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# Stochastic Schrödinger Operators and Jacobi Matrices on the Strip

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**Abstract.** We discuss stochastic Schrödinger operators and Jacobi matrices with wave functions, taking values in $\mathbb{C}^l$ so there are $2l$ Lyaponov exponents $\gamma_1 \geqq \cdots \geqq \gamma_l \geqq 0 \geqq \gamma_{l+1} \geqq \cdots \geqq \gamma_{2l} = -\gamma_1$. Our results include the fact that if $\gamma_1 = 0$ on a set positive measure, then $V$ is deterministic and one that says that $\{E|\text{exactly } 2j\ \gamma\text{'s are zero}\}$ is the essential support of the a.c. spectrum of multiplicity $2j$.

## 1. Introduction

This paper discusses stochastic Schrödinger operators (see [4, 20, 7] for background) on $\mathbb{R}$, that is

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega(x) \tag{1.1}$$

on $L^2(\mathbb{R}, dx)$, and its discrete analog:

$$(h_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n) \tag{1.2}$$

on $l^2(\mathbb{Z})$, where $V_\omega$ is a stochastic process. Several years ago, one of us ($SK$) [10] developed a set of ideas relating $m$-functions, the Lyaponov exponent and absolutely continuous spectrum for (1.1), and subsequently, the other of us ($B.S.$) [19] extended the ideas of [10] to equations of the form (1.2). Among the results were ($\gamma$ = Lyaponov exponent):

($a_0$) $\{E|\gamma(E)=0\} \equiv A$ is the essential support of $d\mu_{\mathrm{ac}}^\omega$.

($b_0$) If $A$ containts an open interval, $I$, then $\sigma(H_\omega)\restriction I$ is purely absolutely continuous

($c_0$) If $|A|>0$ ($|\cdot|$ = Lebesgue measure), then $V_\omega$ is deterministic.

Our goal here is to discuss these results for operators on strips. The basic operator (1.2) on a strip is defined by considering a connected set $S \subset \mathbb{Z}^{\nu-1}$

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(connected means under the notion of joining nearest neighbors). One then considers on $l^2(\mathbb{Z} \times S)$:

$$(h_\omega u)(\alpha) = \sum_{\substack{|\beta-\alpha|=1 \\ \beta\in\mathbb{Z}\times S}} u(\beta) + \tilde{V}_\omega(\alpha)u(\alpha) \tag{1.3}$$

with $\alpha\in\mathbb{Z} \times S \subset \mathbb{Z}^\nu$. Here $\tilde{V}_\omega$ is a process ergodic under the one-dimensional group of translations.

We will actually consider an extension of this class which is more natural for the methods we will use, but as we will explain, there is a price paid for generality. Explicitly, we still study operators of the form (1.1) or (1.2), but now the operators act on $L^2(\mathbb{R};\mathbb{C}^l)$ and $l^2(\mathbb{Z};\mathbb{C}^l)$, i.e., vector functions $u(x)$ (respectively $u(n)$) taking values in $\mathbb{C}^l$. $V_\omega(x)$ (respectively $V_\omega(n)$) is now an $l \times l$ matrix and $V_\omega u$ means applying the matrix to the vector. We suppose that $V$ is real and symmetric. For earlier studies of stochastic Jacobi matrices on the strip, see [22, 23].

The operator (1.3) is of this form, where $\#(S)=l$ and $V$ has $\tilde{V}_\omega(n,\alpha_\perp)$ as diagonal elements and non-random off-diagonal elements (each 0 or 1).

The generalized Schrödinger operators have a $2l \times 2l$ transfer matrix rather than the more usual $2 \times 2$ matrix, and so $2l$ Lyaponov exponents which we label

$$\gamma_1 \geqq \cdots \geqq \gamma_l \geqq \gamma_{l+1} \geqq \cdots \geqq \gamma_{2l}.$$

Because of the constancy of the Wronskian, $\gamma_{2l+1-j} = -\gamma_j$, so $\gamma_l \geqq 0 \geqq \gamma_{l+1}$. We will prove:

(a) $S_j \equiv \{E|\text{exactly } 2j\ \gamma\text{'s are } 0\}$ is the essential support of $dE_{\text{ac}}^{\omega;2j}$, the a.c. spectrum of multiplicity exactly $2j$. There is no a.c. spectrum of odd multiplicity.

(b) If $S_l$ contains an open interval $I$, the spectrum is purely a.c. on $I$.

(c) If $|S_l|>0$ (i.e., if there is a set of measure zero on which $\gamma_1$ vanishes), then $V$ is deterministic.

In some ways, (c) is unsatisfactory. It asserts that if $V$ is non-deterministic, then $\gamma_1>0$, but it does not assert anything about $\gamma_2,\dots,\gamma_l$. One cannot hope to do better because our hypotheses include

$$V_\omega(n) = \begin{pmatrix} V_{\omega,1}(n) & & 0 \\ & \ddots & \\ 0 & & V_{\omega,l}(n) \end{pmatrix}$$

for which the problem decouples into $l$ separate problems. If $l-1$ $V$'s are constant but the $l$th is non-deterministic, then $V$ is non-deterministic, but for suitable $E, \gamma_2 = \cdots = \gamma_l = 0$. Clearly, to go beyond (c), one needs some kind of hypothesis that all components of $V$ are coupled together. An indication that under such a hypothesis non-determinancy implies $\gamma_1,\dots,\gamma_l>0$ is the fact that, in the case where the $V_\omega$'s are iid's with a suitable coupling hypothesis (e.g., (1.3) with the $V(\alpha)$ iid, see [8]), Furstenberg's theorem implies that all $\gamma$'s are non-zero.

In Sect. 2, we introduce the strip analogs of the Jost solutions and the Weyl $m$-functions, and in Sect. 3, we describe the spectral measures. Section 4 discusses the inverse problem for the strip. In these sections, which mirror well-known theory for the usual $l=1$ case, we present only one of the continuum and discrete cases in detail, and the other one briefly.

Section 5 extends the Pastur–Ishii [13, 9] theorem to the strip. Section 6 presents the basic bounds on the $m$-function, and Sect. 7 proves a, b, c. The three sections are the main ones of this paper. In an appendix, we prove the Thouless formula for the strip case of (1.2).

One might reasonably ask why it took five years from [10, 19] to extend the proofs of these results for the strip. Each of us has worked on this problem during that period, so it appears that the problem must be difficult. In a real sense, the difficulty is that the $m$-function becomes a non-commutative $m$-matrix. Thus, simple results like (in the discrete case)

$$m(T^{n-1}\omega)\cdots m(\omega)=\exp\left(\left[\sum_{j=0}^{n-1}\ln m(T^j\omega)\right]\right)$$
$$\sim\exp(n\int\ln m(\omega)d\mu(\omega))$$

fail. Of course, the result is true for $\det m(\omega)$, and in some sense, we have found the right combinations of det and Tr to push the theory through. In retrospect, with these appropriate functions, the theory is not so difficult.

To avoid unessential technical difficulties, we suppose throughout that our potentials are bounded. It should not be hard to extend this to suitable unbounded potentials with the standard hypotheses. Thus, we suppose that $T_x$ (respectively $T_n$) is an ergodic group of operators on a measure space $\Omega$ indexed by $x\in\mathbb{R}$ (respectively $n\in\mathbb{Z}$), and that $F:\Omega\to l\times l$ matrices is bounded and measurable, with values in the real symmetric matrices, and we let $V_\omega(x)=F(T_x\omega)$.

## 2. Jost Solutions and *m*-Functions for the Strip

The results of this section are deterministic, i.e., hold for each operator of the form

$$H=-\frac{d^2}{dx^2}+V(x) \tag{2.1}$$

with $V(x)$ a bounded $l\times l$ real symmetric matrix function on $\mathbb{R}$, and $H$ acts on $L^2(\mathbb{R};\mathbb{C}^l,dx)$, $l$-component $L^2$ functions on $\mathbb{R}$. At the end, we will indicate what can be similarly proven for the discrete case.

$\mathbb{C}_+$ will denote $\{z\in\mathbb{C}\,|\,\mathrm{Im}\,z>0\}$. For $E\in\mathbb{C}_+$, define $J_+(H,E)=\{f$ functions locally in $D(-d^2/dx^2)\,|\,Hf=Ef$ as a pointwise statement; $\int_0^\infty|f(x)|^2dx<\infty\}$.

**Theorem 2.1.** (a) *$J_+$ has dimension $l$.*

(b) *$f\in J_+$ with $f\neq0$ implies that $f(x)\neq0\neq f'(x)$ for all $x$.*

(c) *For each $x$, $f\mapsto f(x)$ is a bijection of $J_+$ and $\mathbb{C}^l$, and similarly for $f\mapsto f'(x)$.*

*Proof.* (a) Consider the operator $A=H\restriction C_0^\infty(0,\infty)$ as a symmetric operator on $L^2(0,\infty)$. Since $H$ is formally self-adjoint, $J_+$ is precisely $\mathrm{Ker}(A^*-E)$, whose dimension is the deficiency index of $A^*$. Since $V$ is bounded, if $A_0=-d^2/dx^2\,|\,C_0^\infty(0,\infty)$, then

$$\dim(\mathrm{Ker}(A^*-E))=\dim(\mathrm{Ker}(A_0^*-E))=l,$$

since one can easily write down all solutions if $-f''-Ef=0$.

(b) Let $u, v$ pointwise solve $Hu = Eu, Hv = E'v$ and let $\langle u, v\rangle_x$ denote the $\mathbb{C}^l$ inner product of $u(x)$ and $v(x)$. Then

$$\frac{d}{dx}[\langle u', v\rangle_x - \langle u, v'\rangle_x] = (E' - \bar{E})\langle u, v\rangle_x,$$

or integrating,

$$\langle u', v\rangle_y - \langle u, v'\rangle_y - [\langle u', v\rangle_x - \langle u, v'\rangle_x] = (E' - \bar{E})\int_x^y \langle u, v\rangle_x dx. \tag{2.2}$$

There are two cases of particular interest. First, since $V$ is real, if $Hv = Ev$, then $H\bar{v} = \bar{E}\bar{v}$, so this yields constancy of the Wronskian:

$$Hu = Eu, \quad Hv = Ev \Rightarrow \langle \bar{u}', v\rangle_x - \langle \bar{u}', v'\rangle_x \quad \text{is constant.} \tag{2.3}$$

Secondly, if $u \in J_+$, then $u'' = (V - E)u \in L^2$ at infinity, so $u'$ is in $L^2$, and thus $[|u'|^2]' \in L^1$, so $u' \to 0$ at infinity. Similarly, $u \to 0$ at infinity, so taking $v = u$ and $y \to \infty$ in (2.2), we obtain:

$$2\,\mathrm{Im}\,\langle u, u'\rangle_x = 2\,\mathrm{Im}\,E \int_x^\infty \langle u, u\rangle_x dx. \tag{2.4}$$

Equation (2.4) says that if $u$ or $u'$ vanishes at $x$, then $u$ vanishes on $(x, \infty)$ and so on all of $\mathbb{R}$. This is what was to be proven.

