Commun. Math. Phys. 118, 171–176 (1988)

Communications in
Mathematical
Physics

Lie Group Exponents and $SU(2)$ Current Algebras

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Abstract. Due to the Cappelli-Itzykson-Zuber classification, the minimal conformally invariant quantum field theories with $SU(2)$ currents are classified by the ADE Lie algebras. Here I give a conceptual proof of the empirically valid relation between their partition functions and the Lie algebra exponents.

Consider a conformally invariant quantum field theory on $S_1 \times R$ with left and right $SU(2)$ currents. Let the Hilbert space of the theory decompose into a finite number of irreducible level k representations of the Kac-Moody current algebra $A_1^{(1)} \times A_1^{(1)}$. Then the partition function is of the form

$$Z(w, \bar{w}) = \sum_{i,j=1}^{k+1} \chi_i^{(k)}(w) a_{ij} \chi_j^{(k)}(w)^* , \tag{1}$$

where the a_{ij} are non-negative integers. The $\chi_i^{(k)}$, $i = 1, \ldots, k+1$ are the characters of the irreducible unitary positive energy representations of level k of $A_1^{(1)}$. The label i is the dimension of the subspace of lowest energy, which forms an irreducible representation of the $SU(2)$ charge algebra.

If the vacuum state is non-degenerate, one must have $a_{11} = 1$. As the partition function can be written as a functional integral over a torus, it must be invariant under modular transformations. The partition functions of this type are in one-to-one correspondence with the Lie algebras of ADE type Lie groups G. In particular, $k+2$ is the Coxeter number of G, and a_{ii} is the number of G exponents equal to i [1].

So far this fact had not been explained in a conceptual way, though in [2] I gave the following construction of the $SU(2)$ modular invariants in terms of G. For a given G fix a set $\Delta^+(G)$ of positive roots and consider the subgroup H of G which leaves the highest root α invariant. Moreover, consider the $SU(2)$ subgroup of G generated by E_α and $E_{-\alpha}$. The coset space $G/(H \times SU(2))$ is the unique quaternionic symmetric space with symmetry group G. More precisely, for adjoint groups G, H the holonomy group is $(\tilde{H} \times SU(2)/Z_2$, where $\tilde{H}$ is a double cover of H. Ignoring

Table 1. The list of quaternionic symmetric spaces

G	H	k
SU(N+2), $N \geqq 1$	$SU(N) \times U(1)$	N
$SO(2N+4)$, $N \geqq 2$	$SO(2N) \times SU(2)$	$2N$
E_6	$SU(6)$	10
E_7	$SO(12)$	16
E_8	E_7	28

this subtlety for the moment, the list of symmetric spaces of this type is given by Table 1.

Consider a free Majorana fermion field taking values in the tangent space $TG/(H \times SU(2))$ and sum over periodic and anti-periodic boundary conditions. The resulting model is isomorphic to the level 1 WZW sigma model on $SO(4k)$, where

$$4k = \dim G - \dim H - 3\,. \tag{2}$$

The model has natural $H \times SU(2)$ currents, and the Hilbert space decomposes into a finite number of irreducible representations of the corresponding current algebra [3, 4]. After gauging the H currents with infinite coupling constant most states become infinitely massive and drop out of the Hilbert space. There remain the states in the lowest energy spaces of the irreducible $H_L \times H_R$ current algebra representations which are singlets under the global H_{L-R} group [5]. One is left with a theory with $SU(2)$ currents of level k, whose Hilbert space decomposes into a finite number of irreducible representations of the $SU(2)$ current algebra. Its partition function is of the form

$$2\left(\operatorname{rank} G \sum_{i=1}^{k+1} |\chi_i^{(k)}(w)|^2 + Z^G(w, \bar{w})\right), \tag{3}$$

where $Z^G(w, \bar{w})$ can be written as in Eq. (1), with coefficients a_{ij}^G. Up to a sign, it is the modular invariant partition function corresponding to G.

Only for $G = SU(N+2)$ the sign is negative and one finds

$$Z^{SU(N+2)}(w, \bar{w}) = -\sum_{i=1}^{N+1} |\chi_i^{(N)}(w)|^2\,. \tag{4}$$

This case is somewhat more complicated than the others, as the symmetric space is hermitian. Moreover, the structure of the modular invariant is very simple. Thus not much is gained by generalizing the calculations to include it, though this is not too difficult [2]. Here it will not be considered any further, i.e. H will be assumed to be semisimple.

