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The Extended Phase Space of the BRS Approach

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Abstract. The origin of the classical BRS symmetry is discussed for the case of a first class constrained system consisting of a $2n$-dimensional phase space S with free action of a Lie gauge group G of dimension m. The extended phase space S_{ext} of the Fradkin-Vilkovisky approach is identified with a globally trivial vector bundle over S with fibre $L^*(G) \oplus L(G)$, where $L(G)$ is the Lie algebra of G and $L^*(G)$ its dual. It is shown that the structure group of the frame bundle of the supermanifold S_{ext} is the orthosymplectic group $\mathrm{OSp}(m, m; 2n)$, which is the natural invariance group of the super Poisson bracket structure on the function space $C^\infty(S_{\text{ext}})$. The action of the BRS operator Ω is analyzed for the case $S = R^{2n}$ with constraints given by pure momenta. The breaking of the $\mathrm{osp}(m, m; 2n)$-invariance down to $\mathrm{sp}(2n-2m)$ occurs via an intermediate "$\mathrm{osp}(m; 2n-m)$." Starting from a $(2n+2m)$-dimensional system with orthosymplectic invariance, different choices for the BRS operator correspond to choosing different $2n$-dimensional constraint supermanifolds in S_{ext}, which in general characterize different constrained systems. There is a whole family of physically equivalent BRS operators which can be used to describe a particular constrained system.

I. Introduction. Use of BRS Methods and Ghosts

A prominent feature in the use of BRS methods in field theory is the appearance of so-called ghosts. Ghosts were first introduced in the context of a path integral approach to gauge field theory. According to Faddeev and Popov [FAD/POP] the functional measure (after gauge fixing) has to contain a determinantal factor which accounts for the fact that we have to factor out by the volume of the gauge group. This way we get rid of the redundancy in the dynamics which is due to the presence of unphysical gauge degrees of freedom.

Ghost fields appear during the computation of Feynman diagrams when one rewrites this determinant in exponential form (using Berezin integration for anticommuting Grassmann fields) to arrive at an effective action. As was first noticed by Becchi, Rouet, and Stora [B/R/S], this effective action possesses a new global symmetry, subsequently called BRS symmetry, which mixes the original bosonic fields of the theory with the new ghost fields. BRS symmetry provides a useful tool for constructing physical quantities and deriving field theoretic identities.

In an attempt to improve the Faddeev-Popov description to allow for a wider class of gauge fixing terms, including covariant ones, Fradkin and Vilkovisky [FRA/VIL] set up a superhamiltonian framework to handle constrained systems within a path integral approach. They succeeded in constructing a unitary S-matrix (unitary on the physical subspace) for such theories by introducing additional bosonic and fermionic degrees of freedom, where the number of anticommuting ghost fields is doubled in comparison with the Faddeev-Popov approach. The resulting effective action is independent of the gauge fixing and invariant under BRS transformations.

Translating these statements into an operator-based approach (see, for example, [KUG/OJI]), one finds the well-known features associated with a BRS quantum theory, such as the existence of a (pseudo-)hermitian and nilpotent BRS operator $\hat{\Omega}$, an indefinite metric "Hilbert" space and the annihilation condition $\hat{\Omega}\Psi=0$ for physical wave functions Ψ.

On the other hand various attempts have been made to develop a classical (=non-quantum) understanding of the BRS approach, see [HEN, MCM]. One finds that there is a classical BRS operator Ω associated with every classical Hamiltonian system with first class constraints. This generator Ω has odd Grassmann parity, is nilpotent and has ghost number one. It is an element of an extended phase space which, due to the existence of Grassmannian degrees of freedom, is not a phase space in the usual sense.

However, so far no closer investigation of the geometric nature of the extended phase space and the symmetry underlying the BRS construction has been made.

To achieve this aim we will focus on the analysis of finite-dimensional Hamiltonian gauge systems where the general setting is as follows: we start with a $2n$-dimensional phase space S, i.e. a manifold with a symplectic structure and local coordinates (q^i, p_i), $i=1,\ldots,n$, and a set of m first class constraints, $\Phi_\alpha(q^i,p_i)=0$, $\alpha=1,\ldots,m$, satisfying Poisson bracket relations

$$\{\Phi_\alpha, \Phi_\beta\} = C_{\alpha\beta}{}^{\gamma}\Phi_\gamma \quad \text{and} \quad \{\Phi_\alpha, H\} = V_\alpha{}^{\beta}\Phi_\beta\,, \tag{1.1}$$

where H is the Hamiltonian of the system and the C's and V's are in general phase space dependent.

From the Dirac-Bergmann analysis (see, for example, [SUN] and references therein) we know that for such a system the motion is constrained to take place in a submanifold S_c of the phase space S. Furthermore the system exhibits gauge invariance, arbitrary parameters appear in the equations of motion, and one has to try and fix them by appropriate gauge choices. Unfortunately, for a generic first class constrained system, general mathematical theorems tell us that there is no such global gauge choice, i.e. in general we cannot identify the symplectic space $S_{\text{phys}}=S_c/G$ of the true degrees of freedom, the so-called reduced phase space, explicitly, in terms of, say, $(n-m)$ pairs $(q_{\text{phys}}, p_{\text{phys}})$. In general one therefore has to keep part or all of the "unphysical" phase space variables on S to describe the system.

We will consider the case where the phase space S is of the form of a principal fibre bundle $G \to S \to S/G$ (i.e. the action of the gauge group G on S is free, which means nothing other than that we have a neat fibration of S into G-orbits). This models the case of a typical gauge theory like Yang-Mills theory, where the gauge group can be made to act freely. The group G is taken to act on S by symplectic transformations, with a global Hamiltonian vector field corresponding to each of the constraints. The constraints form a genuine Lie algebra, the structure constants $C_{\alpha\beta}{}^{\gamma}$ being those of the Lie group G.

Since we are not interested in relativistic covariance, the Fradkin-Vilkovisky formalism reduces to the introduction of m ghost/antighost pairs $(\eta^{\alpha}, P_{\alpha})$, $\alpha = 1, \ldots, m$, obeying anticommutation relations $[\eta^{\alpha}, P_{\beta}] = 0$.

The situation can be depicted as follows:

$$\begin{tikzcd}
S_{\text{ext}}^{2n+2m} \arrow[r, leftrightarrow] \arrow[drr] & S^{2n} \arrow[r, leftrightarrow] & S_c^{2n-m} \arrow[r, leftrightarrow] & S_{\text{phys}} = S_c/G^{2n-2m}\,. \arrow[dl] \\
 & & \Omega? &
\end{tikzcd}$$

Just counting variables (as indicated by the superscripts) it seems that the ghost variables should in some sense count as "negative" degrees of freedom. To make this statement more meaningful, one needs to know more about the relationship between these spaces, in particular about the role played by the classical BRS operator,

$$\Omega = \eta^{\alpha} \Phi_{\alpha} - 1/2 C_{\alpha\beta}{}^{\gamma} \eta^{\alpha} \eta^{\beta} P_{\gamma}\,. \tag{1.2}$$

We will identify the ghosts and the extended phase space with well-defined mathematical objects within a finite-dimensional classical framework in order to get a coherent mathematical picture and a better understanding of the physical symmetry that is encoded into the use of BRS methods and in particular ghosts.