(c) By (b), the map is injective. Since, by (a), $\dim J_+ = \dim \mathbb{C}^l$, it is a bijection. □

**Corollary 2.2.** *There is a unique $l \times l$ matrix valued function, $F_+(x, E)$, obeying*

$$-F_+'' + VF_+ = EF_+,$$
$$F_+(0, E) = \mathbb{1},$$
$$\int^\infty \|F_+(x, E)\|^2 dx < \infty.$$

*Proof.* Let $e_1, \ldots, e_l$ be the canonical Kronecker basis for $\mathbb{C}^l$. By the theorem, there are unique solutions $f_{j,+}$ of $-f'' + Vf = Ef$ obeying $f_{j,+}(0) = e_j$ and $\int^\infty |f_{j,+}|^2 dx < \infty$. $F_+$ is then determined by the conditions $F_+ e_j = f_{j,+}$. □

Because of the bijective condition of Thm. 2.1(c), for each $x, F(x, E)$ is an invertible matrix.

We define the $m$-functions by

$$M_+(E) = \frac{d}{dx} F_+(0, E).$$

This is a matrix determined by $b = M_+(E)a$, given $a$, is the unique vector in $\mathbb{C}^l$ so that there is a solution $u \in J_+$ with $u(0) = a, u'(0) = b$.

Similarly, we define $J_-$ at $-\infty, F_-$ and

$$M_-(E) = -\frac{d}{dx} F_-(0, E).$$

If we need to make the $V$ dependence of $F$ and $M$ explicit, we will refer to $F_+(x,E,V)$ and $M_+(E,V)$. We also let $(T_x V)(y)=V(x+y)$.

**Proposition 2.3.** (a) $(\operatorname{Im} E)^{-1}\operatorname{Im} M_+(E)=\int\limits_0^\infty F_+(x,E)^*F_+(x,E)dx$,

(b) $F_+(x,E,T_yV)F_+(y,E,V)=F_+(x+y,E,V)$,

(c) $F'_+(x,E,V)=M_+(E,T_xV)F_+(x,E,V)$,

(d) $\pm(d/dx)M_\pm(E,T_xV)=V(x)-E-M_\pm(E,T_xV)^2$,

(e) $M_+$ *is symmetric*, i.e., $\langle\bar{b},M_+a\rangle=\langle\overline{M_+b},a\rangle$.

*Proof.* (a) The expectation value of this equality of Hermitian matrices in the vector $a$ is just (2.4) for $u=F_+a$.

(b) $F_+(x,E,V)a$ is just the element, $u$, in $J_+$ with $u(0)=a$. $F_+(x-y,E,T_yV)\,a$ is just the element of $J_+$ with $u(y)=a$. This, with uniqueness, proves (b).

(c) is just translation invariance and the definition of $M$.

(d) This Ricatti equation is proven just like the ordinary Ricatti equation by using (c) to write $M_+(E,T_xV)=F'_+(x)F_+(x)^{-1}$ using $(F_+^{-1})'=-F_+^{-1}F'_+F_+^{-1}$ and $F''_+F_+^{-1}=(V-E)$.

(e) Fix $a,b\in\mathbb{C}^l$. Let $u(x)=F_+(x)b, v(x)=F_+(x)a$. The Wronskian $\langle\bar{u}',v\rangle_x-\langle\bar{u},v'\rangle_x$ is constant. As $x\to\infty$, it goes to zero, so it is zero. But at $x=0$,

$$\langle\bar{u}',v\rangle_0-\langle\bar{u},v'\rangle_0=\langle\overline{Mb},a\rangle-\langle\bar{b},Ma\rangle.\quad\square$$

Since $F$ is invertible, (a) implies that $\operatorname{Im} M_+$ is strictly positive, and so it and $M_+$ are invertible.

$F_\pm, M_\pm$ can obviously be defined also for $E\in\mathbb{C}_-=\bar{\mathbb{C}}_+$. Clearly, since $V$ is real symmetric, $F_\pm(x,\bar{E},V)=\overline{F_\pm(x,E,V)}$, so we have that

$$M_+(\bar{E},V)=\overline{M_+(E,V)}.\tag{2.5}$$

In addition to the solutions $F_\pm$ regular at $\pm\infty$, we will need the solutions $\Phi$, $\Psi$ with boundary conditions at the origin, i.e., $\Phi(x,E,V)$ and $\Psi(x,E,V)$ are $l\times l$ matrices obeying, for any $E$ in $\mathbb{C}$.

$$Hu=Eu\tag{2.6}$$

with the boundary conditions

$$\Phi(x=0)=\mathbb{1},\quad \Phi'(x=0)=0;\quad \Psi(x=0)=0,\quad \Psi'(x=0)=\mathbb{1}.$$

Clearly, for $E\in\mathbb{C}_+\cup\mathbb{C}_-$,

$$F_\pm(x,E)=\Phi(x,E)\pm\Psi(x,E)M_\pm(E).\tag{2.7}$$

The $2l\times 2l$ transfer matrix

$$U(x,E)=\begin{pmatrix}\Psi'(x,E) & \Phi'(x,E)\\ \Psi(x,E) & \Phi(x,E)\end{pmatrix}$$

is a fundamental matrix for (2.6) in that, if $u$ solves (2.6) with $u(0)=a, u'(0)=b$, then

$$\begin{pmatrix}u'(x)\\ u(x)\end{pmatrix}=U(x,E)\begin{pmatrix}b\\ a\end{pmatrix}.\tag{2.8}$$

As usual, $U$ obeys the equation

$$U' = \begin{pmatrix} 0 & E - V(x) \\ \mathbb{1} & 0 \end{pmatrix} U. \tag{2.9}$$

Introduce the $2l \times 2l$ operator

$$J = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}.$$

Then, the constancy of the Wronskian, (2.3), implies that $\det U = 1$ and

$$U(x,E)^t J U(x,E) = J, \tag{2.10a}$$

which is to say that $U(x,E)$ lies in $Sp(2l)$, the symplectic group.

Since $J^{-1} = -J$, (2.10a) says that

$$(-JU^t J)U = \mathbb{1},$$

so since one-sided finite matrix inverses are two-sided,

$$U(-JU^t J) = \mathbb{1},$$

or

$$U(x,E) J U(x,E)^t = J. \tag{2.10b}$$

This will be used in the next section. Because of its property (2.8), $U$ clearly obeys

$$U(x,E;T_y V)U(y,E;V) = U(x+y,E;V) \tag{2.11}$$

* * *

The theory for the discrete case is described in parallel to the above theory. $J_+$ is defined in the same way, but the Wronskian becomes

$$\langle \bar{u}(n+1), v(n)\rangle - \langle \bar{u}(n), v(n+1)\rangle,$$

and (2.4) becomes

$$2\,\mathrm{Im}\,\langle u(n), u(n+1)\rangle = 2\,\mathrm{Im}\,E \sum_{j=n+1}^{\infty} \|u(n)\|^2.$$

$F_+$ is defined by a direct analog of Cor. 2.2, and $M_\pm$ by

$$M_\pm(E,V) = -F_\pm(\pm 1, E, V).$$

The analog of Proposition 2.3 is

**Proposition 2.3′.** (a) $(\mathrm{Im}\,E)^{-1} M_+(E) = \sum_{n=1}^{\infty} F_+^*(n,E)F_+(n,E)$,

(b) $F_+(n,E,T_m V)F_+(m,E,V) = F_+(n+m,E,V)$,

(c) $F_+(n+1,E,V) = M_+(E,T_n V)F_+(n,E,V)$,

(d) $M_\pm(E,T_n V) = E - V(n) - M_\pm(E,T_{n\mp 1}V)^{-1}$.

*$\Phi$, $\Psi$ obey the boundary conditions*

$$\Phi(n=0) = \mathbb{1}, \quad \Phi(n=1) = 0; \quad \Psi(n=0) = 0, \quad \Psi(n=1) = \mathbb{1},$$

*so (2.7) then becomes*

$$F_+(n,E) = \Phi(n,E) - \Psi(n,E)M_+(E), \tag{2.12a}$$

$$F_-(n,E) = \Phi(n,E) + \Psi(n,E)[E - V(0) + M_-(E)]. \tag{2.12b}$$

$U$ has the definition

$$U(n) = \begin{pmatrix} \Psi(n+1) & \Phi(n+1) \\ \Psi(n) & \Phi(n) \end{pmatrix},$$

and now if $u$ solves (2.6) with $u(0) = a, u(1) = b$, then

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = U(n) \begin{pmatrix} b \\ a \end{pmatrix}.$$

An equation like (2.9) continues to hold.

## 3. Green's Functions

As a preliminary for this section, we need to rewrite (2.10):

**Proposition 3.1.** (a) $\Phi(x)\Psi(x)^t = \Psi(x)\Phi(x)^t$,

(b) $\Phi(x)'\Psi(x)^t - \Psi'(x)\Phi(x)^t = -\mathbb{1}$,
(c) $\Psi(x)F_+(x)^t = F_+(x)\Psi(x)^t$,
(d) $\Psi'(x)F_+(x)^t - F'_+(x)\Psi(x)^t = \mathbb{1}$,
(e) $F_-(x)(M_+ + M_-)^{-1}F_+(x)^t = F_+(x)(M_+ + M_-)^{-1}F_-(x)^t$,
(f) $-F'_-(x)(M_+ + M_-)^{-1}F_+(x)^t + F'_+(x)(M_+ + M_-)^{-1}F_-(x)^t = \mathbb{1}$.

*Proof.* (a) and (b) are two of the $4l \times l$ matrix relations made by the $2l \times 2l$ matrix relation (2.10b).

Given Proposition 2.3(e), which says that $M^t_\pm = M_\pm$ and (2.7), (c) and (d) then follow immediately. (e) and (f) follow similarly if one notes that

$$M_-(M_+ + M_-)^{-1}M_+ = (M_-^{-1} + M_+^{-1})^{-1} = M_+(M_+ + M_-)^{-1}M_-,$$

and

$$(M_+ + M_-)^{-1}[M_+ - (-M_-)] = \mathbb{1} = -[(-M_-) - M_+](M_+ + M_-)^{-1}. \quad \square$$

Let $H^\pm$ denote the operators on $L^2(0,\infty)$ (respectively $L^2(-\infty,0)$) with Dirichlet boundary conditions, and let $H_0^\pm$ be the analogous operators with $V=0$.

