Commun. Math. Phys. 117, 569–580 (1988)

Communications in Mathematical Physics

Hyper-Kähler Metrics and a Generalization of the Bogomolny Equations

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Abstract. We generalize the Bogomolny equations to field equations on $\mathbb{R}^3 \otimes \mathbb{R}^n$ and describe a twistor correspondence. We consider a general hyper-Kähler metric in dimension $4n$ with an action of the torus T^n compatible with the hyper-Kähler structure. We prove that such a metric can be described in terms of the T^n-solution of the field equations coming from the twistor space of the metric.

1. Introduction

Let $\tilde{E}$ be a rank k complex vector bundle on $\mathbb{R}^3 \otimes \mathbb{R}^n$ with a connection ∇ and n sections of the adjoint bundle $\Phi^1, \ldots, \Phi^n$, the Higgs fields. Let x^i_α, $i = 1, \ldots, n$, $\alpha = 1, 2, 3$ be the coordinates on $\mathbb{R}^3 \otimes \mathbb{R}^n$ and consider the field equations

$$\left.\begin{aligned} F_{x^i_\alpha x^j_\beta} &= \sum_\gamma \varepsilon_{\alpha\beta\gamma} \nabla_{x^i_\gamma} \Phi^j + \tfrac{1}{2}\delta_{\alpha\beta}[\Phi^i, \Phi^j] \\ \nabla_{x^i_\alpha} \Phi^j &= \nabla_{x^j_\alpha} \Phi^i \end{aligned}\right\}, \tag{1.1}$$

where $F = \sum F_{x^i_\alpha x^j_\beta} dx^i_\alpha \wedge dx^j_\beta$ is the curvature.

In each $\mathbb{R}^3$ obtained by fixing a vector in the $\mathbb{R}^n$ factor of $\mathbb{R}^3 \otimes \mathbb{R}^n$ the field equations reduce to the *Bogomolny equations* by contracting the fields with the vector, [5]. This is the generalization mentioned in the title. We prove that there is a twistor correspondence between solutions to these equations and holomorphic rank k bundles on $T = \mathcal{O}(2) \otimes \mathbb{C}^n$ trivial on real sections of $T \to \mathbb{CP}^1$.

We shall consider the field equations for the abelian torus T^n and their relation to *hyper-Kähler* geometry: Let (M, g) be a $4n$-dimensional Riemannian manifold with three almost complex structures I, J and K satisfying the quaternion algebra identities

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K$$

etc. Assume that g is Hermitian with respect to I, J and K, i.e.

$$g(IX, IY) = g(X, Y), \quad X, Y \in TM$$

etc. Then (M, g) is called a hyper-Kähler manifold iff the complex structures are covariant constant or equivalently iff I, J and K are integrable and the Kähler forms $\omega_1, \omega_2, \omega_3$ are closed, where

$$\omega_1(X, Y) = g(IX, Y), \quad X, Y \in TM$$

etc. [1, 10]. The twistor space Z of such a hyper-Kähler metric is the complex $2n+1$-dimensional manifold consisting of the compatible complex structures on M [6, 15]. It is a generalization of Penrose's non-linear graviton construction [14].

Recently, [11, 16], Hitchin et al described the general hyper-Kähler metric—and its twistor space—in dimension $4n$ with n commuting Killing fields which preserve I, J and K. From their description of the metric it is easily seen that

$$g = \sum_{i,j} [\Phi^{ij} d\bar{x}^i \cdot d\bar{x}^j + (\Phi^{ij})^{-1}(dy^i + A^i)(dy^j + A^j)], \tag{1.2}$$

where $d\bar{x}^i \cdot d\bar{x}^j = \sum_\alpha dx^i_\alpha dx^j_\alpha$,

$$\Phi^i = (\Phi^{i1}, \dots, \Phi^{in}), \quad i = 1, \dots, n$$

are n Higgs fields $\Phi^i : \mathbb{R}^3 \otimes \mathbb{R}^n \to \mathbb{R}^n$ and

$$A = (A^1, \dots, A^n)$$

a 1-form on $\mathbb{R}^3 \otimes \mathbb{R}^n$ with values in $\mathbb{R}^n$. Moreover, (A, Φ^i) satisfy the abelian field equations

$$\left.\begin{aligned} F_{x^i_\alpha x^j_\beta} &= \sum_\gamma \varepsilon_{\alpha\beta\gamma} \nabla_{x^i_\gamma} \Phi^j \\ \nabla_{x^i_\alpha} \Phi^j &= \nabla_{x^j_\alpha} \Phi^i \end{aligned}\right\}. \tag{1.3}$$

The twistor space of the metric is given as a sum of line bundles $L_1 \oplus \cdots \oplus L_n$ over T trivial on holomorphic sections of $T \to \mathbb{CP}^1$, and therefore corresponds to a solution to (1.3). We prove that this solution coincides with the solution appearing in the metric. Finally, we have some remarks on the sheaf cohomological aspects of the computations.