In our search for an interpretation of the classical BRS symmetry we make the following two observations concerning first class constrained systems:

a) the natural invariance group of the kinematics is no longer given by all symplectic transformations in S; consistency with the constrained nature of the system requires that they map the constraint surface into itself.

b) We have a new kinematical invariance of the system (as has repeatedly been emphasized, see, for example, [BRO/MCM], [MCM/PAT]) given by changes of the set of first class constraints $\{\Phi_{\alpha}\}$ to new ones $\{\Phi'_{\alpha}\}$, where

$$\Phi'_{\alpha} = \Lambda_{\alpha}{}^{\beta} \Phi_{\beta} = (\delta_{\alpha}{}^{\beta} + \varepsilon_{\alpha}{}^{\beta}) \Phi_{\beta}\,, \tag{1.3}$$

where Λ is any non-singular $(m \times m)$-matrix and the ε's are m^2 infinitesimal parameters.

Obviously these new constraints describe the same constraint surface, but this symmetry is accounted for within the Dirac-Bergmann algorithm only if it can be realized as a canonical transformation on S.

However, in the BRS approach one finds that even if there is no canonical transformation that produces this change of constraints, there is still a "super-canonical" transformation on the extended phase space yielding the desired result. It is given by a super Poisson bracket with the generator $C = 1/2 \eta^{\alpha} \varepsilon_{\alpha}{}^{\beta} P_{\beta}$.

From a) we might expect that a purely symplectic framework is no longer appropriate when dealing with constrained systems, b) suggests a closer investigation of what kind of supersymmetry we encounter in the BRS approach.

II. Classical BRS: Algebraic Description

Some progress has been made over the years in understanding BRS in a geometric setting, mainly at a field theoretic level in the context of Yang-Mills theory where of course one has a lot of additional complications due to the infinite dimensionality of the phase space and gauge group. Also one has to keep in mind that Yang-Mills theory is special in as far as its phase space is of the form of a cotangent bundle T^*Q of a configuration space Q.

To illustrate how the ghosts come into play we will now go back to our finite-dimensional models with Hamiltonian G-action. We will first take an algebraic point of view, in terms of the function spaces associated with the manifolds S, S_c, and S_c/G.

The classical observables of such a constrained system are the functions $C^\infty(S_c/G)$, i.e. the C^∞-functions on the reduced phase space. Since in general we do not have an explicit description of the reduced phase space, we are interested in a characterization of this function space in terms of $C^\infty(S)$ and the gauge group action. The main points of the following discussion are taken from [KOS/STE], to which we refer the reader for further details.

The functions $C^\infty(S_c)$ are given by equivalence classes of functions on S,

$$C^\infty(S_c) = C^\infty(S)/C^\infty(S)\cdot\{\Phi_\alpha\}\,, \tag{2.1}$$

where $C^\infty(S)\cdot\{\Phi_\alpha\}$ denotes the ideal of $C^\infty(S)$-functions vanishing on the constraint surface, which in our case (assuming the necessary regularity conditions) is generated by the m first class constraint functions $\Phi_a(q,p)$. This means that from the point of view of S_c one cannot distinguish between two functions $f(q,p)$ and $f(q,p)+g_\alpha(q,p)\Phi_\alpha(q,p)$, $f, g_\alpha\in C^\infty(S)$. To arrive at functions $C^\infty(S_c/G)$ we have to restrict $C^\infty(S_c)$ to the subset of functions which are constant along the gauge orbits on S_c. At the manifold level this description corresponds to the two-stage process of restricting to S_c and factoring out by the gauge group action.

One can rephrase these statements in a completely equivalent way, using a cohomological language. Whenever we have a linear representation of a finite-dimensional Lie group G and hence of its Lie algebra $L(G)$ on some vector space V, there is a way of describing elements of V which are invariant under this G-action as elements of the zeroth cohomology group of the corresponding Lie algebra cohomology with values in V.

The complex $\mathrm{Alt}(L(G), V)$ on which this cohomology is defined consists of the alternating multilinear functions on $L(G)$ with values in V, which is isomorphic to the space $\Lambda(L^*(G))\otimes V$, where $\Lambda(L^*(G))=\sum_p \Lambda^p(L^*(G))$ is the exterior algebra of the dual $L^*(G)$ of the Lie algebra $L(G)$.

All we need to know for our purposes is that we can construct from the G-representation on V a coboundary operator $d:\Lambda^p(L^*(G))\otimes V\to\Lambda^{p+1}(L^*(G))\otimes V$

in a natural way and that

$$H_d^0(\Lambda(L^*(G))\otimes V)\equiv\frac{\ker d:\Lambda^0(L^*(G))\otimes V\to\Lambda^1(L^*(G))\otimes V}{\operatorname{Im} d:\Lambda^{-1}(L^*(G))\otimes V\to\Lambda^0(L^*(G))\otimes V}$$
$$\equiv\ker d: V\to L^*(G)\otimes V$$
$$=\{G\text{-invariant elements of } V\}. \tag{2.2}$$

We are interested in the case $V=C^\infty(S)$ or $V=C^\infty(S_c)$, hence typical elements of the above complex are of the form $f(q,p)\eta^1\ldots\eta^p$, where the notation anticipates that the η's will be identified with the ghosts later on; the product between η's is of course the exterior product.

Observing that we have an isomorphism between the exterior algebra $\Lambda(L^*(G))$ and the space Ω^*_{inv} of left-invariant differential forms on G, one can also see how the interpretation of the ghosts η^α as Maurer-Cartan forms (left-invariant one-forms) on the gauge group G comes about. Similarly one finds that there is a description of the functions $C^\infty(S_c)$, in terms of the zeroth homology group of the so-called Koszul complex associated with the momentum map of the Hamiltonian G-action on S.

The relevant complex is $\Lambda(L(G))\otimes C^\infty(S)$, with boundary operator $\delta:\Lambda^p(L(G))\otimes C^\infty(S)\to\Lambda^{p-1}(L(G))\otimes C^\infty(S)$. One can show that

$$H_\delta^0(\Lambda(L(G))\otimes C^\infty(S))\equiv\frac{\ker\delta:\Lambda^0(L(G))\otimes C^\infty(S)\to\Lambda^{-1}(L(G))\otimes C^\infty(S)}{\operatorname{Im}\delta:\Lambda^1(L(G))\otimes C^\infty(S)\to\Lambda^0(L(G))\otimes C^\infty(S)}$$
$$=C^\infty(S)/C^\infty(S)\cdot\{\Phi_\alpha\}=C^\infty(S_c). \tag{2.3}$$

A typical element of $\Lambda(L(G))$, $p_1p_2\ldots p_p$, will be identified with a product of antighosts later on. One can now extend both d and δ to act on the complex $\Lambda(L^*(G))\otimes\Lambda(L(G))\otimes C^\infty(S)$ in such a way that

$$H_d^0(H_\delta^0(\Lambda(L^*(G))\otimes\Lambda(L(G))\otimes C^\infty(S)))=C^\infty(S_c/G). \tag{2.4}$$

Under certain regularity conditions $H_d^0(H_\delta^0(\cdot))$ can be identified with the zeroth cohomology group $H_D^0(\cdot)$ of the complex $\Lambda(L^*(G))\otimes\Lambda(L(G))\otimes C^\infty(S)$, with one single coboundary operator $D=d+(-1)^p2\delta$ ($D^2=0$), which we identify as (twice) the classical BRS operator.

Both $\Lambda(L^*(G))$ and $\Lambda(L(G))$ are commutative superalgebras, i.e. Z_2-graded vector spaces (we have both odd and even elements and the product is given by the exterior product). We can identify canonically $\Lambda(L^*(G))\otimes\Lambda(L(G))$ with the exterior algebra of the direct sum, $\Lambda(L^*(G)\oplus L(G))$, which is also a superalgebra.

