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Etiology of IRF Models

V. Pasquier

Service de Physique Théorique, Institut de Recherche Fondamentale, CEA, CEN-Saclay,
F-91191 Gif-sur-Yvette Cedex, France

Abstract. We show that a class of $2D$ statistical mechanics models known as IRF models can be viewed as a subalgebra of the operator algebra of vertex models. Extending the Wigner calculus to quantum groups, we obtain an explicit intertwiner between two representations of this subalgebra.

Two major progresses have recently been made in the understanding of two dimensional lattice models. The first is the classification of rational and trigonometric solutions of the Yang-Baxter equations for a class of models known as vertex models [1–5]. The second is the discovery of many representatives of another class known as IRF (interacting round a face) models and their study in connection with conformal field theories [6–11]. In the appendix of [11] it was shown that both classes correspond to different representations of the same algebra, and it is the aim of this letter to complete the identification by building an explicit intertwiner between them. The method we use is an application to the quantum group case of Ocneanu's cell technique [12].

Quantum Wigner Calculus. Consider the associative algebra $\hat{U}(SU(2))$ [4, 5] generated by the symbols $q^{h/2}$, J_+, J_- under the following relations:

$$q^{h/2}q^{-h/2}=q^{-h/2}q^{h/2}=1 \ , \quad q^{h/2}J_+q^{-h/2}=qJ_+ \ , \quad q^{h/2}J_-q^{-h/2}=q^{-1}J_- \ ,$$

$$[J_+J_-]=\frac{q^h-q^{-h}}{q-q^{-1}} \ . \tag{1}$$

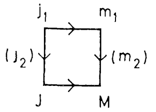
We denote by $\Delta^{(N)}$ the coproduct homomorphism $\Delta^{(N)}\hat{U}\to\hat{U}\otimes^N$ (N fold tensor product)

$$\Delta^{(N)}(q^{h/2})=q^{h/2}\otimes q^{h/2}\ldots\otimes q^{h/2} \ ,$$

$$\Delta^{(N)}(J_\pm)=\sum_{v=1}^{N} q^{h/2}\otimes\ldots\otimes q^{h/2}\otimes \overset{v}{J_\pm}\otimes q^{-h/2}\ldots\otimes q^{-h/2} \ . \tag{2}$$

In what follows, unless we specify it, q is not a root of unity. For $j\in\frac{1}{2}\mathcal{N}$, V_j denotes an irreducible $\hat{U}$ module of spin j and $|jm\rangle$ its canonical bases. Let V_{j_1} V_{j_2} V_J be three such modules, an intertwiner between $V_{j_1}\otimes V_{j_2}$ and V_J is given by the (q) Wigner coefficients

$$\langle (j_1j_2)JM|j_1m_1j_2m_2\rangle = \tag{3}$$



Following the arrows from the upper left to the down right corner, we describe the two states $|(j_1j_2)JM\rangle$ and $|j_1m_1\rangle\otimes|j_2m_2\rangle$. The Wigner coefficients can be deduced from the recursion relations:

$$\begin{aligned}&((J-M+1)(J+M))^{1/2}\langle JM-1|j_1m_1j_2m_2\rangle\\&\quad=q^{-m_2}((j_1-m_1)(j_1+m_1+1))^{1/2}\langle JM|j_1m_1+1j_2m_2\rangle\\&\quad\quad+q^{m_1}((j_2-m_2)(j_2+m_2+1))^{1/2}\langle JM|j_1m_1j_2m_2+1\rangle\ ,\end{aligned}$$

where

$$(n)=\frac{q^n-q^{-n}}{q-q^{-1}}\ . \tag{4}$$

We list them for $j_2=\frac{1}{2}$, 1 Table 1.