In order to calculate the partition function, we have to decompose the partition function of the $SO(4k)$ sigma model into $H \times SU(2)$ characters. The $SO(4k)$ partition function taking values in the characters of $H \times SU(2)$ is

$$\tilde{Z}^G(w, \bar{w}) = |\chi_0^G(w)|^2 + |\chi_1^G(w)|^2 + |\chi_+^G(w)|^2 + |\chi_-^G(w)|^2\,, \tag{5}$$

where

$$\chi_0^G - \chi_1^G = \frac{\hat{D}(G)}{\hat{D}(H \times SU(2))}, \tag{6}$$

$$\chi_+^G - \chi_-^G = \frac{D(G)}{D(H \times SU(2))}, \tag{7}$$

and

$$D(G) = q^{\dim G/24} \prod_{\beta \in \Delta^+(G)} (e^{\beta/2} - e^{-\beta/2}) \prod_{n=1}^{\infty} \times \left[(1-q^n)^{\mathrm{rank}\, G} \prod_{\beta \in \Delta^+(G)} (1-e^{\beta}q^n)(1-e^{-\beta}q^n)\right], \tag{8}$$

$$\hat{D}(G) = q^{-\dim G/48} \prod_{n=1}^{\infty} \times \left[(1-q^{n-1/2})^{\mathrm{rank}\, G} \prod_{\beta \in \Delta^+(G)} (1-e^{\beta}q^{n-1/2})(1-e^{-\beta}q^{n-1/2})\right], \tag{9}$$

and analogously for H and $SU(2)$. Note that

$$\mathrm{rank}\, G = \mathrm{rank}\, H + 1\,. \tag{10}$$

The contributions of the lowest energy subspaces to χ_0^G, χ_1^G, χ_+^G, χ_-^G can be recognized immediately as the characters of $SO(4k)$ scalar, vector and Weyl spinor representations.

The expression for Z^G in Eq. (5) has to be decomposed into irreducible characters of $H \times SU(2)$. To do this we need the Weyl-Kac character and denominator formulas. The denominator formula is

$$D(G) = \sum_{w \in W(G)} \varepsilon(w)\theta_{g(G)}(T(G), w\varrho(G))\,. \tag{11}$$

Here

$$\varrho(G) = \tfrac{1}{2} \sum_{\beta \in \Delta^+(G)} \beta \tag{12}$$

and

$$\theta_n(T, \lambda) = \sum_{\beta \in nT} e^{\beta+\lambda} q^{(\beta+\lambda)^2/2n}\,. \tag{13}$$

$W(G)$ is the Weyl group of G and ε its homomorphism onto $(+1, -1)$. The lattice $T(G)$ is the root lattice and $g(G)$ is the Coxeter number of G. Note that for ADE algebras dual Coxeter number and Coxeter number coincide and all roots are long, which allows the somewhat simplified description. For semisimple Lie groups

$$G' = \prod_i H_i, \qquad H_i \text{ simple}, \tag{14}$$

we shall use the vectorial notation

$$\mathrm{g}(G') = (\mathrm{g}(H_i)), \tag{15}$$

Multiplication by an integral vector (n_i) will map $T(G')$ into $\oplus n_i T(H_i)$. Similarly, the level of the corresponding semisimple affine Kac-Moody algebra will be a vector $k'=(k_i)$. Then the Weyl-Kac character formula is

$$\chi[G,k',\lambda]=D(G')^{-1}\sum_{w\in W(G')}\varepsilon(w)\theta_{g(G')+k'}(T(G'),w\lambda). \tag{16}$$

Here the representation with highest weight $\lambda-\varrho(G')$ is labeled by λ, modulo lattice translation and Weyl group actions. For $SU(2)$ characters we often write

$$\chi[SU(2),k,m\alpha/2]=\chi_m^{(k)} \tag{17}$$

as in Eq. (1).

To decompose $\chi_+^G-\chi_-^G$ into $H\times SU(2)$ characters we need to consider the coset spaces

$$S=W(G)/W(H\times SU(2)) \tag{18}$$

and $T(G)/T(H\times SU(2))$. The latter has two elements, one of which will be represented by zero. In the other one we choose the dominant $H\times SU(2)$ weight τ of shortest length. The cosets in S will be represented by elements s which transform $\varrho(G)$ into a dominant weight of $H\times SU(2)$.

Thus we obtain the decomposition

$$\begin{aligned}\chi_+^G-\chi_-^G=&\sum_{s\in S}\varepsilon(s)\,(\chi[H\times SU(2),(k_H,k),s\varrho(G)]\\&+\chi[H\times SU(2),(k_H,k),g(G)\tau+s\varrho(G)]).\end{aligned} \tag{19}$$

The levels of the $H\times SU(2)$ current algebra representations are

$$(k_i,\ k)=g(G)-g(H\times SU(2))=(g(G)-g(H_i),\ g(G)-2). \tag{20}$$

Note in particular that $g(G)=k+2$.

