Commun. Math. Phys. 119, 543–565 (1988)

Communications in Mathematical Physics

The $A_n^{(1)}$ Face Models

M. Jimbo[1], A. Kuniba[2], T. Miwa[1] and M. Okado[1]

[1] Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

[2] Institute of Physics, College of Arts and Sciences, University of Tokyo, Tokyo 153, Japan

Abstract. Presented here is the construction of solvable two-dimensional lattice models associated with the affine Lie algebra $A_n^{(1)}$ and an arbitrary pair of Young diagrams. The models comprise two kinds of fluctuation variables; one lives on the sites and takes on dominant integral weights of a fixed level, the other lives on edges and assumes the weights of the representations of $sl(n+1, \mathbf{C})$ specified by Young diagrams. The Boltzmann weights are elliptic solutions of the Yang–Baxter equation. Some conjectures on the one point functions are put forth.

1. Introduction

Let us begin with a link between solvable lattice models and the representation theory of $A_1^{(1)}$. It has been found in [1–4] through the computation of the one point functions $P(a)$ of the former; e.g., in Regime III of the models treated in [2–4], they were completely determined by the decomposition of characters

$$\chi_\xi \chi_\eta = \sum_a b_{\xi\eta a} \chi_a$$

for the pair $A_1^{(1)} \oplus A_1^{(1)} \supset A_1^{(1)}$ and the branching coefficients $b_{\xi\eta a}$ appearing therein:

$$P(a) = \frac{b_{\xi\eta a}\chi_a}{\chi_\xi \chi_\eta}.$$

In this paper we take one more step towards a thorough understanding of this phenomenon.

The corner transfer matrix method—the trick in the computation—was originally invented by Baxter [5]. In his study of the hard hexagon model there emerged a remarkable role of the q-series identities of the Rogers–Ramanujan type. A series of models with interactions round a face were then worked out [6] and their critical exponents were identified with those of the minimal conformal field theories [7]. Further studies along this line were pursued by several authors [8–11] until the complete result was obtained in [3, 4]. There appeared the affine

Lie algebra $A_1^{(1)}$ describing the one point functions. A natural question arose: What if $A_n^{(1)}$ replaces $A_1^{(1)}$?

This was answered in [12–15]—but not to the full extent of the problem. If $n=1$ the case covered there contains only the subclass treated in [6]. The whole family considered in [2–4, 11] had two integer parameters L and N. The parameter L was introduced in [6] and $L-2$ means the level of the representations in the Regime III picture presented in [2–4]. The parameter N—going back to the paper by Kulish–Reshetikhin–Sklyanin [16] on vertex models—represents the degree of the symmetric tensor representation of $sl(2,\mathbf{C})$. If we consider $A_n^{(1)}$, there come into the game the representations of $sl(n+1,\mathbf{C})$ that are parametrized by general Young diagrams. The aim of this paper lies, in fact, on this generalization.

Within the category of vertex models this was done by Cherednik [17]. With some modifications his method works well in our case of face models. The distinction between these two types of models is—as the names suggest—that fluctuation variables of vertex models are placed on edges and interact among four round a vertex, while those of face models are placed on sites and interact among four round a face. This last statement is yet untrue for the general case we treat in this paper. The models we consider must have fluctuation variables both on sites and edges. The interactions take place by eight round a face. This is explained most elegantly by Pasquier's argument [18]: In the rational limit, the site variables $a, b, \ldots$ represent the highest weight modules $\mathscr{V}_a, \mathscr{V}_b, \ldots$ of $sl(n+1,\mathbf{C})$ and the edge variables $\alpha, \beta, \ldots$ represent base vectors of $\mathrm{Hom}_{sl(n+1,\mathbf{C})}(\mathscr{V}_a \otimes \mathscr{V}_Y, \mathscr{V}_b)$, where $\mathscr{V}_Y$ is the $sl(n+1,\mathbf{C})$ module specified by the Young diagram Y. If Y is a symmetric tensor or an anti-symmetric tensor, the decomposition of $\mathscr{V}_a \otimes \mathscr{V}_Y$ is multiplicity free and the edge variables are irrelevant. The symmetric case was treated in [13, 14].

In this paper we have constructed a family of solvable lattice models with face interactions parametrized by arbitrary Young diagrams. We call them the $A_n^{(1)}$ face models. Although we state our results only on the restricted models in the sense of [6], a similar treatment is possible for the unrestricted models. For the symmetric or the antisymmetric case computer experiments have brought conjectures which relate the one point functions—we call them the local state probabilities in this paper—with the $A_n^{(1)}$ characters. Their proofs and the computation of the general case are still to be worked out.

The plan of this paper goes as follows. Section 2 gives the framework of the model and the solutions to the symmetric case. The general cases are treated in Sect. 3. The particular case of $Y = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ is explained in Sect. 4 as an illustration of our method. We propose some conjectures on the one point functions in Sect. 5. Appendix A contains some propositions on the symmetric group. Appendix B is a brief exposition of Pasquier's work.

2. Yang–Baxter Equation and $A_n^{(1)}$ Face Model

In this section we formulate the Yang–Baxter equation for the face models. We give its solutions corresponding to the vector and the symmetric tensor representations of $sl(n+1,\mathbf{C})$.

2.1. The Yang–Baxter Equation. Let V, V' and V'' be vector spaces. The Yang–Baxter equation for $W_{VV'}$, $W_{VV''}$, $W_{V'V''}$—which belongs to $\mathrm{Hom}_{\mathbf{C}}(V\otimes V', V'\otimes V)$, etc., respectively—reads as

$$(1\otimes W_{VV'})(W_{VV''}\otimes 1)(1\otimes W_{V'V''}) = (W_{V'V''}\otimes 1)(1\otimes W_{VV''})(W_{VV'}\otimes 1). \quad (2.1)$$

$$\begin{tikzcd}
V''\otimes V\otimes V' \arrow[d, "1\otimes W_{VV'}"] & V\otimes V''\otimes V' \arrow[l, "W_{VV''}\otimes 1"'] \\
V''\otimes V'\otimes V & V\otimes V'\otimes V'' \arrow[u, "1\otimes W_{V'V''}"'] \arrow[d, "W_{VV'}\otimes 1"] \\
V'\otimes V''\otimes V \arrow[u, "W_{V'V''}\otimes 1"] & V'\otimes V\otimes V''. \arrow[l, "1\otimes W_{VV''}"']
\end{tikzcd}$$

For example, the transpositions $P_{VV'}, P_{VV''}, P_{V'V''}$ (i.e., $P_{VV'}(x\otimes y) = y\otimes x$ for $x\in V$ and $y\in V'$, etc.) satisfy the Yang–Baxter equation. In fact, Eq. (2.1) then becomes a disguised form of the relation $(23)(12)(23) = (12)(23)(12)$ in the symmetric group S_3.

In this paper we mainly consider the following more restrictive form of the Yang–Baxter equation: $W_{VV'} = W_{VV'}(u)$, etc., depend upon a complex parameter u and satisfy

$$\begin{aligned}&(1\otimes W_{VV'}(u))(W_{VV''}(u+v)\otimes 1)(1\otimes W_{V'V''}(v))\\ &\quad = (W_{V'V''}(v)\otimes 1)(1\otimes W_{VV''}(u+v))(W_{VV'}(u)\otimes 1). \quad (2.2)\end{aligned}$$

For example, let's consider the case $V = V' = V''$. The choice $W_{VV'}(u) = W_{VV''}(u) = W_{V'V''}(u) = 1 + uP$ (where $P(x\otimes y) = y\otimes x$ as before) solves (2.2). This is explained by an identity in the group ring $\mathbf{C}S_3$,

$$(1 + u(23))(1 + (u+v)(12))(1 + v(23)) = (1 + v(12))(1 + (u+v)(23))(1 + u(12)).$$

2.2. Face Models. We now consider a direct sum decomposition of the form

$$V = \oplus_{a,b\in\mathscr{S}} V_{ab}.$$

An element of $\mathscr{S}$ is called *a local state*. In this paper we construct a family $\mathscr{R}$ of vector spaces $(V = \oplus_{a,b\in\mathscr{S}} V_{ab})_{V\in\mathscr{R}}$ and a family of operators $(W_{VV'}(u))_{V,V'\in\mathscr{R}}$ satisfying

(a-1) the Yang–Baxter equation (2.2) for any triplet $V, V', V''\in\mathscr{R}$,

(a-2) the composition $V_{ab}\otimes V'_{b'c'} \xrightarrow{\iota} V\otimes V' \xrightarrow{W_{VV'}} V'\otimes V \xrightarrow{\pi} V'_{a'd'}\otimes V_{dc}$ is vanishing unless $a = a'$, $b = b'$, $c = c'$, $d = d'$ (*the selection rule*).

We set $W_{VV'}\left(\begin{smallmatrix} a & b\\ d & c\end{smallmatrix}\right) = \pi\circ W_{VV'}\circ\iota$:

$$V_{ab}\otimes V'_{bc} \xrightarrow{\iota} V\otimes V' \xrightarrow{W_{VV'}} V'\otimes V \xrightarrow{\pi} V'_{ad}\otimes V_{dc},$$

and write $W_{VV'} = \oplus_{a,b,c,d\in\mathscr{S}} W_{VV'}\left(\begin{smallmatrix} a & b\\ d & c\end{smallmatrix}\right)$.

Our solution contains two parameters; *the level* l and *the rank* n. We set

$\mathscr{R}_n =$ the set of isomorphism classes of
finite dimensional irreducible representations of $sl(n+1, \mathbf{C})$.

A Young diagram $Y=(f_1,\dots,f_m)$ consisting of m rows ($f_1 \geqq \cdots \geqq f_m > 0$, $m \leqq n$) represents an element $(\rho_Y, \mathscr{V}_Y)$ of $\mathscr{R}_n$. In Sect. 3 we will define $V_Y \in \mathscr{R}$ in one to one correspondence with $Y \in \mathscr{R}_n$—but *not* $V_Y = \mathscr{V}_Y$. We set $L = l+n+1$. We denote by $\mathscr{A}$ the vector space $\mathbf{C}^{n+1}$ spanned by $e_\lambda = (\underbrace{0,\dots,0}_{\lambda},1,\underbrace{0,\dots,0}_{n-\lambda})$ ($\lambda = 0,\dots,n$). Note that $\mathscr{A} \cong \mathscr{V}_\square$, where $\square$ means the Young diagram $Y=(1)$. We call $p=(a^{(0)},\dots,a^{(N)}) \in \mathscr{A}^{N+1}$ an N-path from a to b if $a^{(0)}=a$, $a^{(N)}=b$, and $a^{(i)}-a^{(i-1)}=e_{\lambda_i}$ for some λ_i ($i=1,\dots,N$). We set $e_p^{(N)} = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N} \in \mathscr{A}^{\otimes N}$. The local states are given by

$$\mathscr{S} = \mathscr{S}_{l,n} = \{a = (a_0,\dots,a_n) \in \mathscr{A} \mid \text{(b-1)}, \text{(b-2)}\},$$

(b-1) $L + a_n > a_0 > a_1 > \cdots > a_n$,

(b-2) $a_{\lambda\mu} \overset{\text{def}}{=} a_\lambda - a_\mu \in \mathbf{Z}$ for any $\lambda, \mu = 0,1,\dots,n$.

A vector $a \in \mathscr{S}$ represents the level l dominant integral weight (abb. DIW) of $A_n^{(1)}$

$$(L + a_n - a_0 - 1)\Lambda_0 + (a_0 - a_1 - 1)\Lambda_1 + \cdots + (a_{n-1} - a_n - 1)\Lambda_n.$$

We denote by $(\rho_a, \mathscr{V}_a)$ the irreducible representation of $sl(n+1,\mathbf{C})$ whose highest weight is the classical part of this DIW.