Table 1. Wigner Coefficients[a]

$(jm|j_1m_1\tfrac{1}{2}\mu)$

$j=$	$\mu=\frac{1}{2}$	$\mu=-\frac{1}{2}$
$j_1+\frac{1}{2}$	$q^{(m-j_1-\frac{1}{2})/2}\left(\frac{(j_1+m+\frac{1}{2})}{(2j_1+1)}\right)^{1/2}$	$q^{(m+j_1+\frac{1}{2})/2}\left(\frac{(j_1-m+\frac{1}{2})}{(2j_1+1)}\right)^{1/2}$
$j_1-\frac{1}{2}$	$-q^{(m+j_1+\frac{1}{2})/2}\left(\frac{(j_1-m+\frac{1}{2})}{(2j_1+1)}\right)^{1/2}$	$q^{(m-j_1-\frac{1}{2})/2}\left(\frac{(j_1+m+\frac{1}{2})}{(2j_1+1)}\right)^{1/2}$

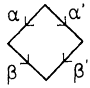
$(jm|j_1m_1 1\mu)$

$j=$	$\mu=1$	$\mu=0$	$\mu=-1$
j_1+1	$q^{m-j_1-1}\left(\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right)^{1/2}$	$q^m\left(\frac{(2)(j_1-m+1)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right)^{1/2}$	$q^{m+j_1+1}\left(\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}\right)^{1/2}$
j_1	$-q^m\left(\frac{(2)(j_1+m)(j_1-m+1)}{(2j_1)(2j_1+2)}\right)^{1/2}$	$\frac{q^{m-j_1-1}(j_1+m)-q^{j_1+m+1}(j_1-m)}{((2j_1)(2j_1+2))^{1/2}}$	$q^m\left(\frac{(2)(j_1-m)(j_1+m+1)}{(2j_1)(2j_1+2)}\right)^{1/2}$
j_1-1	$q^{m+j_1}\left(\frac{(j_1-m)(j_1-m+1)}{(2j_1)(2j_1+1)}\right)^{1/2}$	$-q^m\left(\frac{(2)(j_1-m)(j_1+m)}{(2j_1)(2j_1+1)}\right)^{1/2}$	$q^{m-j_1}\left(\frac{(j_1+m+1)(j_1+m)}{(2j_1)(2j_1+1)}\right)^{1/2}$

[a] The exponents of q are ordinary [not (q)] numbers

Let $V_j V_{g_1} V_{g_2}$ be $\hat{U}$-modules and T a homomorphism of $V_{g_1}\otimes V_{g_2}$ in $V_{g_2}\otimes V_{g_1}$ such that $\Delta^{(2)}(X)T=T\Delta^{(2)}(X)$ for all $X\in\hat{U}$. Matrix elements of T can be described in the bases $\langle g_2\alpha|\otimes\langle g_1\beta|, |g_1\alpha'\rangle\otimes|g_2\beta'\rangle$:

$$\langle g_2\alpha|\otimes\langle g_1\beta|T|g_1\alpha'\rangle\otimes|g_2\beta'\rangle=$$



$$=\sigma_{\alpha\beta,\alpha'\beta'}\ . \tag{5}$$

Alternatively, we can recouple angular momentums:

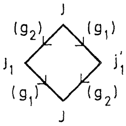
$$V_j\otimes V_{g_2}=\bigoplus_{j_1}V_{j_1}\ ,\qquad V_j\otimes V_{g1}=\bigoplus_{j_1'}V_{j_1'}\ ,$$
$$V_{j_1}\otimes V_{g_1}=\bigoplus_{J}V_J\ ,\qquad V_{j_1'}\otimes V_{g_2}=\bigoplus_{J'}V_{J'}\ ,$$

and describe matrix elements of T as:

$$\langle(jg_2)j_1g_1JM|T|(jg_1)j_1'g_2J'M'\rangle=\delta_{JJ'}\delta_{MM'}\langle(jg_2)j_1g_1J\|T\|(jg_1)j_1'g_2J\rangle\ . \tag{6}$$