Hence we have arrived at an extended (from $C^\infty(S)$) function space $\Lambda(L^*(G)\oplus L(G))\otimes C^\infty(S)$, but we still need a Lie superalgebra structure for the "ghost functions" to model the fermionic anticommutation relations on the extended phase space. Kostant and Sternberg achieve this by making $L^*(G)\oplus L(G)$ into a Lie algebra, taking the semi-direct product of the two, where $L^*(G)$ is now viewed as a commutative vector space algebra. They also introduce an invariant (with respect to this Lie bracket) scalar product which in terms of orthonormal bases $\{P_\alpha\}$ and $\{\eta^\alpha\}$ for $L(G)$ and $L^*(G)$ is of the form

$$(\eta^\alpha,\eta^\beta)=0,\ (P_\alpha,P_\beta)=0,\ (\eta^\alpha,P_\beta)=\eta^\alpha(P_\beta)=\delta^\alpha{}_\beta. \tag{2.5}$$

Since $L^*(G) \oplus L(G)$ is now a vector space with scalar product, there is a unique Clifford algebra $C(L^*(G) \oplus L(G))$ of dimension 2^{2m} associated with it. Via some filtering procedure the supercommutator of the Clifford algebra induces a super Poisson bracket on $\Lambda(L^*(G) \oplus L(G))$ (which however has nothing to do with the Lie algebra bracket defined on $L^*(G) \oplus L(G)$!).

This construction enables them to identify the part d of the BRS operator D explicitly as an element of $\Lambda^2 L^*(G) \otimes L(G)$, which acts on the extended function space by means of the super Poisson bracket. In addition they claim that the Clifford algebra $C(L^*(G) \oplus L(G))$ is the space of quantum observables associated with the classical space $\Lambda(L^*(G) \oplus L(G))$ of the ghost and antighost functions.

The super Poisson bracket of two elements of $\Lambda^1(L^*(G) \oplus L(G))$ is proportional to their scalar product, i.e. we have traced back the origin of the classical canonical relations $\{\eta^\alpha, P_\beta\} = k\delta^\alpha{}_\beta$ between ghosts and antighosts to the natural scalar product structure on $L^*(G) \oplus L(G)$ (here one has made use of the fact that the super Poisson bracket of two odd elements is, like the scalar product, a symmetric object). The choice of a particular value ($\neq 0$) for k doesn't play a role in the classical cohomological considerations. We adopt here $k=2$ (in accordance with [KOS/STE]) which will lead to the usual form for the BRS operator in $\Omega = \eta^\alpha \Phi_\alpha - 1/k C_{\alpha\beta}{}^\gamma \eta^\alpha \eta^\beta P_\gamma$.

This ends the algebraic discussion in terms of function spaces, where we have considered an extension of the function space $C^\infty(S)$ rather than of the phase space S itself.

We will present now a corresponding differential geometric construction which is desirable both from a physical point of view (since phase spaces are more appealing objects to work with than rings of functions) and from a mathematical one (since it may exhibit global properties of the system which are not readily amenable in a function space approach). Knowing about the algebra of ghost functions doesn't yet determine what kind of geometric objects the ghosts themselves are.

The complex $\Lambda(L^*(G) \oplus L(G)) \otimes C^\infty(S)$ will be interpreted as the function space $C^\infty(S_{\text{ext}})$ over an extension S_{ext} of the phase space S. We will construct explicitly the supermanifold S_{ext} and translate statements about cohomology of function spaces into differential geometric statements. In our global differential geometric picture there is a natural way to understand the origin of the super Poisson brackets on $\Lambda(L^*(G) \oplus L(G))$ other than as arising from the supercommutator of the associated Clifford algebra.

III. Classical BRS: Geometric Description

1. The Vertical Bundle

If we were just interested in the gauge invariance of the system and not in describing the restriction to the constraint surface, we would have both an algebraic and a differential geometric picture for what the physical observables are. The first is in terms of functions invariant along the G-orbits in S, but at the same time we can describe S as a principal fibre bundle $G \to S \to S/G$, and hence physical functions as elements of $C^\infty(S/G)$.

In the BRS construction we so far have only an algebraic picture, given in terms of function spaces and the operator for BRS cohomology, D. We haven't specified what kind of geometric object the "extended phase space" really is.

Obviously S_{ext} has to be some supermanifold since it contains both even and odd variables. There are different types of such supermanifolds; however, many of them are vector bundles over ordinary manifolds (see, for example, [BAT]). The reason for considering such a supermanifold in our case also is the following. Take S to be the base manifold of such a vector bundle, with a finite dimensional vector space V attached to it in each point,

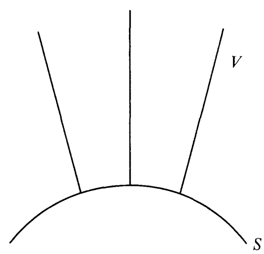

Fig. 1

If we expand out a function f on this supermanifold in local coordinates (s, v), we get an element of $\Lambda V \otimes C^{\infty}(S)$ (where ΛV is the exterior algebra of V); the fact that the fibres are fermionic makes the power series truncate in such a way that the function looks like an element of $\Lambda V \otimes C^{\infty}(S)$.

If the vector bundle is a trivial product bundle of S and V, functions can be written globally this way; in general however this expansion works only within individual coordinate patches.

This suggests that our S_{ext} is a vector bundle over S with fibres given by the vector spaces $L(G) \oplus L^*(G)$. Now the question arises whether this bundle is trivial or whether we also must allow for the possibility of having a twisted vector bundle over S.

This question is relevant when we think of quantizing the theory later on: suppose we know a quantization for both S and $L(G) \oplus L^*(G)$, then only if S_{ext} is a proper product bundle, $S \times (L(G) \oplus L^*(G))$, is its quantum theory the product of the two separate quantum theories, with Hilbert space given by the tensor product $H(S) \otimes H(L(G) \oplus L^*(G))$. We do not know how to relate the quantization of a twisted bundle to the separate quantizations of its base space and its fibre.

One particular vector bundle with fibres $L(G)$ which is naturally associated with the G-action on S is the vertical bundle VS over S. It is the subbundle of TS, the tangent bundle of S, which is spanned by the vector fields tangent to the G-orbits in S, in our case the Hamiltonian vector fields associated with the constraints.

Since the G-action on S is free, S fibres nicely into G-orbits, and the vertical subspaces of TS (in each point s of S) are naturally isomorphic to $L(G)$. Furthermore VS is always a trivial bundle; the vertical vector fields constitute a global basis for the "vertical part of TS."

Another way of describing VS is as the pullback of a vector bundle associated with the principal bundle $G \to S \to S/G$. We use the fact that G acts on $L(G)$ by the adjoint representation to construct the associated bundle $L(G) \to S \times_G L(G) \to S/G$ (as usual, elements of $S \times_G L(G)$ are given by equivalence classes $[s, v]$ of points (s, v) of $S \times L(G)$, where $[s_1, v_1] = [s_2, v_2]$ if $s_2 = s_1 g$ and $v_2 = \mathrm{Ad}(g^{-1}) v_1$ for some $g \in G$). Using the projection map $\pi: S \to S/G$ to pull this bundle back to S we rederive the vertical bundle.