Remarks.

i) For $n = 1$ the field equations have been studied extensively [5, 8, 16].

ii) From (1.2) it follows that the geodesic flow on T^*M is obtained from the hamiltonian

$$H = \sum_{i,j} \Phi^{ij} \sigma^i \sigma^j + \sum_{i,j,\beta} (\Phi^{ij})^{-1} \left(\xi^i_\beta + \sum_k a^{k\beta}_i \sigma^k \right) \left(\xi^j_\beta + \sum_k a^{k\beta}_j \sigma^k \right),$$

where (ξ^i_α, σ^i) are fiber coordinates on T^*M and $A^i = \sum_{k,\alpha} a^{k\alpha}_i dx^k_\alpha$. This may have some physical interpretation in the metrics which arise as asymptotic models of the natural hyper-Kähler metric on the moduli space of k monopoles. Indeed, for $n = 1$ this is the case [7].

2. The Field Equations and the Twistor Correspondence

In this section we shall describe the twistor correspondence between the bundle $T \to \mathbb{CP}^1$ and $\mathbb{R}^3 \otimes \mathbb{R}^n$. Since this is a straightforward generalization of the correspondence between $\mathcal{O}(2)$ and $\mathbb{R}^3$ given in [5] this presentation will omit details.

The space $\mathbb{C}^3 \otimes \mathbb{C}^n$ parametries holomorphic sections of $T \to \mathbb{CP}^1$: If ζ denotes the affine coordinate on the complex line $\mathbb{CP}^1$ and $\eta^i, i = 1, \ldots, n$ are coordinates along the fibre of T, then a holomorphic section of T can be written

$$\eta^i = z^i - x^i\zeta - \bar{z}^i\zeta^2, \tag{2.1}$$

where

$$z^i = x_1^i + ix_2^i, \quad \bar{z}^i = x_1^i - ix_2^i, \quad x^i = 2x_3^i, \tag{2.2}$$

and $x_\alpha^i \in \mathbb{C}$. Since the real structure on T is given by

$$(\zeta, \eta^i) \to (-\bar{\zeta}^{-1}, -\bar{\eta}^i/\bar{\zeta}^2),$$

the real holomorphic sections parametrized by $\mathbb{R}^3 \otimes \mathbb{R}^n$ are given by $x_\alpha^i \in \mathbb{R}$. The sections passing through a point $(\zeta_0, \eta_0^i) \in T$ are parametrized by a $2n$-dimensional affine space $\pi = \pi(\zeta_0, \eta_0^i)$ which is foliated by n-dimensional affine spaces $N = N(\zeta_0, \eta_0^i, \lambda_0^i)$ of sections passing through (ζ_0, η_0^i) in a given direction λ_0^i. Since the metric on $\mathbb{R}^3 \otimes \mathbb{R}^n$ is given by

$$\sum_k (dx^k dx^k + 4dz^k d\bar{z}^k),$$

we see that the leaves N of the foliation are null and given by (2.1) together with

$$\zeta_0^{-1} dz^i + \zeta_0 d\bar{z}^i = 0. \tag{2.3}$$

The space π and its conjugate $\bar{\pi}$ intersect in a real n-dimensional affine space spanned by the n lines

$$x_\alpha^i = \mathring{x}_\alpha^i + tu_\alpha, \tag{2.4}$$

where u_α is the direction related to ζ_0 by stereographic projection

$$u_\alpha = (1 + \zeta_0\bar{\zeta}_0)^{-1}(\zeta_0 + \bar{\zeta}_0, -i(\zeta_0 - \bar{\zeta}_0), 1 - \zeta_0\bar{\zeta}_0).$$

