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Euclidean Formulation of Quantum Field Theory Without Positivity

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Abstract. General properties of local quantum field theories (QFT) without positivity are discussed in connection with their euclidean formulation. Modified euclidean axioms for local QFT's without positivity are presented, which allow us to recover by analytic continuation Wightman functions satisfying the modified Wightman axioms for indefinite metric QFT's.

1. Introduction

With the advent of gauge theories it became clear that it was natural (if not necessary) to consider QFT in which not all the Wightman axioms are satisfied. In particular, it appeared that the introduction of "charged" fields was in conflict with either locality or positivity [1–4]. On the other hand, the success of (perturbative) renormalization theory (also for gauge theories) [5, 6] and the usefulness of keeping a relation with the wisdom gathered from conventional perturbation theory made clear that it could be better, at least at a technical level, to keep locality rather than positivity [3, 4]. Actually, it turned out that even the solution of a long-standing problem like the infrared problem [7] and the construction of charged states in QED was made possible by exploiting the local structure, in a spirit close to the standard Wightman formulation [8]. Even the recent deep results about the geometrical understanding of anomalies in QFT have been made possible by a formulation which kept locality as a basic structure [9, 10]. Also recent attempts of a quantum field theory formulation of string theories with emphasis on the "covariant gauges" suggest that it may be of some interest to investigate the general properties of indefinite metric quantum field theories. Finally, it need not be emphasized here that most of the wisdom gained (on a heuristic level) about "non-perturbative" treatment of covariant gauge field theories and/or covariant string theories heavily rely on the use of the so-called "functional integral techniques," namely of the euclidean formulation of the theory. To our knowledge, a careful discussion of the euclidean formulation of quantum field

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theories which do not satisfy positivity seems to be lacking in the literature. This general question is also at the basis of recent results on the existence of a gauge symmetry breaking order parameter in Higgs models in the so-called α-gauges [24], which do not satisfy Wightman positivity. The aim of this note is to discuss the general properties of local quantum field theories without positivity, in particular their associated Hilbert structures (modified Wightman axioms) [2–4, 11], in connection with their analytic continuation to euclidean points. Modified euclidean axioms are presented which allow us to discuss theories whose infrared singularities violate positivity and lead to a euclidean semigroup for time translation which is not contractive. The modified euclidean axioms allow us to recover by analytic continuation Wightman functions which satisfy the modified Wightman axioms, in particular the Hilbert space structure condition and the Poincaré covariance, compatible with the possible non-unitarity of the space-time translation (and the Lorentz boosts).

To simplify the exposition, we collect here the *notation* and *definitions* for which we largely follow ref. [12]:

$$\mathscr{S}_0(\mathbb{R}^{4n}) = \{f\in\mathscr{S}(\mathbb{R}^{4n}): f \text{ together with all its partial derivatives vanish if } x_i = x_j \text{ for some } 1\leqq i<j\leqq n\},$$

$$\mathscr{S}_+(\mathbb{R}^{4n}) = \{f\in\mathscr{S}_0(\mathbb{R}^{4n}): f \text{ together with all its partial derivatives vanish unless } 0<x_1^0<x_2^0<\cdots<x_n^0<\infty\},$$

$$\mathscr{S}(\mathbb{R}_+) = \{f\in\mathscr{S}(\mathbb{R}) \text{ with } \operatorname{supp} f\subset\mathbb{R}^+\},\ \mathscr{S}(\mathbb{R}_+^4) = \mathscr{S}(\mathbb{R}_+)\otimes\mathscr{S}(\mathbb{R}^3),$$

$$\mathscr{S}(\mathbb{R}_-) = \{f\in\mathscr{S}(\mathbb{R}) \text{ with } \operatorname{supp} f\subset\{x:x\leqq 0\}\},$$

$$\mathscr{S}(\bar{\mathbb{R}}) = \mathscr{S}(\mathbb{R})/\mathscr{S}(\mathbb{R}_-),\ \mathscr{S}(\bar{\mathbb{R}}_+^4) = \mathscr{S}(\mathbb{R}_+)\otimes\mathscr{S}(\mathbb{R}^3).$$

We also introduce the Borchers algebras $\mathscr{B}$, $\mathscr{B}_+$ and $\mathscr{B}(\mathbb{R}^4)$ over $\mathscr{S}(\mathbb{R}^4)$, $\mathscr{S}_+(\mathbb{R}^4)$ and $\mathscr{S}(\mathbb{R}^4)$ respectively, as the algebras generated through sums and products of elements $F=\{f_0,f_1,\ldots,f_n,\ldots\}$ which are terminating sequences of elements $f_0\in\mathbb{C}$, $f_n\in\mathscr{S}(\mathbb{R}^{4n}) = \otimes\mathscr{S}(\mathbb{R}^4)$, $f_n\in\mathscr{S}_+(\mathbb{R}^{4n})$ and $f_n\in\mathscr{S}(\bar{\mathbb{R}}_+^{4n}) = \otimes\mathscr{S}(\bar{\mathbb{R}}_+^4)$ respectively. $\mathscr{B}$, $\mathscr{B}_+$ and $\mathscr{B}(\bar{\mathbb{R}}_+^4)$ are equipped with the direct sum topologies induced by the topologies of $\mathscr{S}(\mathbb{R}^{4n})$, $\mathscr{S}_+(\mathbb{R}^{4n})$ and $\mathscr{S}(\bar{\mathbb{R}}_+^{4n})$ respectively. The *mapping d* is defined by $f_d(x_1,x_2-x_1,\ldots,x_n-x_{n-1})\equiv f(x_1,\ldots,x_n)$ and it is an isomorphism of $\mathscr{S}_+(\mathbb{R}^{4n})$ onto $\mathscr{S}(\mathbb{R}_+^{4n})$. The *mapping v*, $F\to\check{F}$ of $\mathscr{B}_+$ onto a subset $\check{\mathscr{B}}_+$ of $\mathscr{B}(\bar{\mathbb{R}}_+^4)$ is defined by