2. Construction of S_{ext}

It is obvious now how one can construct the extended phase space, which is a bundle with $(L^*(G) \oplus L(G))$-fibres over S. We will define a G-action on these fibres and then proceed as before. The group G acts on $L^*(G)$ by the coadjoint action Ad^* which is defined in terms of the adjoint action Ad:

$$\langle \mathrm{Ad}^*(g)\eta, P \rangle := \langle \eta, \mathrm{Ad}(g) P \rangle, \tag{3.1}$$

where $\eta \in L^*(G)$, $P \in L(G)$, $g \in G$. Identifying the pairing $\langle\ ,\ \rangle$ between $L^*(G)$ and $L(G)$ with the scalar product $(\ ,\ )$ on $L^*(G) \oplus L(G)$ given earlier, (2.5), we see that it is invariant under the "combined" G-action AD on $L^*(G) \oplus L(G)$ which we define by

$$\mathrm{AD}(g)(\eta; P) := (\mathrm{Ad}^*(g^{-1})\eta; \mathrm{Ad}(g) P), \tag{3.2}$$

since then

$$\begin{aligned}(\mathrm{AD}(g)\eta, \mathrm{AD}(g)P) &= \langle \mathrm{Ad}^*(g^{-1})\eta, \mathrm{Ad}(g)P \rangle = \langle \eta, \mathrm{Ad}(g^{-1})\,\mathrm{Ad}(g)P \rangle \\ &= \langle \eta, P \rangle = (\eta, P). \end{aligned} \tag{3.3}$$

This way G acts on $L^*(G) \oplus L(G)$ as a subgroup of the $O(m, m)$-transformations which are just those transformations that leave invariant the metric structure

$$\begin{pmatrix} & \mathbb{1}_m \\ \mathbb{1}_m & \end{pmatrix} \tag{3.4}$$

on $(L^*(G) \oplus L(G))$ [in terms of a basis $(\eta^\alpha; P_\alpha)$; it would be of the more familiar diagonal form had we chosen a basis which mixes ghosts and antighosts].

In fact one can check that the infinitesimal generators H_α associated with this G-action can be written as linear combinations of $O(m, m)$-generators (whose form will be derived in Sect. 4 below):

$$H_\alpha(\cdot) = -1/2 C_{\alpha\beta}{}^{\gamma} \sum_{\beta,\gamma} \{\eta^\beta P_\gamma, \cdot\}, \tag{3.5}$$

where the dot stands for an element of $L^*(G) \oplus L(G)$ (and remember that $\{\eta^\alpha, P_\beta\} = 2\delta^\alpha{}_\beta$), and we have $\{H_\alpha, H_\beta\} = C_{\alpha\beta}{}^{\gamma} H_\gamma$. The AD-invariance of the orthogonal structure on $L^*(G) \oplus L(G)$ is nothing but the invariance of the scalar product on the Lie algebra $L^*(G) \oplus L(G)$. We can now use the G-action defined above to construct the associated bundle

$$L^*(G) \oplus L(G) \to S \times_G (L^*(G) \oplus L(G)) \to S/G. \tag{3.6}$$

Cross sections $\Gamma(S \times_G (L^*(G) \oplus L(G)))$ of this bundle are in one-to-one correspondence with functions $\Psi \in C^\infty(S, L^*(G) \oplus L(G))$ satisfying the equivariance condition $\Psi(sg) = \mathrm{AD}(g^{-1})\Psi(s)$, for all $g \in G$. Using the projection map $\pi: S \to S/G$

to pull back this associated bundle we arrive at a bundle over S:

$$L^*(G) \oplus L(G) \to \pi^*(S \times_G (L^*(G) \oplus L(G))) \to S. \tag{3.7}$$

Note that although the associated bundle over S/G in general is a non-trivial bundle, its pullback to S is always trivial.

Clearly the supermanifold $\pi^*(S \times_G (L^*(G) \oplus L(G)))$ we have constructed this way has the property we were looking for, i.e. functions over it can be written in the form of the complex we derived in the algebraic description, $C^\infty(S) \otimes \Lambda(L^*(G) \oplus L(G))$. Therefore it can serve as a good model for the extended phase space S_{ext}. Note furthermore that S_{ext} has the structure of a "hermitian" vector bundle [we have a smooth assignment $s \to h_s$ of an (indefinite) inner product to $(L^*(G) \oplus L(G))_s$].

3. $\mathrm{OSp}(m, m; 2n)$ *as Structure Group of* S_{ext}

To this phase space extension from S to S_{ext} there corresponds an extension of the structure group of the frame bundle. The existence of a symplectic structure, i.e. a closed, non-degenerate two-form ω on S allows us to choose the symplectic group $\mathrm{Sp}(2n)$ as structure group of the frame bundle of S. One can always choose a local basis for TS such that ω assumes the canonical form

$$\begin{pmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{pmatrix}. \tag{3.8}$$

The value $\omega(X, Y)$ at a point s in S is invariant if we perform any $\mathrm{Sp}(2n)$-transformation on tangent vectors $X, Y \in T_s S$.

The symplectic form can actually be derived from the Poisson bracket structure for functions $C^\infty(S)$. In a similar way we can construct a super two-form ω_{ext} on S_{ext} from the super Lie algebra structure of $C^\infty(S_{\text{ext}})$.

There is a well-defined analogue of tensors and in particular of differential forms, $\Omega(S_{\text{ext}})$, for the case of a graded manifold [KOS]. $\Omega^i_j(S_{\text{ext}})$ has the structure of a bigraded commutative algebra over $C^\infty(S_{\text{ext}})$, with the usual Z-grading (i) for differential forms and the Z_2-grading (j) coming from $C^\infty(S_{\text{ext}})$. There is a unique exterior derivative d of bidegree $(1, 0)$. Applying d on a function $f \in \Omega^0_j(S_{\text{ext}}) \subset C^\infty(S_{\text{ext}})$ we get $df \in \Omega^1_j(S_{\text{ext}})$ which satisfies $df(X) = Xf$, $X \in TS_{\text{ext}}$. Setting

$$T(df_s, dg_s) := \{f, g\}(s) \tag{3.9}$$

defines a two-tensor on S_{ext}; T induces an isomorphism $df \to X_f$ between exact one-forms in $\Omega^1(S_{\text{ext}})$ and elements of TS_{ext}, where we define $-X_f g := T(df, dg)$ (note that assuming homogeneity with respect to the Z_2-grading we have $\|X_f\| = \|f\| = \|df\|$). We have a corresponding non-degenerate two-form in $\Omega^2_0(S_{\text{ext}})$, defined by

$$\omega_{\text{ext}}(X_f, X_g) := \{f, g\}. \tag{3.10}$$

The fact that ω_{ext} is closed, $d\omega_{\text{ext}} = 0$, follows from the generalized Jacobi identity for functions $C^\infty(S_{\text{ext}})$ under super Poisson bracket. Hence ω_{ext} defines a graded symplectic structure on S_{ext}, with

$$\omega_{\text{ext}}(X_f, X_g) = -(-1)^{\|f\| \|g\|} \omega_{\text{ext}}(X_g, X_f). \tag{3.11}$$

Obviously ω_{ext} induces an antisymmetric, symplectic structure on the even directions ($\|X_f\|=0$) and a symmetric, "Riemannian" structure on the odd directions ($\|X_f\|=1$) of tangent space. In an appropriate local basis for T_sS_{ext} this two-form has the form

$$\left(\begin{array}{cc|cc} & \mathbb{1}_m & & \\ \mathbb{1}_m & & & \\ \hline & & & \mathbb{1}_n \\ & & -\mathbb{1}_n & \end{array}\right) \tag{3.12}$$

and has the orthosymplectic group $\mathrm{OSp}(m,m;2n)$ as its invariance group, i.e. we can perform $\mathrm{OSp}(m,m;2n)$-transformations on the tangent vectors in each $(2n+2m)$-dimensional tangent space T_sS_{ext} without changing this metric structure (note that we have to choose matching normalizations for the two metric structures on the odd and the even sector in order to allow also for transformations which mix ghost and bosonic directions in tangent space).

