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A Construction of the $c<1$ Modular Invariant Partition Functions*

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Abstract. Decomposition theorems for certain representations of Kac-Moody algebras which are needed for the construction of modular invariant unitary conformal models are proved. It is shown that all $c<1$ modular invariant models can then be recovered from gauged free fermionic models, including the exceptional cases.

1. Introduction

In a recent paper [6], new two-dimensional conformal models were constructed by tensoring the $c<1$ discrete unitary series of Friedan, Qiu, and Shenker [4] with itself. In this way, one can reach models with central charges larger than one, which are of interest in string theory as well as in two-dimensional statistical mechanics. The building blocks of these new models are the representations of the Virasoro algebra (Vir) with central charge less than one, and they were explicitly constructed in [6] using a technique introduced by Goddard, Kent, and Olive [2] (see also [5]). The construction starts with N free fermions, and a suitable subgroup of the orthogonal group $O(N)\times O(N)$ is gauged to reduce the central charge. It was argued in [6] that, if one starts with a standard set of modular invariant free fermion models, the complete set of minimal modular invariant models discovered by Cappelli, Itzykson, Zuber, and Gepner [8, 9] can be recovered. There were, however, two technical points left incomplete in [6]. A theorem which gave the decomposition of the level one highest weight representations of the affine $O(4N)$ algebra was left unproved. Also, the exceptional solutions of CIZ remained beyond

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the reach of the construction. The purpose of the present paper is to fill in these two gaps.

Using the Weyl-Kac character formula, we first show how to decompose the Ramond sector of $O(4N)$ representations. This is then extended to the Neveu-Schwarz sector, making use of an outer automorphism of the $O(4N)$ affine algebra. Finally, we show that the exceptional solutions of CIZ are obtained by using conformal embeddings associated with exceptional symmetric spaces [10].

While completing this paper, our attention was brought to earlier work on the same subject by Nahm [20], whose ideas are very close to ours. However, we feel our treatment is more detailed and our point of view is somewhat different, so we decided to present it here.

The paper is organised as follows. In Sects. 2 and 3 we briefly recall the main results of [4] and [2, 3], and we fix some notations for the representations of Virasoro and Kac-Moody algebras. For a general introduction to these algebras and their physical applications, see [1]. The mathematical theory is contained in [12]. Sections 4 and 5 explain the construction of modular invariant partition functions. In Sect. 6 we state the decomposition theorems. The proofs are given in Sect. 7. Section 8 illustrates their use in an example. In an appendix we discuss the exceptional solutions.

2. The Discrete Series

Let us recall the commutation relations in the Virasoro algebra:

$$[L_m, L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m,-n}, \tag{2.1}$$

where $m, n \in \mathbb{Z}$. A highest weight representation (HWR) of Vir is generated by a highest weight vector $|hw\rangle$, which satisfies the two conditions:

$$L_0|hw\rangle = h|hw\rangle, \qquad L_n|hw\rangle = 0, \qquad n>0. \tag{2.2}$$

An irreducible HWR (IHWR) is completely specified by the values of h and c. We shall denote it by $V(h,c)$. We define the character of $V(h,c)$ to be:

$$\chi_{h,c}(q)=q^{h-c/24}\sum_{n=0}^{\infty}(\dim V_n)q^n, \tag{2.3}$$

where V_n is the eigenspace of $V(h,c)$ corresponding to the eigenvalue $h+n$ of L_0.

It was realized quite early that $h \geqq 0$, $c \geqq 1$ implies that $V(h,c)$ is unitary, but it took more time to investigate the region $0<c<1$. (An IHWR with $c=0$ is trivial.) For $n=2,3,\ldots$ we set

$$c_n = 1-\frac{6}{(n+1)(n+2)} \tag{2.4}$$

and

$$h_{n,p,q}=\frac{[(n+2)p-(n+1)q]^2-1}{4(n+1)(n+2)} \tag{2.5}$$

with $p \in \{1,2,\ldots,n\}$ and $q \in \{1,2,\ldots,p\}$. Friedan, Qiu, and Shenker [4] proved that if $0<c<1$ and $V(h,c)$ is unitary, then $c=c_n$ and $h=h_{n,p,q}$ for some (n,p,q).

Conversely, Goddard, Kent, and Olive later proved [3] that $V(h_{n,p,q}, c_n)$ is unitary for all (n, p, q).

IHWR in the discrete series (2.4, 2.5) are not only a mathematical curiosity, but also of great physical interest. Namely, to each value of n one can associate an exactly solvable model of statistical mechanics, whose critical exponents are determined by the $h_{n,p,q}$.

It was natural to try to construct string models using the discrete series. One possibility is to take the tensor product of a (finite) number of IHWRs in the discrete series, chosen such that the total central charge is a positive integer. This is possible since the sequence $\{c_n\}$ is rational, strictly increasing, and converging to 1. This was shown in [6].

3. The Coset Construction

We start by reviewing the construction of [2, 3]. The GKO construction uses Kac-Moody algebras, so let's first fix some notations for them. Let g be a simple, finite-dimensional, Lie algebra, and $\{T^i\}$, $1 \leqq i \leqq D$ an orthonormal basis. The affine KM algebra $\hat{g}$ is the Lie algebra with basis k, d, T^i_m, $m \in \mathbb{Z}$ and the commutation relations:

$$[T^i_m, T^j_n] = f_l^{ij} T^l_{m+n} + km\delta_{m,-n}\delta^{ij}, \tag{3.1}$$

$$[d, T^i_m] = mT^i_m, \tag{3.2}$$

$$[k, T^i_m] = [k, d] = 0. \tag{3.3}$$

If the square length of the long roots of g is normalized to 2, as we assume from now on, the eigenvalue of k on a unitary IHWR (UHWR) of $\hat{g}$ is a non-negative integer, called the *level*. UHWRs can be defined as follows. Note first that g is embedded in $\hat{g}$ by $T^i \mapsto T^i_0$. Then a UHWR with highest weight vector $|\Lambda\rangle$ has the following properties:

$$T^i_0|\Lambda\rangle = \tau^i|\Lambda\rangle, \tag{3.4}$$

where τ^i denotes an irreducible finite dimensional representation of g,

$$T^i_n|\Lambda\rangle = 0, \quad n > 0. \tag{3.5}$$

Let $\bar{\Lambda}$ be the highest weight of the representation τ, and θ the highest root of g. A necessary and sufficient condition for unitarity is that

$$(\bar{\Lambda}|\theta) \leqq k. \tag{3.6}$$

A UHWR satisfying (3.4, 3.5, 3.6) is completely determined by $|\Lambda\rangle = |\bar{\Lambda}; k\rangle$. We shall denote it by $L(\Lambda)$. Given a representation $L(\Lambda)$, one can construct on the same space a representation of Vir by the Sugawara formula:

$$L^g_n = \frac{1}{2(k+h_g)} \sum_{m \in \mathbb{Z}} :T^i_{m+n} T^i_{-m}:, \tag{3.7}$$

where h_g is the dual Coxeter number of g, and the colons denote normal ordering:

$$:T^i_m T^j_n: = \begin{cases} T^j_n T^i_m & \text{if} \quad n < 0 \\ T^i_m T^j_n & \text{if} \quad n \geqq 0. \end{cases} \tag{3.8}$$

The representation $L_n \mapsto L_n^g$ of Vir on $L(\Lambda)$ is unitary, because $L(\Lambda)$ has a positive definite hermitian form $\langle\ |\ \rangle$ with respect to which

$$(T_n^i)^\dagger = T_{-n}^i . \tag{3.9}$$

The central charge of this representation is

$$c(g) = \frac{k \dim g}{k + h_g} . \tag{3.10}$$

Another important property of the Sugawara operators is

$$[L_n^g, T_m^i] = -m T_{m+n}^i . \tag{3.11}$$

If g is not simple, but semisimple:

$$g = g_1 \oplus g_2 \oplus \ldots \oplus g_s , \tag{3.12}$$

where the g_i are simple, one defines the action of the T_n^i on

$$L(\Lambda) = L(\Lambda_1) \otimes L(\Lambda_2) \otimes \ldots \otimes L(\Lambda_s) , \tag{3.13}$$

and (3.7) becomes:

$$L_n^g \stackrel{\text{def}}{=} L_n^{g_1} + L_n^{g_2} + \ldots + L_n^{g_s} . \tag{3.14}$$

Likewise,

$$c(g) = c(g_1) + c(g_2) + \ldots + c(g_s) . \tag{3.15}$$

Often we will write

$$L(\Lambda) = (\Lambda_1, \Lambda_2, \ldots, \Lambda_s) \tag{3.16}$$

instead of (3.13). Moreover, when one of the $g_i = su(2)$, we will write (n_i, k_i) with $n_i \leqq k_i$ instead of $(\bar{\Lambda}_i, k_i)$, where $\bar{\Lambda}_i = n_i e$ and e is the fundamental weight of $su(2)$. Sometimes we will also suppress the level k_i if no confusion can arise.