$$\check{f}_n(q_1,\ldots,q_n) = f_{nd}^{FL}(q_1,\ldots,q_n)\restriction\{q_k^0\geqq 0\},$$

where

$$g^{FL}(q_1,\ldots,q_n) = \int g(x_1,\ldots,x_n)e^{-\sum_{k=1}^{n}(q_k^0x_k^0 - i\vec{q}_k\,\vec{x}_k)}d^{4n}x$$

denotes the Fourier–Laplace transform of $g\in\mathscr{S}(\mathbb{R}_+^{4n})$. The mapping $F\to\check{F}$ is continuous with dense range and trivial kernel [12, 17].

2. Wightman and Schwinger Functions for Indefinite Metric QFT Local Structure

We start by discussing QFT's which satisfy all Wightman axioms except positivity [2–4, 11]. For a hermitian scalar field theory they read:

W1 (*Temperedness*): The n-point Wightman function $W_n(x_1,\ldots,x_n)$ are distributions, for simplicity taken of tempered type.[1] They also satisfy the standard hermiticity condition.

W2 (*Poincaré covariance*): For any n, $W_n(x_1,\ldots,n_n)=W(\Lambda x_1-a,\ldots,\Lambda x_n-a)$, $(a,\Lambda)\in P_+^\uparrow$.

W4 (*Weak spectral condition*): By W2, for any n, $W_n(x_1,\ldots,x_n)$ is actually a distribution in the difference variables

$$W_n(x_1,\ldots,x_n)=W_{n-1}^d(x_2-x_1,x_3-x_2,\ldots,x_n-x_{n-1}),$$

which is assumed to have a Fourier transform $\tilde{W}_{n-1}^d\,(q_1,\ldots,q_{n-1})$ with support in $\bar{V}_+^{(n-1)}$

W5 (*Locality*): $W_n(x_1,\ldots,x_i,\,x_{i+1},\ldots,x_n)=W_n(x_1,\ldots,x_{i+1},x_1,\ldots,x_n)$ whenever $(x_i-x_{i+1})^2<0$.

As anticipated, the positivity axiom W3 is not assumed, and furthermore the spectral condition is kept in the weak form (W4) compatible with space-time translations described by operators which are not unitary [11]. In contrast with the standard case therefore, the weak spectral condition is not strictly required by general physical principles (unbroken space-time translations and relativity). Such a condition, which is supported by the perturbative analysis, is essentially the condition which allows an euclidean formulation, and in our opinion this is one of its main motivations[2]. A further axiom which cannot be kept is the clustering behaviour (cluster property); for the relation between the so emerging structure and the essential uniqueness of the vacuum see [11].

We have thus the indefinite metric axioms. Proceeding as in the standard case [16] one can then introduce the Borchers algebra $\mathscr{B}$, the sesquilinear form

$$\begin{aligned}\langle F,G\rangle_W=W(F^*\times G)&\equiv\sum_{n,m}W_{n+m}(f_n^*\times g_m)=\sum_{n,m}\tilde{W}_{n+m-1}^d(\tilde{f}_{nd}^*\times\tilde{g}_{md})\\&\equiv\tilde{W}^d(\tilde{F}_d^*\times\tilde{G}_d)\equiv\langle\tilde{F}_d,\tilde{G}_d\rangle_{\tilde{W}^d},\end{aligned}\tag{2.1}$$

where $\sim$ denotes Fourier transform, $f_n^*(x_1,\ldots,x_n)=\overline{f_n(x_n,\ldots,x_1)}$, and the vector space

$$\mathscr{D}^W=\mathscr{B}/\mathscr{N}_W,\tag{2.2}$$

where $\mathscr{N}_w$ is the kernel of $\langle,\rangle_W$, namely the set of elements F of $\mathscr{B}$, with $\langle F,G\rangle_W=0$ for every $G\in\mathscr{B}$. In general, the inner product $\langle,\rangle_W$ will not be semidefinite and it cannot make $\mathscr{D}^W$ a pre-Hilbert space. One can then recover the fields as operator-valued distributions on the vector space $\mathscr{D}^W$. The Poincaré invariance of Wightman functions defines a representation of the Poincaré group by linear operators $\mathscr{U}(a,\Lambda)$ in $\mathscr{D}^W$ which preserve the inner product.

The above modified axioms in Minkowski space still allow the euclidean

[1] A similar formulation can also be done for a larger class of distributions like those of Jaffe type [13, 14]
[2] The relevance of this condition has been stressed in [3, 4]. See also the recent review by A. S. Wightman [15] for a discussion of this condition

continuation and the definition of the Schwinger functions [12, 17, 18]. They satisfy the following properties (for simplicity we consider the scalar field case):

OS1 (*Temperedness*): For each n, $S_n(x_1,\ldots,x_n)$ belongs to $\mathscr{S}_0(\mathbb{R}^{4n})$, and obey the following hermiticity property:

$$\overline{S_n(f)} = S_n(\theta f^*), \tag{2.3}$$

where $\theta f(x_1,\ldots,x_n) = f(rx_1,\ldots,rx_n)$; $r(x^0,x) = (-x^0,x)$.

