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Rotation Numbers of Periodic Orbits in the Hénon Map

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Abstract. For invertible, area-contracting maps of the plane, it is common for a basin of attraction to have a fractal basin boundary. Certain periodic orbits on the basin boundary are distinguished by being accessible (by a path) from the interior of the basin. A numerical study is made of the accessible periodic orbits for the Hénon family of maps. Theoretical results on rotary homoclinic tangencies are given, which describe the appearance of the accessible saddles, and organize them in a natural way according to the continued fractions expansions of their rotation numbers.

Introduction

Let f denote an area-contracting invertible map of the plane and let U be the basin of attraction of an attractor A. Although U may be easy to describe topologically (for example, U may be connected and simply connected, and thus homeomorphic to a disk), the boundary of U can be extremely complicated. In [7], it is pointed out that much of the basin boundary behavior can be characterized by certain distinguished orbits on the basin boundary, called accessible orbits. A point x is *accessible* from an open set U if there is a path, beginning in U, such that x is the first point not in U reached by the path. For real-valued maps, when the basin boundaries are fractal, most of the points in the basin boundary are not accessible.

When U is connected and simply connected, the map f acts on the accessible points as if they were on a circle. This fact, known to Birkhoff [4], means that one can associate a rotation number to the boundary points accessible from U. We call this number the *accessible rotation number*. It is shown in [2] that if the accessible rotation number is rational, then there exist accessible periodic orbits.

The goal of this paper is to study changes in the accessible rotation number as a map parameter is varied. The results of a computer investigation of a typical two-

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parameter family of plane diffeomorphisms, the Hénon map, are discussed in Sect. 2. Motivated by these results, we single out a specific theoretical phenomenon, the rotary homoclinic tangency, for further study. The main theorem of the paper (Theorem 3.1) shows that this phenomenon underlies many properties of the accessible rotation number observed in the Hénon map.

Some of these properties are illustrated in Fig. 2. First, as the parameters A and B of the Hénon map are varied, the accessible rotation number $\varrho(A, B)$ changes discontinuously. For values of the Jacobian B near one, i.e. as the area contraction becomes weak, these discontinuous jumps occur in more rapid succession, and there is a greater variety of accessible rotation numbers, although they appear to always be rational for $B<1$. Third, the lower the denominator of a rational $\varrho(A, B)$, the broader the region in parameter space in which it is the accessible rotation number. Thus "phase-locking" is more pronounced for small denominators.

By rotary homoclinic tangency we mean the tangency of an unstable manifold of one point in a periodic saddle with the stable manifold of an adjacent point from the same periodic orbit. This definition subsumes the standard definition of homoclinic tangency, which is the tangency of stable and unstable manifolds of the same saddle point (for f or some higher iterate f^k of f).

In studying the dynamics of area-contracting maps of the annulus, Hockett and Holmes [11] use the term "rotary homoclinic points" in reference to transverse intersections of stable and unstable manifolds which encircle the annulus at least once, a situation which is also studied by Aronson et al. [3]. Although we are interested in homoclinic tangencies in the plane, not in the annulus, our use of the term "rotary homoclinic tangency" is suggested by this usage since we are looking at periodic orbits which have a natural rotation number as accessible orbits on a basin boundary.

Gavrilov and Silnikov [6] studied nondegenerate homoclinic tangencies in area-contracting planar maps. They showed that if a fixed point p of a C^3-family f_λ (or of f_λ^k, if p is a periodic point of period k) creates a quadratic homoclinic tangency when $\lambda=\lambda_*$, then there exists a sequence $\lambda_n \to \lambda_*$ such that f_{λ_n} has a saddle node bifurcation at which an attractor of period n and a saddle of period n are born. It was in this situation that Newhouse [12] (see also [13]) proved the existence of infinitely many sinks for a single diffeomorphism. Robinson [13] strengthened Gavrilov and Silnikov's result to hold for the formation of any homoclinic tangency for a real analytic diffeomorphism, even if the intersection is created degenerately.

The sequence $\{p_n\}$ of saddles created at such a tangency plays an important role in metamorphoses, i.e., sudden jumps in basin boundaries at homoclinic tangencies [7, 9]. The discontinuities in rotation numbers mentioned above are a result of these sudden jumps. However, the complexity of rotation numbers and the patterns of metamorphoses which we found in the Hénon study were not fully explained by this theory.

In the present paper, we prove a result (Theorem 3.1) which describes the appearance (and abundance) of rotary homoclinic tangencies. Namely, if a rotary homoclinic tangency of a saddle orbit p is created when $\lambda=\lambda_*$, then there exists a sequence $\lambda_{k_n} \to \lambda_*$ such that for sufficiently large n, the saddle p_n (guaranteed by the Gavrilov-Silnikov result) creates a new rotary homoclinic tangency at λ_{k_n}. Stated informally, the existence of one rotary homoclinic tangency (for instance, the existence of a homoclinic tangency) implies the existence of infinitely many rotary homoclinic tangencies. The theorem is proved under fairly general hypotheses which are satisfied, for instance, by the Hénon map. The advantage of this version is that it is self-replicating. Each of the infinitely many rotary tangencies produces new sequences of saddles with rotary homoclinic tangencies and so on. Thus the original tangency implies the existence of a hierarchy of levels of saddle orbits and tangencies.

The rotation numbers of the new saddles cretaed at one level are related to the rotation number of the saddle at the previous level in a natural way using continued fractions representations of rational numbers. Through an analysis of these continued fractions, we are able to describe the phenomena observed in the Hénon map.