Now it can be shown [1] that $c(g) \geqq \operatorname{rank}(g)$, therefore there is no way to get values of c in the discrete series by the Sugawara formula alone. Following GKO, let p be a subalgebra of g. Clearly, we can form a representation L_n^p of Vir on $L(\Lambda)$. Put

$$L_n^{g/p} = L_n^g - L_n^p . \tag{3.17}$$

Then $L_n^{g/p}$ is another representation of Vir on $L(\Lambda)$ with central charge

$$c(g/p) = c(g) - c(p) . \tag{3.18}$$

Furthermore,

$$[L_n^{g/p}, \hat{p}] = 0 , \tag{3.19}$$

where the action of $\hat{p}$ on $L(\Lambda)$ is defined through the natural embedding $\hat{p} \subset \hat{g}$. From (3.19) we see that $L(\Lambda)$ has the structure of a $(\hat{p} \oplus \mathrm{Vir})$-module.

There are several choices [11] of (g, p, Λ) that lead to the discrete series. We shall use the following one:

$$\begin{aligned} &g = su(2) \oplus su(2) , \qquad p = \text{diagonal}\, su(2) , \\ &L(\Lambda) = (m_1, n-1; m_2, 1) , \end{aligned} \tag{3.20}$$

where $m_1 = 0, 1, \ldots, n-1$, n is the same as in (2.4), and $m_2 \in \{0, 1\}$.

Using character formulas for KM algebras [12] and Vir representations in the discrete series [13], it was proved in [3] that with the choice (3.20), $L(\Lambda)$ decomposes into representations of $\hat{p}\oplus\mathrm{Vir}$ as follows:

$$(m_1, n-1; m_2, 1) = \bigoplus_m (m, n)\otimes[n, m_1+1, m+1], \tag{3.21}$$

where the summation is over $m\in\{0, 1, \ldots, n\}$ such that $m_1 - m \equiv m_2 \pmod 2$, and we used the notation $[n, p, q]$ for $V(h_{n,p,q}, c)$. Note that in (3.21) we implicitly extended the range of q in (2.5) to $n+1$, making use of the fact that the substitutions

$$(p, q)\mapsto(n+1-p, n+2-q) \tag{3.22}$$

leave the value of $h_{n,p,q}$ unchanged.

4. Modular Invariance

In discussing the modular invariance issue, we recall that left and right-moving currents are decoupled, which is reflected in the fact that we actually have *two* commuting copies A_L, A_R of the algebra A acting on the Hilbert space $\mathscr{H}$ of a two-dimensional model defined on a torus (here $A=\hat{g}$, $\hat{p}$ or Vir). A specific model is described by the decomposition of $\mathscr{H}$ into irreducible representations V^i of $A_L\times A_R$:

$$\mathscr{H} = \bigoplus_{i,j\in I}(V_L^i, V_R^j)M_{ij}, \tag{4.1}$$

where M_{ij} is a matrix of multiplicities with non-negative integer entries, and I is some indexing set. The value of c when $A=\mathrm{Vir}$, or k when $A=\hat{g}$, is taken to be the same for all the $V^i_{L,R}$. From now on, we shall omit subscripts referring to c or k, e.g. write $h_{p,q}$ instead of $h_{n,p,q}$.

Let $\chi_i(q)$ be the character of V_R^i and $\chi_i(\bar{q})$ the character of V_L^i, where $\bar{q}$ denotes complex conjugation. The partition function of the model given by (4.1) is:

$$Z(q) = \sum_{i,j}\chi_i(\bar{q})M_{ij}\chi_j(q). \tag{4.2}$$

The character $\chi_{h,c}$ of $V(h,c)$ in the case $A=\mathrm{Vir}$ has already been defined in (2.3). When $A=\hat{g}$ (or $\hat{p}$) we define the character χ_Λ of $L(\Lambda)$ as the function:

$$\chi_\Lambda(q, z) = q^{h_\Lambda - c(g)/24}\sum_{n=0}^{\infty}\mathrm{tr}_n\exp(2\pi i z)q^n, \tag{4.3}$$

where $z\in h$, a Cartan subalgebra of g, and tr_n is the trace on the eigenspace of L_0^g with eigenvalue $h_\Lambda+n$, h_Λ being the least eigenvalue of L_0^g on $L(\Lambda)$:

$$L_0^g|\Lambda\rangle = h_\Lambda|\Lambda\rangle, \tag{4.4}$$

$$h_\Lambda = \frac{c_{\bar{\Lambda}}/2}{k+h_g}, \tag{4.5}$$

and $c_{\bar{\Lambda}}$ is the eigenvalue of the quadratic Casimir operator of g on the representation $L(\bar{\Lambda})$ with highest weight $\bar{\Lambda}$. Comparing (3.2) and (3.11), we find that on $L(\Lambda)$,

$$L_0^g = -d + h_\Lambda. \tag{4.6}$$

Now let $q=e^{2\pi i\tau}$ with $\operatorname{Im}\tau>0$, and let the modular group $SL_2(\mathbb{Z})$ act on τ as usual:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}. \tag{4.7}$$

It was found by Kac and Peterson [12] that $\chi_\Lambda(\tau)$ has very simple transformation properties under (4.7). Cardy [14] and Itzykson and Zuber [7] found similar transformation properties of the $\chi_{h,c}$: if I is the finite set of values of h allowed for a given $c<1$, then the vector space E_c spanned by the $\chi_{h,c}$ with $h\in I$ is stable under the action of $SL_2(\mathbb{Z})$. In other words, the space E_c of characters carries a representation of the modular group. Moreover, this representation is unitary. A similar statement holds for the χ_Λ.

In order to classify all the consistent conformally invariant models on the torus, one has to find all the functions $Z(\tau)$ of the form (4.2) invariant under $SL_2(\mathbb{Z})$. This problem, originally raised by Cardy [14], turns out to be tractable because of these nice transformation properties of the characters. It was solved in [8, 9] for the Virasoro algebra's discrete series. Gepner also found that the problem for Vir is essentially equivalent to the problem for $\widehat{su}(2)$.

In addition to the condition $Z(\sigma\cdot\tau)=Z(\tau)$, $\sigma\in SL_2(\mathbb{Z})$, one also requires non-degeneracy of the vacuum: $M_{00}=1$. The vacuum in the case of Vir is the highest weight vector $|0\rangle$ of the representation $V(h=0,c)$, which together with (2.2) satisfies

$$L_{-1}|0\rangle=0 \tag{4.8}$$

so that $|0\rangle$ is invariant under the group $SL_2(\mathbb{C})$ of projective transformations, whose Lie algebra is spanned by $L_0, L_{\pm 1}$. Similarly, the vacuum in the case of $\hat{g}$ is identified with $|\Lambda_0\rangle$ which satisfies

$$T_0^i|\Lambda_0\rangle=0 \tag{4.9}$$

instead of (3.4).