For any G' with weight τ one has

$$\chi[G',\ k',\ \lambda+(g(G')+k')\tau]\,(e^{ih})=e^{i(k'\tau,h)}q^{k'(\tau,\tau)/2}\chi[G',\ k',\ \lambda]\,(e^{ih}q^{\tau}). \tag{21}$$

Here e^{ih} belongs to the Cartan subgroup of G', and we have identified weights with generators of this Cartan subgroup using the standard bilinear form. Thus q^τ belongs to the complexification of the Cartan group and Eq. (24) makes sense. According to this equation, the translation

$$\lambda\to\lambda+(g(G')+k')\tau \tag{22}$$

yields an automorphism of the Kac-Moody algebra. If τ is a root, the automorphism is inner. Thus the outer automorphisms correspond to the cosets of the weight lattice modulo the root lattice, or equivalently to representations of the center of the covering group of G'. In particular, the translation by $g(G)\tau$ in Eq. (19) yields the outer automorphism of the $H\times SU(2)$ current algebra which corresponds to the isomorphism of the center of $(\tilde{H}\times SU(2))/Z_2$ with $(1,-1)$.

By construction, the highest root α transforms into its negative under the non-trivial element of $W(SU(2))$, whereas

$$w\alpha=\alpha\quad\text{for}\quad w\in W(H). \tag{23}$$

Inversely, $w\alpha \in \{\alpha, -\alpha\}$ implies $w \in W(H \times SU(2))$. Thus,

$$s(\beta)^{-1}\alpha = \beta \tag{24}$$

defines a bijection $\beta \leftrightarrow s(\beta)$ between $\Delta^+(G)$ and S. Using

$$(s(\beta)\varrho(G), \alpha) = (\varrho(G), s(\beta)^{-1}\alpha) = h(\beta), \tag{25}$$

where $h(\beta)$ is the height of the root β, we obtain

$$\begin{aligned}\chi_+^G - \chi_-^G = \sum_{\beta \in \Delta^+(G)} \varepsilon(s(\beta)) \{ & \chi[H, k_H, \pi_H s(\beta)\varrho(G)]\chi_{h(\beta)}^{(k)} \\ & - \chi[H, k_H, \pi_H(g(G)\tau + s(\beta)\varrho(G))]\chi_{k+2-h(\beta)}^{(k)}\}.\end{aligned} \tag{26}$$

Here

$$\lambda = \pi_H \lambda + (\alpha, \lambda)\alpha/2 \tag{27}$$

is the decomposition of G weights into weights of H and $SU(2)$. We also used

$$(\alpha, \tau) = 1. \tag{28}$$

To obtain the corresponding formula for $\chi_0^G - \chi_1^G$, we need an outer automorphism of the $SO(4k)$ current algebra. Under the embedding

$$\tilde{H} \times SU(2)/Z_2 \subset SO(4k) \tag{29}$$

a spinorial weight of $SO(4k)$ is mapped into $\alpha/2$, such that the corresponding outer automorphism of the $SO(4k)$ current algebra just induces the non-trivial outer automorphism of the $SU(2)$ current algebra, but leaves the H current algebra invariant. Thus

$$\begin{aligned}\chi_0^G - \chi_1^G = \sum_{\beta \in \Delta^+(G)} \varepsilon(s(\beta)) \{ & \chi[H, k_H, \pi_H s(\beta)\varrho(G)]\chi_{k+2-h(\beta)}^{(k)} \\ & - \chi[H, k_H, \pi_H(g(G)\tau + s(\beta)\varrho(G))]\chi_{h(\beta)}^{(k)}\}.\end{aligned} \tag{30}$$

This yields

$$a_{hh}^G + \operatorname{rank} G = n(h) + n(k+2-h), \tag{31}$$

where $n(h)$ is the number of G roots of height h. In terms of $m(i)$, the number of G exponents equal to i, one has

$$n(h) = \sum_{i=h}^{k+1} m(i). \tag{32}$$

Due to the other two basic properties of the Lie group exponents

$$\sum_{i=1}^{k+1} m(i) = \operatorname{rank} G \tag{33}$$

and

$$m(i) = m(k+2-i), \tag{34}$$

we obtain

$$a_{hh}^G = m(h), \tag{35}$$

which is the result we were looking for.

Moreover, $a_{ij}^G = 1$ for $i \neq j$, if there are positive roots β, β' of G with heights i, $k+2-j$ respectively, such that $\pi_H s(\beta)\varrho(G)$ and $\pi_H s(\beta')\varrho(G) + g(G)\pi_H \tau$ lie in the same orbit of the semidirect product of $W(H)$ with the lattice translations $T(H)$. It would be nice to have some conceptual understanding of the latter condition. If G is of type D_n, one has $\beta = \beta'$, but not for the exceptional groups.

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Communicated by L. Alvarez-Gaumé

Received March 7, 1988