles of $D^{(0)}$ and those of D. First choose for each value of ν a set of $D^{(0)}$-cocycles $h_i^{(\nu)}$ such that the cohomology classes $[h_i^{(\nu)}]$ of $D^{(0)}$ in the subspace $V^{(\nu)}$ form a basis of $H_0{}^{(\nu)}$. For each $h_i^{(\nu)}$, we specify a particular set of $K_{i,\nu}{}^{(\nu+\tau)}$ such that, as a consequence of the lemma,

$$l_{i,v} = h_i^{(v)} + \sum_{\tau=1}^{\infty} K_{i,v}^{(v+\tau)} \tag{B.9}$$

is a D-cocycle.

We also need a set of vectors $y_i^{(v)}$ such that the $D^{(0)} y_i^{(v)}$ form a basis of the $D^{(0)}$ coboundaries. Then any $D^{(0)}$-cocycle may be written as

$$c = \Sigma_{i,v}[(\lambda^i{}_v h_i^{(v)}) + D^{(0)}(\mu^i{}_v y_i^{(v)})], \tag{B.10}$$

where the $\lambda^i{}_v$ and $\mu^i{}_v$ are parameters. We define the image of c under τ to be

$$\tau(c) = \Sigma_{i,v}[\lambda^i{}_v l_{i,v} + D(\mu^i{}_v y_i^{(v)})], \tag{B.11}$$

and $\tau(c)$ is clearly a D-cocycle. It is also straightforward to show that τ induces a linear relation $\tilde{\tau}$ between the cohomology spaces H_0 and H of $D^{(0)}$ and D respectively.

We shall now show that $\tilde{\tau}$ *is an isomorphism*. Let us first show that τ (and thus $\tilde{\tau}$) is *surjective*, i.e. if C is a D-cocycle, it is the image by τ of a $D^{(0)}$-cocycle. Obviously, if C is a D-cocycle, its lowest component $C^{(0)}$ is a $D^{(0)}$-cocycle. We shall use an important property of τ, which is that *it reduces to the identity at the lowest non-trivial level*, i.e.

$$[\tau(C^{(0)})]^{(0)} = C^{(0)}. \tag{B.12}$$

Thus, $C - \tau(C^{(0)})$ is a D-cocycle beginning at level 1. We shall now show recursively that one can find for any n a $D^{(0)}$-cocycle c_n such that $C - \tau(c_n)$ begins at level $n+1$. This property is true for $n=0$ with $c_0 = C^{(0)}$, and we suppose it to hold up to order N. Then

$$D[C - \tau(c_N)] = 0 \tag{B.13}$$

gives at the lowest non-trivial order,

$$D^{(0)}[C^{(N+1)} - (\tau(c_N))^{(N+1)}] = 0, \tag{B.14}$$

and then

$$c_{N+1} = c_N + C^{(N+1)} - (\tau(c_N))^{(N+1)} \tag{B.15}$$

is a $D^{(0)}$-cocycle. Consider now the quantity

$$C - \tau(c_{N+1}) \equiv C - \tau(c_N) - \tau[C^{(N+1)} - (\tau(c_N))^{(N+1)}]. \tag{B.16}$$

As a consequence of the recursion hypothesis, it vanishes up to level N. Using again the properties of τ, the order $N+1$ is also zero. We have thus constructed a c whose τ-image is C: this completes the proof of the surjectivity of τ.

We now discuss the *injectivity* property of $\tilde{\tau}$: this amounts to proving that any D-coboundary has as inverse τ-image only $D^{(0)}$-coboundaries. Obviously, if $X = \tau(x)$ is a D-coboundary, i.e. $X = D\omega$, its lowest component $X^{(0)} = x^{(0)}$ is a $D^{(0)}$-coboundary, $x^{(0)} = D^{(0)}\omega^{(0)}$, and can be decomposed on our basis,

$$x^{(0)} = D^{(0)} y^{(0)}, \quad \text{with} \quad y^{(0)} = \Sigma_i \mu^i{}_0 y_i^{(0)}. \tag{B.17}$$

As $\omega^{(0)} - y^{(0)}$ is a $D^{(0)}$-cocycle, its τ-image is a D-cocycle. Defining

$$x_n = x - \sum_{v=0}^{n-1} x^{(v)}, \tag{B.18}$$

we get

$$\tau(x_1)=D\omega-\tau(x^{(0)})=D(\omega-y^{(0)})=D[\omega-y^{(0)}-\tau(\omega^{(0)}-y^{(0)})]. \quad \text{(B.19)}$$

The quantity

$$\omega_1=\omega-y^{(0)}-\tau[\omega^{(0)}-y^{(0)}] \quad \text{(B.20)}$$

begins at the level 1.

Assume as recursion hypothesis that one can find two vectors

$$y^{(n)}=\Sigma_i \mu^i{}_n y_i{}^{(n)} \quad \text{(B.21 a)}$$

and ω_{n+1} beginning at level $n+1$, such that

$$x^{(n)}=D^{(0)}y^{(n)} \quad \text{and} \quad \tau(x_{n+1})=D\omega_{n+1}. \quad \text{(B.21 b)}$$

We have just shown that this holds for $n=0$ and we suppose it to be true up to N.

The equation $\tau(x_{N+1})=D\omega_{N+1}$ writes, at the lowest non-trivial level,

$$x^{(N+1)}=D^{(0)}(\omega_{N+1})^{(N+1)}, \quad \text{(B.22)}$$

and $x^{(N+1)}$, being a $D^{(0)}$-coboundary, can be decomposed on our basis,

$$x^{(N+1)}=D^{(0)}y^{(N+1)}, \quad \text{with} \quad y^{(N+1)}=\Sigma_i \mu^i{}_{N+1} y_i{}^{(N+1)}. \quad \text{(B.23)}$$

As $(\omega_{N+1})^{(N+1)}-y^{(N+1)}$ is a $D^{(0)}$-cocycle, its image by τ is a D-cocycle. Then

$$\begin{aligned}\tau(x_{N+2})&=D\omega_{N+1}-\tau(x^{(N+1)})=D(\omega_{N+1}-y^{(N+1)})\\ &\equiv D[\omega_{N+1}-y^{(N+1)}-\tau((\omega_{N+1})^{(N+1)}-y^{(N+1)})]. \end{aligned}\quad \text{(B.24)}$$

The vector

$$\omega_{N+2}=\omega_{N+1}-y^{(N+1)}-\tau[(\omega_{N+1})^{(N+1)}-y^{(N+1)}] \quad \text{(B.25)}$$

begins at the level $N+2$, which shows that the recursion hypothesis holds up to $N+1$.

Therefore x is a $D^{(0)}$-coboundary and consequently $\mathrm{Ker}(\tilde{\tau})$ reduces to the null vector, thus showing the announced injectivity of $\tilde{\tau}$.

Notice that the $\tilde{\tau}$ isomorphism ensures that the cohomology space of D also belongs to the null Faddeev-Popov charge sector.

Acknowledgements. Three of us (C.B., A.B., and R.C.) would like to thank the I.N.F.N. and M.P.I. for their financial support which has permitted them to maintain the collaboration with the L.P.T.H.E. in Paris and CERN during a preliminary draft of the paper. All of us would also like to thank CERN for hospitality and C.B. is grateful to "Université de Paris 7" for granting him the opportunity of spending one month at L.P.T.H.E.

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Communicated by L. Alvarez-Gaumé

Received April 28, 1988