Now, let E be a rank k bundle on T and assume E is trivial on every section (2.1) with $x_\alpha^i \in \mathbb{R}$. Since such a section is isomorphic to the projective line we denote it $\mathbb{CP}_x$. Then E will be trivial on sufficiently close complex sections so we obtain a rank k bundle $\tilde{E}$ on a neighbourhood of $\mathbb{R}^3 \otimes \mathbb{R}^n$ in $\mathbb{C}^3 \otimes \mathbb{C}^n$ by

$$\tilde{E}_x = H^0(\mathbb{CP}_x, \mathcal{O}(E)). \tag{2.5}$$

If we fix a point (ζ_0, η_0^i) in T then we obtain a flat connection ∇_π on $\pi(\zeta_0, \eta_0^i)$ by trivializing $\tilde{E}$

$$\psi : \tilde{E}|_\pi \xrightarrow{\sim} E, \tag{2.6}$$

where ψ evaluates a section on $\mathbb{CP}_x$ in the point (ζ_0, η_0^i). This defines [5], by differentiation at x, a matrix valued function $A = \{a_{ij}\}$ on the set V of vectors at x which are tangent to some N. Moreover, A is homogeneous of degree 1 and holomorphic, i.e.

$$a_{ij} \in H^0(V, \mathcal{O}(1)),$$
$$V = Q_1 \cap \cdots \cap Q_n,$$
$$Q_i = \left\{ [x_1^i, x_2^i, x_3^i] \,|\, \sum_\alpha (x_\alpha^i)^2 = 0 \right\}.$$

It is easily seen that a holomorphic section of $\mathcal{O}(1)|_V$ can be uniquely extended to a section $\hat{a}_{ij} \in H^0(\mathbb{CP}^{3n-1}, \mathcal{O}(1))$. Thus, we obtain a connection ∇ on $\tilde{E}$. Since, by definition, ∇ agrees with ∇_π on π in the directions of N, we have

$$\nabla - \nabla_\pi = \frac{i}{2}\sum_k \Phi^k dt^k \tag{2.7}$$

for some endomorphisms Φ^k, where from (2.3)

$$dt^k = \zeta_0^{-1} dz^k + \zeta_0 d\bar{z}^k. \tag{2.8}$$

Again, it follows from the holomorphic description that each Φ^k are independent of π and so gives a well-defined endomorphism of the bundle $\tilde{E}$. Now, since ∇_π is flat we obtain the equation

$$F|_\pi = \frac{i}{2}\sum_j \nabla\Phi^j \wedge dt^j + \frac{1}{4}\sum_{i,j} [\Phi^i, \Phi^j] dt^i \wedge dt^j, \tag{2.9}$$

where F is the curvature of ∇. This equation together with the coordinate change (2.2), and the fact that on π we have

$$dx^i = \zeta^{-1} dz^i - \zeta d\bar{z}^i \tag{2.10}$$

leads directly to the field equations in (1.1).

To reverse the construction let $\tilde{E} \to \mathbb{R}^3 \otimes \mathbb{R}^n$ be a bundle with connection ∇ and Higgs fields Φ^i, $i = 1, \ldots, n$, satisfying the field equations. We look for a bundle $E \to T$. Since we want the construction to be the inverse we have (2.5). Let π be the $2n$-space associated to a point (ζ, η^i) in T. Then, from (2.6) we have

$$E_{(\zeta,\eta^i)} = \{s \in \Gamma(\pi, \tilde{E}) | \nabla_\pi s = 0\}.$$

Now, since the covariant sections are given by the value at a point, it is sufficient to know s along the real lines in (2.4) generating $\pi \cap \bar{\pi}$. Thus, from (2.7) we are lead to define

$$E_{(\zeta,\eta^i)} = \left\{ s \in \Gamma(\pi \cap \bar{\pi}, \tilde{E}) | \forall j = 1, \ldots, n : \left(\nabla_u - \frac{i}{2}\Phi^j \right) s = 0 \right\}$$

(in the operator $\nabla_u - (i/2)\Phi^j$, u is the vector with coordinates $u^i_\alpha = \delta^{ij} u_\alpha$). In this way we get a C^∞ vector bundle of rank k and we shall proceed to construct a $\bar{\partial}$-operator on $\tilde{E}$: First, we paraphrase the description of T. Consider the double fibration

$$\begin{array}{ccc} S^2 \times (\mathbb{R}^3 \otimes \mathbb{R}^n) & \xrightarrow{\pi} & T \\ \downarrow p & & \\ \mathbb{R}^3 \otimes \mathbb{R}^n & & \end{array}$$

where

$$\pi(u_\alpha, x^i_\alpha) = \left(\zeta = \frac{u_1 + iu_2}{1 + u_3}, \eta^i = z^i - x^i \zeta - \bar{z}^i \zeta^2 \right),$$

$$p(u_\alpha, x^i_\alpha) = x^i_\alpha.$$

Also, consider the vector fields on $S^2 \times (\mathbb{R}^3 \otimes \mathbb{R}^n)$,

$$X^i(u_\alpha, x_\alpha^i) = \sum_\alpha u_\alpha \frac{\partial}{\partial x_\alpha^i}.$$