The reduced matrix element being denoted

$$\sigma^{(jJ)}_{j_1j_1'}=$$



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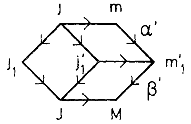
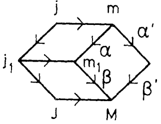
The two representations are related by Wigner coefficients through the following equations:

$$\sum_{j_1'm_1'}$$



$$=\sum_{\substack{\alpha\beta\\ m_1}}$$



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$$\sum_{j_1'm_1'}\sigma^{(jJ)}_{j_1j_1'}\langle j_1'm_1'|jmg_1\alpha'\rangle\langle JM|j_1'm_1'g_2\beta'\rangle$$
$$=\sum_{\alpha\beta m_1}\sigma_{\alpha\beta,a'\beta'}\langle j_1m_1|jmg_2\alpha\rangle\langle JM|j_1m_1g_1\beta\rangle\ . \tag{7}$$

We call them respectively vertex and path representations of T.

Path Representation of the R Matrix. Let V_g be a $\hat{U}$ module. In End$(V_g\otimes V_g)$, the vertex R matrix obeying the Yang Baxter equation [3–5] is characterized by the following equations:

$$\begin{gathered}[R(x),\Delta^{(2)}(X)]=0\quad\forall X\in\hat{U}\ ,\\ R(x)(xj_-\otimes q^{h/2}+q^{-h/2}\otimes j_-)=(j_-\otimes q^{h/2}+xq^{-h/2}\otimes j_-)R(x)\ .\end{gathered} \tag{8}$$

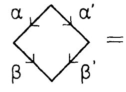
The first equation expresses that $R(x)$ is in the commutant of $\Delta^{(2)}(\hat{U})$. Hence it can be written

$$R(x)=\sum_j\varrho_j(x)P^{(j)}\ . \tag{9}$$

With $P^{(j)}$ projectors onto irreducible components V_j of $V_g \otimes V_g$. The second equation determines $\varrho_j(x)$ and we quote the result from [4]:

$$\frac{\varrho_{j-1}(x)}{\varrho_j(x)}=\frac{x-q^{2j}}{1-xq^{2j}} \ . \tag{10}$$

The vertex matrix elements of $P^{(j)}$ are by definition



$$\sigma_{\alpha\beta,\alpha'\beta'}= \quad =\sum_m \langle g\alpha g\beta | jm\rangle \langle jm | g\alpha' g\beta'\rangle \tag{11}$$

due to the charge conservations, σ breaks into block matrices according to the total charge $Q=\alpha+\beta=\alpha'+\beta'$. The path representation of $P(^{(j)}$ follows from (7).

Examples. (All matrices are symmetric)

$g=\frac{1}{2}$: (6 vertex model)

There are two projectors $P^{(0)}, P^{(1)}=1-P^{(0)}$, the matrix elements of $P^{(0)}$ are:

a) Vertex:

$$Q=1\ , \quad \sigma=0\ ,$$

$$Q=0\ , \quad \begin{pmatrix} \sigma_{1/2-1/2,1/2-1/2} & \sigma_{1/2-1/2,-1/21/2} \\ \sigma_{-1/21/2,1/2-1/2} & \sigma_{-1/21/2,-1/21/2} \end{pmatrix}=\frac{1}{(2)}\begin{pmatrix} q & -1 \\ -1 & q^{-1} \end{pmatrix} .$$

b) Path:

$$J=j\pm 1\ , \quad \sigma=0$$

$$J=j\ , \quad \begin{pmatrix} \sigma_{j-\frac{1}{2},j-\frac{1}{2}} & \sigma_{j-\frac{1}{2},j+\frac{1}{2}} \\ \sigma_{j+\frac{1}{2},j-\frac{1}{2}} & \sigma_{j+\frac{1}{2},j+\frac{1}{2}} \end{pmatrix}=\frac{1}{(2j+1)(2)}\begin{pmatrix} (2j) & * \\ \sqrt{(2j)(2j+2)} & (2j+2) \end{pmatrix} ,$$

$$R(x)=(1-xq^2)P^{(1)}+(x-q^2)P^{(0)} \ . \tag{12}$$

Both expressions of $P^{(0)}$ are known representations of the Temperley and Lieb-Jones algebra [13–14].