1. Preliminaries

Let f be an area-contracting, orientation-preserving diffeomorphism of the plane. The point p is called a *periodic point* of period k if $f^k(p)=p$. Unless otherwise specified, we assume that k is the smallest such number, and thus k is the minimum period of p. For a periodic point p of period k, the *stable manifold* of p is the set $W^s(p)=\{y: f^{kn}(y)\to f^{kn}(p)$, as $n\to\infty\}$; the *unstable manifold* of p is the set $W^u(p) =\{y: f^{-kn}(y)\to f^{-kn}(p)$, as $n\to\infty\}$. The *orbit* of p is the set of k distinct forward images of p. The stable (respectively, unstable) manifold of the orbit of p is the set of k distinct forward images of $W^s(p)$ (respectively, $W^u(p)$). We let $Df(x)$ denote the matrix of partial derivatives of f evaluated at x. The determinant of $Df(x)$ is the *Jacobian* of f at x. A periodic point p of period k is an *attractor* (respectively, *saddle*) if the eigenvalues e_1 and e_2 of $Df^k(p)$ satisfy $|e_1|<1$ and $|e_2|<1$ (respectively, $|e_1|<1$ and $|e_2|>1$).

When f has more than one attractor, the boundaries between respective basins of attraction can be extremely complicated. Such boundaries can be fractal, making final state predictability very difficult, and they can contain infinitely many unstable periodic orbits [7, 1]. As a further obstacle to predictability, it has been shown [9, 1, 7] that, as a parameter in f is continuously varied, sudden discontinuous jumps can occur in the position and structure of a boundary. This phenomenon is called a *basin boundary metamorphosis* and is discussed more fully later in this section and in Sect. 2.

In order to understand the dynamics of f on the boundary, we begin by restricting ourselves to the accessible boundary points. If the basin U is connected and simply connected, then f acts like a circle map on the accessible points (see, for example, [4, 5, or 2]). An orientation-preserving invertible map g of the circle is classified according to a rotation number ϱ which is a measure of the average rotation of the circle under g (see, for example, [8]). Specifically, if $\varrho=h/k$ (reduced), then there are periodic points of period k which go around the circle h times every k iterates. The rotation number is 0 if and only if g has fixed points. In the main result of [2] it was shown that if the rotation number of f on the accessible points is rational, say h/k, then every accessible boundary point is either a periodic point of period k (not necessarily the minimum period) or is on the stable manifold of such a

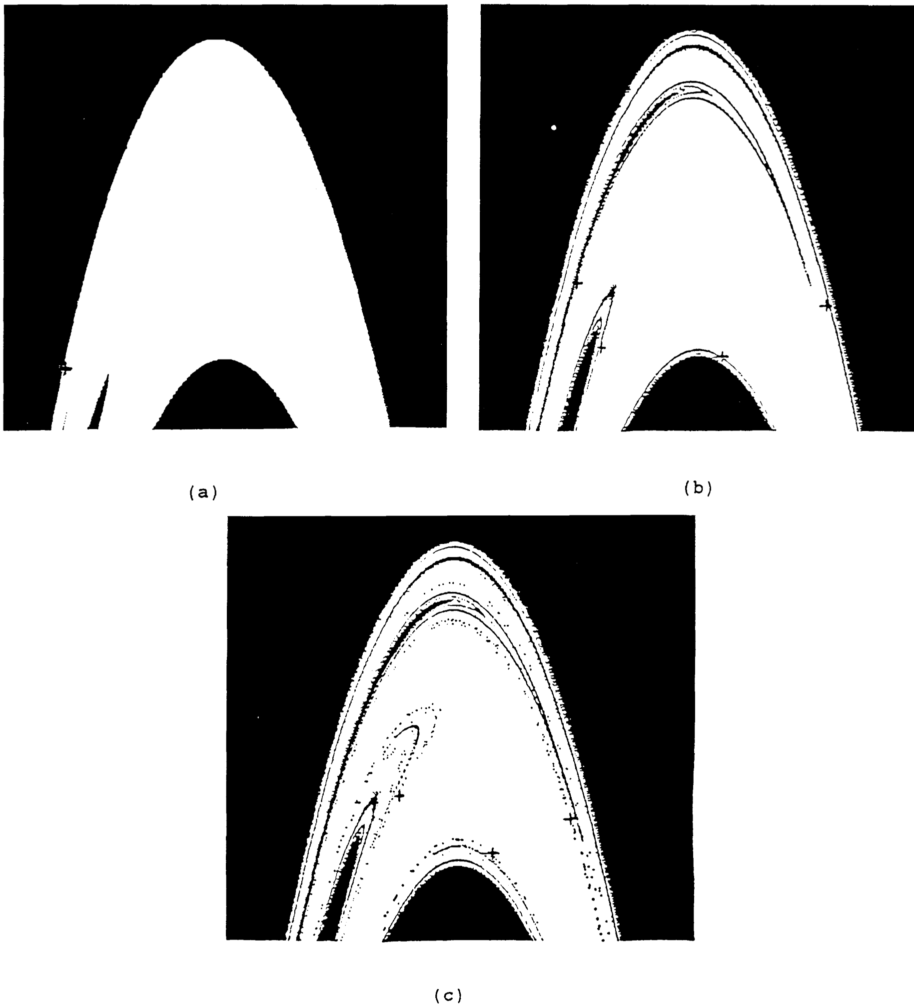

Fig. 1a–c. A portion of the basin of infinity of the Hénon map (1.1) is shown in black for each of three values of the parameter A. The region indicated is the subset $[-3, 3] \times [-4, 12]$ of the plane. In each figure, a cross indicates the position of a point in a saddle orbit accessible from the white region. **a** At $A=1.31$, the set of accessible points consists of a saddle fixed point and its stable manifold. **b** At $A=1.39$, the set of accessible points consists of a period-four saddle and its stable manifold. In **c** At $A=1.40$, the set of accessible points consists of a period-three saddle and its stable manifold

point. Of course, the dynamics of other boundary points are not easily described. Non-accessible periodic orbits can have vastly different (in particular, higher) periods.

Figure 1 shows in black the basin W of attraction of infinity for three different values of the parameter A in the Hénon map [10]

$$f(x, y) = (A - x^2 - By, x). \tag{1.1}$$

The Jacobian of f is B which is fixed at 0.3. (Later we investigate the map for varying B values.) We chose the Hénon map for this study because of its prototypical behavior (when $B=0$, it is the one-dimensional quadratic map $F(x) = A - x^2$), and because calculation of orbits is very efficient. We are interested in the orbits accessible from the white region U which is the complement of the closure of W. Here we are considering a rather broad concept of basin, as in [2]; namely, U is connected and simply connected, and there is an ε-neighborhood N_ε of the boundary of U such that the orbits of a dense set of points in N_ε eventually leave N_ε.