Then, a section s of E corresponds to a section $\hat{s}$ of $p^*\tilde{E}$ which satisfies

$$\left(\nabla_{X^j} - \frac{i}{2}\Phi^j\right)\hat{s} = 0, \quad j = 1, \dots, n,$$

and T is the quotient of $S^2 \times (\mathbb{R}^3 \otimes \mathbb{R}^n)$ by the n commuting vector fields X^i. Now, define vector fields V, Y^j on $S^2 \times (\mathbb{R}^3 \otimes \mathbb{R}^n)$:

$$Y^j = \frac{i}{2(1+\zeta\bar{\zeta})^2}\left\{ i(\zeta^2 - 1)\frac{\partial}{\partial x_1^j} + (1+\zeta^2)\frac{\partial}{\partial x_2^j} + 2i\zeta\frac{\partial}{\partial x_3^j}\right\}$$

$$V = \frac{\partial}{\partial \bar{\zeta}} + 2\sum_j (x_3^j + (x_1^j + ix_2^j)\bar{\zeta})\, Y^j.$$

Then we have

$$\pi_*(Y^j) = \frac{\partial}{\partial \bar{\eta}^j},$$

$$\pi_*(Z) = \frac{\partial}{\partial \bar{\zeta}}.$$

Finally define the $\bar{\partial}$ operator

$$D: \Gamma(E) \to \Omega^{0,1}(E)$$

by

$$\pi^*(Ds) = \sum_j \nabla_{Y^j}\hat{s}\, d\bar{\eta}^j + \nabla_Z \hat{s}\, d\bar{\zeta}.$$

In order for this to be well defined we need to show that $\nabla_{Y^k}\hat{s}$ and $\nabla_Z\hat{s}$ are pull back of sections, i.e. we need to prove

$$\left(\nabla_{X^j} - \frac{i}{2}\Phi^j\right)(\nabla_{Y^k}\hat{s}) = 0,$$

$$\left(\nabla_{X^j} - \frac{i}{2}\Phi^j\right)(\nabla_Z\hat{s}) = 0,$$

or since $(\nabla_{X^j} - (i/2)\Phi^j)\hat{s} = 0$, we need to prove that

$$\left[\nabla_{X^j} - \frac{i}{2}\Phi^j, \nabla_{Y^k}\right]\hat{s} = 0,$$

$$\left[\nabla_{X^j} - \frac{i}{2}\Phi^j, \nabla_Z\right]\hat{s} = 0.$$

This, however, follows directly from the generalized Bogomolny equations (and the fact that the connection on $p^*\tilde{E}$ is $p^*\nabla$). Furthermore it is a straightforward computation to see that $D^2 = 0$ and $D(fs) = \bar{\partial}fs + f\,Ds$, so E is a *holomorphic* bundle. Finally, to show that E is trivial on every real section we consider the point $x^i_\alpha = 0$ and the corresponding curve $\mathbb{P}_0$ given by all the real n-spaces $\pi \cap \bar{\pi}$ passing through 0. Fix a basis $(e_1, \dots, e_k)$ of the fibre $\tilde{E}_0$. Take the unique solution satisfying

$$\begin{aligned} (\nabla_{X^j} - i\Phi^j)\hat{s}_i &= 0 \\ \hat{s}_i(\zeta, \bar{\zeta}, 0) &= e_i \end{aligned} \qquad i = 1, \dots, k; j = 1, \dots, n.$$

Then $(\hat{s}_1, \dots, \hat{s}_k)$ defines sections $\hat{s}_i$ of E over $\mathbb{P}_0$, and it is easily seen by uniqueness of solutions to the system of partial differential equations that $\nabla_Z \hat{s}_i = 0$. Also, $\nabla_{Y^j} \hat{s}_i = 0$, so the trivialization is holomorphic. This ends the description of the twistor correspondence.