$g=1$.

There are 3 projectors $P^{(0)}$, $P^{(1)}$, $P^{(2)}=1-P^{(0)}-P^{(1)}$. The matrix elements of $P^{(0)}$ and $P^{(1)}$ are:

a) Vertex:

$P^{(0)}$:

$$Q=1,2\ , \quad \sigma=0\ ,$$

$$Q=0\ , \quad \begin{pmatrix} \sigma_{1-1,1-1} & \sigma_{1-1,00} & \sigma_{1-1,-11} \\ \sigma_{00,1-1} & \sigma_{00,00} & \sigma_{00,-11} \\ \sigma_{-11,1-1} & \sigma_{-11,00} & \sigma_{-11,-11} \end{pmatrix}=\frac{1}{(3)}\begin{pmatrix} q^{-2} & -q^{-1} & 1 \\ -q^{-1} & 1 & -q \\ 1 & -q & q^2 \end{pmatrix} .$$

$P^{(1)}$:

$$Q=1\ ,\quad \begin{pmatrix}\sigma_{10,10} & \sigma_{10,01}\\ \sigma_{01,10} & \sigma_{01,01}\end{pmatrix}=\frac{(2)}{(4)}\begin{pmatrix}q^{-2} & -1\\ -1 & q^2\end{pmatrix},$$

$$Q=-1\ ,\quad \begin{pmatrix}\sigma_{0-1,0-1} & \sigma_{0-1,-10}\\ \sigma_{-10,0-1} & \sigma_{-10,-10}\end{pmatrix}=\frac{(2)}{(4)}\begin{pmatrix}q^{-2} & -1\\ -1 & q^2\end{pmatrix},$$

$$Q=0\ ,\quad \begin{pmatrix}\sigma_{1-1,1-1} & \sigma_{1-1,00} & \sigma_{1-1,-11}\\ \sigma_{00,1-1} & \sigma_{00,00} & \varrho_{00,-11}\\ \sigma_{-11,1-1} & \sigma_{-11,00} & \sigma_{-11,-11}\end{pmatrix}$$

$$=\frac{(2)}{(4)}\begin{pmatrix}1 & q^{-1}-q & -1\\ q^{-1}-q & (q^{-1}-q)^2 & q-q^{-1}\\ -1 & q-q^{-1} & 1\end{pmatrix}.$$

b) Path:

$P^{(0)}$:

$$J=j\pm1\ ,\qquad \sigma=0\ ,$$

$$J=j\ ,\qquad \sigma=\begin{pmatrix}\sigma_{j-1,j-1} & \sigma_{j-1,j} & \sigma_{j-1,j+1}\\ \sigma_{j,j-1} & \sigma_{j,j} & \sigma_{j,j+1}\\ \sigma_{j+1,j-1} & \sigma_{j+1,j} & \sigma_{j+1,j+1}\end{pmatrix}$$

$$=\frac{1}{(3)(2j+1)}\begin{pmatrix}(2j-1) & * & *\\ \sqrt{(2j+1)(2j-1)} & (2j+1) & *\\ \sqrt{(2j+3)(2j-1)} & \sqrt{(2j+3)(2j+1)} & (2j+3)\end{pmatrix}.$$

$P^{(1)}$:

$$J=j+1\begin{pmatrix}\sigma_{j,j} & \sigma_{j,j+1}\\ \sigma_{j+1,j} & \sigma_{j+1,\sigma+1}\end{pmatrix}=\frac{(2)}{(4)(2j+2)}\begin{pmatrix}(2j) & *\\ \sqrt{(2j)(2j+4)} & (2j+4)\end{pmatrix},$$

$$J=j-1\begin{pmatrix}\sigma_{j,j} & \sigma_{j,j-1}\\ \sigma_{j-1,j} & \sigma_{j-1,j-1}\end{pmatrix}=\frac{(2)}{(4)(2j)}\begin{pmatrix}(2j+2) & *\\ -\sqrt{(2j+2)(2j-2)} & (2j-2)\end{pmatrix},$$