Table 1. The list of modular invariant partition functions for $c<1$

X	n	$Z(X)$
A_n	n	$\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n+1}\lvert\chi_{pq}\rvert^2$
$D_{2l+2}, l\geqq 1$	$4l$	$\frac{1}{2}\sum_{p=1}^{4l}\left\{\sum_{\substack{q\text{ odd}=1\\ q\neq 2l+1}}^{4l+1}\lvert\chi_{pq}\rvert^2+2\lvert\chi_{p,2l+1}\rvert^2+\sum_{q\text{ odd}=1}^{2l-1}(\chi_{pq}\bar{\chi}_{4l+1-p,q}+\text{c.c.})\right\}$
$D_{2l+1}, l\geqq 2$	$4l-2$	$\frac{1}{2}\sum_{p=1}^{4l-2}\left\{\sum_{q\text{ odd}=1}^{4l-1}\lvert\chi_{pq}\rvert^2+\lvert\chi_{p,2l}\rvert^2+\sum_{q\text{ even}=1}^{2l-2}(\chi_{pq}\bar{\chi}_{4l-1-p,q}+\text{c.c.})\right\}$
E_6	11	$\frac{1}{2}\sum_{q=1}^{12}\{\lvert\chi_{1q}+\chi_{7q}\rvert^2+\lvert\chi_{4q}+\chi_{8q}\rvert^2+\lvert\chi_{5q}+\chi_{11q}\rvert^2\}$
E_7	17	$\frac{1}{2}\sum_{q=1}^{18}\{\lvert\chi_{1q}+\chi_{17q}\rvert^2+\lvert\chi_{5q}+\chi_{13q}\rvert^2+\lvert\chi_{7q}+\chi_{11q}\rvert^2$ $+\lvert\chi_{9q}\rvert^2+[(\chi_{3q}+\chi_{15q})\bar{\chi}_{9q}+\text{c.c.}]\}$
E_8	29	$\frac{1}{2}\sum_{q=1}^{30}\{\lvert\chi_{1q}+\chi_{11q}+\chi_{19q}+\chi_{29q}\rvert^2+\lvert\chi_{7q}+\chi_{13q}+\chi_{17q}+\chi_{23q}\rvert^2\}$

The CIZ classification [8] is presented in Table 1. There are two infinite series of solutions A_l and D_l and three exceptional solutions, E_6, E_7, E_8. n refers to the central charge c_n of model X, given by Eq. (2.4). χ_{pq} is the character of $V(h_{p,q}, c_n)$. The reason why they are labeled by the simply-laced simple Lie algebras is because the subscripts p of the diagonal terms $|\chi_{pq}|^2$ appearing in the solutions are precisely the exponents of the corresponding Lie algebras. Apart from that, the origin of the ADE pattern is still mysterious.

5. The Branching Functions

The authors of [6] showed how to realize the models from the discrete series starting from $4n$ free fermions, and adding an external gauge field taking values in a subalgebra of $o(4n)$, which we shall specify in a moment. The gauge field has two features. First it implements physically the GKO mechanism. Second, it projects out and regroups states from the free fermionic Hilbert space, preserving modular invariance. More precisely, we start with a modular invariant partition function of free fermions:

$$Z_f(\tau) = \sum_{\lambda,\mu} \chi_\lambda(\bar{\tau}) M^g_{\lambda\mu} \chi_\mu(\tau), \tag{5.1}$$

where $\lambda, \mu \in \{o, v, s, t\}$, the 4 UHWR of $\hat{g} = \hat{o}(4n)$ of level one (Fig. 1). Then we choose a semisimple subalgebra $p \subset g$ such that $c(g/p) = c_n$. The UHWR $L(\lambda)$ of $\hat{g}$ decomposes as

$$L(\lambda) = \bigoplus_{\Lambda} L(\Lambda) \otimes U(\lambda, \Lambda). \tag{5.2}$$

Here $L(\Lambda)$ are the UHWR of $\hat{p}$ of level $j = (j_1, j_2, \ldots)$, j_i being the level for the i-th simple component of p, which is given by the Dynkin [15] index of the embedding $p_i \subset g$. $U(\lambda, \Lambda)$ is the subspace of $L(\lambda)$ spanned by the highest weight vectors of the $\hat{p}$-modules equivalent to $L(\Lambda)$ contained in $L(\lambda)$. From (3.19) we know that $U(\lambda, \Lambda)$ is a (in general reducible) representation of Vir with central charge $c(g/p)$. If $L(\Lambda)$ is not contained in $L(\lambda)$, we set $U(\lambda, \Lambda) = 0$. Taking the characters of (5.2) we find:

$$\chi_\lambda(\tau) = \sum_{\Lambda} b^\lambda_\Lambda(\tau) \chi_\Lambda(\tau), \tag{5.3}$$

where b^λ_Λ is the character of $U(\lambda, \Lambda)$ and has been called the *branching function* in [16]. One has from (3.17),

$$U(\lambda, \Lambda) = \bigoplus_{n \geqq 0} a_n V(h_\lambda - h_\Lambda + n, c(g/p)) \tag{5.4}$$

with $a_n \in Z_+$. a_0 is the multiplicity of occurrence of $L(\Lambda)$ in $L(\lambda)$. If $c(g/p) < 1$, as we assume now, by (2.5) only a finite number of a_n are non-zero.

Let E^g be the vector space spanned by the χ_λ, with $\lambda \in \{o, v, s, t\}$, and E^p the vector space spanned by the χ_Λ with level$(\Lambda) = j$. Then by (5.3) we have a linear map:

$$b(\tau): E^p \to E^g. \tag{5.5}$$

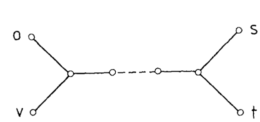

Fig. 1. The Dynkin diagram of $\hat{o}(4n)$

For two branching functions $b_L(\tau), b_R(\tau)$, define the hermitian form:

$$\langle b_L, b_R\rangle = \sum_{\lambda, \Lambda} b^{\lambda}_{L\Lambda}(\bar{\tau}) b^{\lambda}_{R\Lambda}(\tau). \tag{5.6}$$

Let $\varrho^g : SL_2(\mathbb{Z}) \to \mathrm{Aut}(E^g)$ and $\varrho^p : SL_2(\mathbb{Z}) \to \mathrm{Aut}(E^p)$ be the representations of the modular group afforded by the affine characters. Then we have a natural representation $\tilde{\varrho}$ of $SL_2(\mathbb{Z})$ on the space of linear maps $E^p \to E^g$ given by

$$b(\sigma \cdot \tau) = \tilde{\varrho}(\sigma) \cdot b(\tau) = \varrho^g(\sigma) b(\tau) \varrho^p(\sigma^{-1}), \tag{5.7}$$

and since both ϱ^g and ϱ^p are unitary, $\tilde{\varrho}$ is unitary with respect to $\langle\, ,\,\rangle$. In other words, (5.6) is a modular invariant function.

Now let M^g be the matrix defining our free fermion theory $Z_f(\tau)$ in (5.1), and let M^p define a modular invariant partition function for $\hat{p}$. Then M^g and M^p commute with ϱ^g, respectively ϱ^p, and it follows [11] that

$$Z_{g/p}(\tau) = \langle b(\tau) M^p, M^g b(\tau)\rangle \tag{5.8}$$

is a modular invariant partition function for the Virasoro algebra.

Let $Z(X)$ be a modular invariant partition function from the CIZ classification, with $X = A_l$, D_l or E_l. Denote by $\mathscr{H}_{4n}$ the Hilbert space of the simplest modular invariant theory of $4n$ free fermions:

$$\mathscr{H}_{4n} = (o, o) \oplus (v, v) \oplus (s, s) \oplus (t, t). \tag{5.9}$$

We use (5.9) to construct $Z(X)$ when $X = A_l$ or E_l. In the case $X = D_l$, the structure of the Hilbert space depends on n. Define

$$\mathscr{H}'_{4n} = (o, o) \oplus (s, s) \oplus (o, s) \oplus (s, o), \tag{5.10}$$

$$\mathscr{H}''_{4n} = (o, o) \oplus (s, s) \oplus (v, t) \oplus (t, v). \tag{5.11}$$

The free fermion Hilbert space for the case $X = D_l$ is then given when n is even by

$$\mathscr{H} = \begin{cases} \mathscr{H}'_{4n} & \text{if} \quad n \equiv 0 \,(\mathrm{mod}\, 4) \\ \mathscr{H}''_{4n} & \text{if} \quad n \equiv 2 \,(\mathrm{mod}\, 4). \end{cases} \tag{5.12}$$

From (5.9) and (5.12) a matrix M^g can be determined, which when inserted in (5.8) will produce all the invariants $Z(X)$ in Table 1. However, for each $Z(X)$ in Table 1, there is another conformal invariant $Z'(X)$ whose central charge n differs by one unit [8]. For $X = A_n$, $Z'(X) = Z(A_{n+1})$, so this is not very interesting, but for the D_l series, one obtains in this way invariants for odd n. They can also be constructed from fermi fields [6], by first decomposing $o(4n)$ into $o(4) \oplus o(4n-4)$ – see (5.16) below – and with the following choice of boundary conditions:

$$\mathscr{H} = \mathscr{H}_4 \otimes \begin{cases} \mathscr{H}'_{4n-4} & \text{if} \quad n \equiv 1 \,(\mathrm{mod}\, 4) \\ \mathscr{H}''_{4n-4} & \text{if} \quad n \equiv 3 \,(\mathrm{mod}\, 4). \end{cases} \tag{5.13}$$

For E_6, E_7, E_8, there are three other conformal invariants at $n = 10, 16, 28$. They can be easily constructed starting from modular invariant partition functions for $\widehat{su}(2)$ by interchanging $su(2)^c$ and $su(2)^d$ – see Eq. (5.18) below – as in [11]. But it is not possible to obtain them by the kind of fermionic realization which we use.