Hence we have excluded the possibility of global twists in the BRS construction, but at the same time have arrived at a supersymmetric situation with the orthosymplectic supergroup $\mathrm{Osp}(m,m;2n)$ acting as structure group.

4. BRS *Condition Singles Out Equivariant Functions*

Within our geometric picture there also emerges an interpretation for the BRS operator Ω which is used to project out physical functions from $C\,(S_{\text{ext}})$. Observe that we have a G-action on the 2^{2m}-dimensional vector space $\Lambda(L^*(G)\oplus L(G))$ with generators H_α which are of exactly the same form as the ones we defined above for the action of G on $L^*(G)\oplus L(G)$. They act again by the super Poisson bracket and generate orthogonal transformations in the sense that they leave invariant the indefinite, but non-degenerate scalar product that we can extend naturally from $L^*(G)\oplus L(G)$ to other "sectors" $\Lambda^p(L^*(G)\oplus L(G))$, for example on $\Lambda^2(L^*(G)\oplus L(G))$:

$$(\eta^\alpha\eta^\beta, P_\gamma P_\delta):=(\eta^\alpha, P_\gamma)(\eta^\beta, P_\delta)-(\eta^\alpha, P_\delta)(\eta^\beta, P_\gamma)\,. \tag{3.13}$$

Hence we can define an associated vector bundle in the same way as before, but now with this exterior algebra as fibre:

$$\Lambda(L^*(G)\oplus L(G))\to S\times_G\Lambda(L^*(G)\oplus L(G))\to S/G\,. \tag{3.14}$$

The equivariance condition for cross sections of this bundle is

$$\begin{aligned}\Gamma(S\times_G\Lambda(L^*(G)\oplus L(G))=\{&\Psi\in C^\infty(S,\Lambda(L^*(G)\oplus L(G)))\,\|\,\Psi(sg)\\ &=\mathrm{AD}(g^{-1})\Psi(s),\ \text{for all } g\in G\}\,,\end{aligned} \tag{3.15}$$

using the same notation for the G-action as previously. The left-hand side of the equivariance equation can be written as $\Psi(sg)=:[U(g)\Psi](s)$, where the infinitesimal generators T_α corresponding to this (right) G-representation are just given by the Poisson bracket with the constraints Φ_α.

Hence we can rewrite the equivariance condition in infinitesimal form:

$$\begin{aligned}
&[(1+T_\alpha)\Psi](s)=(1-H_\alpha)\Psi(s), \qquad \alpha=1,\ldots,m \\
&\Rightarrow \{\Phi_\alpha, \Psi(s)\}=1/2C_{\alpha\beta}{}^{\gamma}\{\eta^\beta P_\gamma, \Psi(s)\} \\
&\Rightarrow \{\Phi_\alpha-1/2C_{\alpha\beta}{}^{\gamma}\eta^\beta P_\gamma, \Psi(s)\}=0\,. \qquad (3.16)
\end{aligned}$$

Comparing this with the form of the BRS operator Ω we see that the usual BRS condition on physical functions

$$\Psi\in C^\infty(S)\otimes\Lambda(L^*(G)\oplus L(G))\simeq C^\infty(S,\Lambda(L^*(G)\oplus L(G)))$$

can be interpreted as an equivalence condition that singles out functions compatible with the projection $S\to S/G$ plus the statement that these functions shouldn't depend on the P_α's:

$$\begin{aligned}
\{\Omega,\Psi\}&=\{\Phi_\alpha\eta^\alpha-1/2C_{\alpha\beta}{}^{\gamma}\eta^\alpha\eta^\beta P_\gamma,\Psi\}=\eta^\alpha\{\Phi_\alpha-1/2C_{\alpha\beta}{}^{\gamma}\eta^\beta P_\gamma,\Psi\} \\
&\quad+(\Phi_\alpha-1/2C_{\alpha\beta}{}^{\gamma}\eta^\beta P_\gamma)\{\eta^\alpha,\Psi\}=0\,. \qquad (3.17)
\end{aligned}$$

5. *Algebraic Description of Physical Functions*

We only remark here that there is a completely algebraic characterization of the functions $C^\infty(S\times_G(L^*(G)\oplus L(G)))$ on the associated vector bundle, in terms of the function rings $C^\infty(S)$ and $C^\infty(L^*(G)\oplus L(G))$. It is given by the tensor product $C^\infty(S)\otimes_{R[G]}C^\infty(L^*(G)\oplus L(G))$, where the tensor product however is taken over the group algebra $R[G]$ of G, and not over the complex numbers. We define the group algebra $R[G]$ to be the tensor product $Z[G]\otimes_Z R$, where the group ring $Z[G]$ consists of all formal linear combinations $\tilde{g}=\sum_i n_i g_i$ of elements of G with integer coefficients (for more details see [KIR]).

$C^\infty(S)$ ($C^\infty(L^*(G)\oplus L(G))$ has the structure of a right (left) G-space, induced from the G-action on S ($L^*(G)\oplus L(G)$). Since these G-representations on the function spaces are linear they can naturally be extended to representations of the group ring. The following identities characterize elements $\psi_S\otimes\psi_L$ of $C^\infty(S)\otimes_{R[G]}C^\infty(L^*(G)\oplus L(G))$:

$$\begin{aligned}
&\text{a)}\ \psi_S\tilde{g}\otimes\psi_L=\psi_S\otimes\tilde{g}\psi_L\,, \\
&\text{b)}\ (\psi_S+\chi_S)\otimes\psi_L=\psi_S\otimes\psi_L+\chi_S\otimes\psi_L\,, \\
&\text{c)}\ \psi_S\otimes(\psi_L+\chi_L)=\psi_S\otimes\psi_L+\psi_S\otimes\chi_L\,. \qquad (3.18)
\end{aligned}$$

Here a) is equivalent to the old equivariance condition, whereas b) and c) license us to use a decomposition of our physical functions according to a different ghost number as:

$$\begin{aligned}
C^\infty(S)\otimes_{R[G]}C^\infty(L^*(G)\oplus L(G))&=C^\infty(S)\otimes_{R[G]}\sum_i\Lambda^i(L^*(G)\oplus L(G)) \\
&=\sum_i C^\infty(S)\otimes_{R[G]}\Lambda^i(L^*(G)\oplus L(G))\,. \qquad (3.19)
\end{aligned}$$

In going to the group OSp we of course have extrapolated even further away from the physical degrees of freedom. Eventually we will have to restrict the OSp-group to the much smaller invariance group of the kinematical structure of the space of physical degrees of freedom. How this is achieved with the help of the BRS operator will be explained in the following section.

IV. Action of the BRS Operator Ω

1. A Simple Example

To simplify the analysis we will assume from now on that the phase space S is given by the flat Euclidean space R^{2n}, hence we have global coordinates $(\eta^\alpha, P_\alpha, q^i, p_i)$ for the extended phase space S_{ext}, $\alpha=1,\dots,m$, $i=1,\dots,n$, which will be denoted collectively by (z^A), $A=1,\dots,2m+2n$. The orthosymplectic group $\mathrm{OSp}(m,m;2n)$ acts globally, leaving the metric (3.12) invariant.