**Theorem 3.2.** *If $E \in \mathbb{C}_+ \cup \mathbb{C}_-$, $(H-E)^{-1}$, $(H_\pm - E)^{-1}$ have jointly continuous integral kernels $G_E(x,y)$ and $G_E^+(x,y)$ given by*

$$\begin{aligned} G_E^+(x,y) &= \Psi(x)F_+(y)^t && 0 \leqq x \leqq y \\ &= F_+(x)\Psi(y)^t && 0 \leqq y \leqq x. \\ G_E(x,y) &= -F_-(x)(M_+ + M_-)^{-1}F_+(y)^t && x \leqq y \\ &= -F_+(x)(M_+ + M_-)^{-1}F_-(y)^t && y \leqq x, \end{aligned}$$

*Proof.* We prove the formula for $G_E$; the proof for $G_E^+$ is similar. Call the putative formula $\tilde{G}_E$. By Proposition 3.1(e) and (f), $\tilde{G}_E$ for $y$ fixed is continuous and $C^1$ away from $x = y$ with

$$\frac{\partial}{\partial x}\tilde{G}_E(x,y)\bigg|_{x=y+0} - \frac{\partial}{\partial x}\tilde{G}_E(x,y)\bigg|_{x=y-0} = -1. \tag{3.1}$$

Moreover, since $F_\pm$ are $L^2$ at $\pm\infty$, for any $f\in C_0^\infty$,

$$g(x)=\int \tilde{G}_E(x,y)f(y)dy$$

is in $L^2$. By (3.1), $g$ obeys

$$(H-E)g=f.$$

Since $H-E$ is invertible, it follows that $\tilde{G}_E(x,y)$ is the required integral kernel. □

**Proposition 3.3.** *For* $E\in\mathbb{C}_+$, *let* $k=\sqrt{E}$, *the square root with* $\operatorname{Im}k>0$. *Then* (*with* $\|\cdot\|$ *the norm in* $\mathbb{C}^l$):

(a) $\|M_+(E,V)-ik\mathbb{1}\|\leqq(2\operatorname{Im}k)^{-1}\|V\|_\infty[1+(\operatorname{Im}E)^{-1}\|V\|_\infty]$,
(b) $\operatorname{Im}M_+(E,V)\geqq(\operatorname{Re}k)[\operatorname{Im}E(\operatorname{Im}E+\|V\|_\infty)^{-1}]^2\mathbb{1}$,
(c) $\|G_E(x,y;V)-(2ik)^{-1}e^{ik(x-y)}\mathbb{1}\|\leqq a$, *where* $a=(4|E|\operatorname{Im}k)^{-1}\|V\|_\infty[1+(\operatorname{Im}E)^{-1}\|V\|_\infty]$,
(d) $\|G_E^+(x,y;V)+(2ik)^{-1}[e^{ik(x-y)}-e^{ik(x+y)}]\mathbb{1}\|\leqq 3a$,
(e) $\|G_E(x,y;V)\|\leqq C_1\exp(-C_2|x-y|)$; $\|F_+(x,E;V)\|\leqq C_1\exp(-C_2|x|)$; $x>0$ *for some* $C_2>0$.

*Proof.* (a) Let $F_+^0$ be $F_+$ for $H_0$, and let $F=F_+-F_+^0$. Then

$$-F''+VF-EF=-VF_+^0.$$

It follows that

$$F=-(H^+-E)^{-1}VF_+^0, \tag{3.2}$$

i.e.,

$$F_+(x,E,V)=e^{ikx}\mathbb{1}-\int G_E^+(x,y;V)V(y)e^{iky}dy.$$

Using Theorem 3.2, if we differentiate and set $x$ to zero, we get

$$M_+(E,V)=ik\mathbb{1}-\int_0^\infty F_+(y,E,V)^t V(y)e^{iky}dy,$$

so

$$\|M_+(E,V)-ik\mathbb{1}\|\leqq\|V\|_\infty\|e^{iky}\|_2\|F_+\|_2, \tag{3.3}$$

but

$$\|e^{iky}\|_{L^2(0,\infty)}=(2\operatorname{Im}k)^{-1/2}. \tag{3.4}$$

By (3.2) and $\|(H^+-E)^{-1}\|\leqq(\operatorname{Im}E)^{-1}$ we have that

$$\begin{aligned}\|F_+\|_2&\leqq(\operatorname{Im}E)^{-1}\|V\|_\infty\|F_+^0\|_2\\&\leqq(\operatorname{Im}E)^{-1}\|V\|_\infty(2\operatorname{Im}k)^{-1/2}\end{aligned} \tag{3.5}$$

by (3.4). Equations (3.3)–(3.5) prove (a).

(b) By interchanging the roles of $F_+$ and $F_+^0$ we get, for any unit vector $\zeta\in\mathbb{C}^l$:

$$F_+^0\zeta=F_+\zeta+(H_0^+-E)^{-1}VF_+\zeta,$$

so

$$\|F_+^0\zeta\|_2=(2\operatorname{Im}k)^{-1/2}\leqq\|F_+\zeta\|_2(1+(\operatorname{Im}E)^{-1}\|V\|_\infty).$$

Since $\operatorname{Im}E=2(\operatorname{Im}k)(\operatorname{Re}k)$, this implies

$$\|F_+\zeta\|_2^2\geqq(\operatorname{Im}E+\|V\|_\infty)^{-2}(\operatorname{Im}E)(\operatorname{Re}k).$$

(b) follows if one uses Proposition 2.3(a) that

$$\operatorname{Im}(\zeta, M_+(E)\zeta) = \operatorname{Im} E \,\| F_+ \zeta \|_2^2.$$

(c) Note that

$$\int_{-\infty}^{\infty} \| e^{ik|x-y|}(2ik)^{-1} \|^2 dx = \tfrac{1}{4}|E|^{-1}(\operatorname{Im} k)^{-1} \equiv b. \tag{3.6}$$

Since

$$\begin{aligned}(H-E)^{-1} - (H_0-E)^{-1} = & -(H_0-E)^{-1}V(H_0-E)^{-1} \\ & + (H_0-E)^{-1}V(H-E)^{-1}V(H_0-E)^{-1}\end{aligned}$$

and $\|(H-E)^{-1}\| \leqq (\operatorname{Im} E)^{-1}$, we easily obtain

$$|G_E(x,y) - G_E^0(x,y)| \leqq b(1 + \|V\|_\infty (\operatorname{Im} E)^{-1})\|V\|_\infty = a.$$

(d) The proof is just like (c), except we must estimate

$$\int_0^{\infty} \|(2ik)^{-1}[e^{ik(x-y)} - e^{ik(x+y)}]\|^2 dx.$$

We settle for the crude estimate of

$$\left[\left(\int_0^{\infty} \|(2ik)^{-1} e^{ik(x-y)}\|^2\right)^{1/2} + \left(\int_0^{\infty} \|(2ik)^{-1} e^{ik(x-y)}\|^2\right)^{1/2}\right]^2,$$

and note that the first term is $b$, while the second is, at most, $\frac{1}{2}b$, and then that $(1+\sqrt{\frac{1}{2}})^2 \leqq 3$.

(e) The bound on $G_E$ is a standard Combes–Thomas estimate (see e.g. [16]). It implies the bound on $F_+$ by Theorem 3.2 and the invertibility of $F_-$. □

*Remark.* Everywhere that $(\operatorname{Im} E)^{-1}$ occurs in upper bounds, it can be replaced by $[\operatorname{dist}(E, \operatorname{spec}(H))]^{-1}$ which, as $\operatorname{Re} E \to -\infty$, goes as $|E|^{-1}$.

Next, we need the spectral measures. In the continuum case, we will use the Herglotz representation theorem, which says that:

**Herglotz Representation Theorem.** *Let $F(z)$ be a matrix valued function on $\mathbb{C}_+$ with $\operatorname{Im} F > 0$. Then there is a positive matrix valued measure $dH(x)$ on $\mathbb{R}$ and self-adjoint matrices $A, B$ with $B > 0$, so that*

(a) $\displaystyle\int \frac{dH(x)}{1+x^2} < \infty$

(b) $\displaystyle F(z) = A + Bz + \int \left(\frac{1}{x-z} - \frac{x}{1+x^2}\right) dH(x).$

*If, moreover,*

$$|F(iy)| \leqq Cy^{-\theta}, \quad 0 < \theta < 1, \quad y > 0,$$

*then*

(a′) $\displaystyle\int \frac{dH(x)}{(1+|x|)^\alpha} < \infty$ *if* $\alpha > 1-\theta$, (b′) $F(z)=\int \frac{dH(x)}{x-z}$.

*In either case,*

$$dH(x)=\text{weak-limit}_{\varepsilon\downarrow 0}\frac{1}{\pi}\operatorname{Im} F(x+i\varepsilon)dx.$$

For a proof in the scalar case, see [1]. The matrix extension follows by taking expectation values.

It will be useful to define $2l\times 2l$ matrices $M(E)$ for $E\in\mathbb{C}_+$ by

$$M(E)=\begin{pmatrix} -(M_+(E)+M_-(E))^{-1} & \frac{1}{2}(M_+(E)+M_-(E))^{-1}(M_-(E)-M_+(E)) \\ \frac{1}{2}(M_-(E)-M_+(E))(M_+(E)+M_-(E))^{-1} & M_+(E)(M_+(E)+M_-(E))^{-1}M_-(E) \end{pmatrix}.$$

**Lemma 3.4.** $\operatorname{Im} M(E)>0$ *for* $E\in\mathbb{C}_+$.

*Proof.* Let $\varphi\in C_0^\infty$ with $\varphi\geqq 0$ and $\int\varphi(x)dx=1$, and let $\varphi_k(x)=k^{-1}\varphi(k^{-1}x)$. Let

$$M_k(E)=\begin{pmatrix} \int G_E(x,y)\varphi_k(x)\varphi_k(y)dx\,dy & \int G_E(x,y)\varphi_k(x)\varphi_k'(y)dx\,dy \\ \int G_E(x,y)\varphi_k'(x)\varphi_k(y)dx\,dy & \int G_E(x,y)\varphi_k'(x)\varphi_k'(y)dx\,dy \end{pmatrix}.$$

It is obvious that $\operatorname{Im} M_k(E)>0$ since $\operatorname{Im}(H-E)^{-1}>0$. But, by Theorem 3.2, $\operatorname{Im} M_k(E)\to\operatorname{Im} M(E)$ as $k\to 0$. □

**Lemma 3.5.** (a) *For* $x\geqq y$, *we have that*

$$G_E(x,y)=(\Phi(x,E),\Psi(x,E))M(E)(\Phi(y,E)\,\Psi(y,E))^t$$
$$-\tfrac{1}{2}\Phi(x,E)\,\Psi(y,E)^t-\tfrac{1}{2}\Phi(x,E)\,\Phi(y,E)^t.$$

(b) *For* $x\geqq y\geqq 0$,

$$G_E^+(x,y)=\Psi(x,E)M_+(E)\,\Psi(y,E)^t+\Phi(x,E)\,\Psi(y,E)^t.$$

*Proof.* Follows immediately from Theorem 3.2 and (2.7). □

**Theorem 3.6.** *There exist measures* $d\Sigma_+(x)$ *and* $d\Sigma(x)$ *which are respectively* $l\times l$ *matrix valued, and* $2l\times 2l$ *matrix valued so that*

(a) $\int\frac{\Psi(x,E)d\Sigma_+(E)\,\Psi(x,E)^t}{1+|E|^\alpha}<\infty$,

$\int\frac{(\Phi(x,E),\,\Psi(x,E))d\Sigma(E)(\Phi(x,E),\,\Psi(x,E))^t}{1+|E|^\alpha}<\infty$,

*for* $\alpha>\frac{1}{2}$ *and all* $x$.