OS2 (*Euclidean covariance*): For each n, $S_n(f) = S_n(f_{\{a,R\}})$ for all $R\in SO(4)$, $a\in\mathbb{R}^4$, where $f_{\{a,R\}}(x_1,\ldots,x_n) = f(R^{-1}(x_1-a),\ldots,R^{-1}(x_{n-a}))$.

OS1' (*Laplace transform condition*): By OS2, for each n, S_n is a distribution in the difference variables

$$S_n(x_1,\ldots,x_n) \equiv S^d_{n-1}(x_2-x_1,\ldots,x_n-x_{n-1}) \tag{2.4}$$

and the condition reads

$$|S^d_{n-1}(f)| \leqq \|\check{f}\|_{\mathscr{S}}; \quad \forall f\in S(\mathbb{R}_+^{4(n-1)}), \tag{2.5}$$

where $\|\cdot\|_{\mathscr{S}}$ is a Schwartz seminorm on $S(\bar{\mathbb{R}}_+^{4(n-1)})$; this implies that $S^d_{n-1}\in\mathscr{S}'(\bar{\mathbb{R}}_+^{4(n-1)})$.

OS4 (*Symmetry*): $S_n(x_{\pi(1)},\ldots,x_{\pi(n)}) = S_n(x_1,\ldots,x_n)$ for all n and all permutations π of n elements.

Similarly to the Minkowski case we do not have either the analog of the Wightman positivity (i.e. the OS-positivity or OS3 in the notation of ref. [17]) or the cluster property (OS5) in [17].

As in the Minkowski case, the euclidean axioms OS1, OS2, OS4 allow the construction of a vector space, and of a representation of three dimensional rotations, space and time translations.

Theorem 2.1. *Let the set* $\{S_n\}$ *of Schwinger functions be given satisfying* OS1, OS2 *and* OS4. *Then there exist a vector space* $\mathscr{D}^S$ *with a hermitian non-degenerate inner product* $\langle,\rangle_S$; *a group* $\mathscr{U}(\vec{a},\vec{R})$, $\vec{a}\in\mathbb{R}^3$, ($\vec{R}$ = *three dimensional rotation*) *of inner product preserving operators on* $\mathscr{D}^S$, *representing three dimensional space translations and three dimensional rotations; a semigroup* $P(t)$, $t\geqq 0$ *of* $\langle,\rangle_S$-*symmetric operators on* $\mathscr{D}^S$ *representing time translations for* $t\geqq 0$.

Proof. The proof is a simple adaptation of the standard argument. Let $\mathscr{B}_+$ be the Borchers algebra over $\mathscr{B}_+(\mathbb{R}^{4n})$. Given $F, G\in\mathscr{B}_+$ we define

$$\langle F,G\rangle_S \equiv S(\theta F^*\times G) \equiv \sum_{n,m} S_{n+m}(\theta f_n^*\times g_m), \tag{2.6}$$

where θF is the element with components $(\theta f)_n(x_1,\ldots,x_n) = f_n(rx_1,\ldots,rx_n)\cdot\langle,\rangle_S$ is a sesquilinear form on $\mathscr{B}_+\times\mathscr{B}_+$ which is hermitian as a consequence of (2.3):

$$\langle F,G\rangle_S = S(\theta(F^*\times\theta G)) = \overline{S(\theta G^*\times F)} = \overline{\langle G,F\rangle_S}.$$

Let $\mathscr{N}_S$ be the kernel of $\langle,\rangle_S$, i.e. $\mathscr{N}_S = \{F\in\mathscr{B}_+ : \langle F,G\rangle_S = 0, \forall G\in\mathscr{B}_+\}$. Then $\mathscr{D}^S = \mathscr{B}_+/\mathscr{N}_S$ is a linear space. If $[F]_S$ denotes the class in $\mathscr{D}^S$ containing $F\in\mathscr{B}_+$, then

$$\langle [F]_S, [G]_S \rangle_S \equiv \langle F, G \rangle_S \tag{2.7}$$

defines a hermitian, non-degenerate inner product on $\mathscr{D}^S$. For $\vec{a} \in \mathbb{R}^3$ and $\vec{R}$ a three dimensional rotation, we define

$$\mathscr{U}(\vec{a}, \vec{R})[F]_S \equiv [F_{\{\vec{a},\vec{R}\}}]_S; \quad f_{n\{\vec{a},\vec{R}\}}(x_1, \ldots, x_n) = f_n(\vec{R}^{-1}(x_1 - \vec{a}), \ldots, \vec{R}^{-1}(x_n - \vec{a})). \tag{2.8}$$

By OS2, (2.8) is well defined and

$$\langle \mathscr{U}(a,R)[F]_S, \mathscr{U}(a,R)[G]_S \rangle_S = \langle [F]_S, [G]_S \rangle_S.$$

For $t \geqq 0$ we define

$$P(t)[F]_s \equiv [F_t]_s, f_{nt}(x_1, \ldots, x_n) = f_n(x_1 - t, \ldots, x_n - t). \tag{2.9}$$

Since $\langle F_t, G \rangle_S = \langle F, G_t \rangle_S$, the mapping $F \to F_t$ maps $\mathscr{N}_S$ into $\mathscr{N}_S$ and (2.9) is well defined. Moreover, for $t_1, t_2 \geqq 0$ $P(t_1)P(t_2) = P(t_1 + t_2)$. Hence $P(t)$, $t \geqq 0$ is a semigroup of operators on the inner product space $(\mathscr{D}^S, \langle , \rangle_S)$ which is $\langle , \rangle_S$-symmetric, i.e. $\langle P(t)[F]_S, [G]_S \rangle_S = \langle [F]_S, P(t)[G]_S \rangle_S$.