Recall that the function space $\Lambda(L^*(G)\oplus L(G))\otimes C^\infty(R^{2n})$ over S_{ext} has the structure of a super Poisson algebra, being the tensor product of the infinite-dimensional symplectic algebra of functions $C^\infty(R^{2n})$ and the finite-dimensional super Poisson algebra $\Lambda(L^*(G)\oplus L(G))$, which (modulo the constants) is isomorphic to the $(2^{2m}-1)$-dimensional simple Cartan-type Lie superalgebra in Kac's classification.

The new and interesting observation is that we have identified the well-known superalgebra $\mathrm{osp}(m,m;2n)$ as a finite-dimensional subalgebra of the super Poisson algebra $\Lambda(L^*(G)\oplus L(G))\otimes C^\infty(R^{2n})$ with respect to this super Poisson bracket, which (modulo constant factors) is given by the quadratic homogeneous polynomials in the basic variables z^A. The orthosymplectic algebra $\mathrm{osp}(m,m;2n)$ is a simple finite-dimensional Lie superalgebra. We define its generators to be those real $(2m+2n)\times(2m+2n)$-matrices M which satisfy $M^TX+XM=0$, where X is the metric form (3.12).

Note that (due to the "superstructure") the transpose of a matrix $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ has the unusual form $M^T=\begin{pmatrix} A^T & -C^T \\ B^T & D^T\end{pmatrix}$. The algebra $\mathrm{osp}(m,m;2n)$ has $2n^2+n+2m^2-m$ bosonic and $4mn$ fermionic generators, i.e. the composition law for two matrices M_1, M_2 is the usual commutator, unless both M_1 and M_2 are fermionic in which case it is the anticommutator, $M_1M_2+M_2M_1$.

The quadratic polynomials z^Az^B act on functions $C^\infty(S_{\text{ext}})$ via the super Poisson bracket $\{z^Az^B,\cdot\}$, the $GL(2m+2n,R)$-matrix $M(z^Az^B)$ – acting on the $(2m+2n)$-dimensional vector space S_{ext} – corresponding to such an infinitesimal generator can easily be computed starting from the canonical commutation and anti-commutation relations for the basic variables:

$$\{q^i,p_j\}=\delta^i{}_j=-\{p_j,q^i\},\ \{\eta^\alpha,P_\beta\}=2\delta^\alpha{}_\beta=\{P_\beta,\eta^\alpha\} \tag{4.1}$$

(and all other super Poisson brackets vanishing). The bosonic generators of $\mathrm{osp}(m,m;2n)$ correspond to $O(m,m)$- or $\mathrm{Sp}(2n)$-transformations, whereas the fermionic generators mix the two sectors.

2. How Ω Breaks the Kinematical Invariance of S_{ext}

To analyze further the action of the BRS operator, we turn to the easiest case possible where the constraints $\Phi_\alpha=0$ are the first m momenta p_j, $j=1,\dots,m$. In this

Table 1

Generator G	I Number	II Fulfilling $\{\Omega, G\}=0$	III Not of form $G=\{\Omega, G'\}$
$1/2\eta^\alpha\eta^\beta$	$m(m-1)/2$	$m(m-1)/2$	0
$1/2P_\alpha P_\beta$	$m(m-1)/2$	0	0
$1/2\eta^\alpha P_\beta$	m^2	0	0
$q^a p_b$	n^2	$n(n-m)$	$(n-m)^2$
$q^a q^b$	$n(n+1)/2$	$(n-m)(n-m+1)/2$	$(n-m)(n-m+1)/2$
$p_a p_b$	$n(n+1)/2$	$n(n+1)/2$	$(n-m)(n-m+1)/2$
$1/\sqrt{2}q^a\eta^\alpha$	mn	$m(n-m)$	0
$1/\sqrt{2}p_a\eta^\alpha$	mn	mn	0
$1/\sqrt{2}p_a P_\alpha$	mn	0	0
$1/\sqrt{2}q^a P_\alpha$	mn	0	0

case the gauge group is commutative and the BRS operator is of the form $\Omega = \sum_\alpha p_\alpha \eta^\alpha$. Note that Ω itself now has the form of a homogeneous quadratic polynomial, also one can easily check that $M(\Omega)^2 = 0$.

We know that a function F on S_{ext} can only be a classical observable if it satisfies $\{\Omega, F\} = 0$, and in addition that two such observables are physically equivalent if they differ by a function of form $F' = \{\Omega, F\}$.

It is instructive to write down explicitly what this means if we restrict our attention to the quadratic polynomials (Table 1), which can be viewed both as phase space functions and as OSp-generators. Column I of Table 1 gives the number of all generators of the type indicated, column II the number of "physical" ones, and to arrive at column III we have subtracted the number of generators which generate transformations within physical equivalence classes. The factors $1/2$ and $1/\sqrt{2}$ in front of the generators are a consequence of the choice $k=2$ in $\{\eta^\alpha, P_\beta\} = k\delta^\alpha{}_\beta$.

If we restrict ourselves to the Sp(2n)-generators, it is easy to show that the condition $\{\Omega, G\} = 0$ for physical G eliminates all symplectic transformations whose associated vector fields have components along the directions $\partial/\partial p_\alpha$ (i.e. are perpendicular to S_c). In going to column III we factor out by those generators whose associated vector fields have non-vanishing components along the $\partial/\partial q^\alpha$ (these are the directions along gauge orbits).

Hence we end up with functions on the reduced phase space S_c/G, which for this simple example we can identify explicitly with the (2n–2m)-dimensional space of the (q^i, p_i), $i = m+1, \ldots, n$. Equivalently we can characterize them as the generators

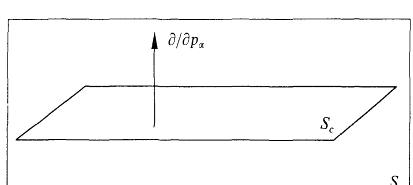

Fig. 2

of $\mathrm{Sp}(2n-2m)$ which is the natural invariance group for the kinematics of the physical subspace.

Let us turn now to the bosonic generators containing ghosts [they generate $O(m,m)$-transformations]. Transformations generated by polynomials of the form $1/2\eta^i\eta^j$ and $1/2P_iP_j$ mix up ghosts and antighosts, whereas transformations generated by the $1/2\eta^iP_j$ do not. The crucial observation is that the m^2 polynomials η^iP_j generate the changes of first class constraints, the supercanonical transformation corresponding to (1.3) is given by the super Poisson bracket with the linear combination $1/2\varepsilon_i{}^j\eta^iP_j$ of these quadratic polynomials.

This can be understood by looking at the form of the BRS operator, $\Omega=p_\alpha\eta^\alpha$, which is the key for the description of the constraint surface. Redefining the constraints amounts here to a linear transformation among the m momenta p_α. This may not in every case be realizable as a canonical transformation on S, but is always equivalent to performing a pseudo-symplectic (=orthogonal) transformation on the η's and P's. The polynomials $1/2\eta^iP_j$ are in one-to-one correspondence with $GL(m,R)$ generators since their corresponding matrices (with respect to a basis $(\eta^\alpha; P_\alpha)$ of the vector space $L^*(G)\oplus L(G)$) are of the form

$$\begin{pmatrix} G & 0 \\ 0 & -G^T \end{pmatrix}, \tag{4.2}$$

where G runs through all m^2 generators of $GL(m,R)$.

Hence we see that in the BRS description this symmetry of the dynamics is explicitly built into the theory; it can't be incorporated if we stick to a strictly symplectic language. The η^iP_j are unphysical functions and as such the condition $\{\Omega,\cdot\}=0$ not only breaks the $\mathrm{Sp}(2n)$-symmetry, but also fixes a choice of first class constraints.