$$J=j\begin{pmatrix}\sigma_{j-1,j-1} & \sigma_{j-1,j} & \sigma_{j-1,j+1}\\ \sigma_{j,j-1} & \sigma_{j,j} & \sigma_{j,j+1}\\ \sigma_{j+1,j-1} & \sigma_{j+1,j} & \sigma_{j+1,j+1}\end{pmatrix}$$

$$=\frac{(2)}{(4)}\begin{pmatrix}\left(1-\dfrac{(2)}{(2j)(2j+1)}\right) & * & *\\ -\left(\dfrac{(2j-1)}{(2j+1)}\right)^{1/2}\dfrac{q^{2j+1}+q^{-2j-1}}{(2j)}, & \dfrac{2+q^{4j+2}+q^{-4j-2}}{(2j)(2j+2)} & *\\ -\dfrac{((2j-1)(2j+3))^{1/2}}{(2j+1)}, & \left(\dfrac{(2j+3)}{(2j+1)}\right)^{1/2}\dfrac{q^{2j+1}+q^{-2j-1}}{(2j+2)}, & \left(1-\dfrac{(2)}{(2j+1)(2j+2)}\right)\end{pmatrix},$$

$$R(x)=(1-xq^4)(1-xq^2)P^{(2)}+(x-q^4)(1-xq^2)P^{(1)}+(x-q^2)(x-q^4)P^{(0)} \tag{13}$$

Path Algebra. Let us fix V_g an irreducible $\hat{U}$ module. We define paths $(j)_{(g)}=(j_0=0,j_1,\dots,j_N)$ by a sequence $j_k\in\frac{1}{2}\mathcal{N}$ such that $V_{j_{k+1}}\subset V_{j_k}\otimes V_g$. We consider the matrix algebra generated by matrix units $((j)_{(g)},(j')_{(g)}), j_N=j'_N$ under the following multiplication law:

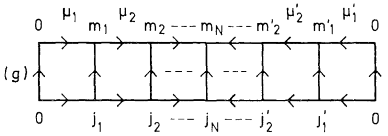
$$((j),(j'))((k),(k'))=\delta_{(j'),(k)}((j),(k')) \ . \tag{14}$$

In a similar way, we define a base of matrix units of $\mathrm{End}(V_g\otimes^N)$ by $((\mu),(\mu'))$; $(\mu)=(\mu_1,\mu_2,\dots,\mu_N)\ \mu_k\in\{-g,-g+1,\dots,g\}$. The first algebra is isomorphic to the commutant $\Delta^{(N)}(\hat{U})'$ of $\Delta^{(N)}(\hat{U})$ in $\mathrm{End}(V_g\otimes^N)$ and the inclusion (i) in $\mathrm{End}(V_g\otimes^N)$ can be described with Wigner coefficients:

$$i((j),(j'))=\sum_{((\mu),(\mu'))}C^{((\mu),(\mu'))}_{((j),(j'))}((\mu),(\mu')) \ ,$$

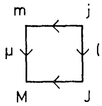
with

$$C^{((\mu),(\mu'))}_{((j),(j'))}=\sum_{(m)(m')}$$



,

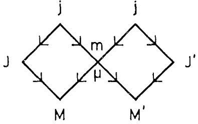
where



$$=(jmg\mu|(jg)JM) \ . \tag{15}$$

The fact that (i) defines a homomorphism requires the orthogonality relations:

$$\sum_{m,\mu}(JM|jmg\mu)(jmg\mu|J'M')=\delta_{JJ'}\delta_{MM'}=\sum_{m,\mu}$$



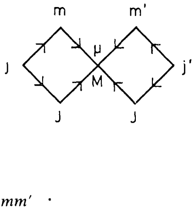
. (16)

A trace on $\Delta^{(N)}(U)'$ is defined [14, 15] by

$$\begin{aligned}\mathrm{tr}((\mu),(\mu'))&=\delta_{(\mu),(\mu')}\prod_{k=1}^{N}q^{-2\mu_k} \ ,\\ \mathrm{tr}((j),(j'))&=\delta_{(j),(j')}(2j_N+1) \ .\end{aligned} \tag{17}$$

Both expressions define the same trace due to the identity

$$(2j+1)\sum_{M,\mu}(JM|jmg\mu)(j'm'g\mu|JM)q^{-2\mu}=(2j+1)\sum_{M,\mu}$$



$$=(2J+1)\delta_{jj'}\delta_{mm'} \ . \tag{18}$$

The proofs can easily be done using the graphical representations.