It was argued in [6] that the zero mode of the gauge field projects out the states from the free fermion Hilbert space which are invariant under the diagonal subalgebra of $p_L \oplus p_R$. Accordingly, we obtain for M^p the invariant:

$$M^p_{\Lambda\Lambda'} = \delta_{{}^t\Lambda, \Lambda'}, \tag{5.14}$$

where ${}^t\Lambda = (\bar{\Lambda}^*, k)$ and $\bar{\Lambda}^*$ is the highest weight of the contragredient p-module $L(\bar{\Lambda})^* = L(\bar{\Lambda}^*)$ of $L(\bar{\Lambda})$.

Now we give the embeddings $p \subset g$ which we use in order to produce the invariants $Z(X)$ of the CIZ classification. The general form of p is

$$p = q \oplus su(2)^a \oplus su(2)^d \tag{5.15}$$

with q semisimple. We use superscripts $a, b, \ldots$ to distinguish between the $su(2)$ factors. The embedding $g \supset p$ is composed out of three more elementary ones:

$$o(4n) \supset o(4n-4) \oplus su(2)^a \oplus su(2)^b, \tag{5.16}$$

$$o(4n-4) \supset q \oplus su(2)^c, \tag{5.17}$$

$$su(2)^b \oplus su(2)^c \supset su(2)^d. \tag{5.18}$$

Equation (5.16) is just the regular embedding, (5.18) the diagonal embedding. (5.17) is obtained as follows. Given a simple Lie algebra g', one chooses an involution such that $q \oplus su(2)^c = p'$ is its fixed point set. In other words g'/p' is a symmetric space, $\dim(g'/p') = 4n-4$. Then (5.17) is the natural embedding of the isotropy group into $o(4n-4)$.

Table 2 contains the choices of q and g' that have to be done to construct a given $Z(X)$. The column labeled "levels" gives the Dynkin indices of the embedding of the three simple factors q, $su(2)^a$ and $su(2)^d$ of p into g. Equivalently, representations of $\hat{p}$, $\widehat{su}(2)^a, \widehat{su}(2)^d$ to be considered later will have the indicated levels.

Table 2. Subalgebras assignments

X	g'	q	Levels
A_n, D_n	$sp(n)$	$sp(n-1)$	$1, 1, n$
E_6	E_6	$su(6)$	$6, 1, 11$
E_7	E_7	$so(12)$	$8, 1, 17$
E_8	E_8	E_7	$12, 1, 29$

Equation (5.18) is nothing but the embedding (3.20) used in the GKO construction. Equations (5.16) and (5.17) are conformal embeddings. This means that they are of the type $g_1 \supset g_2$ such that $L_n^{g_1/g_2} = 0$ or, equivalently, $c(g_1/g_2) = 0$. Notice that branching functions for conformal embeddings are constants, and the coefficients b_Λ^λ are nothing but the multiplicities for $L(\Lambda)$ occurring in $L(\lambda)$.

To obtain $Z(X)$ as a summand in $Z_{g/p}$, the only thing we have to do is to compute the branching functions $b(\tau)$, which are composed of the branching functions for the embeddings (5.16, 5.17, 5.18). In fact, the branching functions for (5.18) are already given by (3.21).

The branching functions for (5.16) are easily found. They are given by

$$\begin{aligned} o&=(o;0,0)\oplus(v;1,1)\\ v&=(o;1,1)\oplus(v;0,0)\\ s&=(s;0,1)\oplus(t;1,0)\\ t&=(s;1,0)\oplus(t;0,1) \end{aligned} \tag{5.19}$$

with the notational conventions explained above and omitting the levels which all are equal to one.

The branching functions for (5.17) will be given by the decomposition theorems of the next section. These theorems, together with (3.21), (5.19), and the modular invariants M^g and M^p defined above lead to the relations:

$$Z_{g/p}=4lZ(X) \tag{5.20}$$

when $X=A_l$, D_l and

$$Z_{g/p}=4[lZ(A_l)+Z(X)] \tag{5.21}$$

when $X=E_l$, $l=6,7,8$.

6. The Decomposition Theorems

The non-trivial part of the computation is to find the branching functions for (5.17). For this, we first have to introduce more definitions. Let $\bar{h}, h, \bar{h}_0, h_0$ be Cartan subalgebras of $g', \hat{g}', p', \hat{p}'$ respectively, such that $\bar{h}_0\subset\bar{h}$, $h_0\subset h$. (h is the direct sum of $\bar{h}$ and the two-dimensional space spanned by k and d.) Denote by $\bar{h}^*$, h^* etc. ... the duals. Also let $\bar{\Delta}, \Delta, \bar{\Delta}_0, \Delta_0$ denote the corresponding sets of roots. We fix a choice of positive roots $\bar{\Delta}_+$, Δ_+ of g' and $\hat{g}'$. Note that $\bar{\Delta}_0\subset\bar{\Delta}$ because the symmetric spaces given in Table 2 are such that g' and p' have the same rank, which implies that the embedding $p'\subset g'$ is regular. We set $\bar{\Delta}_{0+}=\bar{\Delta}_0\cap\bar{\Delta}_+$ and $\bar{\Delta}_1=\bar{\Delta}-\bar{\Delta}_0$.

Let $\bar{W}$, W, $\bar{W}_0$, W_0 be the respective Weyl groups. $\bar{W}_0$ and W_0 are contained in $\bar{W}$ and W, but they are not invariant subgroups. Recall that W is a semidirect product

$$W=\bar{W}\ltimes T, \tag{6.1}$$

where T, the invariant subgroup of "translations" is the lattice M spanned by the long roots of g'.

For $\lambda\in h^*$, denote by $\bar{\lambda}$ its projection on $\bar{h}^*$. Then

$$\lambda=\bar{\lambda}+m\Lambda_0+a\delta\,. \tag{6.2}$$

δ and Λ_0 correspond by duality to k and d. One has $\delta(d)=\Lambda_0(k)=1$, $\delta(k)=\Lambda_0(d)=0$. The invariant bilinear form on h^* reads:

$$(\lambda|\lambda')=(\bar{\lambda}|\bar{\lambda}')+ma'+m'a\,. \tag{6.3}$$

Note that in (6.2), $m=(\lambda|\delta)=\text{level}(\lambda)$. The action of $t_\alpha\in T$, $\alpha\in M$ on h^* is given [12] by:

$$t_\alpha(\lambda)=\lambda+(\lambda|\delta)\alpha-[(\lambda|\alpha)+\tfrac{1}{2}|\alpha|^2(\lambda|\delta)]\delta\,. \tag{6.4}$$

Let

$$\varrho=h_{g'}\Lambda_0+\tfrac{1}{2}\sum_{\alpha\in\bar{\Delta}_+}\alpha \tag{6.5}$$

and ϱ_0 the analog for p'. Later we shall also need the so-called *formal* character ch_Λ:

$$\mathrm{ch}_\Lambda = \sum_\lambda \mathrm{mult}(\lambda) e^\lambda , \tag{6.6}$$

where $\mathrm{mult}(\lambda)$ is the multiplicity of λ, and the sum runs over all the weights of $L(\Lambda)$. For any $\lambda \in h^*$, $x \in h$, e^λ is the function on h defined by

$$e^\lambda(x) = e^{\lambda(x)} . \tag{6.7}$$

The definitions (4.3) and (6.6) are related by:

$$\chi_\Lambda(\tau, z) = q^{h_\Lambda - c(g')/24} \exp(-2\pi i m u)\, \mathrm{ch}_\Lambda(x) , \tag{6.8}$$

where

$$x = 2\pi i(z - \tau d + uk) \tag{6.9}$$

and $q = e^{-\delta}(x) = \exp(2\pi i \tau)$. The series (6.6) converges absolutely in the region $\mathrm{Im}\,\tau > 0$.