Note furthermore that for our example the final set of functions (column III) doesn't contain any ghosts and hence we do not need a condition "ghost number = 0." The polynomials in columns I, II, and III form a series of subalgebras

$$\mathrm{osp}(m,m;2n)\supset\text{“}\mathrm{osp}(m;2n-m)\text{”}\supset\mathrm{sp}(2n-2m), \tag{4.3}$$

where we have defined "$\mathrm{osp}(m;2n-m)$" to be the subalgebra of polynomials in column II. Of course $(2n-m)$ can be an odd number and there is no way of defining this algebra by a matrix condition as in the case of $\mathrm{osp}(m,m;2n)$ or $\mathrm{sp}(2n)$. [However, if we use the formula for the number of bosonic and fermionic generators for the osp-algebra, we get the correct numbers for "$\mathrm{osp}(m;2n-m)$".]

This analysis shows explicitly how the breaking of the kinematical invariance occurs in two steps, by applying the BRS operator Ω, which itself is a generator of fermionic orthosymplectic transformations. Another interesting point is that we do not get a cross product $\mathrm{o}(m,m)\times\mathrm{sp}(2n)$ as an intermediate step in this symmetry breaking (not all fermionic generators are eliminated by requiring BRS invariance, $\{\Omega,\cdot\}=0$).

3. Extension to General Elements of $C^\infty(S_{\mathrm{ext}})$

Clearly the quadratic polynomials as generators of the OSp-action play a central role in the set of all functions on the extended phase space, but in general one would like to extend the preceding discussion to other elements of $C^\infty(S_{\mathrm{ext}})$. We will be interested for example in elements of higher order in the ghosts (the BRS operator for a non-abelian group action contains a term cubic in ghosts) and in expressions containing general functions $C^\infty(S)$ [one needs phase space dependent generators $1/2\varepsilon_i^{\,j}(q,p)\eta^i P_j$ if one wants to allow for local redefinitions of the first class constraints]. Table 1 can readily be extended to general elements F of $C^\infty(S_{\mathrm{ext}})$:

a) if $\{\Omega, F\}=0$ then F has the form $F=f(q,p)\eta^{i_1}\dots\eta^{i_j}$, where f is a G-invariant function of $C^\infty(S)$, $\{\Phi_\alpha, f\}=0$;

b) from this subset the following can be written as $F=\{\Omega, F'\}$ for some $F'\in C^\infty(S_{\mathrm{ext}})$:

$$F=p_\alpha g(q,p)\eta^{i_1}\dots\eta^{i_j},$$

where g is G-invariant or

$$F=\partial/\partial q^\alpha g(q,p)\eta^\alpha\eta^{i_1}\dots\eta^{i_j},$$

where

$$\partial g/\partial q^\alpha \neq 0,\ \partial^2 g/\partial q^\alpha\partial q^\beta=0\,.$$

(Note the prominent role played by the η's, the P's have completely disappeared from our description.)

However, we cannot interpret the super Poisson bracket by an arbitrary $C^\infty(S_{\mathrm{ext}})$-function as an orthosymplectic transformation on S_{ext}. In order to understand this, compare the present situation with the case of a symplectic $2n$-dimensional manifold S. Here the algebra of inhomogeneous quadratic polynomials is a maximal subalgebra of the space of all polynomials under the Poisson bracket (a fact that would be expected to be of importance in any quantization), i.e. if we add a single cubic polynomial to this subalgebra, we generate the whole algebra of polynomials.

Nevertheless any function can be regarded as a generator of symplectic transformations on S, since we can use the symplectic form on S to associate a Hamiltonian vector field with any element of $C^\infty(S)$.

Similarly, the inhomogeneous quadratic polynomials on $L^*(G)\oplus L(G)$ form a maximal subalgebra of $\Lambda(L^*(G)\oplus L(G))$, but because of the superalgebra character of $\Lambda(L^*(G)\oplus L(G))$ (this is not simply an orthogonal algebra!) it is not clear how to give a similar geometric meaning to the action by the super Poisson bracket of a general element of $C^\infty(S_{\mathrm{ext}})$, in general this will definitely not be an orthosymplectic transformation.

4. *Geometric Interpretation of the Action of Ω*

We have seen how Ω operates on functions, but we would like to give its action a more explicit geometric meaning. Suppose we start from a $(2n+2m)$-dimensional extended flat phase space S_{ext} of the form described above, i.e. one which is the trivial product of R^{2n} with a $2m$-dimensional Grassmannian vector space V. We now observe that the choice of an (abelian) BRS operator corresponds to selecting a $2n$-dimensional sub-supermanifold $S_{\mathrm{ext},c}$ of S_{ext}: take for example $\Omega=p_\alpha\eta^\alpha$, then $S_{\mathrm{ext},c}$ defined by $p_\alpha=0$, $\eta^\alpha=0$, $\alpha=1,\dots,m$, is associated with it as a constraint manifold in the sense that the condition $\{\Omega, G\}=0$ eliminates all orthosymplectic generators whose associated vector fields have components along the directions $\partial/\partial\eta^\alpha$, $\partial/\partial p_\alpha$ perpendicular to $S_{\mathrm{ext},c}$:

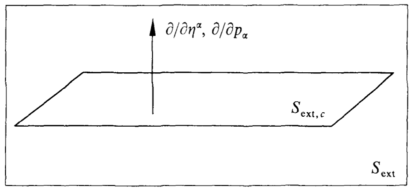

Fig. 3

All other generators (forming an "osp$(m; 2n-m)$"-algebra) generate transformations of $S_{\mathrm{ext},c}$ into itself and hence correspond to observables (modulo gauge transformations) of the constrained system $(R^{2n}, \{p_\alpha\})$.

Another observation is that ghosts and antighosts enter the mathematical description of S_{ext} completely symmetrically. The situation becomes asymmetric only when we start operating with the BRS generator $\Omega=\eta^\alpha p_\alpha$. In fact we could have chosen as the BRS generator for the constrained system $(R^{2n}, \{p_\alpha\})$ any function of the form $\Omega=p_\alpha\chi_\alpha$, where $\{\chi_\alpha\}$, $\alpha=1,\ldots,m$, denotes a subset of m mutually anticommuting elements of $\{\eta^\alpha, P_\alpha\}$, $\{\chi_\alpha, \chi_\beta\}=0$, e.g. something like an anti-BRS operator $\Omega'=p_\alpha P_\alpha$.

These different BRS operators correspond to different choices of submanifold $S_{\mathrm{ext},c}$ in S_{ext}, but they lead to exactly the same results for the physical subspace, only the intermediate step ("osp$(m; 2n-m)$") looks different.

Hence we have a whole family F of BRS charges that can be used for the description of a particular constrained system. However, elements of F do not in general commute, and to get a closing algebra we have to add bosonic generators (in our case the m generators $p_\alpha{}^2$, $\alpha=1,\ldots,m$).

Of course there are also choices for Ω which correspond to physically different situations, for example $\Omega=q^\alpha\eta^\alpha$ describes a constrained system $(R^{2n}, \{q^\alpha\})$, where $S_{\mathrm{ext},c}$ is defined by $q^\alpha=0$, $\eta^\alpha=0$. The transformations mediating between different choices of $S_{\mathrm{ext},c}$, and hence of BRS generators, are given by bosonic orthosymplectic transformations. However, none of these $O(m,m)$-transformations (which include the ones generating changes of the first class constraints) changes physical results, although they may not map $S_{\mathrm{ext},c}$ into itself. Only Sp$(2n)$-transformations perpendicular to $S_{\mathrm{ext},c}$ can lead to different physical situations.