We use the following notations:

(c-1) $a \overset{N}{\rightarrow} b \Leftrightarrow$ there exists an N-path from a to b staying in $\mathscr{S}$—we call it an N-admissible path. In this case a and b in particular belong to $\mathscr{S}$.

(c-2) $a \overset{N}{\Rightarrow} b \Leftrightarrow a \overset{N}{\rightarrow} b$ and any path from a to b is admissible. The pair (a,b) is called *strongly* admissible.

(c-3) The action of $\tau \in S_N$ on $\mathscr{A}^{\otimes N}$ is $x = v_1 \otimes \cdots \otimes v_N \mapsto \tau(x) = v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(N)}$. We denote by $(S_N)_x$ the subgroup of S_N which fixes x. The symmetrizer $S: \mathscr{A}^{\otimes N} \to \mathscr{A}^{\otimes N}$ is defined by $Sx = \sum_{\tau \in S_N/(S_N)_x} \tau(x)$. We set $e_{ab}^{(N)} \overset{\text{def}}{=} S e_p^{(N)}$ if $a \overset{N}{\Rightarrow} b$ and p is a path from a to b. We also set $e_{ab}^{(N)} = 0$ if (a,b) is not strongly admissible. We abbreviate $e_{ab}^{(1)}$ to e_{ab}.

(c-4) $\sigma_{l,n}: \mathscr{A} \to \mathscr{A}$ is a bijection defined by

$$\sigma_{l,n}(a) = (a_n + l + n, a_0 - 1, \dots, a_{n-1} - 1).$$

It implements a Dynkin diagram automorphism of $A_n^{(1)}$ on $\mathscr{S}$.

(c-5) For $a \in \mathscr{S}_{l,n}, a' \in \mathscr{S}_{l',n}$ we set

$$a \oplus a' \overset{\text{def}}{=} a + a' - (n, n-1, \dots, 0) \in \mathscr{S}_{l+l',n}.$$

We have then $\sigma_{l+l',n}(a \oplus a') = \sigma_{l,n}(a) \oplus \sigma_{l',n}(a')$.

(c-6) For $a \in \mathscr{A}$ we set

$$|a|_{l,n} \overset{\text{def}}{=} \frac{1}{2(l+n+1)} \sum_{i=0}^{n} \left(a_i - \frac{1}{n+1} \sum_{j=0}^{n} a_j \right)^2.$$

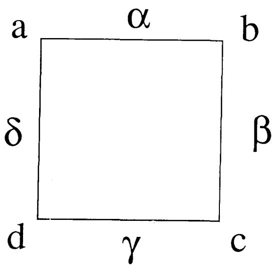

Fig. 1. A configuration round a face

Let us explain how to relate the operator $W_{VV'}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)$ to statistical weights of a lattice model. For simplicity we restrict to the case when $V = V'$. We consider two kinds of fluctuation variables, l_s placed on each site s and v_k placed on each bond k. The variable l_s runs over the local states $l_s = a, b, \ldots \in \mathscr{S}$. We fix a set of base vectors $B_{ab} = \{\alpha, \beta, \ldots\}$ of the space V_{ab}. If k is a horizontal bond and $l_s = a$ and $l_{s'} = b$, where s (respectively s') is the site at the left (respectively right) end of k, then the variable v_k runs over the base vectors $v_k = \alpha, \beta, \ldots \in B_{ab}$. If k is a vertical bond the term "left" (respectively "right") is replaced by "upper" (respectively "lower"). Now consider a configuration a, b, c, d and $\alpha, \beta, \gamma, \delta$ round a face as in Fig. 1. We associate to this configuration the $(\alpha \otimes \beta, \delta \otimes \gamma)$ matrix element of $W_{VV'}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)$ as the Boltzmann weight representing the interaction round the face. By this formulation both vertex and interaction-round-face models are treated on the same footing.

2.3. Vector Representation. The $A_n^{(1)}$ face model introduced in [12] is contained in our family in the following sense. The vector space $V_\square \in \mathscr{R}$ corresponding to the Young diagram $\square = (1) \in \mathscr{R}_n$ is given by

$$V_\square = \oplus_{a, b \in \mathscr{S}} (V_\square)_{ab}, \quad \text{where} \quad (V_\square)_{ab} = \left\{ \begin{matrix} \mathbf{C} e_{ab} & \text{if } a \overset{1}{\Rightarrow} b \\ 0 & \text{otherwise} \end{matrix} \right\}.$$

This is consistent with the definition of the admissibility in [12]. We interpret also the Boltzmann weights. The operator $W_{\square \otimes \square} \overset{\text{def}}{=} W_{V_\square V_\square}$ satisfying (2.2) for the triplet $V_\square$, $V_\square$, $V_\square$ is given as follows: We assume $a \overset{1}{\Rightarrow} b$, $b \overset{1}{\Rightarrow} c$, $a \overset{1}{\Rightarrow} d$, $d \overset{1}{\Rightarrow} c$ (otherwise $W_{\square \otimes \square}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right) = 0$). Let $e_{ab} = e_\lambda, e_{bc} = e_\nu, e_{ad} = e_\mu, e_{dc} = e_\kappa$ and define a scalar ${}_{\mu}{}^{a}_{d}\underset{\kappa}{\overset{\lambda}{\square}}{}^{b}_{c}{}_{\nu}$ by

$$W_{\square \otimes \square}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)(e_{ab} \otimes e_{bc}) = {}_{\mu}{}^{a}_{d}\underset{\kappa}{\overset{\lambda}{\boxed{u}}}{}^{b}_{c}{}_{\nu}\, e_{ad} \otimes e_{dc}.$$

We abbreviate ${}_{\mu}{}^{a}_{d}\underset{\kappa}{\overset{\lambda}{\boxed{u}}}{}^{b}_{c}{}_{\nu}$ to ${}_{\mu}\underset{\kappa}{\overset{\lambda}{\square}}{}_{\nu}$ or ${}^{a}_{d}\square^{b}_{c}$. We define ${}^{a}_{d}\square^{b}_{c} = 0$ unless $a \overset{1}{\Rightarrow} b$, etc., for convenience. In terms of the elliptic theta function with nome p

$$[u] = 2|p|^{1/8} \sin(\pi u/L) \prod_{k=1}^{\infty} (1 - 2p^k \cos(2\pi u/L) + p^{2k})(1 - p^k),$$

the Boltzmann weights, that satisfy (2.2), are written as

$$ {}_{\lambda}\overset{\lambda}{\underset{\lambda}{\square}}{}_{\lambda}=\frac{[1+u]}{[1]},\quad {}_{\lambda}\overset{\lambda}{\underset{\mu}{\square}}{}_{\mu}=\frac{[a_{\lambda\mu}-u]}{[a_{\lambda\mu}]},\quad {}_{\lambda}\overset{\mu}{\underset{\mu}{\square}}{}_{\lambda}=\frac{[u][a_{\lambda\mu}+1]}{[1][a_{\lambda\mu}]}(\lambda\neq\mu). \tag{2.3}$$

Then satisfy the Dynkin diagram symmetry

$$ {}^{a'}_{d'}\square^{b'}_{c'}={}^{a}_{d}\square^{b}_{c},\quad \text{where } a'=\sigma_{l,n}(a),\quad b'=\sigma_{l,n}(b),\quad c'=\sigma_{l,n}(c),\quad d'=\sigma_{l,n}(d).$$

In the limit $L\to\infty$ and $|a_{\lambda\mu}|\to\infty$ $(\lambda\neq\mu)$ the operator $W_{\square\otimes\square}$ reduces to $1+uP$ in the sense that $(1+uP)(e_\lambda\otimes e_\nu)=\sum_{\mu,\kappa}{}_{\mu}\overset{\lambda}{\underset{\kappa}{\square}}{}_{\nu}\, e_\mu\otimes e_\kappa$.

2.4. Fusion Procedure. Before going to the construction of V_{ab} and $W_{VV'}(u)$ let us explain the basic idea of the fusion procedure [17]. We define $(\tilde{V}^N)_{ab}$ to be the subspace of $\mathscr{A}^{\otimes N}$ spanned by $e_p^{(N)}$ for all N-admissible paths p from a to b. Note that $\tilde{V}^1=V_\square$. We define $W_{NN'}(u^{(N)},u^{(N')})=\bigoplus_{a,b,c,d\in\mathscr{S}}W_{NN'}\left(\begin{smallmatrix}a&b\\d&c\end{smallmatrix}\middle|u^{(N)},u^{(N')}\right)\in \mathrm{End}_C(\tilde{V}^{N+N'})$ by

$$W_{NN'}\left(\begin{smallmatrix}a&b\\d&c\end{smallmatrix}\middle|u^{(N)},u^{(N')}\right)(e_p^{(N)}\otimes e_q^{(N')})=\sum_{(a^{ij})}\left(\prod_{\substack{i=1,\dots,N\\ j=1,\dots,N'}}\boxed{u_{ij}}\right)e_r^{(N')}\otimes e_s^{(N)}. \tag{2.4}$$

The notations are as follows. $u^{(N)}=(u_1,\dots,u_N)$ and $u^{(N')}=(u'_1,\dots,u'_{N'})$ are parameters attached to $\tilde{V}^N$ and $\tilde{V}^{N'}$, respectively. The argument in the box is $u_{ij}\overset{\mathrm{def}}{=}u_i-u'_j$. The four DIWs round the box—located clockwise from the north-west corner—are $a^{i-1\,j-1}$, a^{ij-1}, a^{ij}, $a^{i-1\,j}$. The sum is over $a^{ij}\in\mathscr{S}$ such that the paths $p=(a^{00},\dots,a^{N0})$ and $q=(a^{N0},\dots,a^{NN'})$ are fixed along with $a^{00}=a$, $a^{N0}=b$, $a^{NN'}=c$, $a^{0N'}=d$. The paths r and s are $r=(a^{00},\dots,a^{0N'})$, $s=(a^{0N'},\dots,a^{NN'})$. With these definitions the Yang–Baxter equation (2.1) is satisfied for any triplet $\tilde{V}^N$, $\tilde{V}^{N'}$, $\tilde{V}^{N''}$.

Suppose that a set of data $(V,\alpha_V(u))_{V\in\mathscr{R}}$ is given in such a way that

(d-1) $V=\bigoplus_{a,b\in\mathscr{S}}V_{ab}$ is a subspace of $\tilde{V}^N$ satisfying $V_{ab}\subset(\tilde{V}^N)_{ab}$,

(d-2) $\alpha_V(u)$ is a one parameter family of N-vectors and

$$W_{VV'}(u,u')=W_{NN'}(\alpha_V(u),\alpha_{V'}(u'))$$

satisfies

$$W_{VV'}\left(\begin{smallmatrix}a&b\\d&c\end{smallmatrix}\middle|u,u'\right)(V_{ab}\otimes V'_{bc})\subset V'_{ad}\otimes V_{dc}.$$

Then $W_{VV'}(u,u')$, $W_{VV''}(u,u'')$ and $W_{V'V''}(u',u'')$ satisfy the Yang–Baxter equation (2.1). We abbreviate $W_{V_YV_{Y'}}(u,u')$ to $W_{Y\otimes Y'}(u-u')$ (the general definition of which is given in Sect. 3) since the dependence on u,u' is through the difference $u-u'$.