Remark. For the boosts one can essentially reproduce the results of [19] namely get the analogue of the virtual representation. In our case, the lack of positivity makes the domain problem more delicate. Actually, Sect. 4 will somewhat clarify this problem.

Proposition 2.2. *For a given set of Schwinger functions* $\{S_n\}$ *satisfying* OS1, OS2, OS1′ *and* OS4, *there exist Wightman functions* $\{W_n\}$ *satisfying* W1, W2, W4 *and* W5. *Moreover, one can establish a one-to-one mapping between* $\mathscr{D}^S$ *and a subspace* $\check{\mathscr{D}} \equiv \check{\mathscr{B}}_+ / \mathscr{N}_{\check{W}^d}$, *where* $\check{\mathscr{B}}_+$ *is the image of* $\mathscr{B}_+$ *under the mapping* $^\vee$ *and* $\mathscr{N}_{\check{W}^d}$ *is the kernel of* $\langle , \rangle_{\check{W}^d}$ *(Eq. (2.1)).* $\check{\mathscr{D}}$ *is dense in* $\mathscr{D}^W$ *with respect to the quotient topology induced by the topology of* $\mathscr{B}(\bar{\mathbb{R}}^4_+)$. *Furthermore, this mapping preserves the corresponding inner products.*

Proof. As a consequence of the Laplace transform condition the Schwinger functions of the difference variables S^d_n are the Fourier–Laplace transforms of distributions $\tilde{W}^d_n \in \mathscr{S}(\bar{\mathbb{R}}^{4n}_+)$, [17, 18]:

$$S^d_{n-1}(\xi_1, \ldots, \xi_{n-1}) = \int e^{\sum_{k=1}^{n-1} (\xi^0_k q^0_k - i\vec{\xi}_k \vec{q}_k)} \tilde{W}^d_{n-1}(q_1, \ldots, q_{n-1}) dq^{4(n-1)}. \tag{2.10}$$

The Wightman functions are then defined by [12]:

$$W_n(x_1, \ldots, x_n) = \int e^{-i \sum_{k=1}^{n} (x_{k+1} - x_k) q_k} \tilde{W}^d_{n-1}(q_1, \ldots, q_{n-1}) dq^{4(n-1)}. \tag{2.11}$$

The so-defined Wightman functions are Poincaré covariant (as a consequence of OS2) and by the support properties of $\tilde{W}^d_{n-1}$, W_n satisfy the spectral condition in the weak form. Similarly, locality follows from OS4 (see Sect. 4 of ref. [12]). Furthermore, as in [12], for any $F_+, G_+ \in \mathscr{B}_+$ we have

$$\begin{aligned} S(\theta F^*_+ \times G_+) = \sum_{n,m} \int \bar{\check{f}}_{+n}(q_n, \ldots, q_1) \check{g}_{+m}(q_n, \ldots, q_{n+m-1}) \\ \cdot \tilde{W}^d_{n+m-1}(q_1, \ldots, q_{n+m-1}) dq^{4(n+m-1)} = \tilde{W}^d(\check{F}^*_+ \times \check{G}_+). \end{aligned} \tag{2.12}$$

It is then natural to consider the vector space $\check{\mathscr{D}} = \check{\mathscr{B}}_+/\mathscr{N}_{\tilde{W}^d}$. Since $\check{\mathscr{B}}_+$ is dense in $\mathscr{B}(\bar{\mathbb{R}}^4_+)$ in the $\mathscr{B}(\bar{\mathbb{R}}^4_+)$-topology [12], $\check{\mathscr{D}}$ is dense in $\mathscr{B}(\bar{\mathbb{R}}^4_+)/\mathscr{N}_{\tilde{W}^d}$ in the induced quotient topology. We then define the map w of $\check{\mathscr{B}}_+$ onto $\mathscr{D}^s$ by [12]

$$w(\check{F}_+) = [F_+]_S. \tag{2.13}$$

Actually w maps equivalent elements in $\check{\mathscr{B}}_+/\mathscr{N}_{\tilde{W}^d}$ into the same vectors of $\mathscr{D}^S$, hence it defines a mapping of $\check{\mathscr{D}}$ onto $\mathscr{D}^S$. Furthermore, for any $F_+, G_+ \in \mathscr{B}_+$ by (2.10) we have

$$\begin{aligned}\langle w(\check{F}_+), w(\check{G}_+)\rangle_S &= \langle [F_+]_S, [G_+]_S\rangle_S = S(\theta F_+^* \times G_+)\\ &= \tilde{W}^d(\check{F}_+^* \times \check{G}_+) = \langle [\check{F}_+]_{\tilde{W}^d}, [\check{G}_+]_{\tilde{W}^d}\rangle_{\tilde{W}^d},\end{aligned}$$

and the last equality defines the inner product on $\check{\mathscr{D}}$ which is preserved by the map w.

3. The Hilbert Space Structure Condition and Its Implications for Schwinger Functions

The structure discussed above does not require the Osterwalder–Schrader positivity of the Schwinger functions and as such it applies also to theories for which this property is not fulfilled. However, when one looks for a physical interpretation of the theory the identification of the physical states becomes a crucial issue. In general, they do not belong to the vector spaces $\mathscr{D}^W$ or $\mathscr{D}^S$, and therefore it becomes a crucial physical problem to find a Hilbert topology on $\mathscr{D}^W$ or on $\mathscr{D}^S$ such that the physical states can be approximated as closely as we like by states of $\mathscr{D}^W$ or $\mathscr{D}^S$, i.e. they belong to the closures $\mathscr{D}^W$, $\mathscr{D}^S$ with respect to such topologies. This problem arises in particular in gauge quantum field theories, where the physical charged states do not belong [1] to the vector space $\mathscr{D}^W$ of local states. The construction of the physical charged states crucially relies on the use of a Hilbert topology or Hilbert structure [8]; this shows the relevance of the Hilbert space structure, which can be associated to a set of Wightman functions. They "parametrize" the possible "infrared behaviour" of the states which can be obtained by closures of the local states through Hilbert topologies, i.e. they parametrize the possible charge content of the physical states. It is well-known that in interacting QED the standard choice of Hilbert topology leads to a physical set of states with zero charge [20], and to get physical charged states one needs a careful choice of Hilbert topology [8].[3]