The picture we find here is really that of an extended constrained system $(R^{2n} \times V^{2m}, \{p_\alpha; \chi_\alpha\})$, where we have replaced the original phase space R^{2n} by the cartesian product $R^{2n}\times V^{2m}$ with a $2m$-dimensional (Grassmannian) vector space V, and at the same time doubled the number of first-class constraints by adding m (Grassmannian) constraints $\chi_\alpha=0$.

V. Conclusions

We have learned that when dealing with a first class constrained system $(S, \{\Phi_\alpha\})$, it is neither profitable to think of it in purely symplectic terms (emphasis on Sp$(2n)$), nor to look just at its principal bundle structure $G\to S\to S/G$ (emphasis on G). The

BRS description combines these two structures in an elegant way by introducing a set of odd ghost variables.

There is a well-known way of describing the classical observables of such a theory – usually characterized as being the weakly invariant functions on S – in a cohomological language, as zeroth cohomology group H_D^0 of the BRS complex.

We have shown that in a global geometric setting one could view them as a particular subalgebra of the functions $C^\infty(S_{\text{ext}})$, where S_{ext} is a supermanifold, a vector bundle over S with fibres $L^*(G)\oplus L(G)$. The super Poisson bracket for odd functions from $C^\infty(S_{\text{ext}})$ has its origin in the duality between $L(G)$ and $L^*(G)$. Since the bundle S_{ext} has no global twists, the algebraic description in terms of rings of functions is valid globally; however, the idea of considering the orthosymplectic group as a natural invariance group suggested itself when we were looking at the description of the system in terms of fibre bundles. In turn we were able to identify $\operatorname{osp}(m,m;2n)$ as a subalgebra of the super Poisson algebra $C^\infty(S_{\text{ext}})$ under the super Poisson bracket.

The final picture is as follows: operating with a single BRS operator is mathematically equivalent to starting with a system that is invariant under the action of the orthosymplectic group. We have seen explicitly how the reduction of the extended $(2n+2m)$-dimensional phase space S_{ext} to the physical phase space of dimension $(2n-2m)$ takes place, and hence in what sense the ghosts constitute negative degrees of freedom (unfortunately it is not as easy as putting a minus instead of a plus sign somewhere in the equations!). Attaching two types of fibres, $L^*(G)$ and $L(G)$, to S accounts for the symmetry of changing the first class constraints and staying at the same time within a formalism that still looks symplectic.

This introduces additional unphysical degrees of freedom into the theory, which is reflected in the fact that also the new kinematical invariance group, $\operatorname{OSp}(m,m;2n)$, is bigger than before, in fact, going back to the table of quadratic polynomials, all terms that do not appear in the last column correspond either to unphysical or to gauge symmetries of the system. We can choose a BRS operator Ω from a whole family of physically equivalent BRS charges to reduce the system back to the physical degrees of freedom, where different choices of Ω correspond to different choices of constraint supermanifolds $S_{\text{ext},c}\subset S_{\text{ext}}$.

For the general, non-abelian case, $\{\Omega,F\}=0$ can be interpreted as an equivariance condition for cross sections of an associated bundle, and here also orthogonal transformations among the ghosts and antighosts will obviously lead to physically equivalent theories.

For our simple example $S_{\text{ext}}=R^{2n}\times(L^*(G)\oplus L(G))$, of course the ghosts decouple completely, and we could do the whole analysis without ever mentioning odd variables, just substituting the conditions

a) $\{\Omega,\cdot\}=0$ by $\{p_\alpha,\cdot\}=0$ (Poisson brackets on S), $\alpha=1,\dots,m$, and
b) $f=\{\Omega,f'\}$ by $f=p_\alpha\times$(function on S),

and hence recover Dirac's description in terms of weakly invariant functions.

The advantage of the BRS description here is that we have built the symmetry of changing constraints into the theory and thus have a richer symmetry structure.

This ends the analysis of our model with a flat phase space S and abelian constraints p_α. Although this is a special example of a constrained system, we expect that a similar picture will emerge for more complicated cases. Usually one works in a local framework anyway, that is a sort of tangent space approximation (effectively ignoring the fact that the phase space may be non-flat), where one can make use of the possibility of transforming the first class constraints locally into pure momenta (hence our results would apply). For the general issue of abelianization of gauge theory the reader is referred to [VIL].

Of course major problems remain to be solved, the most interesting being the extension of the discussion to a non-abelian group G (the case for which the method was originally invented), where the usefulness of the BRS formalism should become apparent. Finite-dimensional models for this case typically have non-flat phase spaces, and as a consequence functions describing the constraint surface are only defined locally. What is the significance of the orthosymplectic group for this more general case? It will be interesting to see whether the BRS approach allows us to describe a larger class of constrained systems, including cases which cannot be tackled by standard methods. This will also have consequences for the quantization of such systems.

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References

[BAT] Batchelor, M.: Graded manifolds and supermanifolds. In: Mathematical aspects of superspace. Seifert, H.-J. et al. (eds.). NATO ASI, Series C, Vol. 132. Dordrecht: Reidel 1984, p. 91

[BRO/MCM] Browning, A.D., McMullan, D.: The Batalin, Fradkin, and Vilkovisky formalism for higher-order theories. J. Math. Phys. **28** (2), 438 (1987)

[B/R/S] Becchi, C., Rouet, A., Stora, R.: The abelian Higgs Kibble model, unitarity of the S-operator. Phys. Lett. **52** B, 344 (1974)

[FAD/POP] Faddeev, L.D., Popov, V.N.: Feynman diagrams for the Yang-Mills field. Phys. Lett. **25** B, 29 (1967)

[FRA/VIL] Fradkin, E.S., Vilkovisky, G.A.: Quantization of relativistic systems with constraints, equivalence of canonical and covariant formalisms in quantum theory of gravitational field. CERN-preprint Ref. TH. 2332 (1977)

[HEN] Henneaux, M.: Hamiltonian form of the path integral for theories with a gauge freedom. Phys. Rep. **126**, No. 1, 1 (1985)

[KIR] Kirillov, A.A.: Elements of the theory of representations. Berlin, Heidelberg, New York: Springer 1976

[KOS] Kostant, B.: Graded manifolds, graded Lie theory, and prequantization. Lecture Notes in Mathematics, Vol. 570, pp. 177. Berlin, Heidelberg, New York: Springer 1977

[KOS/STE] Kostant, B., Sternberg, S.: Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Ann. Phys. (NY) **176**, 49 (1987)

[KUG/OJI] Kugo, T., Ojima, I.: Local covariant operator formalism of non-abelian gauge theories and quark confinement problem. [Suppl.] Prog. Theor. Phys. **66**, 1 (1979)

[MCM] McMullan, D.: Yang-Mills theory and the Batalin-Fradkin-Vilkovisky formalism. J. Math. Phys. **28** (2), 428 (1987)

[MCM/PAT] McMullan, D., Paterson, J.: Factor ordering and ghost variables. University of Glasgow preprint GUTPA/87/12-1 (1987)

[SUN] Sundermeyer, K.: Constrained dynamics. Lecture Notes in Physics, Vol. 169. Berlin, Heidelberg, New York: Springer 1982

[VIL] Vilkovisky, G.: The Gospel according to DeWitt. In: Quantum theory of gravity. Christensen, S.M. (ed.). Bristol: Hilger 1984

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