2.5. Symmetric Tensors. Let us recapitulate the case of symmetric tensors given in [13, 14]. Consider the Young diagram $(N)=\underbrace{\square\cdots\square}_{N}$. We denote by $S\mathscr{A}^{\otimes N}$ the space of the symmetric tensors of degree N. We set

$$V_{(N)}=\bigoplus_{a,b\in\mathscr{S}}(V_{(N)})_{ab},\quad (V_{(N)})_{ab}=(\tilde{V}^N)_{ab}\cap S\mathscr{A}^{\otimes N}.$$

We have in fact $\dim(V_{(N)})_{ab}\leqq 1$ and $\dim(V_{(N)})_{ab}=1$ if and only if $a\overset{N}{\Rightarrow}b$. The particular choice of the N-vector is

$$\alpha_{V_{(N)}}(u) = (u, u+1, \ldots, u+N-1).$$

Let us prove (d-2) for $W_{(N)\otimes(M)}$. First we show

Lemma 2.1. *Assume that* $a \overset{N}{\Rightarrow} b$, $b \overset{1}{\Rightarrow} c$, $a \overset{1}{\Rightarrow} d$, $d \overset{N}{\rightarrow} c$. *We set*

$$h_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = \sum_p \prod_{i=0}^{N-1} {}^{a^{(i)}}_{d^{(i)}}\boxed{u+i}^{a^{(i+1)}}_{d^{(i+1)}},$$

where the sum is over the paths $p = (a^{(0)}, \ldots, a^{(N)})$ *from* a *to* b, *and* $q = (d^{(0)}, \ldots, d^{(N)})$ *is a path from* d *to* c. *This is independent of the choice of* q. *In fact, we have*

$$h_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = \left(\prod_{j=1}^{N-1} \frac{[u+j]}{[1]}\right) \frac{[u + b_\nu - a_\mu] \prod_{\lambda(\neq\mu)} [b_\nu - a_\lambda + 1]}{\prod_\lambda [c_{\nu\lambda} + \delta_{\nu\lambda}]}, \tag{2.5}$$

where $e_{ad} = e_\mu$, $e_{bc} = e_\nu$. *If* (d, c) *is not strongly admissible then*

$$h_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = 0.$$

Proof. We use the induction on N. The case $N = 1$ follows from (2.3). Assume that the assertion is true for $N - 1$. We have

$$h_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = \mu\,{}^{a}_{d}\overset{\mu}{\underset{\mu}{\boxed{u}}}{}^{a'}_{d'}\,\mu\, h_{N-1}\left(\begin{smallmatrix} a' & b \\ d' & c \end{smallmatrix}\middle| u+1\right) \qquad \text{if} \quad d^{(1)} = d',$$

$$= \mu\,{}^{a}_{d}\overset{\mu}{\underset{\kappa}{\boxed{u}}}{}^{a'}_{d''}\,\kappa\, h_{N-1}\left(\begin{smallmatrix} a' & b \\ d'' & c \end{smallmatrix}\middle| u+1\right)$$

$$+ \mu\,{}^{a}_{d}\overset{\kappa}{\underset{\kappa}{\boxed{u}}}{}^{a''}_{d''}\,\mu\, h_{N-1}\left(\begin{smallmatrix} a'' & b \\ d'' & c \end{smallmatrix}\middle| u+1\right) \quad \text{if} \quad d^{(1)} = d'',$$

where $a' = a + e_\mu$, $a'' = a + e_\kappa$, $d' = d + e_\mu$, $d'' = d + e_\kappa$ and $\mu \neq \kappa$. Applying the standard addition theorem we obtain (2.5). The last assertion is true since if (d, c) is not strongly admissible then $\mu \neq \nu - 1$ and $a_{\nu-1} - b_\nu \equiv 1 \pmod{L}$. ■

Similarly, we have

Lemma 2.2. *Assume that* $b \overset{N}{\Rightarrow} c$, $a \overset{1}{\Rightarrow} b$, $d \overset{1}{\Rightarrow} c$, $a \overset{N}{\rightarrow} d$. *We set*

$$v_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = \sum_p \prod_{i=0}^{N-1} {}^{a^{(i+1)}}_{a^{(i)}}\boxed{u+i}^{b^{(i+1)}}_{b^{(i)}},$$

where the sum is over the paths $p = (b^{(N)}, \ldots, b^{(0)})$ *from* b *to* c, *and* $q = (a^{(N)}, \ldots, a^{(0)})$ *is a path from* a *to* d. *This is independent of the choice of* q. *In fact, we have*

$$v_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = \left(\prod_{j=1}^{N-1} \frac{[u+j]}{[1]}\right) \frac{[u + c_\mu - b_\nu] \prod_{\lambda(\neq\mu)} [c_\lambda - b_\nu + 1]}{\prod_\lambda [a_{\lambda\nu} + \delta_{\lambda\nu}]},$$

where $e_{ab} = e_\nu$, $e_{dc} = e_\mu$. *If* (a, d) *is not strongly admissible then*

$$v_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = 0.$$

The following proposition, which proves (d-2) for $W_{(N)\otimes(M)}$, is a consequence of these lemmas and the definition (2.4).

Proposition 2.3. *Assume that* $a \overset{N}{\Rightarrow} b$, $b \overset{M}{\Rightarrow} c$, $a \overset{M}{\rightarrow} d$, $d \overset{N}{\rightarrow} c$. *Then we have*

$$W_{(N)\otimes(M)}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right)(e^{(N)}_{ab} \otimes e^{(M)}_{bc}) = w_{NM}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) e^{(M)}_{ad} \otimes e^{(N)}_{dc},$$

where

$$w_{NM}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right) = \sum_{q} \prod_{i=0}^{M-1} h_N\left(\begin{smallmatrix} a' & b' \\ d' & c' \end{smallmatrix}\middle| v'\right), \tag{2.6}$$

$$= \sum_{p} \prod_{i=0}^{N-1} v_M\left(\begin{smallmatrix} a'' & b'' \\ d'' & c'' \end{smallmatrix}\middle| v''\right). \tag{2.7}$$

In (2.6), we choose any admissible path $(d^{(0)},\dots,d^{(M)})$ *from a to d, the sum is over the paths* $q = (b^{(0)},\dots,b^{(M)})$ *from b to c such that* $(d^{(i)}, b^{(i)})$ *is strongly admissible, and* $a' = d^{(i)}$, $b' = b^{(i)}$, $c' = b^{(i+1)}$, $d' = d^{(i+1)}$, $v' = u - i$. *In (2.7), we choose any admissible path* $(c^{(N)},\dots,c^{(0)})$ *from d to c, the sum is over the paths* $p = (a^{(N)},\dots,a^{(0)})$ *from a to b such that* $(a^{(i)}, c^{(i)})$ *is strongly admissible, and* $a'' = a^{(i+1)}$, $b'' = a^{(i)}$, $c'' = c^{(i)}$, $d'' = c^{(i+1)}$, $v'' = u + N - M - i$.

3. The General Case

This section is devoted to the construction of the operator $W_{Y\otimes Y'}$ for an arbitrary pair of Young diagrams (Y, Y'). We retain the notations of the previous section; in particular L will denote the positive integer $l + n + 1 \geqq 3$ (l = the level of DIWs). We shall deal with the spaces

$$\mathscr{A}^{\otimes N} \supset (\tilde{V}^N)_{ab} \supset (V_Y)_{ab},$$

where $\mathscr{A} = \mathbf{C}^{n+1}$, and $(\tilde{V}^N)_{ab}$ is the span of $e^{(N)}_p$ with p running over N-admissible paths from a to b. The third one $(V_Y)_{ab}$ will be defined in Sect. 3.3 below.

3.1. Elementary Operators. Fix a positive integer $N < L$. We denote by $W_i(u) \in \mathrm{End}_{\mathbf{C}}(\tilde{V}^N)$ $(i = 1, 2, \dots, N-1)$ the operators

$$W_i(u)(e_{a^{(0)}a^{(1)}} \otimes \cdots \otimes e_{a^{(N-1)}a^{(N)}})$$
$$= \sum_{a'^{(i)}} {}^{a^{(i-1)}}_{a'^{(i)}} \boxed{u}^{a^{(i)}}_{a^{(i+1)}} e_{a^{(0)}a^{(1)}} \otimes \cdots \otimes e_{a^{(i-1)}a'^{(i)}} \otimes e_{a'^{(i)}a^{(i+1)}} \otimes \cdots \otimes e_{a^{(N-1)}a^{(N)}},$$

where the box signifies the Boltzmann weight (2.3) for $\square \otimes \square$, and the sum is over DIWs $a'^{(i)}$ such that $a^{(i-1)} \overset{1}{\Rightarrow} a'^{(i)}$, $a'^{(i)} \overset{1}{\Rightarrow} a^{(i+1)}$. A basic property of the $W_i(u)$ is that they satisfy the Yang–Baxter equation (Fig. 2)

$$W_i(u) W_{i+1}(u+v) W_i(v) = W_{i+1}(v) W_i(u+v) W_{i+1}(u). \tag{3.1}$$

Besides (3.1) we have the inversion relation

$$W_i(-u) W_i(u) = \frac{[1-u][1+u]}{[1]^2}\, id_{\tilde{V}^N}. \tag{3.2}$$

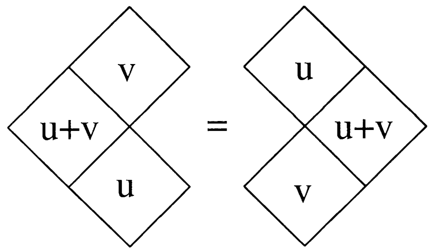

Fig. 2. The Yang–Baxter equation

Using the explicit form (2.3) we can verify also that

$$\begin{aligned}\operatorname{Ker} W_i(-1) &= \operatorname{Im} W_i(1)\\ &= \text{the subspace of } \tilde{V}^N \text{ spanned by}\\ &\quad e_{a^{(0)}a^{(1)}} \otimes \cdots \otimes e^{(2)}_{a^{(i-1)}a^{(i+1)}} \otimes \cdots \otimes e_{a^{(N-1)}a^{(N)}}. \end{aligned} \tag{3.3}$$

In terms of the $W_i(u)$ the operator for the symmetric tensors in Sect. 2 can be written as

$$\begin{aligned} W_{(N)\otimes(M)}(u) &= (W_M(u-M+1)\cdots W_{M+N-1}(u-M+N))\\ &\quad \times \cdots\\ &\quad \times (W_1(u)\cdots W_N(u+N-1)). \end{aligned}$$

For $i<j$ we set

$$S[i,j] = (W_i(1)W_{i+1}(2)\cdots W_{j-1}(j-i))\cdots(W_i(1)W_{i+1}(2))W_i(1). \tag{3.4}$$

By virtue of the Yang–Baxter equation (3.1) the right-hand side can be rearranged in various other ways. In particular for each $k=i,\ i+1,\dots,j-1$ it can be put in the form $S[i,j] = W_k(1)\times(\cdots)$.