(b) $G_z^+(x,y)=\int\frac{\Psi(x,E)d\Sigma_+(E)\,\Psi(y,E)^t}{E-z}$ *for* $x\geqq y$.

(c) $G_z(x,y)=\int\frac{\Phi(x,E),\,\Psi(x,E))d\Sigma(E)(\Phi(y,E),\,\Psi(y,E))^t}{E-z}$ *for* $x\geqq y$.

(d) $d\Sigma_+(E)=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im} M_+(E)dE$,

$$d\Sigma(E) \quad = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} M(E) dE.$$

*Proof.* By the Herglotz representation theorem and Lemma 3.4, $d\Sigma, d\Sigma_+$ as defined in (d) exist. Since $\Phi$ and $\Psi$ are holomorphic, the singularities of $G$ are the same as those of $M$. By Lemma 3.5, $G_E(x,x)$ (respectively $G_E^+(x,x)$) have imaginary boundary values given by the measures in part (d), so by the bounds in Proposition 3.3, (a) holds and (b), (c) hold for $x = y$. Each of the four combinations

$$G_E(x,x) + G_E(y,y) + \omega G_E(x,y) + \bar{\omega} G_E(y,x)$$

is Herglotz for $\omega = \pm 1, \pm i$, so these obey Herglotz representation theorems which yield the required formula for $G, G^+$. $\square$

One can further analyze $d\Sigma$ and $d\Sigma_+$ to develop eigenfunction expansions (see e.g. [2]); all we need is the following theorem, which holds in any dimension (see e.g. [18]):

**Theorem 3.7.** *Let $E_\Delta^{(n)}(H)$ be the spectral projections of spectral multiplicity $n$ (so that $\sum_n E_{\mathbb{R}}^n(H) = \mathbb{1}$ and $HE_{\mathbb{R}}^{(n)}(H)$ has uniform multiplicity $n$). Let $\tilde{\rho}$ be any strictly positive trace class operator, and let $d\rho_n(\lambda) = \operatorname{tr}(\tilde{\rho} E^{(n)}(d\lambda))$. Then, for* a.e. *$E$ with respect to $d\rho_n(\lambda)$*

$$\{\varphi \mid H\varphi = E\varphi; |\varphi(x)| \leqq C(1+|x|)^{1+\varepsilon}, |\varphi'(x)| \leqq C(1+|x|)^{1+\varepsilon}\}$$

*has dimension at least $n$.*

*Remark.* In general dimension with general potential, one only has an $L^2$ bound on $\varphi'$, but since $V$ is bounded, we have $L^\infty$ bounds on $\varphi$ and $\Delta\varphi$ which yields $L^\infty$ bounds on $\nabla\varphi$ in 1-dimension.

* * *

The discrete case can be analyzed in the exact same way; in fact, some significant shortcuts are available:

(i) The formula for $G_E$ has $(M_+ + M_- + E - V(0))^{-1}$ in place of $(M_+ + M_-)^{-1}$. Similarly, a change is needed in the definition of $M$. Everywhere that $M_-$ appears, it must be replaced by $[M_-(E) + E - V(0)]$. These changes are due to (2.12) replacing (2.7). The asymmetry between $M_\pm$ comes from the fact that

$$M(E) = \begin{pmatrix} G_E(0,0) & G_E(0,1) \\ G_E(1,0) & G_E(1,1) \end{pmatrix}$$

with 1 and not $-1$.

(ii) The analog of Proposition 3.3 is simpler because boundedness of the matrix kernel is implied by boundedness of the operator, so we need only expand to second rather than third order, e.g.

$$\| G_E(x,y;V) - G_E(x,y;V=0) \| \leqq (\operatorname{Im} E)^{-2} \| V \|_\infty$$

is immediate. Moreover, $M_+$ is the expectation value of $(H^+ - E)^{-1}$ in the vector $\delta_1$ and $H^+$ is bounded, so the analog of (a) is easy by $(H^+ - E)^{-1} = -E^{-1} + \sum_{n=1}^{\infty} (-E)^{n+1}(H^+)^n$ which implies

$$\| M_+(E, V) + E^{-1} \| \leqq |E|^{-1} (\operatorname{Im} E)^{-1} (2 + \| V \|_\infty).$$

(iii) We need not appeal to the Herglotz representation theorem to define $d\Sigma$ and $d\Sigma_+$. Rather, we can use

$$M_+(E) = (\delta_1, (H^+ - E)^{-1} \delta_1) = \int \frac{d\mu + (w)}{w - E}, \tag{3.7}$$

where $\mu_+$ is the measure guaranteed by the spectral theorem, so that

$$\int w^n d\mu + (w) = (\delta_1, (H^+)^n \delta_1). \tag{3.8}$$

Here $\mu_+$ is an $l \times l$ matrix valued measure and (3.8) is shorthand for

$$\int w^n d(a, \mu_+(w) b) = (\delta_1 \otimes a, (H^+)^n \delta_1 \otimes b)$$

for all $a, b \in \mathbb{C}^l$.

## 4. The Inverse Problem

In this section, we discuss

**Theorem 4.1.** *In the continuum case, $M_+(E)$ determines $\{V(x)\}_{x \geqq 0}$ in the sense that if $V^{(1)}$ and $V^{(2)}$ are two bounded potentials and $M_+^{(1)} = M_+^{(2)}$, then $V^{(1)}(x) = V^{(2)}(x)$ for* a.e. $x \geqq 0$.

**Theorem 4.2.** *In the discrete case, $M_+(E)$ determines $\{V(n)\}_{n \geqq 1}$ in the sense that if $V^{(1)}$ and $V^{(2)}$ are two bounded potentials and $M_+^{(1)} = M_+^{(2)}$, then $V^{(1)}(n) = V^{(2)}(n)$ for all $n \geqq 1$.*

Actually, we need the fact that $V$ is a measurable function of $M_+$ as follows from the actual proof. We'll give the proof of Theorem 4.2 which is a straightforward extension of the $l = 1$ case (see e.g. [19]). In the same way, one can extend either the Marchenko [11] or Gel'fand–Levitan (e.g. [12]) method to the $\mathbb{C}^l$-valued continuum case.

*Proof of Theorem 4.2.* By (3.7) and the fact that $M_+$ has bounded support for $E$ near infinity:

$$M_+(E) = \sum_{n=0}^{\infty} (-E)^{-(n+1)} \int w^n d\mu_+(w),$$

so that, by (3.8), $M_+(E)$ determines

$$(\delta_1, (H^+)^n \delta_1).$$

It is easy to see inductively that

$$(H^+)^n (\delta_1 \otimes a) = \delta_{n+1} \otimes a + \delta_n \otimes V(n) a + \sum_{j=1}^{n-1} \delta_j \otimes f_{n,j}(V),$$

where $f_{n,j}(V)$ is a $\mathbb{C}^l$-valued function of $V(1),\ldots,V(n-1)$ with

$$(b, V(n)a) = (\delta_1 \otimes b, (H^+)^{2n-1}\delta_1 \otimes a) + \text{function of } V(1),\ldots,V(n-1),$$

which means that $(\delta_1, (H^+)^n\delta_1)$ inductively determine $V(j)$. $\square$

## 5. The Pastur–Ishii Theorem for the Strip

Before proceeding with the study of random Schrödinger operators on strips, we recall the notion of Lyaponov exponents and the Osceledec multiplicative ergodic theorem.

Suppose we are given a family of $m \times m$ invertible (complex) matrices $\{A(x), x\in\mathbb{R}\}$. The numbers

$$\gamma_j = \lim_{x\to\infty} \frac{1}{x} \log \mu_j(A(x)), \quad j = 1, 2, \ldots, m \tag{5.1}$$

if the limits exist, are called the *Lyaponov exponents*, where, for a matrix $A, \mu_j(A)$ denotes the eigenvalues of a non-negative definite matrix $(A^*A)^{1/2}$, ordered so $\mu_1(A) \geqq \mu_2(A) \geqq \cdots \geqq \mu_m(A) \geqq 0$. If all these limits exist, we say that $\{A(x):x\in\mathbb{R}\}$ has *Lyaponov behavior* (at $+\infty$). Since $\mu_1(A)\mu_2(A)\ldots\mu_j(A) = \|\Lambda^j A\|$, the limits (5.1) exist if and only if the following limit exists:

$$\lim_{x\to\infty} \frac{1}{x} \log \|\Lambda^j A(x)\| = \gamma_1 + \cdots + \gamma_j, \quad j = 1, 2, \ldots, m, \tag{5.2}$$

and the limits are related as indicated.