In Fig. 1a ($A=1.31$) there is an attractor of period 2 in the white region U to which orbits of almost all points in U tend. At this A value, there is a saddle fixed point p on the boundary. The entire boundary is composed of p and the stable manifold $W^s(p)$ of p. Thus the rotation number ϱ is 0. In Fig. 1b ($A=1.39$), the boundary is fractal. There is an accessible saddle of period 4, giving a rotation number of 1/4. All other accessible points are on the stable manifold of this orbit. In Fig. 1c ($A=1.40$), the accessible points consist of a periodic orbit of period 3 and its stable manifold; hence the rotation number is 1/3. For $B=0.3$ and A between 0 and 2, these rotation numbers – 0, 1/4, and 1/3 – are the only ones observed.

In a one-parameter family of invertible circle maps, the rotation number varies continuously with the parameter. In the present situation, however, we see an entirely different phenomenon. The rotation number of accessible basin boundary points does not vary continuously with the parameter. In Fig. 1a, the stable manifold of the fixed point saddle p on the boundary is visible since it is the basin boundary; the unstable manifold $W^u(p)$ is not visible, but extends from p into the white region. At a critical parameter value $A = A_* \approx 1.315$, the stable and unstable manifolds of p become tangent for the first time and then cross for $A > A_*$. At this parameter value, there is a sudden jump in the boundary, and points which were well within the interior of U at $A = A_*$ go to infinity for all $A > A_*$. In [9] Hammel and Jones prove that at a homoclinic tangency (where $W^s(p)$ and $W^u(p)$ become tangent and then cross) $W^s(p)$ jumps into the region U. Their result holds for a general class of maps, not just for the Hénon map, and implies that a boundary metamorphosis occurs whenever the stable and unstable manifolds of an accessible saddle become tangent and then cross (see [7 and 1]). In our example, there exists a periodic saddle of period 4 well within the interior of U at $A = A_*$. For every value of A greater than A_*, this orbit is in the boundary. Another metamorphosis occurs at $A \approx 1.395$, and the boundary jumps to the period 3 orbit shown in Fig. 1c. This metamorphosis is described in the next section.

2. A Numerical Study of Metamorphoses of the Hénon Map

Figure 2 shows the results of a numerical study of rotation numbers of accessible points for the two-parameter Hénon map (1.1). The basin U is the set described in Sect. 1. For $0 < B < 1$, f is orientation preserving and area contracting. The diagram indicates that the rotation number is a discontinuous function of the parameter. Although not every rotation number which occurs for A and B in the range shown is indicated, the representation is correct for $B=0$ to $B=0.95$ on a

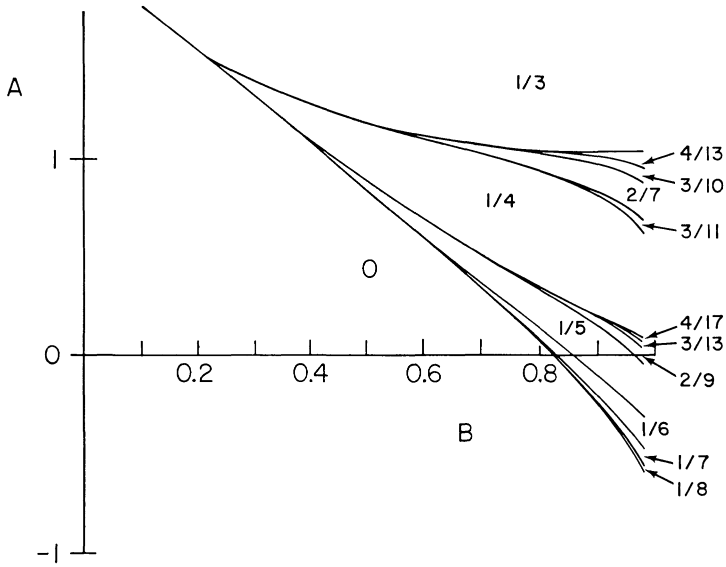

Fig. 2. Rotation numbers of accessible orbits of the two-parameter Hénon map are shown. The fractions represent rotation numbers of orbits accessible from the region U, i.e., the set of points not in the closure of the basin of infinity. In each of Fig. 1a–c, U is the region shown in white

grid size of 0.01×0.01. Beyond $B = 0.95$, the regions are too small and too numerous for all of them to be shown in this picture.

Some qualitative observations can be made about the diagram. First, the rotation number appears always to be rational when $B < 1$. Second, the size of the region in parameter space with rotation number h/k roughly decreases as k increases. Third, the greater the area contraction (i.e., the smaller B), the fewer metamorphoses occur as A is varied and the greater the jump in rotation number at each metamorphoses. For larger B, more distinct rotation numbers are possible. Fourth, for fixed B, the rotation number is increasing (although not continuous) as a function of A. We choose to emphasize these qualitative aspects of the diagram because they are likely to be observed not just for the Hénon map, but in more generality. To explain these phenomena, we need to examine in more detail the mechanism for changing the rotation number of the accessible orbits, which is the metamorphosis mentioned in the previous section.

For $A \geqq -(B+1)^2/4$, (1.1) has fixed points. There is a saddle-node bifurcation at $A = -(B+1)^2/4$ at which point a fixed point attractor θ and a saddle p come into existence. For values of A near the saddle node, the saddle p and its stable manifold form the boundary of the basin U of the attractor θ. Thus the boundary of U starts as a smooth curve on which all points are accessible from U and which has a rotation number of 0. This is the situation illustrated in Fig. 1a and for all values of A and B in Fig. 2 in the region marked 0. The curve which bounds this region indicates the parameter values where a homoclinic tangency occurs. We refer to the tangencies along this curve as *simple* homoclinic tangencies since the intersecting manifolds emanate from a fixed point.