Restricted Algebras. We define the (q) dimension of V_j by $\dim V_j = (2j+1)$. When $q = e^{\pi i/L}$, $L \in \mathcal{N}$ we restrict to values of j such that $(\dim V_j) > 0$. This imposes $j \leqq j_{\max} = \frac{L}{2} - 1$. In this case we define paths by the connection matrices $A^{(g)}_{jj'}$, $0 \leqq j, j', g \leqq j_{\max}$ determined by the recursion relations:

$$\begin{aligned} &A^{(0)} = 1 \ , \\ &A^{(1/2)} = 0 \quad \text{if} \quad |j - j'| \neq \tfrac{1}{2} \ , \\ &1 \quad \text{if} \quad |j - j'| = \tfrac{1}{2} \ , \\ &A^{(g-1)} + A^{(g)} = A^{(g-\frac{1}{2})} A^{(\frac{1}{2})} \ . \end{aligned} \tag{19}$$

A path $(j)_g$ is admissible if $0 \leqq j_k \leqq j_{\max}$ and $A^{(g)}_{j_k, j_{k+1}} = 1$. The path algebra is a in (14) with admissible paths. Then it can be shown that (i) defines an isomorphism between the path algebra and the quotient of the algebra generated by the projectors $P^{(j)}$ (expressed in the vertex representation) by the ideal of operators the product of which with any operator of the algebra is traceless. In particular, the matrix elements of $P^{(j)}$ are obtained by taking the generic expression (12), (13) (setting $q = e^{i\pi/L}$) and restricting them to admissible paths.

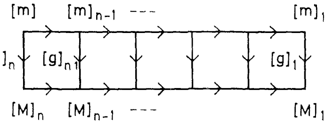
GL(n). What precedes can be extended to an arbitrary Lie group [9, 11]; let us consider $\hat{U}(gl(n))$ defined in [3, 5]. Irreducible $\hat{U}$ modules are characterised by a Young pattern $[m] = (m_{1n} \geqq m_{2n} \dots \geqq m_{nn} \geqq 0)$. The Gelfand Zetlin bases of $[m]$ consists of states (m) which are highest weights $[m]_k$ of $\hat{U}(gl(k))$ for the natural inclusion $gl(1) \subset \dots \subset gl(n)$. They are denoted by the symbol [16]

$$(m) = \begin{bmatrix} m_{1n} & & m_{2n} & \dots & m_{nn} \\ & m_{1,n-1} & & \dots & m_{n-1,n-1} \\ & & m_{11} & & \end{bmatrix} = \begin{bmatrix} [m] \\ [m]_{n-1} \\ [m]_1 \end{bmatrix} ,$$

$$m_{ij} \geqq m_{i,j-1} \geqq m_{i+1,j} \ . \tag{20}$$

Using similar notations as for $\hat{U}(SU(2))$, we denote the Wigner coefficients for $V_{[M]} \subset V_{[m]} \otimes V_{[g]}$ by

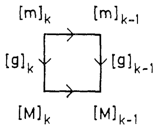
$$(([m]\,[g])\,[M]\,[M]_k|[m]\,[m]_k\,[g]\,[g]_k) =$$



. (21)

They factorize into the product of reduced coefficients

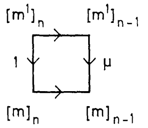
$$(([m]_k\,[g]_k)\,[M]_k\,[M]_{k-1}|[m]_k\,[m]_{k-1}\,[g]_k\,[g]_{k-1}) =$$