[3] The claim appeared in the literature [21] that actually the physical subspace is independent of the choice of the Hilbert topology is contradicted by many examples. The point is that the construction of non-local gauge invariant fields (from local fields), a basic step in the approach of ref. [21, 22], requires to make reference to a Hilbert topology. The class of non-local gauge invariant fields which can be constructed in terms of a local field depends on the (sometimes implicit) choice of a Hilbert topology. The requirement of gauge invariance for the above non-local fields is not enough to uniquely identify a set of states corresponding to irreducible representations of the local observable algebra (sec ref. [8], Sect. 4). In general, in attempting to construct a sufficiently large set of non-local gauge invariant fields one ends up with highly reducible representations of the local observable algebra, as in ref. [22]

As discussed in [11] a necessary and sufficient condition for the existence of a Hilbert space topology in Minkowski QFT is that the Wightman functions satisfy the following property[4] which replaces the axiom of positivity:

W3′ (Hilbert space structure condition): There exists a Hilbert seminorm p_W on $\mathscr{B}$ such that

$$W(F^* \times G)| \leqq p_W(F) p_W(G).$$

From a distributional point of view it is usually better and practically more convenient (to avoid pathologies which do not seem to occur in QFT, at least at the perturbative level) to have that the above seminorm is *continuous on* $\mathscr{B}$ with respect to the $\mathscr{S}$-topology (*briefly* $\mathscr{B}$*-continuous*). In the following we will always consider W3′ with such an additional requirement.

From W3′ it follows that the Hilbert seminorm p_W defines a semi-definite inner product $(,)_W$ on $\mathscr{B}$. Clearly the kernel of such a product is contained in $\mathscr{N}_W$ as a consequence of W3′ but there may be elements $F \in \mathscr{N}_W$ such that $p_W(F) \neq 0$ (*degenerate metric*). However, one may define [11] a new seminorm p'_W

$$p'_W(F) = \operatorname*{Inf}_{N \in \mathscr{N}_W} p_W(F + N), \tag{3.1}$$

such that if $F \in \mathscr{N}_W$, $p'_W(F) = 0$; hence $\ker p'_W = \mathscr{N}_W$ (non-degenerate seminorm). Furthermore, the Hilbert space structure condition W3′ remains true also for the new seminorm p'_W, since obviously $\forall N, M \in \mathscr{N}_W, F, G \in \mathscr{B}$,

$$W((F + N)^* \times (G + M)) = W(F^* \times G).$$

Moreover, if p_W is a $\mathscr{B}$ continuous Hilbert seminorm, so is p'_W. In fact, if (F, G) denotes the Hilbert product which defines p_W on $\mathscr{B}$ the sesquilinear form

$$(F, G)' = \operatorname*{Inf}_{N, M \in \mathscr{N}_W} (F + N, G + M)$$

also defines a Hilbert product on $\mathscr{B}$,

$$p'_W(F)^2 = (F, F)' \leqq p_W(F)^2$$

and

$$|(F, G)'| \leqq p_W(F) p_W(G).$$

Hence, $\mathscr{B}$ continuity of p'_W follows from $\mathscr{B}$ continuity of p_W and $(.,.)'$ is jointly $\mathscr{S}$-continuous, and therefore it defines a tempered distribution on $\mathscr{B} \times \mathscr{B}$. Summarizing we have:

Proposition 3.1. *If p is a $\mathscr{B}$-continuous Hilbert seminorm fulfilling* W3′, *then we can always construct a non-degenerate seminorm p' still fulfilling* W3′ *which is also a $\mathscr{B}$-continuous Hilbert seminorm. Furthermore the Hilbert scalar product defining a $\mathscr{B}$-continuous seminorm is defined by a kernel which is a tempered distribution.*

From now on we will always assume that p_W is non-degenerate. The non-degeneracy of the seminorm p_W implies that the extension of $\langle \cdot, \cdot \rangle_W$ to the

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[4] See also [23] for the algebraic formulation of a stronger condition

closure K^W of $\mathscr{B}/\mathscr{N}_W$ with respect to the Hilbert topology of p_W, defines a metric η

$$\langle\cdot,\cdot\rangle_W = (\cdot, \eta\cdot),$$

which is non-degenerate.

By a standard procedure [11] we can also obtain a metric with bounded inverse so that $\mathscr{K}^W$ is actually a Krein space, but this will not be used in the following.

We will now discuss the implication of W3′ for the Schwinger functions in analogy with the correspondence between Wightman positivity and OS-positivity in the standard case [12].