When there is a homoclinic tangency of the stable and unstable manifolds of p (under certain non-degeneracy assumptions), infinitely many horseshoes are formed near a point q of tangency (see, for example, [8]). Specifically, there exists an integer $M>0$ such that for $n \geqq M$, f^n forms a horseshoe. For ease of exposition we describe here the case where horseshoes form prior to the actual tangency. Although this is not always the case, it is the pertinent situation for the consideration of metamorphoses, and we are more specific about this assumption in the statement and proof of the theorem in Sects. 3 and 4. The domains and images of two horseshoe maps in the sequence are illustrated in Fig. 3.

In the horseshoe formed by f^n there are two saddle orbits of minimum period n. One is a "flipped" or "Möbius" saddle (i.e., Df^n evaluated at a point in this orbit has an eigenvalue in $(-\infty, -1)$), and the other is a "regular" saddle [i.e., Df^n evaluated at a point in the orbit has an eigenvalue in $(1, \infty)$]. We call the regular saddle orbit of period n the *principal saddle* of period n. Given a point q of tangency of $W^s(p)$ with $W^u(p)$, there exists a sequence $\{p_n\}$ of principal saddles converging to q.

For n sufficiently large, the stable and unstable manifolds of successive principal saddles in the sequence cross forming a complicated web. In particular, there exists an $N>0$ such that for $n \geqq N$, $W^u(p_n)$ crosses $W^s(p_{n+1})$ (see [1]). This crossing implies $W^u(p_{n+1}) \subset \overline{W^u(p_n)}$ (see, e.g., [8] for a discussion of the λ-lemma). Since the saddles p_n converge to q, this implies that q is in $\overline{W^u(p_n)}$, for all $n \geqq N$. Thus when $W^s(p)$ crosses $W^u(p)$, it simultaneously crosses $W^u(p_n)$ for all $n \geqq N$. Then $W^s(p_n) \subset \overline{W^s(p)}$ for all $n \geqq N$. Since $W^s(p)$ is in the boundary of U at the homoclinic tangency, when $W^s(p)$ and $W^u(p)$ cross, p_n and $W^s(p_n)$ are in the boundary for all $n \geqq N$ (this fact was pointed out in [7]).

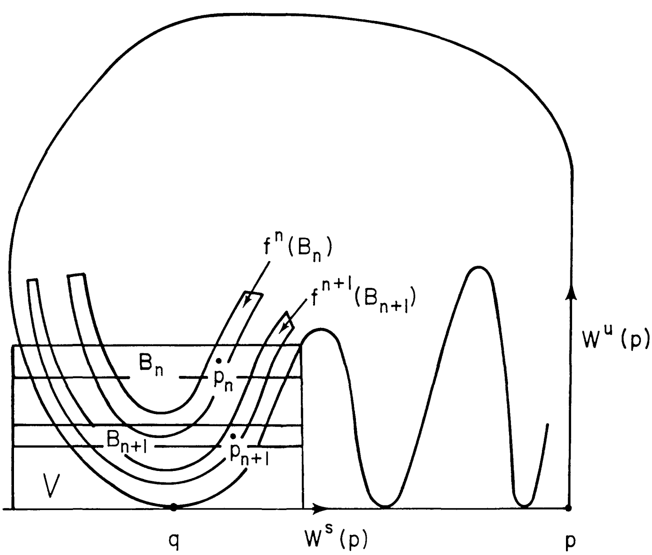

Fig. 3. A homoclinic tangency of the stable and unstable manifolds of a saddle p is shown, along with the domains and ranges of two horseshoe maps in the infinite sequence of horseshoe maps formed near a point q of tangency

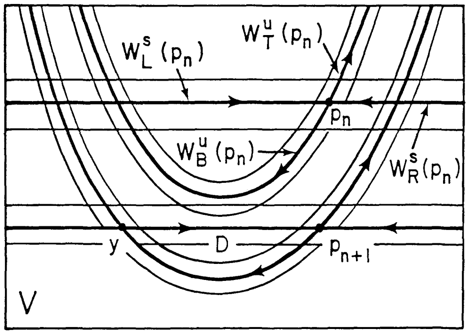

Fig. 4. Portions of the stable and unstable manifolds of two adjacent principal saddles are shown. The lower boundary of the region V contains a point of homoclinic or rotary homoclinic tangency. The point y and the region D are described in the proof of Theorem 3.1

The value of N where this process starts determines the rotation number of the new accessible orbits. For the homoclinic tangency at $B=0.3$ and $A=A_* \approx 1.315$, N equals 4, and the subsequent metamorphosis results in a jump in the rotation number from 0 to 1/4. Notice, however, that for $A > A_*$ all the principal saddles p_n, for $n \geqq 4$, and their stable manifolds are in the boundary (along with the stable manifolds of other points in the invariant sets of the horseshoes). Figure 4 shows portions of the stable and unstable manifolds of two successive principal saddles. As shown, part of $W^s(p_n)$ extends from p_n through the rectangular domain B_n, and part of $W^u(p_n)$ extends from p_n through the horseshoe-shaped image $f^n(B_n)$. As the Jacobian B increases toward one, indicating weaker area contraction by f, the spacing between successive horseshoe images, and thus between unstable manifolds of the saddles p_n, increases relative to the spacing of the stable manifolds. Thus we expect N (where the process of linking of stable and unstable manifolds begins) to become larger with B. As shown in Fig. 2, this is in fact the case.

Subsequent metamorphoses are possible. Jumps in the boundary occur when there is a crossing of $W^s(p_N)$ (the accessible points) by another invariant manifold. Thus if r is a periodic saddle in U and if $W^u(r)$ crosses $W^s(p_N)$, then the boundary jumps to contain r and its stable manifold. For example, when $B=0.3$ in (1.1), $W^u(p_3)$ crosses $W^s(p_4)$ at $A \approx 1.395$, at which point the rotation number jumps from 1/4 to 1/3. See Fig. 1c and Fig. 2. As B is increased, Fig. 2 shows regions in parameter space with rotation numbers which are of the form h/k, where $h > 1$. In such regions there are accessible periodic orbits which are not simple; that is, the orbit winds around the basin h times every k iterates. Surprisingly, these rotation numbers can also be understood in terms of homoclinic tangencies and horseshoe formation.