. (22)

If $[g] = (1, 0, 0, \dots)$ is the fundamental representation, $[g]_k = (\mu_k, 0, \dots) = \mu_k$ with $\mu_n = 1 \geqq \mu_{n-1} \dots \geqq \mu_1 \geqq 0$. The corresponding reduced coefficients are listed in

Table 2. Reduced Wigner Coefficients of $\hat{U}(gl(n))$[a]



$$\begin{aligned} m_{j,n} &= \qquad\qquad m'_{j,n}+\delta_{i,j} \\ m_{j,n-1} &= \quad m'_{j,n-1} \quad \text{if} \quad \mu=0 \\ &= m'_{j,n-1}+\delta_{j,k} \quad \text{if} \quad \mu=1 \end{aligned}$$

μ	
0	$q^{1/2\left(\sum\limits_{\substack{j=1\\j\neq i}}^{n} l'_{jn}-\sum\limits_{j=1}^{n-1} l_{j,n-1}+n-1\right)}\left[\dfrac{\prod\limits_{j=1}^{n-1}(l_{j,n-1}-l'_{in}-1)}{\prod\limits_{\substack{j=1\\j\neq 1}}^{n}(l'_{jn}-l'_{in})}\right]^{1/2}$
1	$q^{1/2(l'_{in}-l_{k,n-1}+1)}S(k-i)\left[\prod\limits_{\substack{j=1\\j\neq k}}^{n-1}\dfrac{(l'_{in}-l_{j,n-1}+1)}{(l_{k,n-1}-l_{j,n-1})}\prod\limits_{\substack{j=1\\j\neq i}}^{n}\dfrac{(l'_{jn}-l_{k,n-1}+1)}{(l'_{jn}-l'_{in})}\right]^{1/2}$

$$l_{ij}=m_{ij}-i$$

$$S(i)=sign\ of\ i$$

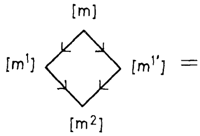
[a] The exponents of q are ordinary [not (q)] numbers

Table 2. Since $V^{(1,0,\dots)}\otimes V^{(1,0,\dots)}=V^{(2,0,\dots)}\oplus V^{(1,1,0,\dots)}$. We can build two projectors $P^{(1,1,0,\dots)}$, $P^{(2,0,\dots)}=1-P^{(1,1,0,\dots)}$. Let us denote by α the state $\mu_k=1$ $k\geqq n-\alpha$, $\mu_k=0$ $k<n-\alpha$. Then, the vertex matrix elements of $P^{(1,1,\dots)}$ are denoted by $\sigma_{\alpha\beta,\alpha'\beta'}$. Due to the charge conservations, $\sigma_{\alpha\beta,\alpha'\beta'}=0$ unless $\{\alpha,\beta\}=\{\alpha',\beta'\}$, and we have for $\alpha>\beta$:

$$\begin{pmatrix}\sigma_{\alpha\beta,\alpha\beta} & \sigma_{\alpha\beta,\beta\alpha}\\ \sigma_{\beta\alpha,\alpha\beta} & \sigma_{\beta\alpha,\beta\alpha}\end{pmatrix}=\frac{1}{(2)}\begin{pmatrix}q & -1\\ -1 & q^{-1}\end{pmatrix} . \tag{23}$$

From the $gl(n)$ generalisation of (7) we deduce the path matrix elements

$$\sigma_{[m^1][m^{1'}]}=$$



$$=\sigma^{([m][m^2])}_{[m^1],[m^{1'}]} . \tag{24}$$

We set

$$i_1=[m^1]=(m_{kn}+\delta_{ki_1}) ,$$

$$i'_1=[m^{1'}]=(m_{kn}+\delta_{ki'_1}) ,$$

$$(i,j)=[m^2]=(m_{kn}+\delta_{ki}+\delta_{kj}) , \tag{25}$$

then $i_1, i_1' \in \{i,j\}$ and:

$$\begin{aligned} &i=j, && \sigma_{i,i}=0 \\ &j=i+1,\ m_{in}=m_{jn}: && \sigma_{i,i}=1 \\ &i<j,\ m_{in}>m_{jn}: && \end{aligned}$$

$$\begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix} = \frac{1}{(2)(l_i-l_j)} \begin{pmatrix} (l_i-l_j+1) & * \\ \sqrt{(l_i-l_j+1)(l_i-l_j-1)} & (l_i-l_j-1) \end{pmatrix} \tag{26}$$

with $l_i = m_{in} - i$.