Lemma 3.1. Im $S[i,j]$ *consists of all the symmetric tensors on the interval* $[i,j]$; *namely it is spanned by*

$$e_{a^{(0)}a^{(1)}} \otimes \cdots \otimes e^{(j-i)}_{a^{(i-1)}a^{(j)}} \otimes \cdots \otimes e_{a^{(N-1)}a^{(N)}}.$$

Proof. From the remark above the image is contained in such a sub space. To see that the two coincide, we use the induction on $j-i$. The proof then reduces to the following statement: if (a,c) is $(N+1)$-strongly admissible, then there exists a DIW b such that $h_N\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\Big| 1\right) \neq 0$. This can be checked by using the explicit expression (2.5). ∎

3.2. The Operator F. Let $Y=(f_1,\dots,f_m)$ be a Young diagram with N nodes ($f_1 \geqq \cdots \geqq f_m > 0$, $N=f_1+\cdots+f_m$). A node at the i^{th} row and the j^{th} column is represented by (i,j). We inscribe on (i,j) the number $(i-1)z+j-i$, where z is an auxiliary parameter [17]. Denote the resulting sequence by (Fig. 3).

$$\begin{aligned} &v_1(z),\dots,v_N(z)\\ &\quad = \underbrace{0,1,\dots,f_1-1}_{f_1},\dots,\underbrace{(m-1)z+1-m,\dots,(m-1)z+f_m-m}_{f_m}. \end{aligned} \tag{3.5}$$

0	1	2	3	4
z-1	z	z+1	z+2	z+3
2z-2	2z-1	2z	2z+1	2z+2
3z-3	3z-2	3z-1	3z	3z+1

Fig. 3. The sequence $v_i(z)$ for $Y=(3,2,2)$

Set $v_i=v_i(z)$, $v_{ij}=v_i-v_j$, and $g_i=f_1+\cdots+f_i$.

We shall introduce an operator $F(z)$ that has the property of reversing the order of v_i's in the sense

$$F(z)(W_1(v_1)W_2(v_2)\cdots W_N(v_N))=(W_1(v_N)\cdots W_{N-1}(v_2)W_N(v_1))F(z).$$

Explicitly it is given by

$$F(z)=W_1(v_{21})(W_2(v_{31})W_1(v_{32}))\cdots(W_{N-1}(v_{N1})\cdots W_2(v_{NN-2})W_1(v_{NN-1})). \quad (3.6)$$

Lemma 3.2. *We have*

$$F(z)=z^\kappa O(1) \quad \textit{as} \quad z\to 0,$$

where $\kappa=\#\{((i_1,j_1),(i_2,j_2))|i_1<i_2,\, j_1-i_1=j_2-i_2\}$ *denotes the number of pairs on the diagonal lines.*

Proof. By the Yang–Baxter equation (3.1), (3.6) can be brought to the form

$$F(z)=F^{(1)}(z)F^{(2)},$$

where $F^{(2)}$ reverses the order of v_i's within each block, while so does $F^{(1)}(z)$ blockwise (Fig. 4); more precisely

$$\begin{aligned} F^{(1)}(z)&=B_{12}(z)(B_{13}(z)B_{23}(z))\cdots(B_{1m}(z)\cdots B_{m-2\,m}(z)B_{m-1\,m}(z)),\\ F^{(2)}&=S_1S_2\cdots S_m. \end{aligned} \quad (3.7)$$

Here $S_i=S[g_m-g_i+1,\, g_m-g_{i-1}]$, and $B_{ij}(z)$ denotes the operator

$$B_{ij}(z)=id_{g_{j-1}-g_i}\otimes W_{(f_j)\otimes(f_i)}((j-i)(z-1))\otimes id_{N-g_j+g_{i-1}}|_{\tilde V^N}$$

considered on the space $\tilde V^N\subset \tilde V^{g_{j-1}-g_i}\otimes(\tilde V^{f_j}\otimes \tilde V^{f_i})\otimes \tilde V^{N-g_j+g_{i-1}}$. By id_k we mean the identity operator on $\tilde V^k$. From Lemma 3.1, Im $F^{(2)}$ is spanned by blockwise symmetric tensors

$$e^{(f_m)}_{a^{(m)}a^{(m-1)}}\otimes\cdots\otimes e^{(f_1)}_{a^{(1)}a^{(0)}}, \quad e^{(f_i)}_{a^{(i)}a^{(i-1)}}\in V_{(f_i)}.$$

Hence each $B_{ij}(z)$ in $F(z)$ actually operates on the symmetric tensors. By virtue of Lemma 2.1 and Proposition 2.3, it is then divisible by

$$\beta_{ij}(z)=\prod_{r=2}^{f_j}\prod_{s=1}^{f_i}[(j-i)(z-1)+r-s].$$

It is easy to see that $\sum_{i<j}\beta_{ij}(z)$ has exactly κ zeros at $z=0$. ■

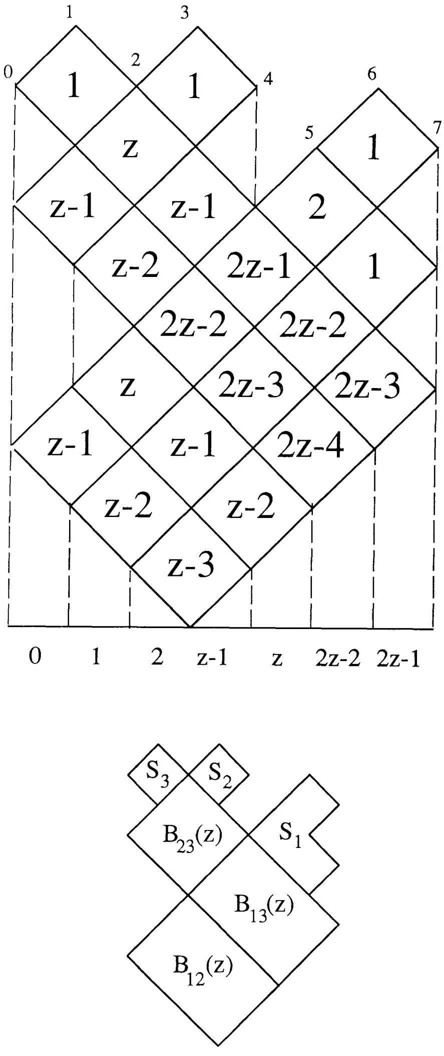

Fig. 4a and b. The operator $F(z)$ for $Y=(3,2,2)$ as a composition of $W_i(u)$ **b** Block structure of $F(z)$

Removing this factor we define $F \in \mathrm{End}_{\mathbf{C}}(\tilde{V}^N)$ by

$$F = \left(F(z) \prod_{i<j} \beta_{ij}(z)^{-1} \right)\Bigg|_{z=0}. \tag{3.8}$$

From the proof above and Proposition 2.3 it is clear that the image of F are blockwise symmetric:

$$\operatorname{Im} F \subset V_{(f_1)} \otimes \cdots \otimes V_{(f_m)}. \tag{3.9}$$

We remark that in the limit $L \to \infty$ and $|a_{\mu\nu}| \to \infty \, (\lambda \neq \mu)$, F becomes $\mathbf{c}_T \times$ (invertible operator), where $\mathbf{c}_T$ is the Young symmetrizer for a standard tableau T on the diagram Y (see Appendix A).

3.3. $W_{Y\otimes Y'}$ for General Young Diagrams. With each Young diagram Y we now associate the data $(V_Y, \alpha_Y(u))$, $V_Y = \oplus_{a,b\in\mathcal{S}}(V_Y)_{ab}$, $u\in\mathbf{C}$. Let F be as above, and set

$$\begin{aligned}(V_Y)_{ab} &\overset{\text{def}}{=} F((\tilde{V}^N)_{ab}) \subset (\tilde{V}^N)_{ab},\\ \alpha_Y(u) &= (u + v_1(0), \ldots, u + v_N(0)),\end{aligned} \tag{3.10}$$

where $v_i(z)$ are given by (3.5).

Let (Y, Y') be a pair of Young diagrams. As in Sect. 2 we put

$$W_{Y\otimes Y'}\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\middle| u-u'\right) = W_{NN'}\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\middle| \alpha_Y(u), \alpha_{Y'}(u')\right).$$

Let $\alpha_Y(u) = (u_1, \ldots, u_N)$, $\alpha_{Y'}(u') = (u'_1, \ldots, u'_{N'})$. In terms of $W_i(u)$ we can write as

$$\begin{aligned} W_{Y\otimes Y'}(u-u')|_{\tilde{V}^{N+N'}} &= w_{N'}(u-u')\cdots w_2(u-u')w_1(u-u'),\\ w_i(u-u') &= W_i(u_1-u'_i)W_{i+1}(u_2-u'_i)\cdots W_{i+N-1}(u_N-u'_i).\end{aligned} \tag{3.11}$$

Proposition 3.3. $W_{Y\otimes Y'}\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\middle| u\right)((V_Y)_{ab}\otimes(V_{Y'})_{bc}) \subset (V_{Y'})_{ad}\otimes(V_Y)_{dc}$.

Proof. From the definition (3.6)–(3.8) it is easy to see that

$$W_1(u_1)\cdots W_N(u_N)(F\otimes \mathrm{id}) = (\mathrm{id}\otimes F)W_1(u_N)\cdots W_N(u_1)\in \mathrm{End}_{\mathbf{C}}(\tilde{V}^{N+N'}).$$

Applying this repeatedly to (3.11) we get

$$W_{Y\otimes Y'}\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\middle| u\right)((V_Y)_{ab}\otimes(\tilde{V}^{N'})_{bc}) \subset (\tilde{V}^{N'})_{ad}\otimes(V_Y)_{dc}.$$

By a similar argument we have

$$W_{Y\otimes Y'}\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\middle| u\right)((\tilde{V}^{N})_{ab}\otimes(V_{Y'})_{bc}) \subset (V_{Y'})_{ad}\otimes(\tilde{V}^{N})_{dc}.$$

Whence follows Proposition 3.3. ■

According to the scheme in Sect. 2, Proposition 3.3 guarantees that for any triplet Y, Y', Y'' the operators $W_{Y\otimes Y'}|_{V_Y\otimes V_{Y'}}$, $W_{Y\otimes Y''}|_{V_Y\otimes V_{Y''}}$ and $W_{Y'\otimes Y''}|_{V_{Y'}\otimes V_{Y''}}$ solve the Yang–Baxter equation.

3.4. The Dimension of $(V_Y)_{ab}$. Let us study the dimensionality of the space $(V_Y)_{ab}$ defined in (3.10). For this purpose we introduce further the following operators:

$$\begin{aligned} G_i &= W_i(-1) \quad \text{if} \quad i \neq g_1, \ldots, g_m,\\ &= W_{g_{j-1}+1}(1)W_{g_{j-1}+2}(2)\cdots W_{g_j}(f_j) \quad \text{if} \quad i = g_j.\end{aligned}$$

Let

$$(V'_Y)_{ab} = \{v\in(\tilde{V}^N)_{ab} | G_i v = 0 \quad \text{for} \quad i = 1, \ldots, N-1\}.$$

Proposition 3.4. $(V_Y)_{ab} \subset (V'_Y)_{ab}$.