The following theorem is due to Kac and Peterson [18], with a correction by Nahm [20], and gives the decomposition of the half-spin representations s and t (the Ramond sector). It is in fact an easy generalisation of the finite-dimensional analog, which was proved by Parthasarathy [19].

Theorem 1. *Let g' be a simple Lie algebra, $p' \subset g'$ a semisimple subalgebra of the same rank, such that $g' = p' \oplus V$ defines a symmetric space, i.e. $[V, V] \subset p'$. Let W_1 be a set of coset representatives of W/W_0, such that $w(\varrho)$ is a dominant weight for any $w \in W_1$. Then under $\hat{o}(V) \supset \hat{p}'$ we have*

$$s = \bigoplus_{w \in W_1^+} L(w(\varrho) - \varrho_0) , \tag{6.10}$$

$$t = \bigoplus_{w \in W_1^-} L(w(\varrho) - \varrho_0) , \tag{6.11}$$

where $W_1^\pm = \{w \in W_1 \mid \det(w) = \pm 1\}$.

Thus the problem of finding the branching functions for s and t is solved by the preceding theorem, provided we have an explicit description of the set W_1 in the cases of interest. Observe that

$$W_1 = (\bar{W} \ltimes T)/(\bar{W}_0 \ltimes T_0) = (\bar{W}/\bar{W}_0) \times (T/T_0) . \tag{6.12}$$

One finds that T/T_0 is trivial in the classical case $g' = sp(n)$, but $T/T_0 \approx Z_2$ in the exceptional cases [20].

Corollary 1. *If $g' = sp(n+1)$, $p' = sp(n) \oplus su(2)$, then* (6.10) *and* (6.11) *become*

$$s = \bigoplus_{j=0,\ j\ \mathrm{even}}^{n} (\Lambda_j, n-j) , \tag{6.13}$$

$$t = \bigoplus_{j=1,\ j\ \mathrm{odd}}^{n} (\Lambda_j, n-j) , \tag{6.14}$$

where Λ_j refers to the j-th fundamental representation of $\widehat{sp}(n)$.

In the exceptional cases, $g' = E_6, E_7, E_8, q = su(6), so(12), E_7$, the set W_1 has to be found with the help of a computer program. The results are given in the appendix.

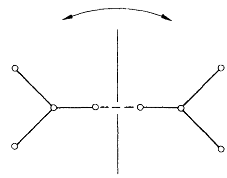

Fig. 2. The automorphism μ

The decomposition of o and v, the Neveu-Schwarz sector, can be deduced by means of the next theorem.

Theorem 2. *For each q in Table 2, consider $\hat{o}(4N)\supset\hat{q}\oplus\widehat{su}(2)$, where $4N=\dim(g'/p')$. The $su(2)$ factor has level N. Denote the branching functions for this embedding by $b^{\lambda}_{\Lambda,j}$, where $\lambda\in\{o,v,s,t\}$, Λ is the highest weight of a UHWR of $\hat{q}$ and $j\in\{0,1,\dots,N\}$ specifies the UHWR of $\widehat{su}(2)$. Let μ be the automorphism of the Dynkin diagram of $\hat{o}(4N)$ given by Fig. 2. Then the following relations hold:*

$$b^{\mu(\lambda)}_{\Lambda,N-j}=b^{\lambda}_{\Lambda,j}\,. \tag{6.15}$$

Corollary 2. *Under $\hat{o}(4n)\supset\widehat{sp}(n)\oplus\widehat{su}(2)$ we have*

$$o=\bigoplus_{j=0,\ j\ \mathrm{even}}^{n}(\Lambda_j,j)\,, \tag{6.16}$$

$$v=\bigoplus_{j=1,\ j\ \mathrm{odd}}^{n}(\Lambda_j,j)\,. \tag{6.17}$$

7. Proofs

Proof of Theorem 1. Since p' and g' have the same rank, $\dim V=2n$ is even. Let $\{\pm\varepsilon_i\}$, $1\leqq i\leqq n$ be the weights of the $o(2n)$-module V, and let $i:p'\to o(2n)$ be the inclusion map. One can choose $\bar{h}_0$ such that $i(\bar{h}_0)$ is contained in a Cartan subalgebra $\bar{u}$ of $o(2n)$. Looking at the transposed map $i^*\overset{\mathrm{def}}{=}\pi:\bar{u}^*\to\bar{h}_0^*$, one sees that there is a bijection between $\{\pm\varepsilon_i\}$ and $\bar{\Delta}_1$, because $\sigma\circ i=\varrho$, where σ is the standard representation of $o(2n)$ on V and ϱ is the representation of p' on V induced by the adjoint action of g'. So we can arrange that

$$\pi(\varepsilon_i)=\alpha_i\,, \tag{7.1}$$

where $\{\alpha_1,\alpha_2,\dots,\alpha_n\}=\bar{\Delta}_{1+}$.

Using the construction of s and t by fermionic oscillators [17, 18], we easily compute the formal character of the Ramond representation $s\oplus t$ of $\hat{o}(2n)$:

$$\mathrm{ch}(s\oplus t)=e^{\Lambda_0^g}\prod_{i=1}^{n}(e^{\varepsilon_i/2}+e^{-\varepsilon_i/2})\prod_{m=1}^{\infty}(1+e^{\varepsilon_i}q^m)(1+e^{-\varepsilon_i}q^m) \tag{7.2}$$

with $q=e^{-\delta}$. Λ_0^g is the analog of Λ_0 for $\hat{g}=\hat{o}(2n)$. Because s and t are, respectively, subspaces of even and odd fermion numbers, from (7.2) one derives:

$$\mathrm{ch}(s)-\mathrm{ch}(t)=e^{\Lambda_0^g}\prod_{i=1}^{n}(e^{\varepsilon_i/2}-e^{-\varepsilon_i/2})\prod_{m=1}^{\infty}(1-e^{\varepsilon_i}q^m)(1-e^{-\varepsilon_i}q^m)\,. \tag{7.3}$$

Recall the Weyl-Kac character formula for $L(\Lambda)$:

$$\mathrm{ch}_\Lambda = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\varrho)-\varrho}}{\prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{\mathrm{mult}\,\alpha}}, \tag{7.4}$$

where $\varepsilon(w)=\det(w)$. Putting $\Lambda=0$ in (7.4) one obtains the character of the trivial, one dimensional representation, $\mathrm{ch}_0=1$, and the denominator identity:

$$\sum_{w\in W} \varepsilon(w) e^{w(\varrho)} = e^{\varrho} \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{\mathrm{mult}\,\alpha}. \tag{7.5}$$

Inserting (7.5) back into (7.4), one gets another form of the character formula:

$$\mathrm{ch}_\Lambda = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\varrho)}}{\sum_{w\in W} \varepsilon(w) e^{w(\varrho)}}. \tag{7.6}$$

The roots Δ of $\hat{g}'$ are given in terms of $\bar{\Delta}$ and δ by:

$$\Delta_+ = \bar{\Delta}_+ \cup \{\alpha + m\delta \mid \alpha\in\bar{\Delta},\ m\in\mathbb{Z},\ m>0\} \cup \{m\delta \mid m\in\mathbb{Z},\ m>0\}. \tag{7.7}$$