To this purpose, we note that for any F, $G\in\mathscr{B}$, putting $\hat{F} = \tilde{F}_d \upharpoonright \{q_k \geqq 0\}$ and by using the weak spectral condition we have

$$W^d(\tilde{F}_d^* \times \tilde{G}_d) = W^d(\hat{F}^* \times \hat{G}). \tag{3.2}$$

Proposition 3.2. *The Hilbert space structure condition* W3′ *with Hilbert seminorm* p_W *which is a* $\mathscr{B}$*-continuous and non-degenerate* (*namely* $\ker p_W = \mathscr{N}_W$) *implies that there exists seminorm* $\hat{p}$ *on* $\mathscr{B}(\bar{\mathbb{R}}_+^4)$ *which is continuous in the* $\mathscr{S}(\bar{\mathbb{R}}_+^4)$*-topology* (*briefly* $\mathscr{B}(\bar{\mathbb{R}}_+^4)$*-continuous*) *such that for any* $\hat{F}, \hat{G}\in\mathscr{B}(\bar{\mathbb{R}}_+^4)$

$$|\tilde{W}^d(\hat{F}^* \times \hat{G})| \leqq \hat{p}(\hat{F})\hat{p}(\hat{G}). \tag{3.3}$$

Proof. For a given seminorm p_W on $\mathscr{B}$ which is $\mathscr{B}$-continuous one defines a new seminorm p_d on $\mathscr{B}$ by

$$p_d(F) = p_W(F_{d^{-1}}). \tag{3.4}$$

Since d, d^{-1} are $\mathscr{B}$-continuous maps, p_d is also $\mathscr{B}$-continuous and one may define a seminorm $\tilde{p}_d$ on the Fourier transforms by

$$\tilde{p}_d(\tilde{F}) = p_d(F). \tag{3.5}$$

One also has that if p_W is a Hilbert seminorm derived from the Hilbert scalar product (F, G) on $\mathscr{B}$, then also

$$[\tilde{F}, \tilde{G}] \equiv (F_{d^{-1}}, G_{d^{-1}}) \tag{3.6}$$

is a Hilbert scalar product on $\mathscr{B}$ and

$$\tilde{p}_d(\tilde{F})^2 = [\tilde{F}, \tilde{F}]. \tag{3.7}$$

Furthermore, the non-degeneracy of p_W implies that

$$\ker \tilde{p}_d = \mathscr{N}_{\tilde{W}^d}. \tag{3.8}$$

In fact, if $\tilde{F}\in\ker\tilde{p}_d$, then by the above definition $F_{d^{-1}}\in\ker p_W = \mathscr{N}_W$ and by (3.2) $\tilde{F}\in\mathscr{N}_{\tilde{W}^d}$. Conversely if $\tilde{F}\in\mathscr{N}_{\tilde{W}^d}$, then by (3.2) $F_{d^{-1}}\in\mathscr{N}_W = \ker p_W$, and therefore by (3.4) $F\in\ker p_d$. Hence, by (3.5) $F\in\ker p_d$. Now the Hilbert product [,] corresponding to $\tilde{p}_d$ defines a tempered distribution on $\mathscr{B}\times\mathscr{B}$ with support in $\{q_k^0 \geqq 0\}$ (by the weak spectral condition and (3.8)), i.e. a Hilbert product on $\mathscr{B}(\bar{\mathbb{R}}_+^4)$ which is $\mathscr{B}(\bar{\mathbb{R}}_+^4)$-continuous. Thus it defines a $\mathscr{B}(\bar{\mathbb{R}}_+^4)$-continuous seminorm $\hat{p}$ on $\mathscr{B}(\bar{\mathbb{R}}_+^4)$,

$$\hat{p}(\tilde{F}_d \upharpoonright \{q_k^0 \geqq 0\}) = \tilde{p}_d(\tilde{F}_d \upharpoonright \{q_k^0 \geqq 0\}). \tag{3.9}$$

In conclusion, since the set $\{\tilde{F}_d|\{q_k^0 \geqq 0\}: F \in \mathscr{B}\}$ coincides with $\mathscr{B}(\bar{\mathbb{R}}_+^4)$ [12], $\hat{p}$ is defined for any $\hat{F} \in \mathscr{B}(\bar{\mathbb{R}}_+^4)$ and (3.3) is satisfied.

Proposition 3.3. *Given a set of Wightman functions satisfying the Hilbert space structure condition* W3′ *with non-degenerate $\mathscr{B}$-continuous seminorm p_W, there exists a non-degenerate Hilbert seminorm p_S on $\mathscr{B}_+$ such that for any $F_+ \mathscr{B}_+$,*

$$p_S(F_+) \leqq \|\check{F}_+\|_{\mathscr{S}}, \tag{3.10}$$

where $\|\cdot\|_{\mathscr{S}}$ is some Schwartz seminorm, and furthermore the corresponding Schwinger function satisfy the following Hilbert space structure condition: $\forall F_+ G_+ \in \mathscr{B}_+$

$$|S(\theta F_+^* \times G_+)| \leqq p_S(F_+)p_S(G_+). \tag{3.11}$$

Proof. By Eq. (2.12), the fact that for every $F_+ \in \mathscr{B}_+$, $\check{F}_+ = F_{+d}^{FL} \upharpoonright \{q_k^0 \geqq 0\} \in \mathscr{B}(\bar{\mathbb{R}}_+^4)$ and Proposition 3.2 we have

$$|S(\theta F_+^* \times G_+)| = |\tilde{W}^d(\check{F}_+^* \times \check{G}_+)| \leqq \hat{p}(\check{F}_+)\hat{p}(\check{G}_+).$$

So putting $p_S(F_+) \equiv \hat{p}(\check{F}_+)$, we have (3.11). The bound (3.10) follows from $\mathscr{B}(\bar{\mathbb{R}}_+^4)$-continuity of $\hat{p}$; the $\mathscr{B}_+$-continuity of p_S follows from the continuity of the mapping $F_+ \to \check{F}_+$.