**Lemma 5.1.** (Osceledec–Ruelle [17]) *Let $\{A(x):x\in\mathbb{R}\}$ be a family of $m\times m$ matrices obeying*

$$\lim_{n\to\infty} \frac{1}{n} \log\left\{ \sup_{0\leqq\delta\leqq 1} \|A(\delta+n)A(n)^{-1}\| \right\} = \lim_{n\to\infty} \frac{1}{n} \log\left\{ \sup_{0\leqq\delta\leqq 1} \|A(n)A(n+\delta)^{-1}\| \right\} = 0. \tag{5.3}$$

*Suppose $\{A(x):x\in\mathbb{R}\}$ has Lyaponov behavior at $+\infty$. Then there exist subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = \mathbb{C}^m$ so that*

$$\dim V_j = j, \quad j = 0, 1, 2, \ldots, m, \tag{5.4}$$

*if $u \in V_j \setminus V_{j-1}$ for some $j = 1, 2, \ldots, m$, then*

$$\lim_{x\to\infty} \frac{1}{x} \log \|A(x)u\| = \gamma_{m+1-j}. \tag{5.5}$$

Now we consider random matrices. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space admitting a $\mu$-preserving one-parameter group of transformations $\{T_x:x\in\mathbb{R}\}$ on $\Omega$. We always assume the ergodicity of $\{T_x:x\in\mathbb{R},\mu\}$. If a measurable map $A:\mathbb{R}\times\Omega\to\mathbb{C}^n\otimes\mathbb{C}^n$ satisfies

$$\begin{cases} A(x+y,\omega) = A(x, T_y\omega)A(y,\omega) \\ A(0,\omega) \quad = 1 \end{cases} \tag{5.6}$$

then we call it a multiplicative cocycle with respect to $\{T_x\}$. It is easy to see that if $A$ is a multiplicative cocycle, so is $A(-x,\omega)$ with respect to $\{T_{-x}\}$. Moreover, $A(x,\omega)$ is invertible and $A(x,\omega)^{-1}=A(-x,T_x\omega)$, and hence $\{A(x,\omega)^*\}^{-1}$ is also a multiplicative cocycle with respect to $\{T_x\}$.

**Lemma 5.2.** *Suppose a multiplicative cocycle $A$ satisfies*

$$\mathbb{E}\left[\log\left\{\sup_{0\leqq\delta\leqq 1}\|A(\delta,\omega)\|\right\}+\log\left\{\sup_{0\leqq\delta\leqq 1}\|A(\delta,\omega)^{-1}\|\right\}\right]<\infty. \tag{5.7}$$

*Then, for* a.e. $\omega\in\Omega$ *with respect to* $\mu$, $\{A(x,\omega)\}$ *has Lyaponov behavior at* $\pm\infty$, *and the condition* (5.3) *holds. Moreover, the Lyaponov exponents are independent of* $\omega$. *Let* $\gamma_j^\pm$ $(j=1,2,\dots,m)$ *be the Lyaponov exponents at* $\pm\infty$ *respectively. Then*

$$\gamma_j^+=-\gamma_{m-j+1}^-,\quad j=1,2,\dots,m. \tag{5.8}$$

*Proof.* The proof can be done by applying the subadditive ergodic theorem to $g(x,\omega)=\log\|\Lambda^j A(x,\omega)\|$. Equation (5.7) assures $\mathbb{E}|g(x,\omega)|\leqq cx$ and the condition (5.3). The independence of the Lyaponov exponents comes from the ergodicity. The identity (5.8) is shown as

$$\begin{aligned}\gamma_1^+ +\dots+\gamma_j^+ &= \lim_{x\to+\infty}\frac{1}{x}\mathbb{E}\log\{\|\Lambda^j A(x,\omega)\|\}\\ &= \lim_{x\to+\infty}\frac{1}{x}\mathbb{E}\log\{\|\Lambda^j A(-x,T_x\omega)^{-1}\|\}\\ &= \lim_{x\to-\infty}\frac{1}{-x}\mathbb{E}\log\{\|\{\Lambda^j A(x,\omega)\}^{-1}\|\}\\ &= -\gamma_m^- -\gamma_{m-1}^- -\dots-\gamma_{m-j+1}^-\end{aligned}$$

for $j=1,2,\dots,m$, which implies (5.8). The interchange of $\mathbb{E}$ and lim is guaranteed by the subadditive ergodic theorem. □

Now we apply these ideas to a random Schrödinger operator. Let $V$ be a measurable function from $\omega$ to the $l\times l$ real symmetric matrices. By Proposition 2.3(b), $F_\pm$ are multiplicative cocycles with $m=l$, and by (2.11), $U$ is a cocycle with $m=2l$. Therefore, applying Lemma 5.2, $U,F_+$ and $F_-$ have Lyaponov exponents at $+\infty$ which we denote at energy $E$ by:

$$\gamma_1(E)\geqq\gamma_2(E)\geqq\dots\geqq\gamma_{2l}(E),\quad -\gamma_l^+(E)\geqq-\gamma_{l-1}^+(E)\geqq\dots\geqq-\gamma_1^+(E),$$

and

$$\gamma_1^-(E)\geqq\gamma_2^-(E)\geqq\dots\geqq\gamma_l^-(E).$$

$\gamma_i$ are defined for all $E$; $\gamma_i^\pm$ only for $E$ with $\operatorname{Im}E>0$. Since $F_\pm(\cdot,E,\omega)\in L^2(\mathbb{R}_\pm;\mathbb{C}^l)$, we see

$$\gamma_j^\pm(E)\geqq 0,\quad j=1,2,\dots,l,\quad E\in\mathbb{C}\backslash\mathbb{R}. \tag{5.9}$$

Proposition 3.3(e) shows that $\gamma_j^\pm$ are strictly positive on $\mathbb{C}\backslash\mathbb{R}$.

**Lemma 5.3.**

$$\begin{cases} \gamma_j(E) = \gamma_j^-(E), & j = 1, 2, \dots, l \\ \gamma_j(E) = -\gamma_{2l-j+1}^+(E), & j = l+1, \dots, 2l \end{cases} \tag{5.10}$$

$$\begin{cases} \gamma_j(E) = -\gamma_{2l-j+1}(E) & j = 1, 2, \dots, 2l \\ \gamma_j^+(E) = \gamma_j^-(E) & j = 1, 2, \dots, l \quad \text{on} \quad \mathbb{C}\backslash\mathbb{R}. \end{cases} \tag{5.11}$$

*Proof.* To see (5.10), we have only to observe

$$\tilde{U}(x) = \begin{pmatrix} F'_+(x, E, \omega) & F'_-(x, E, \omega) \\ F_+(x, E, \omega) & F_-(x, E, \omega) \end{pmatrix}$$

also satisfies (2.9) except the initial condition, and $\tilde{U}(0)$ is non-singular owing to the facts that $\operatorname{Im} M_\pm(E) > 0$ on $\mathbb{C}\backslash\mathbb{R}$, and that for all $a, b, F_+ a \neq F_- a$, since otherwise $H$ would have an $L^2$ solution. Equation (5.11) follows from $U^{-1} = J^{-1} U^t J$. $\square$

We are interested in this section in some results where $E \in \mathbb{R}$ where $\gamma^\pm$ are not defined. We are heading towards

**Theorem 5.4.** *Let $S_j = \{E \in \mathbb{R} \mid \text{exactly } 2j\ \gamma\text{'s vanish}\}$. Then the multiplicity of the* a.c. *spectrum on $S_j$ is at most $2j$.*

Eventually we will show that $S_j$ is precisely the essential support of the a.c. spectrum of multiplicity $2j$. Theorem 5.4 is the strip analog of the Pastur [13]–Ishii [9] theorem. It follows from Pastur's argument (see e.g. p. 180 of [7]) and

**Lemma 5.5.** *Suppose that $E \in \mathbb{R}$ is fixed and in $S_j$. Let $\omega$ be such that $H_\omega$ has Lyaponov behavior with the typical Lyaponov exponents* (i.e., *those that hold for* a.e. $\omega$). *Then any subspace of $\{\varphi(x) \mid H_\omega \varphi = E\varphi, \varphi \notin L^2$ and $\varphi$ polynomially bdd$\}$ has dimension at most $2j$.*

*Proof.* Clearly, it suffices to prove that

$$\dim P \leqq 2j + \dim L, \tag{5.12}$$

where

$$P = \{\varphi \mid H\varphi = E\varphi, \varphi \text{ polynomially } bdd\},$$
$$L = \{\varphi \mid H\varphi = E\varphi, \varphi \in L^2\}.$$

The usual case $l = 1$ is not typical in that simple dimension counting works there, but consider the case $l = 2, j = 1$. Then there are 3 solutions which might be bounded at $\pm\infty$, one decaying and two with 0 rate of exponential growth. Thus, a priori there might be a solution decaying at $+\infty$ linked to a bounded solution at $-\infty$, one decaying at $-\infty$ linked to a bounded solution at $+\infty$ and one bounded at both ends. In that case, one would have $\dim P = 3, \dim L = 0$ and (5.12) would be violated. It is this possibility that is eliminated by the Lagrangian subspace argument below.

Since $V$ is real, we will only look at real valued solutions and real dimensions. Let $\dim L = d$ and $\dim P = p$. For $\varphi, \psi$ solving $H\varphi = E\varphi$, let

$$\omega(\varphi, \psi) = (\varphi'(0), \psi(0)) - (\varphi(0), \psi'(0)).$$

Since $\varphi$ is determined by $(\varphi'(0), \varphi(0))$, and

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is non-singular, $\omega$ is a non-degenerate antisymmetric form. In particular, if $A, B$ are two subspaces with

$$\varphi \in A, \psi \in B \Rightarrow \omega(\varphi, \psi) = 0,$$

then

$$\dim A + \dim B \leqq 2l \tag{5.13}$$

(since $JA$ is orthogonal to $B$).

Let

$$G_{\pm} = \{\varphi \mid \varphi \text{ decays exponentially at } \pm\infty\}.$$

By the Osceledec theorem and the fact that $2j$ of the $2l$ exponents are 0,

$$\dim G_{\pm} = l - j.$$

Clearly $G_+ \cap G_- \subset L$, so

$$\dim(G_+ + G_-) \geqq 2l - 2j - d. \tag{5.14}$$

But if $\varphi \in P$ and $\psi \in G_{\pm}$, then $\omega(\varphi, \psi) = 0$ by evaluating the Wronskian in the limit $\pm\infty$, so, by (5.13),

$$p + \dim(G_+ + G_-) \leqq 2l. \tag{5.15}$$

Equations (5.14) and (5.15) prove (5.12). $\square$

## 6. Bounds on Expectation Values of $M$

In this section, we will prove various bounds on expectation values of $\operatorname{Im} M$, especially analogs of the key inequality of [10, 19] (actually an equality in the continuum case):

$$\mathbb{E}(\operatorname{tr}(\operatorname{Im} M_{\pm})^{-1}) \leqq 2\gamma(E)/\operatorname{Im} E.$$

We will provide proofs in both the continuum and discrete cases, with parallel theorems indicated by Theorem 6.xC and Theorem 6.xD for the continuum and discrete cases.

**Lemma 6.1.** (a) (Craig–Simon [5]) *For $j = 1, \ldots, l, \gamma_1(E) + \cdots + \gamma_j(E)$ is subharmonic on $\mathbb{C}$.*

(b) *For almost all $E_0 \in \mathbb{R}$,*

$$\lim_{\varepsilon \downarrow 0} \gamma_j(E_0 + i\varepsilon) = \gamma_j(E_0), \quad j = 1, \ldots, l.$$

*Remark.* The results are true for $j = 1, \ldots, 2l$, but since $\gamma_{2l-j+1} = -\gamma_j$, there is nothing new in the results for $j = l+1, \ldots, 2l$.