Proof. Using the Yang–Baxter equation for the symmetric tensors, we can rearrange for each j the right hand-side of (3.7) to get $F = B_{jj+1}\times(\cdots)$, where $B_{jj+1} = (\beta_{jj+1}(z)^{-1})\mathrm{id}_{g_{j-1}}\otimes W_{(f_{j+1})\otimes(f_j)}(z-1)\otimes \mathrm{id}_{N-g_{j+1}}|_{z=0}$. Hence it suffices to show that $G_i B_{jj+1}=0$ for $g_{j-1}<i<g_{j+1}$. It is obvious for $i \neq g_j$ by virtue of (3.9) and (3.3). Suppose $i=g_j$, and write $f=f_j$, $f'=f_{j+1}$. By the inversion relation (3.2) we have

$$\begin{aligned}&(W_1(-z+1)\cdots W_f(-z+f))W_{(f')\otimes(f)}(z-1)\\&=\left(\prod_{i=1}^{f}\frac{[z-i+1][-z+i+1]}{[1]^2}\right)(\mathrm{id}\otimes W_{(f'-1)\otimes(f)}(z)).\end{aligned}\tag{3.12}$$

The first factor in the right-hand side contains one zero for $z=0$. Noting that $f \geqq f'$ and using Lemma 2.2 we find that when operated on the symmetric tensors the second factor is divisible by $[z]^{f'-1}$. Dividing (3.12) by $\beta_{jj+1}(z)=O(z^{f'-1})$ and letting $z \to 0$ we get $G_{g_j}B_{jj+1}=0$. ■

Proposition 3.5. *Fix* DIWs *a and b. If L is sufficiently large, we have*

$$\dim(V_Y)_{ab}=\dim(V'_Y)_{ab}=[\rho_a \otimes \rho_Y : \rho_b],$$

where $[\rho_a \otimes \rho_Y : \rho_b]$ *signifies the multiplicity of* $\mathscr{V}_b$ *in* $\mathscr{V}_a \otimes \mathscr{V}_Y$.

Proof. First note that if L is large (e.g., $L+a_n>N+a_0$) then the space $(\tilde{V}^N)_{ab}$ is independent of L. Fixing a, b, N we regard F, G_i as finite dimensional matrices on $(\tilde{V}^N)_{ab}$ having L as a parameter. Proposition 3.4 states that $\mathrm{Im}\,F \subset \bigcap_i \mathrm{Ker}\,G_i$ for all L. Consider the limit $L \to \infty$, and put $F^0=\lim F$, $G_i^0=\lim G_i$. For L large enough we have

$$\mathrm{rank}\,F^0 \leqq \mathrm{rank}\,F \leqq \dim\bigcap_i \mathrm{Ker}\,G_i \leqq \dim\bigcap_i \mathrm{Ker}\,G_i^0.$$

On the other hand,

$$\mathrm{rank}\,F^0=\dim\bigcap_i \mathrm{Ker}\,G_i^0=[\rho_a \otimes \rho_Y : \rho_b]$$

holds (Proposition B.2). This proves Proposition 3.5. ■

3.5. Remarks. It can be shown that

$$[\rho_a \otimes \rho_Y : \rho_b] \leqq m_Y(b-a),$$

where $m_Y(w)$ denotes the multiplicity of a weight w in $\mathscr{V}_Y$. If $w=\sum_\lambda k_\lambda e_\lambda$, $\sum_\lambda k_\lambda=N$, then $m_Y(w)$ is the number of the ways of assigning N integers $\underbrace{0,\ldots,0}_{k_0},\ldots,\underbrace{n,\ldots,n}_{k_n}$ on the nodes of Y so that the integers $n(i,j)$ in the (i,j) position satisfy $n(i-1,j)<n(i,j) \leqq n(i,j+1)$ for all i,j.

Because of the invariance of the Boltzmann weights under any Dynkin diagram automorphism σ of $A_n^{(1)}$, we have $(V_Y)_{\sigma(a)\sigma(b)} \cong (V_Y)_{ab}$. Proposition 3.5 implies that in general

$$\dim(V_Y)_{ab} \leqq \min_\sigma[\rho_{\sigma(a)} \otimes \rho_Y : \rho_{\sigma(b)}],\tag{3.13}$$

hence in particular $\dim(V_Y)_{ab} \leqq m_Y(b-a)$.

Conjecture. *The equality in (3.13) holds for all $L > N$ and p with $-1 < p < 1$ ($p =$ the elliptic nome).*

4. An Example—The Case of $Y = \yng(2,1)$

Our purpose here is to illustrate the construction of Sect. 3 on the example $Y = \yng(2,1)$. Throughout this section distinct Greek letters $\lambda, \mu, \ldots$ will represent distinct numbers.

4.1. The space $(V_{\yng(2,1)})_{ab}$. First let us examine the space $(V_{\yng(2,1)})_{ab}$ for a and b satisfying $a \xrightarrow{3} b$. We distinguish the three cases

(e-1) $b - a = 3e_\lambda$,
(e-2) $b - a = 2e_\lambda + e_\mu$,
(e-3) $b - a = e_\lambda + e_\mu + e_\nu$.

By the remarks in Sect. 3.5 we have $\dim(V_{\yng(2,1)})_{ab} \leqq 0, 1, 2$, respectively. Hereafter the trivial case (e-1) will be omitted from the consideration. First consider the case (e-2). The action of F on $(\tilde{V}^3)_{ab}$ is as follows:

$$Fe_\lambda \otimes e_\lambda \otimes e_\mu = \frac{[a_{\lambda\mu}]}{[a_{\lambda\mu} + 1]} f^{\lambda\mu}_{a, 2\lambda+\mu},$$

$$Fe_\lambda \otimes e_\mu \otimes e_\lambda = \frac{[a_{\lambda\mu} + 2]}{[a_{\lambda\mu} + 1]} f^{\lambda\mu}_{a, 2\lambda+\mu},$$

$$Fe_\mu \otimes e_\lambda \otimes e_\lambda = -\frac{[2]}{[1]} f^{\lambda\mu}_{a, 2\lambda+\mu},$$

where

$$f^{\lambda\mu}_{a, 2\lambda+\mu} = \frac{[a_{\lambda\mu} - 1]}{[a_{\lambda\mu}]} (e_\lambda \otimes e_\mu \otimes e_\lambda + e_\mu \otimes e_\lambda \otimes e_\lambda) - \frac{[2][a_{\lambda\mu} + 2]}{[1][a_{\lambda\mu}]} e_\lambda \otimes e_\lambda \otimes e_\mu.$$

Therefore we have

(e-2) $(V_{\yng(2,1)})_{ab} = \mathbf{C} f^{\lambda\mu}_{a, 2\lambda+\mu}$.

A similar calculation shows

(e-3) $(V_{\yng(2,1)})_{ab} = \mathbf{C} f^{\mu\nu}_{a, \lambda+\mu+\nu} + \mathbf{C} f^{\nu\lambda}_{a, \lambda+\mu+\nu} + \mathbf{C} f^{\lambda\mu}_{a, \lambda+\mu+\nu}$.

Here the notations are

$$\begin{aligned} f^{\mu\nu}_{a, \lambda+\mu+\nu} &= A^{\lambda}_{\mu\nu}(A^{\nu}_{\lambda\mu}(e_\lambda \otimes e_\nu \otimes e_\mu + e_\nu \otimes e_\lambda \otimes e_\mu) \\ &\quad - A^{\mu}_{\nu\lambda}(e_\lambda \otimes e_\mu \otimes e_\nu + e_\mu \otimes e_\lambda \otimes e_\nu)), \\ A^{\lambda}_{\mu\nu} &= \frac{[a_{\lambda\mu} + 1][a_{\lambda\nu} + 1]}{[a_{\lambda\mu}][a_{\lambda\nu}]}. \end{aligned}$$

Table. The Boltzmann weights for $\begin{smallmatrix}\square\square\\ \square\ \ \end{smallmatrix} \otimes \square$. In the Table, $W = W_{\begin{smallmatrix}\square\square\\ \square\ \ \end{smallmatrix}\otimes\square}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle| u\right)$ and $2\lambda+\mu$ signifies $2e_\lambda + e_\mu$, etc

---

$$\begin{array}{ccc} & 2\lambda+\mu & \\ \lambda & \square & \lambda \\ & 2\lambda+\mu & \end{array} \qquad W f^{\lambda\mu}_{a,2\lambda+\mu} \otimes e_\lambda = \frac{[2+u][a_{\lambda\mu}+1]}{[1][a_{\lambda\mu}]} e_\lambda \otimes f^{\lambda\mu}_{d,2\lambda+\mu}$$

$$\begin{array}{ccc} & 2\lambda+\mu & \\ \mu & \square & \mu \\ & 2\lambda+\mu & \end{array} \qquad W f^{\lambda\mu}_{a,2\lambda+\mu} \otimes e_\mu = \frac{[1+u][a_{\lambda\mu}-1]^2}{[1][a_{\lambda\mu}]^2} e_\mu \otimes f^{\lambda\mu}_{d,2\lambda+\mu}$$

$$\begin{array}{ccc} & 2\lambda+\mu & \\ \nu & \square & \nu \\ & 2\lambda+\mu & \end{array} \qquad W f^{\lambda\mu}_{a,2\lambda+\mu} \otimes e_\nu = \frac{[u][a_{\lambda\nu}-1][a_{\mu\nu}-1]}{[1][a_{\lambda\nu}+1][a_{\mu\nu}]} e_\nu \otimes f^{\lambda\mu}_{d,2\lambda+\mu}$$

$$\begin{array}{ccc} & 2\lambda+\mu & \\ \lambda & \square & \mu \\ & 2\mu+\lambda & \end{array} \qquad W f^{\lambda\mu}_{a,2\lambda+\mu} \otimes e_\mu = -\frac{[a_{\lambda\mu}+1][a_{\lambda\mu}-1-u]}{[a_{\lambda\mu}]^2} e_\lambda \otimes f^{\mu\lambda}_{d,2\mu+\lambda}$$

$$\begin{array}{ccc} & 2\lambda+\mu & \\ \mu & \square & \nu \\ & 2\lambda+\nu & \end{array} \qquad W f^{\lambda\mu}_{a,2\lambda+\mu} \otimes e_\nu = \frac{[a_{\lambda\mu}-1][a_{\lambda\nu}][a_{\mu\nu}-u]}{[a_{\lambda\mu}][a_{\lambda\nu}+1][a_{\mu\nu}]} e_\mu \otimes f^{\lambda\nu}_{d,2\lambda+\nu}$$

$$\begin{array}{ccc} & 2\lambda+\mu & \\ \lambda & \square & \nu \\ & \lambda+\mu+\nu & \end{array} \qquad W f^{\lambda\mu}_{a,2\lambda+\mu} \otimes e_\nu = \frac{[a_{\lambda\nu}-u]}{[a_{\lambda\nu}+2]}\left(\frac{[a_{\lambda\mu}+1]^2[a_{\mu\nu}+2]}{[a_{\lambda\mu}]^2[a_{\mu\nu}+1]} e_\lambda \otimes f^{\nu\lambda}_{d,\lambda+\mu+\nu} + \frac{[2][a_{\lambda\mu}+1][a_{\lambda\nu}+1]}{[1][a_{\lambda\mu}][a_{\lambda\nu}]} e_\lambda \otimes f^{\lambda\mu}_{d,\lambda+\mu+\nu}\right)$$

$$\begin{array}{ccc} & \lambda+\mu+\nu & \\ \nu & \square & \lambda \\ & 2\lambda+\mu & \end{array} \qquad W f^{\nu\lambda}_{a,\lambda+\mu+\nu} \otimes e_\lambda = 0$$

$$W f^{\lambda\mu}_{a,\lambda+\mu+\nu} \otimes e_\lambda = \frac{[a_{\mu\nu}-1][a_{\lambda\nu}-1][a_{\lambda\nu}+1+u]}{[a_{\mu\nu}][a_{\lambda\nu}]^2} e_\nu \otimes f^{\lambda\mu}_{d,2\lambda+\mu}$$

$$\begin{array}{ccc} & \lambda+\mu+\nu & \\ \lambda & \square & \lambda \\ & \lambda+\mu+\nu & \end{array} \qquad W f^{\alpha\lambda}_{a,\lambda+\mu+\nu} \otimes e_\lambda = \frac{[1+u][a_{\lambda\beta}-1][a_{\lambda\beta}+1]^2[a_{\lambda\alpha}+1]}{[1][a_{\lambda\beta}]^3[a_{\lambda\alpha}]} e_\lambda \otimes f^{\alpha\lambda}_{d,\lambda+\mu+\nu}$$