4. Modified Euclidean Axioms for Indefinite Metric OFT

The above discussion allows us to solve the problem of characterizing the euclidean formulation of indefinite metric quantum field theories *beyond* the vector space (or local) structure discussed in Sect. 2. (For the need of a Hilbert space structure and its relation with the construction of "charged" states see [8] and the brief discussion in Sect. 3.) It should be remarked that, as discussed above, the weak spectral condition allows us to get the euclidean formulation of QFT (by analytic continuation) even in the case in which the infrared singularities are so severe (e.g. of so-called confining type [3, 11]) that the Fourier transform of the Wightman functions $\langle \Psi, U(x)\Psi \rangle$, Ψ a local state, are no longer measures and therefore the space-time translations cannot be described by unitary operators (*unbounded representations of space-time translations*) [3, 11]. The problem however arises for the converse way, namely the problem of characterizing those properties of the Schwinger functions which allows us to recover the Wightman functions and the possibly unbounded representations of the space-time translations. In particular, in this general case, the standard treatment based on the existence of a contractive semi-group in the euclidean space of states can no longer be used, since the latter property characterizes the existence of a unitary group in Minkowski space. The purpose of this section is to provide a solution for this problem.

As a consequence of the result of Sect. 3 we are led to consider the following *modified euclidean axioms* for indefinite metric QFT's:

OS1 (*Temperedness*): For each n, $S_n \in \mathscr{S}_0(\mathbb{R}^{4n})$ and obeys the hermicity property

$$\overline{S_n(f)} = S_n(\theta f^*).$$

OS2 (*Euclidean covariance*): $S_n(f) = S_n(f_{\{a,R\}})$ for all $R \in SO(4)$, $a \in \mathbb{R}^4$.

OS3′ (*Hilbert space structure*): There exists non-degenerate Hilbert seminorm p_S on $\mathscr{B}_+$ such that for every $F_+ \in \mathscr{B}_+$,

$$p_S(F_+) \leqq \| F^{FL}_{+d} \upharpoonright \{q^0_k \geqq 0\} \|_{\mathscr{S}} = \| \check{F}_+ \|_{\mathscr{S}} \tag{4.1}$$

for some Schwartz seminorm $\| \ \|_{\mathscr{S}}$ on $\mathscr{B}(\bar{\mathbb{R}}^4_+)$ (briefly: p_S is $\mathscr{B}(\bar{\mathbb{R}}^4_+)$-continuous) and furthermore, for any $F_+, G_+ \in \mathscr{B}_+$,

$$|S(\theta F^* \times G_+)| = \left| \sum_{n,m} S_{n+m}(\theta f^*_n \times g_m) \right| \leqq p_S(F_+) p_S(G_+). \tag{4.2}$$

OS4 (*Symmetry*): $S_n(x_{\pi(1)}, \ldots, x_{\pi(n)}) = S_n(x_1, \ldots, x_n)$ for all n and all permutations π of n elements.
Now we have:

Proposition 4.1. *The Hilbert space structure condition* OS3′ *implies the Laplace transform condition* OS1′.

Proof. We start by considering the case in which $F_+ = 1$, G_+ has only the n-component g_{+n} non-vanishing and it is of the form

$$g_{+n}(x_1, \ldots, x_n) = g_1(x_1) g_{n-1}(x_2 - x_1, \ldots, x_n - x_{n-1}) \tag{4.3}$$

with $\tilde{g}_1(0) = 1$ and $g_1 \in \mathscr{S}(\mathbb{R}^4_+)$, $g_{n-1} \in \mathscr{S}(\mathbb{R}^{4(n-1)}_+)$. Clearly such functions define elements in $\mathscr{B}_+$. Then the $\mathscr{B}(\bar{\mathbb{R}}^4_+)$-continuity (4.1) gives

$$p_S(g_{+n}) \leqq \| \check{g}_1 \|_{\mathscr{S}} \| \check{g}_{n-1} \|_{\mathscr{S}},$$

and (4.2) yields

$$\begin{aligned} |S(G_+)| &= \left| \int S_n(x_1, \ldots, x_n) g_1(x_1) g_{n-1}(x_2 - x_1, \ldots, x_n - x_{n-1}) d^4x_1 \cdots d^4x_n \right| \\ &= \left| \int g_1(x_1) d^4x_1 \right| \left| \int S^d_{n-1}(x_2 - x_1, \ldots, x_n - x_{n-1}) \right. \\ &\quad \left. \cdot g_{n-1}(x_2 - x_1, \ldots, x_n - x_{n-1}) d^4x_2 \cdots d^4x_n \right| \\ &\leqq p_S(g_1 g_{n-1}) \leqq \| \check{g}_1 \|_{\mathscr{S}} \| \check{g}_{n-1} \|_{\mathscr{S}}. \end{aligned}$$

Since $\tilde{g}_1(0) = 1$ we have

$$|S^d_{n-1}(g_{n-1})| \leqq \| \check{g}_1 \|_{\mathscr{S}} \| \check{g}_{n-1} \|_{\mathscr{S}}. \tag{4.4}$$

This means that condition OS1′ holds.

Now we can establish the link between the Euclidean and the Minkowski space formulation.

Theorem 4.2. 1. *Given a set of Schwinger function satisfying* OS1, OS2, OS3′ *and* OS4, *we can associate to them a Hilbert space of states $\mathscr{H}^S$ with a positive scalar product $(,)_S$ and a non-degenerate metric operator has such that*

i) $S(\theta F^*_+) \times G_+) = \langle F_+, G_+ \rangle_S = \langle [F_+]_S, [G_+]_S \rangle_S = ([F_+]_S, \eta_S [G_+])_S,$

where $[\ \ ]_S$ is the equivalence class with respect to the Schwinger ideal.

ii) *the set of euclidean "local" state $\mathscr{D}^S$ is dense in $\mathscr{H}^S$.*

2. *The Wightman functions corresponding to the Schwinger functions via Laplace*

transform ((2.10), (2.11)) satisfy the modified Wightman axioms W1, W2, W3′, W4 *and* W5 (*Sect. 2 and 3*) *with* p_W *defined in terms of* p_S.