The multiplicities are: $\mathrm{mult}(m\delta)=\mathrm{rank}\, g'$ and $\mathrm{mult}(\alpha)=1$ in all other cases. Recall that $p'=q\oplus su(2)$. Let Λ_0' and Λ_0'' be the analogs of Λ_0 for $\hat{q}$ and $\widehat{su}(2)$. We extend the map π to u^* by

$$\pi(\Lambda_0^g) = j'\Lambda_0' + j''\Lambda_0'', \tag{7.8}$$

where j' and j'' are the Dynkin indices of $p'\subset o(2n)$. In the same way as we defined π, we define $\phi: h^* \to h_0^*$. Note that

$$\phi(\Lambda_0) = \Lambda_0' + \Lambda_0'', \tag{7.9}$$

because $p'\subset g'$ is a regular embedding. Let $L(\Lambda)$ be a representation of $\hat{p}'$ with highest weight $\Lambda=(\Lambda',\Lambda'')$. We define

$$\mathrm{ch}_\Lambda = \mathrm{ch}_{\Lambda'}\, \mathrm{ch}_{\Lambda''}. \tag{7.10}$$

Comparison of (7.3) and (7.5, 7.7) yields using (7.1):

$$\pi[\mathrm{ch}(s)-\mathrm{ch}(t)] = \frac{\phi\left(\sum_{w\in W} \varepsilon(w) e^{w(\varrho)}\right)}{\sum_{w_0\in W_0} \varepsilon(w_0) e^{w_0(\varrho_0)}} = \frac{\sum_{w\in W_1} \varepsilon(w) \sum_{w_0\in W_0} \varepsilon(w_0) e^{w_0 w(\varrho)}}{\sum_{w_0\in W_0} \varepsilon(w_0) e^{w_0(\varrho_0)}}. \tag{7.11}$$

By the Weyl-Kac formula (7.6) for $\hat{p}'$, the theorem is now proven, provided we show that s and t have no irreducible component in common, whose contribution would cancel in the difference $\mathrm{ch}(s)-\mathrm{ch}(t)$. To show this, we consider first the finite dimensional analogs $\bar{s}$ and $\bar{t}$ of s and t. We know that the weights of the $o(2n)$-module $\bar{s}$ are of the form

$$\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \dots \pm \varepsilon_n) \tag{7.12}$$

with an even number of minuses, while those of $\bar{t}$ have an odd number of minuses. We prove that the sets of weights of $\bar{s}$ and $\bar{t}$, considered as p'-modules, do not intersect. Supposing the contrary, we would have an equality such as:

$$\bar{\varrho}_1-\alpha'_1-\alpha'_2-\ldots-\alpha'_l=\bar{\varrho}_1-\alpha''_1-\alpha''_2-\ldots-\alpha''_m. \tag{7.13}$$

where $\alpha'_i,\alpha''_i\in\bar{\Delta}_{1+}$, $\bar{\varrho}_1=\bar{\varrho}-\bar{\varrho}_0$, l is even and m is odd. But when a root in $\bar{\Delta}_1$ is expressed as a linear combination of simple roots, the sum of the coefficients in front of the simple roots which belong to $\bar{\Delta}_1$ must be odd, by the properties of a symmetric space. Therefore (7.13) is a contradiction.

Since g' is simple and p' semisimple, it follows that the p'-module V is irreducible. As a consequence the embedding $p'\subset o(2n)$ is *integral*, in the terminology of Dynkin [15]. This means that for any irreducible $o(2n)$-module $L(\bar{\lambda})$, its decomposition into irreducible p'-modules:

$$L(\bar{\lambda})=\bigoplus_{i\in I}L(\bar{\Lambda}_i) \tag{7.14}$$

is such that $\bar{\Lambda}_i-\bar{\Lambda}_j\in Q$, where Q is the root lattice of p', for any $i,j\in I$. Let $P(\bar{\Lambda}_i)$ denote the set of weights of $L(\bar{\Lambda}_i)$. $P(\bar{\Lambda}_i)$ is the intersection of $\bar{\Lambda}_i+Q$ with the solid polyhedron whose vertices are the elements of the orbit $\bar{W}_0\cdot\bar{\Lambda}_i$. Let $\mathcal{M}(\bar{\Lambda}_i)$ be the set of points of $\bar{\Lambda}_i+Q$ whose distance to the origin is minimal. Then $\mathcal{M}(\bar{\Lambda}_i)\subset P(\bar{\Lambda}_i)$, but $\mathcal{M}(\bar{\Lambda}_i)=\mathcal{M}(\bar{\Lambda}_j)$. So we have proved

$$\bar{\Lambda}_i-\bar{\Lambda}_j\in Q\ \Rightarrow\ P(\bar{\Lambda}_i)\cap P(\bar{\Lambda}_j)\neq\emptyset. \tag{7.15}$$

The relation $\bar{\Lambda}_i-\bar{\Lambda}_j\in Q$ defines equivalence classes of weights. Denote by $[\bar{s}]$ and $[\bar{t}]$ the equivalence class of weights of $\bar{s}$ and $\bar{t}$, respectively. They are well-defined because $p'\subset o(2n)$ is integral. Equation (7.15) implies that $[\bar{s}]\neq[\bar{t}]$ because the sets of weights of $\bar{s}$ and $\bar{t}$ are disjoint.

Now we come back to the infinite-dimensional theory and s,t. $\bar{s}$ is a subspace of s, and all the weights of s are in the same equivalence class, $[\bar{s}]$, and likewise for t. The converse of statement (7.15) is clear, so $[\bar{s}]\neq[\bar{t}]$ implies that the sets of weights of s and t are disjoint. This concludes the proof of the theorem. Observe that from (7.8) and (7.9), by looking at the coefficient of ϱ_0 in (7.11) we get:

$$j'=h_{g'}-h_q;\qquad j''=h_{g'}-2. \tag{7.16}$$

Proof of Corollary 1. In this case one has $W_1=\bar{W}/\bar{W}_0$. Let $\{\pm e_i\}i=1,\ldots,n+1$ be the weights of the fundamental representation of $sp(n+1)$. The e_i form an orthonormal basis of $\bar{h}^*$. The structure of the group $\bar{W}$ is:

$$\bar{W}=S_{n+1}\ltimes\mathbb{Z}_2^{n+1}, \tag{7.17}$$

where S_{n+1} is the permutation group on the set $\{e_1,\ldots,e_n\}$ and $\mathbb{Z}_2^{n+1}$ acts by $e_i\mapsto\pm e_i$. We can choose the same basis $\{e_i\}$ for $\bar{h}_0$, with $i=1$ corresponding to $su(2)$ and $i=2,\ldots,n+1$ to $sp(n)$. Then we have:

$$\bar{\varrho}=(n+1)e_1+ne_2+(n-1)e_3+\ldots+e_{n+1}, \tag{7.18}$$

$$\bar{\varrho}_0=e_1+ne_2+(n-1)e_3+\ldots+e_{n+1}. \tag{7.19}$$

It can be checked that

$$W_1=\{1,(12),(123),\ldots,(12\ldots n+1)\} \tag{7.20}$$

is a set of coset representatives with the required property. Here $(12 \ldots n)$ is the cycle $1 \to 2 \to \ldots \to n \to 1$. Computing $w(\varrho) - \varrho_0$ for $w \in W_1$ proves the corollary.

Proof of Theorem 2. For definiteness we take $q = sp(N)$. The exceptional cases may be treated similarly. Put $g = o(4N)$. Let ω_i, $i = 1, \ldots, 2N$ and $\dot{\omega}_i$, $i = 1, \ldots, N$ be the fundamental weights of g and q. Then the Cartan subalgebra u^* of $\hat{g}$ is spanned by Λ_0 (here we do not use the superscript g), δ, and the ω_i; h_0^* is spanned by $\Lambda'_0, \Lambda''_0, \delta, e$ and the $\dot{\omega}_i$, where $'$ refers to the $sp(N)$ component of p', $''$ to the $su(2)$ component and e is the fundamental weight of $su(2)$. The map $\pi : u^* \to h_0^*$ defined before is explicitly given by

$$\begin{aligned}
\pi(\Lambda_0) &= \Lambda'_0 + N\Lambda''_0 , \\
\pi(\delta) &= \delta , \\
\pi(\omega_i) &= \dot{\omega}_i + ie , \qquad 1 \leqq i \leqq N , \\
\pi(\omega_i) &= \dot{\omega}_{2N-i} + ie , \qquad N+1 \leqq i \leqq 2N-2 , \\
\pi(\omega_{2N-1}) &= \dot{\omega}_1 + (N-1)e , \\
\pi(\omega_{2N}) &= Ne .
\end{aligned} \tag{7.21}$$

π projects a weight of $\hat{g}$ onto a weight of $\hat{p}'$. In principle, one could find the decomposition of a representation of $\hat{g}$ by applying π to all the weights of the representation and doing an appropriate bookkeeping of the multiplicities.