*Proof.* (a) $\Lambda^j U(x, E)$ is analytic in $E$, so $\mathbb{E}(\log \|\Lambda^j U\|) \equiv G_j(x, E)$ subharmonic. Since $x^{-1} G_j(E, x)$ converges to $\gamma_1 + \cdots + \gamma_j$, this function is subharmonic.

(b) Obviously we need only show

$$\lim_{\varepsilon \downarrow 0} (\gamma_1 + \cdots + \gamma_j)(E_0 + i\varepsilon) = (\gamma_1 + \cdots + \gamma_j)(E_0), \quad j = 1, \ldots, l.$$

But any subharmonic function on $\mathbb{C}$ has non-tangential continuity at a.e. points on $\mathbb{R}$ (see e.g. [14], p. 141; any point of $\mathbb{R}$ is a regular point for two-dimensional Brownian motion). $\square$

For $E \in \mathbb{C} \setminus \mathbb{R}$, define

$$w(E) = \tfrac{1}{2}\mathbb{E}(\operatorname{tr}(M_+ + M_-)), \tag{6.1c}$$

and

$$\gamma(E) = \sum_1^l \gamma_j(E) = \sum_1^l \gamma_j^+(E) = \sum_1^l \gamma_j^-(E). \tag{6.2}$$

**Theorem 6.2C.** (a) $w(E) = \mathbb{E}(\operatorname{tr}(M_+)) = \mathbb{E}(\operatorname{tr}(M_-))$,

(b) $w'(E) = \mathbb{E}(\operatorname{tr}(G_E(0, 0; \omega)))$,

(c) $-\operatorname{Re} w(E) = \gamma(E)$,

(d) $\mathbb{E}[\operatorname{tr}((\operatorname{Im} M_\pm(E, \omega)^{-1}))] = 2\gamma(E)/\operatorname{Im} E$.

*Remark.* We do not usually bother to check integrability explicitly. The bounds in Proposition 3.3 always suffice to prove integrability of objects like $M_+$ or $(\operatorname{Im} M_+)^{-1}$.

*Proof.* (a) By Proposition 2.3(d), with $M_\pm(x) = M_\pm(E, T_x V)$ and $' = d/dx$,

$$(M_+ + M_-)' = M_-^2 - M_+^2 = M_-(M_+ + M_-) - (M_+ + M_-)M_+, \tag{6.3}$$

so

$$\frac{d}{dx} \operatorname{tr} \ln(M_+ + M_-) = \operatorname{tr}(M_-) - \operatorname{tr}(M_+). \tag{6.4}$$

If $g(x) = (d/dx) f(T_x V)$ with $f$ bounded:

$$\mathbb{E}(g) = \lim_{n \to \infty} \frac{1}{n} \int_0^n g(x)\,dx = \lim_{n \to \infty} \frac{1}{n} [f(T_n V) - f(T_0 V)] = 0,$$

so (6.4) implies (a).

(b) Let $\dot{} = d/dE$ and $' = d/dx$. Proposition 2.3(d) implies:

$$\dot{M}'_+ - \dot{M}'_- = -2 - M_+ \dot{M}_+ - \dot{M}_+ M_+ - \dot{M}_- M_- - M_- \dot{M}_-.$$

Straightforward but tedious calculation shows that

$$[\operatorname{tr}[(M_+ + M_-)^{-1}(\dot{M}_+ - \dot{M}_-)]]' = -2\operatorname{tr}(M_+ + M_-)^{-1} - \operatorname{tr}(\dot{M}_+ + \dot{M}_-),$$

which, given Theorem 3.2, implies (b).

(c) We have that

$$F'_+ F_+^{-1} = M_+,$$

so

$$\operatorname{tr}(\operatorname{Re} M_+) = (\ln|\det F_+|)'.$$

Since

$$\frac{1}{x}\ln|\det F_+| \to -\gamma(E),$$

by Lemma 5.3, we obtain the desired result.

(d) Write

$$M_+ = X + iY \tag{6.5}$$

with $X = X^*$, $Y = Y^* > 0$. Taking imaginary parts of Proposition 2.3(d):

$$Y' = -\operatorname{Im} E - XY - YX, \tag{6.6C}$$

so

$$[\operatorname{tr}(\log Y)]' = \operatorname{tr}(Y'Y^{-1}) = -\operatorname{Im} E \operatorname{tr}(Y^{-1}) - 2\operatorname{tr}(X)$$

which, given (c), yields (d). □

*Remark.* In terms of the integrated density of states,

$$\mathbb{E}(\operatorname{tr}(G_E(0,0;\omega))) = \int \frac{dk(E')}{E' - E}.$$

Thus (b) and (c) provide a proof of the Thouless formula for the continuum, multicomponent wave functions (see [3] for a statement of the continuum Thouless formula which needs a subtraction).

In the discrete case, we define

$$w(E) = \mathbb{E}\operatorname{tr}(\ln(M_+)) = \mathbb{E}\ln(\det(M_+)). \tag{6.1D}$$

As noted at the end of Sect. 3 (using the remark that $(\operatorname{Im} E)^{-1}$ can be replaced by $\operatorname{dist}(E, \operatorname{spec}(H))$ in bounds), as $\operatorname{Re} E \to -\infty$, $\operatorname{Im} E = c > 0$, $\|M_+ + E^{-1}\mathbb{1}\| \to 0$, so $\det(M_+) \sim (-E)^{-l}$, and we take the branch of ln with $\operatorname{Im}\ln(\det(M_+)) \to 0$ in this limit.

We also define

$$N_\pm(E) = \operatorname{Im} M_\pm + \tfrac{1}{2}\operatorname{Im} E.$$

**Theorem 6.2D.** (a) $w(E) = \mathbb{E}(\operatorname{tr}\ln(M_-))$,

(b) $w'(E) = \mathbb{E}(\operatorname{tr}(G_E(0,0;\omega)))$,

(c) $-\operatorname{Re} w(E) = \gamma(E)$,

(d) $\mathbb{E}([\operatorname{tr}(N_\pm(E,\omega))]^{-1}) \leqq 2\gamma(E)/\operatorname{Im} E$,

(e) $\mathbb{E}\left(\ln\left[\det\left(1 + \dfrac{\operatorname{Im} E}{\operatorname{Im} M_+}\right)\right]\right) = 2\gamma(E)$.

*Proof.* (a, b, c) We begin with (c), which follows from

$$\ln[\det F_+(n)] = \sum_{j=0}^{n-1} \ln\det M_+(T^j n).$$

Using the Thouless formula (Theorem A.1 in the appendix), we see that

$$\operatorname{Re} w(E) = -\int \ln|E - E'|\,dk(E') = \operatorname{Re}[-\int \ln(E' - E)\,dk(E')].$$

It follows that

$$w(E) = -\int \ln(E' - E)dk + ia$$

for some real number $a$. As $\operatorname{Re} E \to -\infty$, $\operatorname{Im} w(E) \to 0$ while $\operatorname{Im}[-\ln \cdots] \to -i\pi$. It follows that

$$w(E) = i\pi - \int \ln(E' - E)dk(E'),$$

and so

$$w'(E) = \int \frac{1}{E' - E} dk(E'),$$

proving (b). Since the same argument holds if we start with $M_-$, (a) holds.

(e) Taking imaginary parts of $M_+(T^n\omega) = V(n) - E_- M_+(T^{n-1}\omega)^{-1}$, we obtain

$$\operatorname{Im} M_+(T^n\omega) = -\operatorname{Im} E + [M_+(T^{n-1}\omega)^*]^{-1} \operatorname{Im} M_+(T^{n-1}\omega)[M_+(T^{n-1}(\omega))]^{-1}. \tag{6.6D}$$

Thus

$$\ln\det\left(1 + \frac{\operatorname{Im} E}{\operatorname{Im} M_+(T^n\omega)}\right) = -\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4,$$

where $\alpha_i = \ln|\det(\beta_j)|$ and

$$\beta_1 = M_+(T^{n-1}\omega)^*, \quad \beta_2 = M_+(T^n\omega), \quad \beta_3 = \operatorname{Im} M_+(T^{n-1}\omega); \quad \beta_4 = \operatorname{Im} M_+(T^{n-1}\omega).$$

Since $\mathbb{E}(\alpha_3) = \mathbb{E}(\alpha_4)$, and by (c), $\mathbb{E}(\alpha_1) = \mathbb{E}(\alpha_2) = -\gamma(E)$, (e) holds.

(d) follows from (e) as in [19] using

$$\ln\det(1 + C) = \operatorname{tr}(\ln(1 + C)),$$

and

$$\ln(1 + x) \geqq x/(1 + \tfrac{1}{2}x). \quad \square$$

**Corollary 6.3.** *Let*

$$N_\pm = \operatorname{Im} M_\pm,$$
$$B = (M_+ + M_-),$$

*in the continuum case, and*

$$N_\pm = \operatorname{Im} M_\pm + \tfrac{1}{2}\operatorname{Im} E$$
$$B = M_+ + M_- + E - V(0)$$

*in the discrete case. Then*

$$\mathbb{E}(\operatorname{tr}[N_+^{-1} + N_-^{-1} + 4\operatorname{Im}(B^{-1})]) \leqq 4\left[(\operatorname{Im} E)^{-1}\gamma(E) - \frac{\partial\gamma}{\partial \operatorname{Im} E}\right].$$

*Proof.* By Theorem 6.2(b) and (c) and formulae in Sect. 3 for $G$:

$$\mathbb{E}(\operatorname{tr}[\operatorname{Im}(B^{-1})]) = -\frac{\partial\gamma}{\partial \operatorname{Im} E}.$$

Since $\mathbb{E}(\operatorname{tr} N_\pm^{-1}) \leqq 2\gamma(E)/\operatorname{Im} E$ by Theorem 6.3(d), we have the desired result. $\square$

In the $l=1$ case, we used

$$N_+^{-1}+N_-^{-1}+4\,\mathrm{Im}(B^{-1})=(N_+^{-1}+N_-^{-1})[\{(N_+-N_-)^2+(\mathrm{Re}\,B)^2\}/|B|^2].$$

While this is false in the present case due to non-commutativity, the following lemma suffices for our proof in the next section.