3. *The Hilbert topologies defined by* p_S *and* p_W *turn* $\mathscr{D}^S$ *and* $\mathscr{D}^W$ *into Hilbert spaces* $\mathscr{H}^S$ *and* $\mathscr{H}^W$ *with a natural identification between them, which preserves the inner product structures.*

Proof. 1. By OS3′, the Hilbert product which defines p_S can be used to define a positive scalar product and therefore a norm on $\mathscr{D}^S$. Hence by completing $\mathscr{D}^S$ with respect to such a norm, we get a Hilbert space $\mathscr{H}^S$ and i) and ii) follow.

2. The existence of Wightman functions satisfying W1, W2, W4 and W5 is guaranteed by the Laplace transform condition which is implied by OS3′ (Proposition 4.1), so Proposition 2.2 applies. As far as W3′ is concerned we remark that the mapping w defined by (2.13) is $\mathscr{B}(\bar{\mathbb{R}}^4_+)$-continuous, since

$$p_S(w(\check{F}_+)) = p_S([F_+]_S) = p_S(F_+) \leqq \|\check{F}_+\|_{\mathscr{S}}. \tag{4.5}$$

Hence, the continuous extension $\bar{w}$ maps $\mathscr{B}(\bar{\mathbb{R}}^4_+)$ onto a dense subspace $\mathscr{D}^S$ of $\mathscr{H}^S$; furthermore since $\check{\mathscr{B}}_+$ is dense in $\mathscr{B}(\bar{\mathbb{R}}^4_+)$ and $\tilde{W}^d$ is a distribution in $\mathscr{B}(\bar{\mathbb{R}}^4_+)'$, the continuous extension of the equation

$$\tilde{W}^d(\check{F}^*_+ \times \check{G}_+) = \langle w(\check{F}_+), w(\check{G}_+)\rangle_S \tag{4.6}$$

gives for any $H_1, H_2 \in \mathscr{B}(\bar{\mathbb{R}}^4_+)$

$$\tilde{W}^d(H^*_1 \times H_2) = \langle \bar{w}(H_1), \bar{w}(H_2)\rangle_S, \tag{4.7}$$

i.e. by Eqs. (2.1) and (3.2), for $F, G \in B$,

$$W(F^* \times G) = \langle \bar{w}(\hat{F}), \bar{w}(\hat{G})\rangle_S \tag{4.8}$$

(see [12] Sect. 4). Condition OS3′ then gives

$$|W(F^* \times G)| \leqq p_W(F), p_W(G)\rangle_S, \tag{4.9}$$

where $p_W(F) \equiv p_S(\bar{w}(\hat{F}))$. One has that
a) p_W is a Hilbert seminorm
b) p_W is $\mathscr{B}$-continuous as a consequence of the $\mathscr{B}_+$-continuity of p_S, the $\mathscr{B}(\bar{\mathbb{R}}^4_+)$-continuity of $\bar{w}$ and the $\mathscr{B}$-continuity of the mapping $F \to \hat{F}$.
c) p_W is non-degenerate, i.e. $\ker p_W = \mathscr{N}_W$, in fact if $F \in \mathscr{N}_W$, then by (4.8) and the non-degeneracy of p_S we have $\bar{w}(\hat{F}) \in \ker p_S$, and furthermore $p_W(F) = p_S(\bar{w}(\hat{F})) = 0$, i.e. $F \in \ker p_W$.

3. We remark that the seminorm $\hat{p}$ on $\mathscr{B}(\bar{\mathbb{R}}^4_+)$ constructed from p_W (Proposition 3.2) has the form

$$\hat{p}(\hat{F}) = p_S(\bar{w}(\hat{F})). \tag{4.10}$$

Since (4.10) is $\mathscr{B}(\bar{\mathbb{R}}^4_+)$-continuous, $\check{\mathscr{D}} = \check{\mathscr{B}}_+/\mathscr{N}_{\tilde{W}^d}$ is $\hat{p}$-dense in $\mathscr{B}(\bar{\mathbb{R}}^4_+)/\mathscr{N}_{\tilde{W}^d} = \mathscr{D}^W$. By Proposition 2.2 $\bar{\check{\mathscr{D}}}^p = \mathscr{H}^W$ can be naturally identified with $\bar{\mathscr{D}}^S = \mathscr{H}^S$, and this identification preserves the inner product structures.

Remark. As a consequence of the lack of OS-positivity, the euclidean invariance of the Schwinger functions does not give rise in general to unitary representations of the three dimensional space translations and rotations, nor to a contractive semi-group for time translations. Actually, for the three dimensional euclidean group we get representation by operators $\mathscr{U}(\vec{a}, \vec{R})$ which preserve the indefinite inner product $\langle , \rangle_S$ (η_S-unitary representation). For the time translation we get a semigroup $P(t), t \geqq 0$ of operators which are symmetric with respect to the inner product $\langle , \rangle_S$ (η_S-symmetric semi-group). Also for the boosts the situation is in general different from the standard case and strictly speaking one does not get the same type of virtual representations of ref. [19]. The counterpart of these phenomena for the Poincaré group is that the Poincaré covariance of the Wightman functions only guarantees the existence of η-unitary representation of the Poincaré group; in general one expects that the operator $\mathscr{U}(a)$, $a \in \mathbb{R}^4$ as well as the $\mathscr{U}(\Lambda)$ (Λ-Lorentz boost) are unbounded.

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