Let α_i, $i = 0, 1, \ldots, 2N$ denote the simple roots of $\hat{o}(4N)$. The fundamental weights Λ_i of $\hat{o}(4N)$ are defined by the relations:

$$\frac{2(\Lambda_i|\alpha_j)}{(\alpha_j|\alpha_j)} = \delta_{ij} . \tag{7.22}$$

They are linked to the ω_i:

$$\Lambda_i = m_i \Lambda_0 + \omega_i , \tag{7.23}$$

where $m_1 = m_{2N-1} = m_{2N} = 1$ and $m_i = 2$ for $2 \leqq i \leqq 2N-2$. Note that

$$\{o, v, s, t\} = \{\Lambda_0, \Lambda_1, \Lambda_{2N-1}, \Lambda_{2N}\} . \tag{7.24}$$

From the automorphism μ of the Dynkin diagram shown in Fig. 2 one can construct an automorphism of u^* [21]:

$$\mu = t_\omega \circ w , \tag{7.25}$$

where $\omega = \omega_{2N-1} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{2N})$, t_ω is defined as in (6.4) and w is the Weyl group element of $o(4N)$ given by:

$$w(\varepsilon_i) = -\varepsilon_{2N-i+1} . \tag{7.26}$$

The action of μ on the fundamental weights is

$$\mu(\Lambda_i) = \Lambda_{2N-i} \bmod \mathbb{C}\delta . \tag{7.27}$$

We also have an automorphism μ' of the Dynkin diagram of $\widehat{su}(2)$ (see Fig. 3).

Fig. 3. The automorphism μ'

Correspondingly, we define an automorphism of h_0^* by:

$$\mu'(n\Lambda_0''+me)=n\Lambda_0''+(n-m)e-(n/4-m/2)\delta\,, \tag{7.28}$$

and $\mu'=$ identity on $\Lambda_0', \delta, \dot{\omega}_i$.

Note that μ (and μ') can be extended to an automorphism of the whole affine algebra $\hat{o}(4N)$ [respectively $\widehat{su}(2)$] by putting

$$\mu(e_i)=e_{\mu(i)}\,,\qquad \mu(f_i)=f_{\mu(i)}\,, \tag{7.29}$$

where $e_i, f_i, i=0,1,\ldots,2N$ are the Chevalley generators. From that it follows that μ and μ' preserve weight multiplicities.

The reason why we introduced π, μ, and μ' is because we have the commutative diagram:

$$\begin{array}{ccc} u^* & \xrightarrow{\pi} & h_0^* \\ \mu\big\downarrow & & \big\downarrow\mu' \\ u^* & \xrightarrow{\pi} & h_0^* \end{array} \tag{7.30}$$

i.e. we have $\pi\circ\mu=\mu'\circ\pi$. It is trivial to verify this using (7.21), (7.25), and (7.28).

By considering (7.30) the theorem follows immediately. Indeed, applying $\pi\circ\mu$ on $s\oplus t$ corresponds to mapping the Ramond sector to the Neveu-Schwarz sector, and then looking at the decomposition of the Neveu-Schwarz sector, which is what we want to know. On the other hand, applying first π, i.e. decompose the Ramond sector – which is easy to do by Theorem 1 – and then μ', amounts to the same thing.

8. An Example

To illustrate the use of decomposition theorems, we will show how to compute the invariant $Z(X)$ for $X=D_l$, $n\equiv 2\,(\mathrm{mod}\,4)$. In this case $\mathscr{H}$ is given by (5.11). We use (5.19), Corollaries 1 and 2 and find the decomposition of the representation o of $\hat{o}(4n)$ into irreducible representations of $\hat{p}'\oplus\widehat{su}(2)^a\oplus\widehat{su}(2)^b$:

$$o=\bigoplus_{j\ \mathrm{even}=0}^{n-1}(\Lambda_j,j,0,0)\oplus\bigoplus_{j\ \mathrm{odd}=1}^{n-1}(\Lambda_j,j,1,1)\,. \tag{8.1}$$

With (3.21) this can be further decomposed into representations of $\hat{q}\oplus\widehat{su}(2)^a\oplus\widehat{su}(2)^d\oplus\mathrm{Vir}$:

$$\begin{aligned} o=&\bigoplus_{j\ \mathrm{even}=0}^{n-1}\ \bigoplus_{m\ \mathrm{even}=0}^{n}(\Lambda_j,0,m,[j+1,m+1])\\ &\oplus\bigoplus_{j\ \mathrm{odd}=1}^{n-1}\ \bigoplus_{m\ \mathrm{even}=0}^{n}(\Lambda_j,1,m,[j+1,m+1])\,. \end{aligned} \tag{8.2}$$

Now consider (o_L, o_R). According to (5.14) we have to select the representations which satisfy $\Lambda_{j,L}=\Lambda_{j,R}$, $m_L=m_R$ etc. Setting $p=j+1$ and $q=m+1$ we find that the contribution of (o_L, o_R) is

$$n\sum_{p=1}^{n}\ \sum_{q\ \mathrm{odd}=1}^{n}|\chi_{pq}|^2\,. \tag{8.3}$$

Similarly one can find the decomposition of v and t:

$$v = \bigoplus_{j\ \text{even}=0}^{n-1} \bigoplus_{m\ \text{odd}=1}^{n} (\Lambda_j, 1, m, [j+1, m+1]) \oplus \bigoplus_{j\ \text{odd}=1}^{n-1} \bigoplus_{m\ \text{odd}=1}^{n} (\Lambda_j, 0, m, [j+1, m+1]), \tag{8.4}$$

$$t = \bigoplus_{j\ \text{even}=0}^{n-1} \bigoplus_{m\ \text{odd}=1}^{n} (\Lambda_j, 1, m, [n-j, m+1]) \oplus \bigoplus_{j\ \text{odd}=1}^{n-1} \bigoplus_{m\ \text{odd}=1}^{n} (\Lambda_j, 0, m, [n-j, m+1]), \tag{8.5}$$

As a result, the contribution of (v_L, t_R) is n times

$$\sum_{p=1}^{n} \sum_{q\ \text{even}=1}^{n} \chi_{pq}\bar{\chi}_{n+1-p,q} = |\chi_{p,2l}|^2 + \sum_{p=1}^{n} \sum_{q\ \text{even}=1}^{2l-2} (\chi_{pq}\bar{\chi}_{4l-1-p,q} + \text{c.c.}). \tag{8.6}$$

The terms (s_L, s_R) and (t_L, v_R) produce exactly the same expressions, (8.3) and (8.6). Thus we get the relation (5.20) in a particular case.

Appendix: The Exceptional Solutions

The explicit decompositions of $s \oplus t$ resulting from applying Theorem 1 when $g' = E_6, E_7, E_8$, $q = su(6), so(12), E_7$ are given in Tables 3–5. Each entry in these tables corresponds to an irreducible representation $L(\Lambda)$ of $\hat{q} \oplus \widehat{su}(2)$. The first column contains a + or a − depending on whether the representation is contained in s or t. The second and third column are the Dynkin coordinates of Λ for the q and the $su(2)$ components, respectively. The entries whose second column is

Table 3. The decomposition in the case $g' = E_6$

+	00000	10	+	01202	2	−	10050	9	+	20040	10
+	00006	0	−	01302	3	+	10104	2	−	20102	5
+	00030	6	−	02001	7	+	10112	4	+	20202*	0
+	00060	10	+	02004	0	+	10120	8	+	20202*	6
−	00100	9	+	02020*	4	−	10212	5	+	20210	2
−	00130	7	+	02020*	10	−	10301	1	−	20310	3
+	00200	10	−	02031	7	+	10401	2	+	21012	2
−	00203	3	+	02101	8	+	11011	6	+	21101	4
+	00303	4	−	02120	5	−	11030	9		21201	5
+	00400	0	+	03000	6	−	11103	1	+	30003	4
−	00500	1	−	03011	9	−	11111*	3	−	30111	1
+	00600	0	+	03030	6	−	11111*	7		30200	3
−	01005	1	−	03100	7	+	11211	4	+	30300	4
+	01010	8	+	04002	10	−	12010	5	−	31002	3
−	01021	5	+	04010	8	+	12021	8	+	40020	0
+	01040	8	−	05001	9	+	12110	6	+	40101	2
−	01110	9	+	06000	10	−	13020	7	−	50010	1
+	01121	6	−	10020	7	−	20013	3	+	60000	0