3. Rotary Homoclinic Tangencies

In studying the metamorphosis described above where the rotation number jumps from 1/4 to 1/3, it was observed [7] that just prior to the crossing of $W^u(p_3)$ with $W^s(p_4)$ there is a tangency of the unstable manifold of p_4 with the stable manifold of

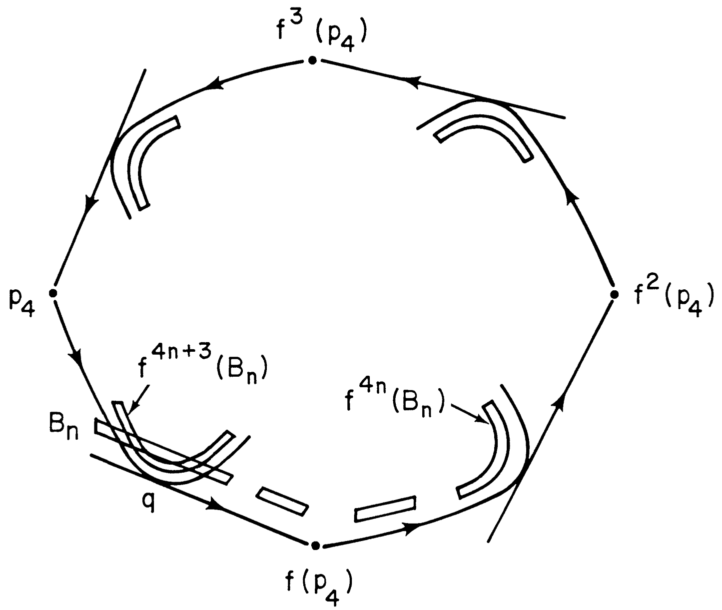

Fig. 5. The domain B_n and the range $f^{4n+3}(B_n)$ of a horseshoe map formed near a rotary homoclinic point q are shown. The point p_4 is a periodic saddle of period four

another point in the orbit of p_4. Again, there is recurrent behavior, and horseshoes are formed in a neighborhood of a point of tangency. In this case, however, the principal saddle in each horseshoe is not a simple orbit. Figure 5 depicts the horseshoe formation as a rectangular domain B_n near a point q of tangency moves around the circuit formed by the ring of stable and unstable manifolds. Every four iterates of f moves B_n along the stable and then the unstable manifold of p_4 to another point of tangency. As shown in Fig. 5, this occurs in $4n$ iterates. Then another three iterates moves $f^{4n}(V)$ back to q, so that $f^{4n+3}(B_n)$ forms a horseshoe-shaped image intersecting B_n. In particular, there exists $N>0$ such that for $n \geqq N$ there is a domain B_n for which f^{4n+3} forms such a horseshoe. There is a principal saddle in B_n with minimum period $4n+3$. This orbit is not simple since each point in the orbit goes around the circuit $n+1$ times every $4n+3$ iterates.

As in the case of the simple homoclinic tangeny, when the invariant manifolds of the accessible period-four orbit pass through the tangency at $A=A_4$, there is a metamorphosis. There will be a jump in the basin boundary so that the saddles of period $4n+3$ (for n sufficiently large) described above, together with their stable manifolds, will be in the boundary for $A>A_4$. Hence we expect accessible periodic orbits of period $4N+3$ and rotation number $(N+1)/(4N+3)$, for some $N \geqq 0$. Again, as in the case of a simple homoclinic tangency, as B increases and there is less area contraction, the larger we expect N to be.

We have described two levels in the hierarchy of horseshoes: horseshoes with rotation number $1/n$, for n sufficiently large, in the case of the simple homoclinic tangency; and between horseshoes of rotation number $1/N$ and $1/(N-1)$ at the first level, a secondary level of horseshoes with rotation numbers $(n+1)/(nN+N-1)$, for n sufficiently large. In fact, there are infinitely many levels. Before stating the theorem which guarantees the existence of this hierarchy, we give more formal definitions of some of the concepts described above.

Let f be an orientation-preserving, area-contracting, invertible map of the plane, and let p be a periodic point of period k. Suppose that $W^u(p)$ is tangent with $W^s(f^j(p))$, for some j, $1 \leqq j \leqq k$. Let q be a point of tangency, let α be the segment of $W^u(p)$ from p to q (inclusive), and let β be the segment of $W^s(f^j(p))$ from q to $f^j(p)$ (inclusive). We say that the orbit of p (or sometimes, by abuse of notation, the point p) forms a *rotary homoclinic tangency* at q if $\Gamma(p) = \bigcup_{i=1}^{k} f^i(\alpha \cup \beta)$ forms a simple closed curve (cf. Fig. 5).

If p forms a rotary homoclinic tangency, let $\pi_1 = p$, and let $\{\pi_1, \pi_2, \ldots, \pi_k\}$ be the orbit of p, where the points are listed in order of their appearance along the circle $\Gamma(p)$, say in the counter-clockwise sense. We say that p has rotation number h/k with respect to $\Gamma(p)$, if, for all $1 \leqq i \leqq k$, $f(\pi_i) = \pi_j$, where $j = (i+h) \bmod k$. Informally speaking, f restricted to the orbit of p moves the points around $\Gamma(p)$ as if they were an orbit of an order-preserving circle map of rotation number h/k.