Remark. In the rest of this paper we shall only consider the abelian case where the term $[\Phi^i, \Phi^j)$ disappears. We hope to consider the non-abelian equations in a later paper, in particular their possible relation to the moduli space of monopoles in $\mathbb{R}^3$.

3. The Metric and the Twistor Space

We shall review briefly the work of Hitchin et al. [11, 6]: Let M be a $4n$-dimensional hyper-Kähler manifold with a free action of $\mathbb{R}^n$ on it. It is assumed that this action extends to a free holomorphic action of $\mathbb{C}^n$ on the twistor space Z. Then Z becomes a principal $\mathbb{C}^n$ bundle over T.

Remark. Strictly, Z is a $\mathbb{C}^n$ bundle only over some open subset of T. For example for $n = 1$ we seek solutions on $\mathbb{R}^3$ to the equations

$$dA = *d\Phi,$$

and since Φ is contained in the metric

$$g = \Phi d\bar{x}\cdot d\bar{x} + \Phi^{-1}(d\tau + A)^2,$$

we need Φ to be positive (or negative). But if Φ is positive and harmonic then Φ must be constant. This is easily seen using Poisson's integral formula and Gauss' law of the arithmetic mean. Thus, global non-trivial solutions do not exist.

The projection of the twistor lines in Z to T gives the $3n$-parameter family of sections (2.1) of $T \to \mathbb{CP}^1$. To find the full $4n$-parameter family of twistor lines we describe Z in terms of transition functions: Let $U, \tilde{U}$ be the usual cover of T. Then we have coordinates (ξ^i, η^i, ζ) on $\mathbb{C}^n \times U$ and $(\tilde{\xi}^i, \tilde{\eta}^i, \tilde{\zeta})$ on $\mathbb{C}^n \times \tilde{U}$ related by

$$\tilde{\xi}^i = \xi^i + \frac{\partial H}{\partial \eta^i}(\eta^j, \zeta), \quad \tilde{\eta}^i = \zeta^{-2}\eta^i, \quad \tilde{\zeta} = \zeta^{-1} \tag{3.1}$$

on $\mathbb{C}^n \times U \cap \tilde{U}$. Here H is a holomorphic function defined on $U \cap \tilde{U} = \mathbb{C}^n \times \mathbb{C}^*$. It is the Hamiltonian function for a symplectic vector field with respect to a symplectic form ω on the fibres of $Z \to T \to \mathbb{CP}^1$. Now, we seek holomorphic functions $\tilde{\xi}^i$ of $\tilde{\zeta}$ and functions ξ^i of ζ which satisfy

$$\tilde{\xi}^i(\zeta^{-1}) = \xi^i(\zeta) + \frac{\partial H}{\partial \eta^i}(\eta^j(\zeta), \zeta), \tag{3.2}$$

where $\eta^i(\zeta)$ is given in (2.1), i.e. we seek curves in Z which project to a fixed section of T under the projection $Z \to T$. Expanding in power series

$$\tilde{\xi}^i = \sum_{n=0}^{\infty} a_n^i \zeta^{-n}, \quad \xi^i = \sum_{n=0}^{\infty} b_n^i \zeta^n, \quad \frac{\partial H}{\partial \eta^i} = \sum_{n=-\infty}^{\infty} c_n^i \zeta^n, \tag{3.3}$$

we obtain from (3.2) and the residue theorem

$$b_n^j = -c_n^j = \frac{-1}{2\pi i} \int_\Gamma \zeta^{-(n+1)} \frac{\partial H}{\partial \eta^j} d\zeta, \quad n = 1, 2, \ldots \tag{3.4}$$

(and similarly with a_n^j)

$$a_0^j - b_0^j = \frac{1}{2\pi i} \int_\Gamma \zeta^{-1} \frac{\partial H}{\partial \eta^j} d\zeta. \tag{3.5}$$

Thus, from (3.5) we see that the coefficients a_0^i, b_0^i are not uniquely determined and this 1-dimensional ambiguity gives the remaining n parameters of twistor lines (in order for the lines to be real we must demand that $a_0^i = -\bar{b}_0^i$).