Both expressions correspond to known representations of the Hecke algebra [5, 17]. The R matrix:

$$R(x) = (1-xq^2)P^{(2,0,\dots)} + (x-q^2)P^{(1,1,0,\dots)} \tag{27}$$

is obtained in [18] in the vertex representation and studied in [8, 11] in the path representation. For $V^{[g]}$ an irreducible $\hat{U}$ module, the path algebra is defined as for $SU(2)$. The Wigner coefficients intertwine the path algebra and $\Delta^{(N)}(U)'$ in $\mathrm{End}(V_g)\otimes^N$. The expression of the trace in the path algebra is:

$$\mathrm{tr}\,(([m]),([m'])) = \delta_{[m],[m']} \prod_{i,j} \frac{(l_i - l_{i+j})}{(j)} ,$$

where

$$([m]) = ([m]_1, \dots, [m]_N) \quad \text{denotes a path} . \tag{28}$$

When $q = e^{i\pi/L}$, we must restrict $[m]$ to values such that

$$\dim V_{[m]} = \prod_{i,j} \frac{(l_i - l_{i+j})}{(j)} > 0 . \tag{29}$$

This imposes $m_{1n} - m_{nn} \leqq L - n$ and the connection matrices defining admissible paths are determined by the recursion relations:

$$\begin{aligned} &A^{(0)} = 1 , \\ &A^{(1,0,\dots)}_{[m][m']} = \begin{array}{l} \text{the matrix of the generic model restricted to } [m][m'] \\ \text{such that } \dim V_{[m]},\ \dim V_{[m']} > 0 , \end{array} \\ &A^{[g]} A^{[1,0,\dots,0]} = \sum_{[g']} A^{[1,0,\dots]}_{[g][g']} A^{[g']} . \end{aligned} \tag{30}$$

Discrete Groups. We consider the limit $q=1$, where the R matrix degenerates to its rational limit. Let Γ be a discrete group, V_g a Γ module and $\Delta^{(2)}$ the homomorphism of Γ in $\mathrm{End}(V_g \otimes V_g)$. Assume that $[R(x), \Delta^2(\gamma)] = 0$ for all $\gamma \in \Gamma$. The preceding method yields a path representation of R with intertwiner given by the Γ Wigner coefficients. The case where Γ is a discrete subgroup of $SU(2)$ and V_g its two dimensional module is considered in [19–21]. The connection matrix of the path algebra: A^g is the incidence matrix of an extended Coxeter diagram and the components of its Perron-Frobenius vector are the dimensions of the representations of Γ [22]. Other models associated with ordinary Coxeter diagrams have been discussed [10]. It is natural to ask whether they correspond to a Hopf algebra [3].

Then each vertex of the Coxeter diagram should be associated to a $\hat{U}$ module V^a of (q) dimension S^a with S^a the component of the Perron-Frobenius vector normalised so that its smallest component $S^0=1$. One must therefore define commuting matrices A^a with positive integer coefficients characterising the paths built on V^a and determined by the identities (expressing the associativity of the tensor product):

$$A^{(0)}=1 \ ,$$

$$A^{(1)}=\text{the incidence matrix of the Coxeter diagram} \ ,$$

$$A^{(a)}A^{(1)}=\sum_{(b)} A^{(1)}_{ab}A^{(b)} \ . \tag{31}$$

Remarkably, this is only possible for A_n, D_{2n}, E_6, E_8 as in Ocneanu's classification of subfactors [12]. We hope to be able to discuss this point further separately.

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