$$(\{\alpha,\beta\} = \{\mu,\nu\})$$

$$\begin{array}{ccc} & \lambda+\mu+\nu & \\ \kappa & \square & \kappa \\ & \lambda+\mu+\nu & \end{array} \qquad W f^{\alpha\beta}_{a,\lambda+\mu+\nu} \otimes e_\kappa = \frac{[u][a_{\kappa\lambda}+1][a_{\kappa\mu}+1][a_{\kappa\nu}+1]}{[1][a_{\kappa\lambda}][a_{\kappa\mu}][a_{\kappa\nu}]} e_\kappa \otimes f^{\alpha\beta}_{d,\lambda+\mu+\nu}$$

$$(\alpha,\beta = \lambda,\mu,\nu)$$

$$\begin{array}{ccc} & \lambda+\mu+\nu & \\ \lambda & \square & \kappa \\ & \mu+\nu+\kappa & \end{array} \qquad W f^{\alpha\lambda}_{a,\lambda+\mu+\nu} \otimes e_\kappa = \frac{[a_{\lambda\beta}+1][a_{\lambda\beta}-1][a_{\lambda\alpha}+1][a_{\beta\kappa}][a_{\lambda\kappa}-u]}{[a_{\lambda\beta}]^2[a_{\lambda\alpha}][a_{\beta\kappa}+1][a_{\lambda\kappa}]} e_\lambda \otimes f^{\alpha\kappa}_{d,\mu+\nu+\kappa}$$

$$(\{\alpha,\beta\} = \{\mu,\nu\})$$

These vectors obey the linear relation

$$f^{\mu\nu}_{a,\lambda+\mu+\nu}+f^{\nu\lambda}_{a,\lambda+\mu+\nu}+f^{\lambda\mu}_{a,\lambda+\mu+\nu}=0.$$

To sum up, we have

(e-2) $\dim(V_{\yng(2,1)})_{ab}=1$,

(e-3) $\dim(V_{\yng(2,1)})_{ab}\leqq 2$.

In the latter case the equality fails if $A^{\mu}_{\nu\lambda}A^{\nu}_{\lambda\mu}A^{\lambda}_{\mu\nu}=0$. If, e.g., $A^{\mu}_{\nu\lambda}=0$ and $A^{\nu}_{\lambda\mu}A^{\lambda}_{\mu\nu}\neq 0$ then $f^{\nu\lambda}_{a,\lambda+\mu+\nu}=0$, $\mathbf{C}f^{\mu\nu}_{a,\lambda+\mu+\nu}=\mathbf{C}f^{\lambda\mu}_{a,\lambda+\mu+\nu}$ and $\dim(V_{\yng(2,1)})_{ab}=1$. We have $(V_{\yng(2,1)})_{ab}=0$ if two of $A^{\mu}_{\nu\lambda}$, $A^{\nu}_{\lambda\mu}$, $A^{\lambda}_{\mu\nu}$ are zero.

4.2. The Boltzmann Weights for $\yng(2,1)\otimes\square$. We give in Table the whole list of non-zero Boltzmann weights $W_{\yng(2,1)\otimes\square}\left(\begin{smallmatrix}a & b\\ d & c\end{smallmatrix}\middle| u\right)$ with the common factor $[u+1][u-1]/[1]^2$ dropped.

4.3. A Boltzmann Weights for $\yng(2,1)\otimes\yng(2,1)$. The Boltzmann weights for $\yng(2,1)\otimes\yng(2,1)$ can be obtained by composing those for $\yng(2,1)\otimes\square$ given in Table. Here we write only one case as an example. This time the factor $[u]^2[u+1][u-1][u+2][u-2]/[1]^7[3]$ will be dropped.

$$\begin{array}{ccc} & 2\lambda+\mu & \\ 2\lambda+\mu & \boxed{} & \lambda+\mu+\nu \\ & \lambda+\mu+\nu & \end{array}$$

$$Wf^{\lambda\mu}_{a,2\lambda+\mu}\otimes f^{\mu\nu}_{b,\lambda+\mu+\nu}=\frac{[a_{\lambda\nu}+2-u][1+u][3+u]}{[a_{\lambda\nu}+2]}f^{\lambda\mu}_{a,2\lambda+\mu}\otimes f^{\mu\nu}_{d,\lambda+\mu+\nu}$$

$$-\frac{[a_{\lambda\mu}+2]^2[a_{\lambda\nu}+3]^2[a_{\mu\nu}+1]^2[u][1+u][a_{\lambda\nu}-1-u]}{[a_{\lambda\mu}+1]^2[a_{\lambda\nu}+1]^2[a_{\mu\nu}]^2[a_{\lambda\nu}+2]}f^{\lambda\mu}_{a,2\lambda+\mu}\otimes f^{\lambda\mu}_{d,\lambda+\mu+\nu}$$

$$Wf^{\lambda\mu}_{a,2\lambda+\mu}\otimes f^{\lambda\mu}_{b,\lambda+\mu+\nu}$$

$$=-\frac{[a_{\lambda\mu}+1]^2[a_{\mu\nu}][a_{\mu\nu}+2][u][3+u][a_{\lambda\nu}+1-u]}{[a_{\lambda\mu}][a_{\lambda\mu}+2][a_{\mu\nu}+1]^2[a_{\lambda\nu}+3]}f^{\lambda\mu}_{a,2\lambda+\mu}\otimes f^{\mu\nu}_{d,\lambda+\mu+\nu}$$

$$+\left(\frac{[1]^2[a_{\lambda\nu}][a_{\mu\nu}+2][a_{\lambda\mu}-u][a_{\lambda\nu}+1-u][a_{\lambda\mu}+3+u]}{[a_{\lambda\mu}][a_{\lambda\mu}+1][a_{\lambda\nu}+1][a_{\lambda\nu}+2][a_{\mu\nu}]}\right.$$

$$\left.+\frac{[a_{\lambda\mu}][a_{\lambda\nu}+3][a_{\mu\nu}-1][1+u][2+u][a_{\lambda\nu}+1-u]}{[a_{\lambda\mu}+1][a_{\lambda\nu}+1][a_{\lambda\nu}+2][a_{\mu\nu}]}\right)f^{\lambda\mu}_{a,2\lambda+\mu}\otimes f^{\lambda\mu}_{d,\lambda+\mu+\nu}$$

5. Conjectures on Local State Probabilities

The local state probability (abb. LSP) is the probability that a lattice site assumes a given state $a\in\mathscr{S}_{l,n}$. We denote it by $P(a)$. Let us state our conjectures on the LSPs of the models corresponding to the Young diagram $Y=(N)$ or $Y=(1^N)$.

Hereafter we deal exclusively with the case $0 < p < 1$, $-(n+1)/2 < u < 0$, which corresponds to Regime III in [6]. We use the notations (c-4) $\sim$ (c-6) in Sect. 2.

Let $\mathscr{L}_a$ denote the irreducible $A_n^{(1)}$ module with the highest weight a and χ_a its character. Consider the character identity describing the irreducible decomposition of the tensor module $\mathscr{L}_\xi \otimes \mathscr{L}_\eta (\xi \in \mathscr{S}_{l-M,n}, \eta \in \mathscr{S}_{M,n})$

$$\chi_\xi(q, z_1, \ldots, z_n)\chi_\eta(q, z_1, \ldots, z_n) = \sum_{a \in \mathscr{S}_{l,n}} b_{\xi\eta a}(q)\chi_a(q, z_1, \ldots, z_n). \tag{5.1}$$

The branching coefficients $b_{\xi\eta a}(q)$ are power series in q (with some overall fractional power) and their linear span is stable under the change $q = e^{2\pi i\tau} \mapsto \bar{q} = e^{-2\pi i/\tau}$. They are the characters of the GKO-Virasoro algebra [19] (not necessarily irreducible ones) arising from the affine Lie algebra pair $A_n^{(1)} \oplus A_n^{(1)} \supset A_n^{(1)}$.

The identity (5.1) with the choice $M = N$ or $M = 1$ is related with our models corresponding to the N-symmetric $(Y = (N))$ or the N-antisymmetric $(Y = (1^N))$ tensors, respectively. We have the following conjectured form of the LSP expression:

$$P(a) = \lim_{m \to \infty} P_m(a, b^{(m+1)}, b^{(m+2)}), \tag{5.2a}$$

$$P_m(a, b, c) = u_a X_m(a, b, c; x^{n+1}) \Big/ \sum_{a \in \mathscr{S}_{l,n}} u_a X_m(a, b, c; x^{n+1}), \tag{5.2b}$$

$$u_a = x^{\sum_{\mu=0}^{n}(\mu - n/2)a_\mu} \prod_{\mu < \nu} E(x^{a_{\mu\nu}}, x^L), \tag{5.2c}$$

$$E(z, q) = \prod_{k=1}^{\infty} (1 - zq^{k-1})(1 - z^{-1}q^k)(1 - q^k), \tag{5.2d}$$

$$X_m(a, b, c; q) = \sum q^{\sum_{j=1}^{m} jH(a^{(j)}, a^{(j+1)}, a^{(j+2)})}, \tag{5.2e}$$

$$H(a, a', a'') = \min_{\tau \in S_N} \#\{i \mid \mu_i \geqq \nu_{\tau(i)}\} \quad \text{for } Y = (N), \tag{5.2f}$$

$$= \max_{\tau \in S_N} \#\{i \mid \mu_i \geqq \nu_{\tau(i)}\} \quad \text{for } Y = (1^N), \tag{5.2g}$$

where

$$a' = a + \sum_{i=1}^{N} e_{\mu_i}, \quad a'' = a' + \sum_{i=1}^{N} e_{\nu_i}.$$

Here the parameter x is defined by $p = e^{-\varepsilon}$, $x = e^{-4\pi^2/L\varepsilon}$ and the sum in (5.2e) is over $a^{(2)}, \ldots, a^{(m)} \in \mathscr{S}_{l,n} (a^{(1)} = a,\ a^{(m+1)} = b,\ a^{(m+2)} = c)$ under the condition that $(V_Y)_{a^{(j)}a^{(j+1)}} \neq 0$ for $1 \leqq j \leqq m$. In (5.2a) we choose $(b^{(j)})_{j=1}^{\infty}$ to be

$$b^{(j)} = \xi \oplus \sigma_{M,n}^{(j-1)K}(\eta), \tag{5.3}$$

where $(K, M) = (1, N)$ or $(N, 1)$ according to whether $Y = (N)$ or $Y = (1^N)$.

We conjecture the large m limit of the combinatorial q-polynomial (5.2e) as follows.

$$\lim_{m \to \infty} q^{\gamma(a, \xi, \eta) - H_m} X_m(a, b^{(m+1)}, b^{(m+2)}; q) = b_{\xi\eta a}(q),$$

where

$$\gamma(a,\xi,\eta)=|\xi|_{l-M,n}+|\eta|_{M,n}-|a|_{l,n}-|(n,n-1,\dots,0)|_{0,n},$$
$$H_m=\sum_{j=1}^{m} jH(b^{(j)},b^{(j+1)},b^{(j+2)}).$$

This has been proved for $n=1$, $Y=(N)$ in [3,4] and for n: general, $Y=(1)$ in [15]. We also examined several cases by computer experiments. Admitting the conjectures we obtain the LSP expression

$$P(a)=\frac{b_{\xi\eta a}(x^{n+1})\chi_a(x^{n+1},x,\dots,x)}{\chi_\xi(x^{n+1},x,\dots,x)\chi_\eta(x^{n+1},x,\dots,x)}.$$

We note that the identity (5.1) assures the correct normalization

$$1=\sum_{a\in\mathscr{S}_{l,n}} P(a).$$

The LSPs are not known for the models corresponding to general Young diagrams. We hope to settle this question in a future publication.