**Lemma 6.4.** *Let $Y_+$ and $Y_->0$ and $X=X^*$ be $l\times l$ matrices. Then*

$$Y_+^{-1}+Y_-^{-1}+4\,\mathrm{Im}(X+i(Y_++Y_-))^{-1}\geqq 0,$$

*and it equals zero if and only if $X=0$ and $Y_+=Y_-$.*

*Proof.* By straightforward calculations:

$$Y_+^{-1}+Y_-^{-1}+4\,\mathrm{Im}(Y_++Y_-)^{-1}=(Y_+^{-1}-Y_-^{-1})(Y_+^{-1}+Y_-^{-1})^{-1}(Y_+^{-1}-Y_-^{-1})$$

and, for $Y=Y_++Y_->0$,

$$\mathrm{Im}(X+iY)^{-1}-\mathrm{Im}(iY)^{-1}=(X+iY)^{-1}XY^{-1}X(X-iY)^{-1}.$$

Each of these terms is non-negative and the first is zero if and only if $Y_+^{-1}=Y_-^{-1}$, and the second is zero if and only if $X=0$. □

The inequalities so far suffice to handle the results at energies where all $\gamma_j$ are zero. To get the results when only some of the $\gamma$'s are zero, we need

**Theorem 6.5.** *Let* $\mathrm{Im}\,E>0$. *Let $\mu_j^\pm(E,\omega)$ denote the eigenvalue of* $\mathrm{Im}\,M_\pm(E,\omega)$ *ordered by $\mu_1^\pm(E,\omega)\geqq\cdots\geqq\mu_l^\pm(E,\omega)>0$. Then, on $\mathbb{C}_+$ for $j=1,\dots,l$:*

$$\mathbb{E}\left(\sum_{k=1}^{j}\frac{1}{\mu_k^\pm(E,\omega)}\right)\leqq 2(\mathrm{Im}\,E)^{-1}\sum_{k=1}^{j}\gamma_{l+1-k}(E)$$

*in the continuum case, and*

$$\mathbb{E}\left(\sum_{k=1}^{j}\frac{1}{\mu_k^\pm(E,\omega)+\frac{1}{2}\mathrm{Im}\,E}\right)\leqq 2(\mathrm{Im}\,E)^{-1}\sum_{k=1}^{j}\gamma_{l+1-k}(E)$$

*in the discrete case.*

*Proof.* We begin with the discrete case:

Write, for $E$ fixed,

$$Y_-(n)=\mathrm{Im}\,M_-(E,T^n\omega)$$

and

$$G(n)=Y(n)^{1/2}F_-(n).$$

By multiplying the analog of (6.6D) for $M_-$ by $F_-(n)$ on the right and $F_-(n)^*$ on the right, and using

$$F_-(n+1)=-M_-(n+1)^{-1}F_-(n),$$

we obtain

$$\begin{aligned}G(n+1)^*G(n+1)&=G^*(n)G^*(n)+\mathrm{Im}\,EF_-(n)^*F_-(n)\\&=G^*(n)\left[1+\frac{\mathrm{Im}\,E}{Y(n)}\right]G(n).\end{aligned}$$

Using $\Lambda^{l-j}$ of this equation, we obtain

$$\Lambda^{l-j}(G^* G(n+1)) \leqq \left\| \Lambda^{l-j}\left(1 + \frac{\operatorname{Im} E}{Y(n)}\right) \right\| \Lambda^{l-j}(G^* G(n)).$$

But

$$\left\| \Lambda^{l-j}\left(1 + \frac{\operatorname{Im} E}{Y(n)}\right) \right\| = \prod_{k=1}^{l-j} \left(1 + \frac{\operatorname{Im} E}{\mu_{l+1-k}^{-}(E, T^n \omega)}\right).$$

It follows that

$$\frac{1}{n} \ln \| \Lambda^{l-j}(G^* G(n+1)) \| \leqq \frac{1}{n} \ln \| \Lambda^{l-j}(Y(0)) \| + \sum_{k=1}^{l-j} \frac{1}{n} \sum_{m=0}^{n-1} \ln\left(1 + \frac{\operatorname{Im} E}{\mu_{l+1-k}^{-}(E, T^m \omega)}\right).$$

Since $F_{-} = Y^{-1/2} G$, we have

$$\| \Lambda^{l-j}(G^* G(n+1)) \| \geqq \| \Lambda^{l-j}[Y(n+1)]^{-1} \|^{-1} \| \Lambda^{l-j} F_{-}(n+1) \|^2. \tag{6.7}$$

Since $Y$ is bounded above and below, we obtain, by taking $n \to \infty$:

$$2(\gamma_1 + \cdots + \gamma_{l-j}) \leqq \sum_{k=1}^{l-j} \mathbb{E}\left( \ln\left(1 + \frac{\operatorname{Im} E}{\mu_{l+1-k}^{-}(E, \omega)}\right)\right).$$

Since Theorem 6.2D(e) can be rewritten

$$2(\gamma_1 + \cdots + \gamma_l) = \sum_{k=1}^{l} \mathbb{E}\left( \ln\left(1 + \frac{\operatorname{Im} E}{\mu_{l+1-k}^{-}(E, \omega)}\right)\right),$$

we obtain

$$\sum_{k=1}^{j} \mathbb{E}\left( \ln\left(1 + \frac{\operatorname{Im} E}{\mu_k^{-}(E, \omega)}\right)\right) \leqq 2 \sum_{k=1}^{j} \gamma_{l+1-k}(E).$$

since

$$\ln(1+x) \geqq \frac{x}{1+\frac{1}{2}x}, \quad \ln\left(1 + \frac{\operatorname{Im} E}{\mu_{-}}\right) \geqq \frac{\operatorname{Im} E}{\mu_{-} + \frac{1}{2}\operatorname{Im} E},$$

yielding the result for the discrete case.

The continuum case is similar. From (6.6C), we obtain

$$\begin{aligned}(F_{-}^* Y F_{-})' &= F_{-}^*(-X + iY) Y F_{-} + F_{-}^*(\operatorname{Im} E + XY + YX) F_{-} + F_{-}^* Y(-X - iY) F_{-} \\ &= \operatorname{Im} E (F_{-}^* F_{-}).\end{aligned}$$

Define, for $A: \mathbb{C}^l \to \mathbb{C}^l$, a map

$$d\Lambda^{l-j}(A): \Lambda^{l-j}(\mathbb{C}^l) \to \Lambda^{l-j}(\mathbb{C}^l)$$

by

$$d\Lambda^{l-j}(A)(e_1 \Lambda \cdots \Lambda e_{l-j}) = \sum_{k=1}^{l-j} (e_1 \Lambda \cdots A e_k \Lambda \cdots e_{l-j})$$

(so $d\Lambda(A) = (d/dt)\Lambda(e^{tA})|_{t=0}$). Then with $G = \sqrt{Y} F_{-}$, we find

$$\begin{aligned}[\Lambda^{l-j}(G^* G)]' &= \operatorname{Im} E \Lambda^{l-j}(G^*) d\Lambda^{l-j}(Y^{-1}) \Lambda^{l-j}(G) \\ &\leqq \operatorname{Im} E \| d\Lambda^{l-j}(Y^{-1}) \| \Lambda^{l-j}(G^* G),\end{aligned}$$

so

$$\Lambda^{l-j}(G^* G(x)) \leqq \exp\left(\int_0^x \operatorname{Im} E \,\|d\Lambda^{l-j}(Y^{-1}(y))\|\, dy\right)\Lambda^j(G^* G(0)).$$

Thus, taking logs, dividing by $x$ and taking $x$ to infinity, and using (6.7),

$$2(\gamma_1(E) + \cdots + \gamma_{l-j}(E)) \leqq \mathbb{E}\left(\sum_{k=1}^{l-j} \operatorname{Im} E[\mu_{l+1-k}^-(E,\omega)]^{-1}\right).$$

Theorem 6.2C(d) says that

$$2(\gamma_1 + \cdots + \gamma_l) = \sum_{k=1}^{l} \mathbb{E}(\operatorname{Im} E[\mu_k^-(E,\omega)]^{-1}),$$

yielding the required result. $\square$

We will need another version of Theorem 6.5 which is not strictly weaker but is often weaker as $\gamma_k(E_0 + i\varepsilon)$ is normally monotone decreasing in $\varepsilon$.

**Theorem 6.6.** *Fix $E_0$ real with $\lim_{\varepsilon\downarrow 0} \gamma_k(E_0 + i\varepsilon) = \gamma_k(E_0)$ for all $k$. The inequalities in Theorem 6.5 remain true if $E = E_0 + i\varepsilon$ and $\sum_{k=1}^{j} \gamma_{l+1-k}(E)$ is replaced by*

$$\sum_{k=1}^{j} \gamma_{l+1-k}(E_0 + i\varepsilon) + \sum_{k=j+1}^{l} [\gamma_{l+1-k}(E_0 + i\varepsilon) - \gamma_{l+1-k}(E_0)].$$

*Proof.* We describe the continuum case. Since

$$\frac{Y(E_0 + i\varepsilon)}{\varepsilon} = \int \frac{d\Sigma(E')}{(E' - E_0)^2 + \varepsilon^2},$$

we see that $Y(E_0 + i\varepsilon)/\varepsilon$ is monotone decreasing as $\varepsilon$ increases. Thus, $\mu_k(E_0 + i\varepsilon)/\varepsilon$ is monotone decreasing, and so for $\varepsilon > \delta > 0$,

$$\frac{\varepsilon}{\mu_k(E_0 + i\varepsilon)} \geqq \frac{\delta}{\mu_k(E_0 + i\delta)}.$$

Thus

$$\mathbb{E}\left(\sum_{k=1}^{j} \frac{\varepsilon}{\mu_k^\pm(E_0 + i\varepsilon, \omega)}\right) \leqq \mathbb{E}\left(\sum_{k=1}^{l} \frac{\varepsilon}{\mu_k^\pm(E_0 + i\varepsilon, \omega)}\right) - \mathbb{E}\left(\sum_{k=j+1}^{l} \frac{\delta}{\mu_k^\pm(E_0 + i\delta, \omega)}\right).$$

By Theorem 6.2C(d),

$$\mathbb{E}\left(\sum_{k=1}^{l} \frac{\varepsilon}{\mu_k^\pm(E_0 + i\varepsilon, \omega)}\right) = 2\sum_{k=1}^{l} \gamma_{l+1-k}(E_0 + i\varepsilon),$$

and the proof of Theorem 6.5 shows that

$$\mathbb{E}\left(\sum_{k=j+1}^{l} \frac{\delta}{\mu_k^\pm(E_0 + i\delta, \omega)}\right) \geqq 2\sum_{k=j+1}^{l} \gamma_{l+1-k}(E_0 + i\delta),$$

so

$$\mathbb{E}\left(\sum_{k=1}^{l} \frac{1}{\mu_k^{\pm}(E_0+i\varepsilon,\omega)}\right) \leqq 2\varepsilon^{-1} \sum_{k=1}^{j} \gamma_{l+1-k}(E_0+i\varepsilon)$$
$$+ 2\varepsilon^{-1} \sum_{k=j+1}^{l} [\gamma_{l+1-k}(E_0+i\varepsilon) - \gamma_{l+1-k}(E_0+i\delta)].$$

Now take $\delta$ to zero. □

## 7. The Main Theorems and Their Proofs

Given the inequalities in the last section, the main theorems have proofs which are, in essence, the same as the proofs in [10].