At a rotary homoclinic tangency there is recurrent behavior. Just as in the case of a simple homoclinic tangency, horseshoes (and therefore principal saddles) are formed near a point of tangency. (We discuss sufficient non-degeneracy assumptions in Sect. 4). In describing metamorphoses, we are interested in principal saddles which form prior to the rotary tangency. For some types of tangencies, recurrent behavior does not occur prior to tangency but only after the manifolds have crossed. When principal saddles are created prior to tangency and when this tangency is the first interesection of the invariant manifolds involved, we call the tangency an *inner tangency*. Let p be a regular fixed-point saddle of an orientation-preserving map, and let $W_1^u(p)$ and $W_2^u(p)$ denote the two branches of the local unstable manifold of p. If, prior to tangency, $W_1^u(p)$ (respectively, $W_2^u(p)$) approaches $W^s(p)$ on the side from which $W_1^u(p)$ (respectively, $W_2^u(p)$) emanates, then the tangency will be inner. More generally, let p be a periodic orbit of period k, $k \geqq 1$. For a rotary tangency of p, let $\pi_1 = p$ and let $\{\pi_1, \pi_2, \ldots, \pi_k\}$ be the ordering of the orbit of p on $\Gamma(p)$. Then if $W_1^u(\pi_j)$ (respectively, $W_2^u(\pi_j)$) approaches $W^s(\pi_{(j+1) \bmod k})$ $(j = 1, \ldots, k)$ on the side from which $W_1^u(\pi_{(j+1) \bmod k})$ (respectively, $W_2^u(\pi_{(j+1) \bmod k})$) emanates, then the tangency will be inner.

The rotation numbers of accessible orbits at different levels are best described by continued fractions. As explained above, the sequence $\{s_n\}$ of saddles which exists when p_N (the simple orbit of period N) forms a rotary homoclinic tangency have rotation numbers $\{(n+1)/(nN+N-1)\}$, for n sufficiently large. As a continued fraction, $(n+1)/(nN+N-1)$ is $\dfrac{1}{N - 1/(n+1)}$. In general, let $[a_1, a_2, \ldots, a_j]$ denote the rotation number of a periodic saddle p at level j, where for $a_i \geqq 2$, $[a_1, a_2, \ldots, a_j]$ stands for the finite continued fraction

$$[a_1, a_2, \ldots, a_j] = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_j}}}}.$$

Each rational number strictly between 0 and 1 can be written in a unique way as $[a_1, \ldots, a_j]$ for some $j \geqq 1$ and $a_i \geqq 2$. Further, consider the reverse lexicographic ordering on the symbols $[a_1, \ldots, a_j]$. To be precise, we say that $a < b$ if and only if $a = [a_1, \ldots, a_i, a_{i+1}, \ldots, a_j]$, $b = [a_1, \ldots, a_i, b_{i+1}, \ldots, b_j]$, and $a_{i+1} > b_{i+1}$. (If a is shorter than b, fill out a with ∞ signs.) Thus $[4] < [4, 3] < [4, 2] < [4, 2, 2] < [4, 2, 2, 2] < [3]$. One checks that this ordering is identical with the usual ordering of rational numbers. It should be emphasized that these properties hold for the "minus" continued fractions we have described here, and not the "plus" continued fractions often used in the study of circle maps. It is these properties (and Lemmas 4.1–4.2) that make the above continued fraction representation of rational numbers very natural for keeping track of rotation numbers in this context.

We are now ready to state the main theorem, which will be proved in the next section.

Theorem 3.1. *Let f_λ be a C^3-family of real analytic, area-contracting, orientation-preserving maps of the plane. Suppose that a periodic saddle p forms an inner rotary homoclinic tangency at a point q when $\lambda = \lambda_*$. Then there exists a sequence $\{\lambda_n\}$ of parameter values with limit λ_* and a sequence $\{p_n\}$ of principal saddles with limit q such that for n sufficiently large, each principal saddle p_n has an inner rotary homoclinic tangency at λ_n.*

Moreover, if the rotation number of p with respect to $\Gamma(p)$ is $[a_1, \ldots, a_j]$, then the rotation number of p_n with respect to $\Gamma(p_n)$ is $[a_1, \ldots, a_j, n+1]$.

The theorem implies that at any rotary homoclinic tangency there are infinitely many levels of rotary homoclinic tangencies and principal saddles described by the continued fraction representations of their rotation numbers. Since each of these rotary tangencies is an inner tangency, a basin boundary metamorphosis can occur at any level, depending of course on the level of the accessible saddle.

As B is increased in (1.1), (i.e., the area contraction is decreased), not only do we observe larger values of N in the rotation number $1/N$ as the line of simple homoclinic tangencies is crossed in Fig. 2, but we also observe deeper levels of continued fractions. Qualitatively, this phenomenon can be explained by the fact the unstable manifolds of principal saddles at a given level are more spatially separated relative to their stable manifolds when there is less area contraction. This spacing allows metamorphoses which result in accessible orbits at the next level.

As an illustration, we interpret the jumps indicated in Fig. 2 when B is fixed at 0.9 and A is varied. For $A = 0.8$, there is an accessible period 4 saddle orbit, and so the rotation number is 1/4. As A is increased, this accessible saddle has a rotary homoclinic tangency. The sequence of principal saddles converging to the point of tangency has rotation numbers $1/(4 - 1/n)$ for $n \geqq 2$. For values of A slightly greater than the tangency value (approximately $A = 0.815$), the accessible orbit has rotation number $[4, 3] = 1/(4 - 1/3) = 3/11$. The other orbits created by the tangency, of rotation number $[4, n]$ for $n \geqq 4$, are buried in the basin boundary and not accessible. Later (at approximately $A = 0.831$), the 3/11 orbit is buried by a new accessible orbit with rotation number $[4, 2] = 2/7$. As A increases beyond 0.972, the 2/7 orbit has a rotary tangency, resulting in a sequence of principal saddles of periods $[4, 2, n]$ for $n \geqq 2$. It turns out that the $n=2$ orbit is accessible for A immediately beyond $A=0.972$, so the accessible orbit has rotation number $[4, 2, 2]=3/10$. At $A=1.013$, the 3/10 orbit has a rotary tangency, resulting in a new accessible orbit with rotation number $[4, 2, 2, 2]=4/13$. When the 4/13 orbit has a rotary tangency (at approximately $A=1.036$), the entire sequence of orbits converging to the tangency is buried by the stable manifold of the period 3 saddle orbit remaining from the original period 1 homoclinic tangency, and the rotation number jumps to $[3]=1/3$.

We end this section with the following conjectures:

Conjecture 1. Given a nonnegative rational number r, $r \leqq 1/3$, there is a region in (A, B)-parameter space where the rotation number of the accessible orbits of the basin U in (1.1) is r.