Appendix A

In this appendix, we give a characterization of the minimal right ideals of $\mathbf{C}S_N$ following [17]. Fix a Young diagram $Y=(f_1,f_2,\dots,f_m)$ with N nodes. For a standard tableau T on Y [20] the *Young symmetrizer* $\mathbf{c}_T$ is defined to be

$$\mathbf{c}_T=\mathbf{s}_T\mathbf{a}_T,\quad \mathbf{s}_T=\sum_{\tau\in H_T}\tau,\quad \mathbf{a}_T=\sum_{\tau\in V_T}(\operatorname{sgn}\tau)\tau.$$

Here H_T (respectively V_T) denotes the horizontal (respectively vertical) group, i.e., the subgroup of S_N whose elements permute the letters on each row (respectively column) of T among themselves. We use the symbols $S(j_1\cdots j_k)$, $\mathbf{s}(j_1\cdots j_k)$ and $\mathbf{a}(j_1\cdots j_k)$ to denote the permutation group on the letters $\{j_1,\dots,j_k\}$ and the (anti-) symmetrizer on them

$$\mathbf{s}(j_1\cdots j_k)=\sum_{\tau\in S(j_1\cdots j_k)}\tau,\quad \mathbf{a}(j_1\cdots j_k)=\sum_{\tau\in S(j_1\cdots j_k)}(\operatorname{sgn}\tau)\tau. \tag{A.1}$$

Define the element $\mathscr{F}(z)$ of $\mathbf{C}S_N$ to be

$$\begin{aligned}\mathscr{F}(z)&=\mathscr{F}_{N-1}(z)\mathscr{F}_{N-2}(z)\cdots\mathscr{F}_1(z),\\ \mathscr{F}_i(z)&=(1+(v_N(z)-v_i(z))s_{N-i})\\ &\quad\times(1+(v_{N-1}(z)-v_i(z))s_{N-i-1})\cdots(1+(v_{i+1}(z)-v_i(z))s_i).\end{aligned} \tag{A.2}$$

Here $v_i(z)$ $(i=1,2,\dots,N)$ are defined in (3.5), and $s_i=(i\ i+1)$ $(i=1,2,\dots,N-1)$ are the generators of $\mathbf{C}S_N$. As in Lemma 3.2 one can show

Lemma A.1. *There exists an element $\mathscr{F}$ in $\mathbf{C}S_N$ which satisfies*

$$\mathscr{F}(z)=z^\kappa\mathscr{F}+o(z^\kappa)\quad(z\to 0),$$

where κ is defined in Lemma 3.2.

Next we define the elements $\mathscr{G}_i(i=1,2,\dots,N-1)$ as follows:

$$\begin{aligned}\mathscr{G}_i &= 1 - s_i \quad \text{if} \quad i \neq g_1, g_2, \ldots, g_m, \\ &= (1 + s_{g_{j-1}+1})(1 + 2s_{g_{j-1}+2}) \cdots (1 + f_j s_{g_j}) \quad \text{if} \quad i = g_j,\end{aligned} \tag{A.3}$$

where $g_j = f_1 + f_2 + \cdots + f_j$ $(j = 1, 2, \ldots, m)$.

From now on, we fix T to be the standard tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & g_1 \\ \hline g_1 + 1 & \cdots & g_2 & \\ \cline{1-3} \vdots & & \iddots & \\ \cline{1-3} g_{m-1} + 1 & \cdots & g_m & \\ \cline{1-3} \end{array}$$

The following proposition characterizes the minimal right ideal $\mathbf{c}_T \mathbf{C}S_N$.

Proposition A.2.

$$\mathbf{c}_T \mathbf{C}\mathbf{S}_N = \mathscr{F}\mathbf{C}S_N = K,$$

where

$$K = \{x \in \mathbf{C}S_N \mid \mathscr{G}_i x = 0 \quad \textit{for all} \quad i = 1, 2, \ldots, N-1\}.$$

We divide the proof into two steps:

(f-1) $K = \mathbf{c}_T \mathbf{C}\mathbf{S}_N$

(f-2) $K = \mathscr{F}\mathbf{C}S_N$

Proof of (f-1) Let us first prove that

$$\mathscr{G}_i \mathbf{c}_T = 0 \quad \text{for all} \quad i = 1, 2, \ldots, N-1. \tag{A.4}$$

In the case $i \neq g_1, g_2, \ldots, g_m$, (A.4) is clear. Consider the case $i = g_j$. For simplicity, we assume $i = g_1$. (The clear cases are similar). From the definition (A.1) and (A.3), $\mathscr{G}_{g_1}\mathbf{s}(1 \cdots g_1) = g_1!\mathbf{s}(1 \cdots g_1 + 1)$. So we can express $\mathscr{G}_{g_1}\mathbf{c}_T$ as

$$\mathbf{s}(1 \cdots g_1 + 1) x \mathbf{a}(1 g_1 + 1)\mathbf{a}(2 g_1 + 2) \cdots \mathbf{a}(f_2 g_2) y,$$

where $x \in \mathbf{C}S(g_1 + 1 \cdots g_2)$, $y \in \mathbf{C}S_N$. Note that for all $w \in S(g_1 + 1 \cdots g_2)$

$$\begin{aligned}&\mathbf{s}(1 \cdots g_1 + 1) w \mathbf{a}(1 g_1 + 1) \cdots \mathbf{a}(f_2 g_2) \\ &\quad = \mathbf{s}(1 \cdots g_1 + 1)\mathbf{a}(1 w(g_1 + 1)) \cdots \mathbf{a}(f_2 w(g_2)) w = 0.\end{aligned}$$

This proves $\mathscr{G}_{g_1}\mathbf{c}_T = 0$.

Next we prove that if $\mathscr{G}_i x = 0$ for all i, then $x \in \mathbf{c}_T \mathbf{C}S_N$. We use the induction on N. From the assumption of the induction

$$x \in \mathbf{c}_{T'} \mathbf{C}S_N,$$

where T' is the Young tableau obtained by removing the node N from T. The following is a known fact about the induced representation of the symmetric group

[20]:

$$\mathbf{c}_{T'}\mathbf{C}S_N = \bigoplus_{T' \prec S} \mathbf{c}_S \mathbf{C}S_N,$$

where S runs over the standard tableaux obtained by adjoining one more node to T'. Since each $\mathbf{c}_S\mathbf{C}S_N$ is irreducible, the proof finishes if we show

$$\mathscr{G}_{N-1}\mathbf{c}_S \neq 0 \quad \text{for} \quad S \neq T.$$

Case 1. $f_m > 1$. In this case $\mathscr{G}_{N-1} = 1 - s_{N-1}$. If $S \neq T$, then

$$\mathscr{G}_{N-1}\mathbf{c}_S = \mathbf{c}_S - \mathbf{c}_{\tilde{S}} s_{N-1},$$

where $\tilde{S}$ is the standard tableau obtained by interchanging the letters $N-1$ and N in S. Noting that

$$\mathbf{C}S_N = \bigoplus_{S_1\text{-standard tableaux}} \mathbf{c}_{S_1}\mathbf{C}S_N \tag{A.5}$$

as a right $\mathbf{C}S_N$-module [20], we find $\mathscr{G}_{N-1}\mathbf{c}_S \neq 0$.

Case 2. $f_m = 1$.

In this case $\mathscr{G}_{N-1} = (1 + s_{g_{m-2}+1})(1 + 2s_{g_{m-2}+2}) \cdots (1 + f_{m-1}s_{N-1})$. Note that

$$\begin{aligned} &\frac{1}{f_{m-1}!}\mathscr{G}_{N-1}\mathbf{s}_{(g_{m-2}+1\cdots N-1)} \\ &= \mathbf{s}_{(g_{m-2}+1\cdots N)} \\ &= (1 + (N-1N) + \cdots + (g_{m-2}+1N))\mathbf{s}_{(g_{m-2}+1\quad N-1)}. \end{aligned} \tag{A.6}$$

If S is the Young tableau obtained by adding a node in the $(m-1)$-th row in T', $\mathscr{G}_{N-1}\mathbf{c}_S = (f_{m-1}+1)!\mathbf{c}_S \neq 0$. In the other cases, we can write from (A.6) $\mathscr{G}_{N-1}\mathbf{c}_S$ in the following manner:

$$\frac{1}{f_{m-1}!}\mathscr{G}_{N-1}\mathbf{c}_S = \mathbf{c}_S + \sum_{j=1}^{f_{m-1}} (N-jN)\mathbf{c}_S.$$

For each j, $(N-jN)\mathbf{c}_S = (N-1N)\cdots(N-j+1N)(N-jN)\mathbf{c}_S$ belongs to the ideal $\mathbf{c}_{S_j}\mathbf{C}S_N$, where S_j denotes the standard tableau $\tau_j(S)$, $\tau_j = (N-jN-j+1\cdots N)$. Here for $\tau \in S_N$, $\tau(S)$ signifies the tableau obtained by replacing each letter i on S by $\tau(i)$. Since $S = S_0, S_1, \ldots, S_{f_{m-1}}$ are distinct, we again get $\mathscr{G}_{N-1}\mathbf{c}_S \neq 0$ by (A.5). ■

Proof of (f-2). In the same way as in Proposition 3.4, we can show $K \supset \mathscr{F}\mathbf{C}S_N$. Since K is irreducible by (f-1), it suffices to show that $\mathscr{F} \neq 0$. By expanding (A.2), we get the expression

$$\mathscr{F} = \beta w_0 + (\text{linear combination of the elements of length} < N(N-1)/2),$$

where $\beta = (z^{-\kappa}\prod_{i<j}(v_j(z) - v_i(z)))|_{z=0}$ is a non-zero scalar and w_0 is the longest element in S_N. ■

Appendix B

Let $(\rho_\square, \mathscr{V}_\square)$ be the vector representation of $gl(n+1, \mathbf{C})$. For $1 \leqq i \leqq N-1$ we put

$$\mathscr{W}_i(u) = 1 + uP_{ii+1} \in \mathrm{End}_{gl(n+1,\mathbf{C})}(\mathscr{V}_\square^{\otimes N}), \tag{B.1}$$

where P_{ii+1} denotes the transposition of the i^{th} and the $(i+1)^{\text{th}}$ components. For $a = (a_0, \dots, a_n) \in \mathbf{Z}^{n+1}$ with $a_0 > \dots > a_n$, we mean by $\mathscr{V}_a$ the irreducible $gl(n+1, \mathbf{C})$-module generated by a highest weight vector v:

$$E_{\lambda\lambda} v = (a_\lambda - n + \lambda) v \quad (\forall \lambda), \quad E_{\lambda\mu} v = 0 \quad (\lambda < \mu),$$

where $E_{\lambda\mu}$ denotes the standard generators of $gl(n+1, \mathbf{C})$. The purpose of this appendix is to describe the action of $\mathrm{id} \otimes \mathscr{W}_i(u)$ on the module $\mathscr{V}_a \otimes \mathscr{V}_\square^{\otimes N}$ following [18].