Table 4. The decomposition in the case $g' = E_7$

+	000000	16	+	010000	14	−	100202	7	+	202020	0
+	000004	10	+	010004	12	+	100211	8	+	202100	2
+	000008	16	−	010013	13	−	100300	9	−	210111	7
−	000013	11	−	010051	1	+	101001	14	+	210120*	2
−	000017	15	+	010060	2	+	101003	10	+	210120*	8
+	000026	16	+	010102*	8	−	101012	11	+	211010	4
+	000062	0	+	010102*	14	+	101041	2	+	220100	6
−	000071	1	−	010111	9	−	101050	3	+	300031	4
+	000080	0	+	010200	10	+	101101	12	−	300040	5
+	000102	12	−	010211	5	+	101121	4	+	301021	6
+	000106	14	+	010220	6	−	101130	5	−	301030	7
−	000115	15	−	011001	15	−	102000	13	−	301110	1
+	000204	16	−	011003	11	−	103010	1	−	302000	3
+	000302	6	+	011012	12	−	110000	15	−	310020	3
−	000311	7	−	011101	13	−	110102	9	−	311010	5
+	000400	8	+	012000	14	+	110111*	6	−	400031	5
−	001001	13	+	012020	2	+	110111*	10	+	400040	6
−	001003	9	+	020000	16	−	110120	7	+	400200	0
−	001005	13	−	020011	7	−	110200	11	+	401010	2
+	001012	10	+	020020	8	−	111030	1	+	402000	4
+	001014	14	+	020040	0	−	111110	3	+	410020	4
−	001101	11	+	020102	10	+	120011	8	−	500100	1
−	001103	15	−	020111	11	−	120020	9	−	501010	3
+	002000	12	+	020200*	4	−	120100	5	+	600000	0
+	002002	16	+	020200*	12	−	130000	7	+	600100	2
+	002004	12	+	030000	6	+	200000	16	−	700000	1
−	002013	13	−	030011	9	−	200131	3	+	800000	0
−	002031	3	+	030020	10	+	200140	4			
+	002040	4	+	040000	8	+	200202	8			
+	002102	14	−	100000	15	−	200211	9			
−	003001	15	−	100004	11	+	200300	10			
+	004000*	0	+	100013	12	−	201021	5			
+	004000*	16	−	100102	13	+	201030	6			

Table 5. The decomposition in the case $g' = E_8$

+	0000000	28	+	0001200	26	−	0030000	15	+	0120100	18
−	0000003	21	+	0002000*	18	−	0030100	19	−	0130000	17
+	0000006*	0	+	0002000*	28	+	0040000*	0	+	0200000	28
+	0000006*	28	−	0002011	23	+	0040000*	18	+	0200004	0
+	0000012	22	+	0002020	24	+	0100000	26	−	0200031	19
−	0000015	27	+	0003000*	6	−	0100003	23	+	0200040	20
+	0000024	28	+	0003000*	22	−	0100005	1	+	0200200	14
−	0000051	15	−	0010000	25	+	0100012	24	−	0200201	7
+	0000060	16	+	0010003	24	−	0100031	17	+	0201020*	10
−	0000101	23	−	0010012	25	+	0100040	18	+	0201020*	18
+	0000102	20	+	0010021	18	−	0100101	25	+	0202000	4
+	0000104	26	−	0010030	19	+	0100102	22	−	0210100	13
−	0000111	21	+	0010101	26	−	0100111	23	−	0210110	17
−	0000113	27	−	0010102	23	+	0100200	24	+	0220010	16
+	0000200	22	+	0010111	24	+	0100220	12	−	0300021	9

Table 5 (continued)

+	0000202	28	−	0010130	13	+	0101000	26	+	0300200*	6
−	0000203	25	−	0010200	25	−	0101011	21	+	0300200*	16
+	0000212	26	−	0010300	9	+	0101020*	16	+	0301000	12
−	0000301	27	−	0011000	27	+	0101020*	22	−	0310100	15
+	0000400*	10	−	0011010	17	+	0101200	8	−	0400001	11
+	0000400*	28	+	0011011	22	+	0102000	20	+	0400020	8
+	0001000	24	−	0011020	23	−	0102001	5	+	0401000	14
−	0001003	25	−	0012000	21	−	0110000	27	+	0500000	10
−	0001011	19	+	0020000	28	+	0110021	20	−	0500001	13
+	0001012	26	−	0020003	3	−	0110030	21	+	0600000	12
+	0001020	20	−	0020021	21	−	0110110	15	−	1000000	27
+	0001040	14	+	0020030*	12	−	0110120	11	+	1000003	22
−	0001101	27	+	0020030*	22	−	0111010	19	−	1000012	23
+	0001102	24	+	0020100	16	+	0120002	2	+	1000041	16
−	0001111	25	+	0021010	20	+	0120010	14	−	1000050	17
+	1000101	24	−	1110003	1	+	2101002	2	+	4020010	4
−	1000102	21	+	1110110*	12	−	2101011	7	+	4100012	4
+	1000111	22	+	1110110*	16	+	2101100	10	−	4101001	9
−	1000200	23	−	1111001	3	+	2110101	4	+	4110001	6
−	1000310	11	−	1120010	15	−	2110110	13	+	4201000	8
−	1001000	25	+	1200111	8	−	2200101	9	+	5000003	6
+	1001011	20	−	1200200	15	+	2201010*	6	−	5001100	1
−	1001020	21	−	1201010	11	+	2201010*	12	+	5010002	8
−	1001030	15	−	1201100	5	−	2300011	11	−	5010011	3
−	1002000	19	+	1210100	14	+	2300100	8	−	5020000	5
−	1002100	7	+	1300011	10	+	2400010	10	−	5100002	5
+	1010000	26	−	1300110	7	+	3000103	4	−	5110001	7
+	1010004	2	−	1301000	13	−	3000300	13	−	6000003	7
−	1010021	19	+	1400001	12	−	3002000	9	+	6000200	0
+	1010030	20	−	1400010	9	+	3010012	6	+	6001010	2
+	1010120	14	−	1500000	11	+	3010200	12	+	6010001	4
+	1010210	10	+	2000000	28	−	3011001	1	+	6020000	6
+	1011002	4	−	2000041	17	−	3020100	3	+	6100002	6
+	1011010	18	+	2000050	18	−	3100102	3	−	7000110	1
−	1020020	13	+	2000300	12	+	3101001	8	−	7001000	3
−	1020100	17	−	2001003	3	−	3101100	11	−	7010001	5
+	1030000	16	+	2001030	16	−	3110011	5	+	8000020	0
−	1030001	1	+	2002010	8	+	3200101	10	+	8000100	2
−	1100000	27	−	2010102	5	−	3201000	7	+	8001000	4
+	1100031	18	−	2010120	15	−	3300100	9	−	9000010	1
−	1100040	19	−	2010200	11	−	4000013	5	−	9000100	3
−	1100210	13	+	2020002	0	+	4002000*	0	+	10,000000	0
−	1101020	17	+	2020020	14	+	4002000*	10	+	10,000010	2
+	1101101	6	+	2021000	2	−	4010002	7	−	11,000000	1
−	1101110	9	+	2100210	14	+	4010101	2	+	12,000000	0

followed by $*$ give rise to the off-diagonal terms of the exceptional invariants. They come in pairs with the q component of Λ repeating itself. This peculiar phenomenon does not happen in the classical cases.

One can check that the following property holds for each one of these tables: the number of entries whose third column equals j, where j is any integer between zero and $h_{g'}-2$ [see Eq. (7.16)], is equal to the rank l of g', except when $j+1$ is an exponent of g', in which case it is equal to $l+1$. This takes care of the diagonal terms in formula (5.21).

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