Conjecture 2. There are no parameter values with $0<B<1$ where the accessible orbits have an irrational rotation number.

4. Proof of Theorem

We begin with some observations on the continued fractions defined in the previous section. For a fixed continued fraction $[a_1, \ldots, a_j]$, define the lowest-terms numerator and denominator by

$$M_i/N_i=[a_1, \ldots, a_i] \quad \text{for} \quad 1 \leqq i \leqq j.$$

Fact 1. For $i \geqq 3$, $M_i=a_iM_{i-1}-M_{i-2}$ and $N_i=a_iN_{i-1}-N_{i-2}$.

Fact 2. For $i \geqq 2$, $M_iN_{i-1}-M_{i-1}N_i=1$.

Fact 1 is easy to derive by induction, and Fact 2 is a consequence of Fact 1.

Lemma 4.1. *Let $0<M/N=[a_1, \ldots, a_j]<1$ be a rational number with M/N in lowest terms. Then*

(a) *$\Phi(x)=x+M$ is a group automorphism of the additive group $\mathbb{Z}/(N)$, and*

(b) *The unique solution of $\Phi^k(0)=1$ for k in $\mathbb{Z}/(N)$ satisfies $h/k=[a_1, \ldots, a_{j-1}]$ in lowest terms. In fact, $kM=1+hN$.*

Proof. Part (a) is obvious. The key point $kM=1+hN$ of part (b) follows from Fact 2.

For example, consider the rational number $[2, 3, 4]=11/18$. Note that $[2, 3]=3/5$. The content of Lemma 4.1 is that to solve the equation $11k=1$ in $\mathbb{Z}/(18)$ requires $k=5$.

Remark. Lemma 4.1 has a more geometric illustration when applied to the closed orbits of a circle map. Let g be an order-preserving map of the circle with a closed orbit of rotation number $M/N=[a_1, \ldots, a_j]$. Let p_1 be a point of the orbit and let p_2 be the next point of the orbit in the chosen direction of rotation (with respect to a fixed lift). We pose the following question. How many iterates of g are necessary to map p_1 to p_2! The answer, according to Lemma 4.1, is k, where $h/k=[a_1, \ldots, a_{j-1}]$. Moreover, p_1 will have gone around the circle h times to reach p_2.

The second lemma can be most easily expressed in terms of Farey sums. Given two rational numbers h_1/k_1 and h_2/k_2, we will denote the Farey sum by $h_1/k_1 \oplus h_2/k_2 = (h_1+h_2)/(k_1+k_2)$.

Lemma 4.2. (a) *For* $n \geqq 1$, $[a_1, \ldots, a_j] \oplus [a_1, \ldots, a_j, n] = [a_1, \ldots, a_j, n+1]$.

(b) *Let* $h_1/k_1 = [a_1, \ldots, a_j]$, $h_2/k_2 = [a_1, \ldots, a_j - 1]$ *in lowest terms. Then for* $n \geqq 1$, $(nh_1 + h_2)/(nk_1 + k_2) = [a_1, \ldots, a_j, n+1]$.

Proof. Part (a) is proved by induction. The $j=1$ case is that $1/a_1 \oplus n/(na_1 - 1) = (n+1)/(a_1(n+1)-1) = [a_1, n+1]$. In general, let $M_1/N_1 = [a_2, \ldots, a_j]$ and $M_2/N_2 = [a_2, \ldots, a_j, n]$. Then

$$\begin{aligned}
[a_1, \ldots, a_j] \oplus [a_1, \ldots, a_j, n] &= \frac{1}{a_1 - M_1/N_1} \oplus \frac{1}{a_1 - M_2/N_2} \\
&= N_1/(a_1 N_1 - M_1) \oplus N_2/(a_1 N_2 - M_2) \\
&= \frac{1}{a_1 - (M_1 + M_2)/(N_1 + N_2)} \\
&= [a_1, \ldots, a_j, n+1],
\end{aligned}$$

where the last equality follows from the induction hypothesis.

Part (b) follows from the obvious equality $[a_1, \ldots, a_j - 1] = [a_1, \ldots, a_j, 1]$ and repeated application of part (a).

Proof of Theorem 3.1. First, we give a proof of the theorem in the case that p is a fixed point. Then we indicate the changes needed to deal with the general case. We follow a standard construction of the horseshoes which form near a quadratic tangency, (see, for example, [8]). For a real analytic map, tangencies of non-coincident invariant manifolds are not necessarily quadratic, but they are of finite order. In cases where the tangency is not quadratic, Robinson [13] has shown that there are at least hyperbolic principal saddles when the tangency is of finite order. Our proof depends on the existence of these saddles.

Set $f = f_{\lambda*}$. Let W be a neighborhood of p in which f is smoothly conjugate to a linear map. Without loss of generality, we assume that q is in W. Let V be a rectangular neighborhood in W bounded on one side by a segment of $W^s(p)$ containing q and on the opposite side by a line segment joining points of $W^u(p)$, as shown in Fig. 3. For each n sufficiently large, there is a domain B_n extending from the left to the right side of V on which f^n forms a horseshoe-shaped image, as shown in Fig. 3. The set $f^n(B_n)$ lies above $W^u(p)$ and intersects the top boundary of V in two components. The invariant set of the horseshoe is in $f^n(B_n) \cap B_n$. For the particular position of invariant manifolds shown in Fig. 3, the principal saddle p_n is in the right component of $f^n(B_n) \cap B_n$. For n sufficiently large, $W^s(p_n)$ extends through B_n and $W^u(p_n)$ extends through $f^n(B_n)$, (see, for example, [1]). We show the portion of each invariant manifold from p_n to the first intersection with the boundary of V in Fig. 4.

It was shown in [1] that at $\lambda = \lambda_*$ there exists $N > 0$ such that $W^u(p_n)$ crosses $W^s(p_{n+1})$ for $n \geqq N$. (See Remark 1 following the proof.) We will show that this crossing implies that p_{n+1} has a rotary homoclinic point; i.e., for some $\lambda < \lambda_*$, p_{n+1} has a rotary homoclinic tangency, and then its stable and unstable manifolds cross.