Recall that $\mathscr{V}_a$ has a distinguished orthonormal basis $\{|am\rangle\}$ (the Gelfand–Zetlin basis [21]) labeled by an array of integers

$$\begin{array}{c} m_{0n}, \ m_{1n}, \ m_{2n}, \ \dots\dots\dots, \ m_{nn} \\ m_{0n-1}, m_{1n-1}, \dots, m_{n-1n-1} \\ \ddots \qquad\qquad\qquad \iddots \\ m_{01}, m_{11} \\ m_{00} \end{array}$$

such that

(g-1) $m_{\lambda\mu} \geqq m_{\lambda\mu-1} \geqq m_{\lambda+1\mu}$ for all λ, μ,
(g-2) $m_{\lambda n} = a_\lambda - n + \lambda \quad (0 \leqq \lambda \leqq n)$.

The highest weight vector corresponds to the pattern $m = m(a)$: $m(a)_{\lambda\mu} = a_\lambda - n + \lambda$ for all λ, μ. In the tensor module $\mathscr{V}_a \otimes \mathscr{V}_\square$ any irreducible component appears with multiplicity 1. The corresponding Gelfand–Zetlin basis $\{|(ab)m\rangle\}$, belonging to a component of highest weight b, is given in the form

$$|(ab)m\rangle = \sum_{m', \lambda} C(m, m', \lambda) |am'\rangle \otimes e_\lambda. \tag{B.2}$$

Here the $C(m, m', \lambda)$ are the "Wigner coefficients," whose explicit expressions can be found in [18, 21].

We consider now the decomposition of the module $\mathscr{V}_a \otimes \mathscr{V}_\square^{\otimes N}$. Let

$$\Omega_{ab} = \{v \in \mathscr{V}_a \otimes \mathscr{V}_\square^{\otimes N} \mid E_{\lambda\lambda} v = (b_\lambda - n + \lambda) v \quad (\forall \lambda), \quad E_{\lambda\mu} v = 0 \quad (\lambda < \mu)\}$$

be the space of highest weight vectors of highest weight b, so that we have $\mathscr{V}_a \otimes \mathscr{V}_\square^{\otimes N} \cong \sum_b \Omega_{ab} \otimes \mathscr{V}_b$. Let $p = (a^{(0)}, \dots, a^{(N)})$ be a path from a to b. We define the vectors $|(a^{(0)}, \dots, a^{(i)})m\rangle \in \mathscr{V}_a \otimes \mathscr{V}_\square^{\otimes i}$ $(i = 0, 1, \dots, N)$ inductively by

$$|(a^{(0)})m\rangle = |am\rangle,$$
$$|(a^{(0)}, \dots, a^{(i)})m\rangle = \sum_{m', \lambda} C(m, m', \lambda) |(a^{(0)}, \dots, a^{(i-1)})m'\rangle \otimes e_\lambda.$$

Setting $i = N$ and $m = m(b)$ we get a family of vectors

$$\| a^{(0)}, \ldots, a^{(N)} \rangle\!\rangle \stackrel{\text{def}}{=} |(a^{(0)}, \ldots, a^{(N)}) m(a^{(N)}) \rangle$$

labeled by the paths p; they provide an orthonormal basis of Ω_{ab}.

Since the $\mathscr{W}_i(u)$ (B.1) lies in the commutant of $gl(n+1, \mathbf{C})$ one may regard $\mathrm{id} \otimes \mathscr{W}_i(u)$ as acting on Ω_{ab}. We define $W_i^0(u)$ on $\oplus_{a,b} \Omega_{ab}$ by

$$W_i^0(u)|_{\Omega_{ab}} = (\mathrm{id} \otimes \mathscr{W}_i(u))|_{\Omega_{ab}}.$$

Proposition B.1. [18] *With respect to the above basis we have*

$$W_i^0(u) \| a^{(0)}, \ldots, a^{(N)} \rangle\!\rangle = \sum_{a'^{(i)}} {}^{a^{(i-1)}}_{a'^{(i)}} \square^{a^{(i)}}_{a^{(i+1)}} \| a^{(0)}, \ldots, a'^{(i)}, \ldots, a^{(N)} \rangle\!\rangle,$$

where (in the notation of Sect. 2)

$$ {}_{\lambda}\overset{\lambda}{\underset{\lambda}{\square}}{}_{\lambda} = 1 + u, \quad {}_{\lambda}\overset{\lambda}{\underset{\mu}{\square}}{}_{\mu} = 1 - \frac{u}{a_{\lambda\mu}}, \quad {}_{\lambda}\overset{\mu}{\underset{\mu}{\square}}{}_{\lambda} = u \sqrt{\frac{(a_{\lambda\mu}+1)(a_{\lambda\mu}-1)}{a_{\lambda\mu}^2}}. \tag{B.3a}$$

Changing the vector $|(ab)m\rangle$ (see (B.2)) by its scalar multiple $s(a,b)|(ab)m\rangle$ has the effect of multiplying ${}^{a}_{d}\square^{b}_{c}$ by $s(a,b)s(b,c)/s(a,d)s(d,c)$. With the choice

$$s(a, a+e_\lambda) = \left(\prod_{0 \leq \nu < \lambda} a_{\nu\lambda}(a_{\nu\lambda} - 1) \right)^{-1/2}$$

the last expression of (B.3a) is changed to

$$ {}_{\lambda}\overset{\mu}{\underset{\mu}{\square}}{}_{\lambda} = u \frac{a_{\lambda\mu}+1}{a_{\lambda\mu}}. \tag{B.3b}$$

In this form the operators $W_i^0(u)$ are precisely the rational limit of $W_i(u)$ in Sect. 3:

$$W_i^0(u)|_{\Omega_{ab}} = \lim_{L \to \infty} W_i(u)|_{(\tilde{V}^N)_{ab}}.$$

Let $\pi: \mathbf{C}S_N \to \mathrm{End}_{gl(n+1,\mathbf{C})}(\mathscr{V}_{\square}^{\otimes N})$ be the natural map sending s_i to P_{ii+1}. We put

$$F^0 = (\mathrm{id} \otimes \pi(\mathscr{F}))|_{\Omega_{ab}}, \quad G_i^0 = (\mathrm{id} \otimes \pi(\mathscr{G}_i))|_{\Omega_{ab}}.$$

These are the rational limits of F and G_i in Sect. 3.

Proposition B.2. *We have*

$$\mathrm{Im}\, F^0 = \mathrm{Im}\,(\mathrm{id} \otimes \pi(\mathbf{c}_T))|_{\Omega_{ab}} = \bigcap_{i=1}^{N} \mathrm{Ker}\, G_i^0. \tag{B.4}$$

Their dimension is $[\rho_a \otimes \rho_Y : \rho_b]$.

Proof. Proposition A.2 states that (B.4) is true for $\pi =$ the regular representation. Since any irreducible representation of S_N is a subrepresentation of the regular representation, it must be true for all finite dimensional representations by the complete reducibility. Noting that $\mathscr{V}_Y \cong \pi(\mathbf{c}_T)(\tilde{V}^N)$ we have $\dim(\mathrm{id} \otimes \pi(\mathbf{c}_T))(\Omega_{ab}) = [\rho_a \otimes \rho_Y : \rho_b]$. ■

Acknowledgements. We thank I. V. Cherednik and Vincent Pasquier for sending their preprints. We also acknowledge Susumu Ariki, Yukihiro Kanie, Masaki Kashiwara, Shin-ichi Kato, Kazuhiko Koike, Jun Murakami, Itaru Terada and Hiroyuki Yamane for discussions. Last but not least, we thank Keiko Nakahara for her assistance in preparation of the manuscript using T_EX. ■

References

1. Jimbo, M., Miwa, T., Okado, M.: Solvable lattice models with broken $\mathbf{Z}_n$ symmetry and Hecke's indefinite modular forms. Nucl. Phys. **B275** [FS17], 517–545 (1986)
2. Date, E., Jimbo, M., Miwa, T., Okado, M.: Automorphic properties of local height probabilities for integrable solid-on-solid models. Phys. Rev. **B35**, 2105–2107 (1987)
3. Data, E., Jimbo, M., Kuniba, A., Miwa, T., Okada, M.: Exactly solvable SOS models: Local height probabilities and theta function identities. Nucl. Phys. **B290** [FS20], 231–273 (1987)
4. Date, E., Jimbo, M., Kuniba, A., Miwa, T., Okado, M.: Exactly solvable SOS models. II: Proof of the star-triangle relation and combinatorial identities. Adv. Stud. Pure Math. **16** Kinokuniya-Academic 1988
5. Baxter, R. J.: Exactly solved models in statistical mechanics. London: Academic 1982
6. Andrews, G. E., Baxtrer, R. J., Forrester, P. J.: Eight-vertex SOS model and generalized Rogers–Ramanujan-type identities. J. Stat. Phys. **35**, 193–266 (1984)
7. Huse, D. A.: Exact exponents for infinitely many new multicritical points. Phys. Rev. **B30**, 3908–3915 (1984)
8. Jimbo, M., Miwa, T.: A solvable lattice model and related Rogers–Ramanujan type identities. Physica **15D**, 335–353 (1985)
9. Baxter, R. J. Andrews, G. E.: Lattice gas generalization of the hard hexagon model. I. Star-triangle relation and the local densities. J. Stat. Phys. **44**, 249–271 (1986)
10. Kuniba, A., Akutsu, Y., Wadati, M.: Exactly solvable IRF models. I. A three state model. J. Phys. Soc. Jpn. **55**, 1092–1101 (1986)
11. Date, E., Jimbo, M., Miwa, T., Okado, M.: Fusion of the eight-vertex SOS model. Lett. Math. Phys. **12**, 209–215 (1986)
12. Jimbo, M., Miwa, T., Okado, M.: Solvable lattice models whose states are dominant integral weights of $A_{n-1}^{(1)}$. Lett. Math. Phys. **14**, 123–131 (1987)
13. Jimbo, M., Miwa, T., Okado, M.: An $A_{n-1}^{(1)}$ family of solvable lattice models. Mod. Phys. Lett. **B1**, 73–79 (1987)
14. Jimbo, M., Miwa, T., Okado, M.: Symmetric tensors of the $A_{n-1}^{(1)}$ family, preprint RIMS 592. Kyoto Univ. Algebraic Analysis (Festschrift for M. Sato's 60th birthday). New York: Academic Press 1988
15. Jimbo, M., Miwa, T., Okado, M.: Local state probabilities of solvable lattice models. An $A_{n-1}^{(1)}$ family, preprint RIMS 594, Kyoto Univ., Nucl. Phys. **B300** [FS22], 74–108 (1988)
16. Kulish, P. P., Reshetikhin, N. Yu., Sklyanin, E. K.: Yang–Baxter equation and representation theory. I. Lett. Math. Phys. **5**, 393–403 (1981)
17. Cherednik, I. V.: On special bases of irreducible finite-dimensional representations of the degenerated affine Hecke algebra. Funct. Anal. and Appl. **20** N1, 87–89 (1986)
18. Pasquier, V.: Etiology of IRF models. Commun. Math. Phys. **118**, 355–364 (1986)
19. Goddard, P., Kent, A., Olive, D.: Virasoro algebras and coset space models, Phys. Lett. **B152**, 88–92 (1985)
20. James, G., Kerber, A.: The representation theory of the symmetric group. Encyclopedia of mathematics and its applications. Reading. MA: vol **16**. London: Addison–Wesley 1981
21. Baird, G. E., Biedenharn, L. C.: On the representations of the semisimple Lie groups. V. Some explicit Wigner operators for SU_3. J. Math. Phys. **6**, 1847–1854 (1965)

Communicated by H. Araki

Received April 18, 1988; in revised form June 6, 1988