**Lemma 7.1.** *Let*

$$Q = \left\{ E \in \mathbb{R} \,\middle|\, \gamma(E) = \lim_{\varepsilon \downarrow 0} \gamma(E+i\varepsilon); \left\{ \frac{1}{\varepsilon}[\gamma(E+i\varepsilon) - \gamma(E)] - \frac{\partial \gamma}{\partial \varepsilon}(E+i\varepsilon) \right\} \right.$$
$$\left. \text{goes to } 0 \text{ as } \varepsilon \downarrow 0; \lim_{\varepsilon \downarrow 0} \frac{\partial \gamma}{\partial \varepsilon}(E+i\varepsilon) \text{ exists.} \right\}$$

*Then* $|\mathbb{R} \backslash Q| = 0$.

*Proof.* $w'$ is Herglotz, so by the standard theory of such functions, $\partial\gamma/\partial\varepsilon(E_0+i\varepsilon)$ has a limit for almost all $E_0 \in \mathbb{R}$. Moreover, since $w$ is Herglotz and $\gamma$ is subharmonic, $\gamma(E_0+i\varepsilon) \to \gamma(E_0)$ for almost all $E_0$. For $E_0$ in both sets, the desired results are valid. □

**Theorem 7.2.** *For a stochastic Schrödinger operator or Jacobi matrix with $\mathbb{C}^l$-valued wave functions, let*

$$S_j = \{E \in \mathbb{R} \,|\, \textit{exactly } 2j \ \gamma\text{'s are } 0\}.$$

*Then*

(a) *$S_j$ is the essential support of* a.c. *spectrum of multiplicity* $2j$.
(b) *There is no odd multiplicity* a.c. *spectrum.*

*Proof.* Since $\gamma_{2l-j+1} = -\gamma_j$, the $S_j$'s cover $\mathbb{R}$ so (a) implies (b). By Theorem 5.4, the a.c. spectral multiplicity on $S_j$ is at most $2j$, so if we show it is at least $2j$, we have the result if we note again that the $S_j$'s cover $\mathbb{R}$.

By the general theory of the Stieltjes transform,

$$d\Sigma_{\mathrm{ac}}^{+}(E) = \frac{1}{\pi} \operatorname{Im} M_{\pm}(E+i0)\,dE,$$

where $M_{\pm}(E+i0)$ exists for a.e. $E$ and $\omega$. For such an $E$ and $\omega$, $\mu_k^{\pm}(E+i\varepsilon,\omega) \to \mu_k^{\pm}(E+i0,\omega)$. By Fubini's theorem, $Q_1 = \{E \,|\, \text{a.e. } \omega, M_{\pm}(E+i\varepsilon) \to M_{\pm}(E+i0)\}$ has full measure. If $E_0$ is also in the set $Q$ of Lemma 7.1, and in $S_j$, then

$$\varepsilon^{-1}\left[\sum_{k=1}^{j}\gamma_{l+1-k}(E_0+i\varepsilon)+\sum_{k=j+1}^{l}(\gamma_{l+1-k}(E_0+i\varepsilon)-\gamma_{l+1-k}(E_0))\right]$$
$$=\varepsilon^{-1}[\gamma(E_0+i\varepsilon)-\gamma(E_0)]$$

has a finite limit. Thus, by Theorem 6.6 and Fatou's lemma:

$$\mathbb{E}([\mu_k^{\pm}(E+i0,\omega)]^{-1})<\infty, \quad k=1,\dots,j.$$

It follows that a.e. $E\in Q\cap Q_1\cap S_j$ and a.e. $\omega, \mu_k^{\pm}(E+i0,\omega)>0$, and thus that $\operatorname{Im} M_{\pm}(E+i0)$ has rank at least $j$. Thus, $d\Sigma_{\mathrm{ac}}^{\pm}\restriction Q\cap Q_1\cap S_j$ has multiplicity at least $j$. Since $H-(H^+\oplus H^-)$ has finite rank, $H$ has a.c. multiplicity at least $2j$ (since $B-C$ finite rank implies $B_{\mathrm{ac}}$ is unitarily equivalent to $C_{\mathrm{ac}}$ by the Kato–Rosenbljum theorem, see e.g. [15]). □

**Theorem 7.3.** *For* a.e. *$E$ in $S_l$ and $\omega$,*

$$M_-(E+i0,\omega)=-M_+(E+i0,\omega)^*$$

*in the continuum case, and*

$$M_-(E+i0,\omega)=-(M_+(E+i0,\omega)+E-V(0))^*$$

*in the discrete case.*

*Proof.* By the argument in the last theorem for a.e. $E\in S_l$, $\operatorname{Im} M_{\pm}$ are strictly positive. If $E\in Q_1$, Fatou's lemma, Corollary 6.3 and Lemma 6.4 imply that

$$\mathbb{E}([(\operatorname{Im} M_+)(E+i0)]^{-1}+[(\operatorname{Im} M_-)(E+i0)]^{-1}+4\operatorname{Im}(B(E+i0)^{-1}))=0,$$

where $B=M_++M_-$ (respectively $M_++M_-+E-V(0)$) in the continuum (respectively discrete) case. By Lemma 6.4, it follows that

$$\operatorname{Im} M_+=\operatorname{Im} M_-;\quad \operatorname{Re} B=0,$$

which is the required result. □

**Theorem 7.4.** *If $S_l$ contains an open interval, $I$, then $H$ has purely* a.c. *spectrum on $I$.*

*Proof.* By Theorem 7.3, $\operatorname{Re} B(E+i0)=0$ on $I$. By the reflection principle for Herglotz functions, $B$ is analytic through $I$. Moreover, $\operatorname{Im} B(E+i0)\geqq 0$ in a complex neighborhood of $I$, so by the open mapping theorem, $\operatorname{Im} B(E+i0)>0$ on $I$. It follows that $-B^{-1}$ is analytic on $I$, and so the Green's function has an analytic continuation through $I$, which implies that there is no singular spectrum. □

**Theorem 7.5.** *If $|S_l|>0$, then $V$ is deterministic.*

*Proof.* $V$ on $\{x\leqq 0\}$ (respectively $\{n<0\}$) in the continuum (respectively discrete) case determines $M_-$. By Theorem 7.3, this determines $M_+$ on $S_l$ (with $V(0)$ in the discrete case). Since $|S_l|>0$, this determines $M_+$ on all of $\mathbb{C}_+$ by analytic continuation for Herglotz functions. By Theorem 4.1 (respectively Theorem 4.2), $V$ is determined on $\{x\geqq 0\}$ (respectively $\{n>0\}$). □

## Appendix. The Thouless Formula in the Discrete Case

In this appendix, we will prove the Thouless formula for the discrete operator (1.2), with $V$ matrix-valued. Such a formula for the strip was first proven by Craig–Simon [6]. We give a proof here for two reasons. First, their statement and proof is not for the general class of $V$ we consider in this paper. Second, by using the techniques from this paper, we can give a more transparent proof than [6].

We need the following from the paper itself:

(1) The basic definition of $F_{\pm}, \Phi$ from Sect. 2.

(2) The relation between the Lyaponov exponents for $H$ and the decay (respectively growth) of $F_{\pm}$ (Lemma 5.3), in particular, that for $\operatorname{Im} E > 0$:

$$\lim_{n\to\infty} n^{-1} \ln|\det F_-(n,E)| = \sum_{j=1}^{l} \gamma_j(E), \tag{A.1}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln \|F_+(n,E)\| = \lim_{n\to-\infty} \frac{1}{n} \ln \|F_-(n,E)\| = -\gamma_l(E). \tag{A.2}$$

(3) For $\operatorname{Im} E > 0$, $\gamma_l(E) > 0$ (Proposition 3.3 and (A.2) above).

**Theorem A.1.** *Let $V_\omega$ be an $l \times l$ real-symmetric matrix-valued bounded ergodic process, and let $h_\omega$ be given by (1.2). Let $\gamma_1, \dots, \gamma_l$ be the first $l$ Lyaponov exponents, and let $k(E)$ be the integrated density of states. Then, (the integral is for $E' \in \mathbb{R}$):*

$$\gamma_1(E) + \cdots + \gamma_l(E) = \int \ln|E - E'|\, dk(E')$$

*for all $E \in \mathbb{C}$.*

*Proof.* By the subharmonicity argument of Craig–Simon [5], we need only prove the result for $\operatorname{Im} E \neq 0$ and thus only for $\operatorname{Im} E > 0$, so fix $E_0$ with $\operatorname{Im} E_0 > 0$.

Since $H$ does not have $E_0$ as eigenvalue, $\{F_+(x)a \mid a \in \mathbb{C}^l\}$ and $\{F_-(x)a \mid a \in \mathbb{C}^l\}$ are disjoint, and so their sum is the entire $2l$ dimensional family of solutions. Thus, for $l \times l$ matrices $A, B$:

$$\Psi(x) = F_-(x)A + F_+(x)B.$$

If $Aa = 0$ for some $a$, then $\Psi a = F_+(x)Ba$ would lie in $l^2(0,\infty)$, so $H^+$ would have $E$ as eigenvalue, which is impossible. Thus, $\operatorname{Ker}(A) = 0$, so $A$ is invertible. Writing

$$\Psi = (1 + F_+ BA^{-1} F_-^{-1}) F_- A,$$

and using (A.2), we conclude

$$\lim_{n\to+\infty} \frac{1}{n} \ln|\det \Psi| = \lim_{n\to\infty} \frac{1}{n} \ln|\det F_-|$$

$$= \gamma_1 + \cdots + \gamma_l$$

by (A.1).

Next, consider $\Psi(n,E)$ as a function of $E$. Since

$$\Psi(n+1,E) = (E - V(n))\,\Psi(n,E) - \Psi(n-1,E)$$

and $\Psi(1,E) = \mathbb{1}$, we see inductively that $\Psi(n+1,E)$ is a monic $l \times l$ matrix polynomial of degree $n$ so $\det(\Psi(n+1, E))$ is a monic (scalar) polynomial of degree $nl$. On the other hand, $\Psi(n+1)a = 0$ for some $a$ if and only if the Hamiltonian with boundary conditions $\Psi(0) = \Psi(n+1) = 0$ has eigenvalue $E$. Thus, the zeros of $\det \Psi(n+1)$ are exactly those eigenvalues. It is easy to see that the multiplicities even agree. One can now use the argument of Thouless [21] (see [3, 7]). □

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