Let $W^s_L(p_n)$ and $W^s_R(p_n)$ denote the two branches of $W^s(p_n)$, as shown in Fig. 4. Similarly, let $W^u_T(p_n)$ and $W^u_B(p_n)$ denote the two branches of $W^u(p_n)$. Since $W^u_T(p_{n+1})$ extends from p_{n+1} to the top boundary of V, $W^u_T(p_{n+1})$ crosses $W^s_R(p_n)$. This crossing implies $W^u(p_n) \subset \overline{W^u_T(p_{n+1})}$, and hence that $W^u(p_n) = W^u(f^n(p_n)) \subset \overline{W^u_T(f^n(p_{n+1}))}$. For $n \geqq N$, $W^s_L(p_{n+1})$ crosses $W^u_B(p_n)$ [1]; since $W^u(p_n) \subset \overline{W^u_T(f^n(p_{n+1}))}$, $W^s_L(p_{n+1})$ must also cross $W^u_T(f^n(p_{n+1}))$, resulting in a rotary homoclinic point.

Clearly, there are parameter values prior to the tangency of $W^s(p)$ with $W^u(p)$ for which $W^s_L(p_{n+1})$ and $W^u_T(f^n(p_{n+1}))$ do not intersect. Thus there is a parameter value at which they are tangent for the first time. We claim that this tangency is an inner tangency, i.e., that principal saddles are formed prior to tangency. We need to show that $W^u_T(f^n(p_{n+1}))$ approaches $W^s_L(p_{n+1})$ on the side from which $W^u_T(p_{n+1})$ emanates. Suppose otherwise, and let y be a point at which $W^u_B(p_{n+1})$ and $W^s_L(p_{n+1})$ intersect (see Fig. 4), and let D be a region bounded by the segment of $W^u_B(p_{n+1})$ between p_{n+1} and y (inclusive) and by the segment of $W^s_L(p_{n+1})$ between p_{n+1} and y (inclusive). Then part of $W^u_T(f^n(p_{n+1}))$ must be inside D before tangency, implying that $W^u(f^n(p_{n+1}))$ intersects $W^s_L(p_{n+1})$ or $W^u_B(p_{n+1})$ before tangency, both of which are impossible. Thus $W^u(f^n(p_{n+1}))$ forms an inner tangency with $W^s(p_{n+1})$.

In the general case, we begin with a rotary homoclinic tangency of a point p of rotation number $h/k = [a_1, \ldots, a_j]$ with respect to $\Gamma(p)$. Consider the rational number $c/d = [a_1, \ldots, a_j - 1]$. We will show that the rotation numbers of the sequence of principal saddles p_n as above are given by $(nh+c)/(nk+d)$. Then it follows from Lemma 4.2(b) that the rotation number of p_n on $\Gamma(p_n)$ is $[a_1, \ldots, a_j, n+1]$.

Let $\pi_1 = p$, and let $\{\pi_1, \pi_2, \ldots, \pi_k\}$ be the ordering of the orbit of p along the closed curve $\Gamma(p)$. A principal saddle is mapped from a neighborhood U_1 of the point q of tangency of $W^u_T(\pi_k)$ with $W^s_L(p)$ along $W^s_L(p)$ toward p and then out $W^u_T(p)$ from p to a neighborhood U_2 of the point of tangency of $W^u_T(p)$ with $W^s_L(\pi_2)$. Let p_n be the principal saddle for which this process takes n steps, for n sufficiently large. Since the period of p is k, n steps accounts for nk iterates of f. Also, each step accounts for h circuits of the curve $\Gamma(p)$. From the remark following Lemma 4.1 and Lemma 4.2(a), it follows that $f^{nk}(p_n)$ will map from U_2 back to U_1 (and thus to p_n) in another d iterates of f and c circuits of $\Gamma(p)$.

The rest of the proof proceeds as in the fixed point case, except that the period of p_n is $nk+d$ instead of n. Thus we get, for sufficiently large n, a crossing of $W^s_L(p_{n+1})$ with $W^u_T(f^{nk+d}(p_{n+1}))$. By the remark after Lemma 4.1, $f^{nk+d}(p_{n+1})$ is the point in the orbit of p_{n+1} adjacent to p_{n+1}, as needed. $\square$

Remarks 1. In [1] it was shown that when $\lambda = \lambda_*$ there exists an $N \geqq 0$ such that for $n \geqq N$, $W^u(p_n)$ crosses $W^s(p_{n+1})$, under the hypothesis that the tangency is quadratic. The proof there depends on the fact that the distance from $f^n(B_n)$ to $W^u(p)$ relative to the distance of B_n to $W^s(p)$ goes to 0, as n goes to ∞. This fact follows from an analysis of distances in W (where the map is linear) which depend on the eigenvalues of $Df(p)$ and n. Outside W, a fixed number k of iterates, independent of n, maps $f^{n-k}(B_n)$ (the last iterate of B_n in W) to $f^n(B_n)$. As long as f is Lipschitz, the ratio of the distances in the limit as $n \to \infty$ is preserved with the addition of k iterates. Thus, since the existence of the principal saddles is guaranteed for finite order tangencies (see [13]), that hypothesis is sufficient for the result in [1].

2. Let p be a periodic point of period k which forms an inner rotary homoclinic tangency when $\lambda = \lambda_*$. It is easily seen that if the invariant manifolds forming the tangency cross transversally (say for $\lambda > \lambda_*$), then the stable manifold of p crosses the unstable manifold of p transversally for $\lambda > \lambda_*$. Resulting points of intersection of $W^s(p)$ with $W^u(p)$ are transverse homoclinic points in the ordinary sense, i.e., points whose forward and backward iterates are asymptotic to p under f^k.

3. Under non-degeneracy assumptions, the existence of homoclinic or rotary homoclinic tangencies implies the existence of infinitely many horseshoe maps, i.e., invariant sets which are conjugate to the full two-shift. Under such hypotheses, each principal saddle guaranteed by Theorem 3.1 represents an entire horseshoe.

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