The manifold M which parametrizes the twistor lines is diffeomorphic to the fibres of $Z \to \mathbb{CP}^1$ and the twistor lines intersect the fibre at $\zeta = 0$ at a point with I-complex coordinates,

$$\eta^i(0) = z^i, \quad \xi^i(0) = b_0^i \equiv u^i. \tag{3.6}$$

To find x^i as a function of u^j and z^j we consider the solution to the equations in $\mathbb{R}^3 \otimes \mathbb{R}^n$,

$$F_{x^i x^j} + F_{z^i \bar{z}^j} = 0, \tag{3.7}$$

defined by [9]

$$F(x^j, z^j, \bar{z}^j) = \frac{1}{2\pi i} \int_\Gamma \zeta^{-2} H(z^j - x^j \zeta - \bar{z}^j \zeta^2, \zeta) d\zeta. \tag{3.8}$$

Then, it follows from (3.5), (3.6), (3.8), and the reality condition, that

$$F_{x^i} = u^i + \bar{u}^i, \tag{3.9}$$

which determines x^i implicitly as a function of u^j and z^j.

To determine the metric we use the fact that the symplectic form ω on the fibres of Z is given in terms of the symplectic forms $\omega_1, \omega_2, \omega_3$:

$$\omega = (\omega_2 + i\omega_3) + 2\omega_1 \zeta - (\omega_2 - i\omega_3)\zeta^2. \tag{3.10}$$

From this it is shown that

$$\omega_1 = i(\bar{\partial}(F_{z^j}) \wedge dz^j + du^j \wedge \bar{\partial} x^j). \tag{3.11}$$

By implicit differentiation of (3.9) we find that

$$\sum_k F_{x^i x^k} x^k_{\bar{z}^j} + F_{x^i \bar{z}^j} = 0, \tag{3.12}$$

$$\sum_k F_{x^i x^k} x^k_{\bar{u}^j} = \delta_{ij}.$$

Then from (3.7), (3.11), and (3.12) we get the metric

$$g = 2\,\mathrm{Re}\sum_{i,j}\Bigg[\bigg(F_{x^ix^j} + \sum_{k,l} F_{z^ix^k}(F_{x^kx^l})^{-1}F_{x^l\bar{z}^j}\bigg)dz^i \otimes d\bar{z}^j$$
$$- \sum_k F_{z^iz^k}(F_{x^kx^j})^{-1}\,dz^i \otimes d\bar{u}^j - \sum_k (F_{x^ix^k})^{-1}F_{x^k\bar{z}^j}du^i \otimes d\bar{z}^j$$
$$+ (F_{x^ix^j})^{-1}\,du^i \otimes d\bar{u}^j\Bigg]. \tag{3.13}$$

This ends the summary of [11, 6].

Now let us prove that the metric has the form in (1.2): Introduce real coordinates

$$y^j = i(\bar{u}^j - u^j), \tag{3.14}$$

then from (3.9) we obtain

$$du^j = \tfrac{1}{2}(dF_{x^j} + i\,dy^j). \tag{3.15}$$

Let Φ^{ij}, $i, j = 1, \ldots, n$ be the functions on $\mathbb{R}^n \otimes \mathbb{R}^3$ given by

$$\Phi^{ij} = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right). \tag{3.16}$$

The matrix Φ represents n-Higgs fields for the group $\mathbb{R}^n$,

$$\Phi^i = (\Phi^{i1}, \ldots, \Phi^{in}). \tag{3.17}$$

Let $\theta = (\theta^1, \ldots, \theta^n)$ be the connection in the principal $\mathbb{R}^n$-bundle M given by

$$\theta^i = \sum_j \Phi^{ij} g\left(\frac{\partial}{\partial y_j}, \cdot\right) = dy^i + A^i, \tag{3.18}$$

where $A = (A^1, \ldots, A^n)$ is a 1-form on $\mathbb{R}^n \otimes \mathbb{R}^3$ with values in $\mathbb{R}^n$. Then we easily get

$$(\Phi^{ij}, A^j) = \Big(2F_{x^ix^j}, \sum_l i(F_{x^jz^l}dz^l - F_{x^j\bar{z}^l}d\bar{z}^l)\Big). \tag{3.19}$$

Furthermore, we compute the quotient metric

$$g - \sum_{i,j} g\left(\frac{\partial}{\partial y^i}, \cdot\right)\Phi^{ij} g\left(\frac{\partial}{\partial y^j}, \cdot\right) = \sum_{i,j} \Phi^{ij}[\mathrm{Re}(dz^i \otimes d\bar{z}^j) + \tfrac{1}{4}dx^i dx^j] = \sum_{i,j} \Phi^{ij} d\bar{x}^i \cdot d\bar{x}^j.$$

Hence, the metric has the form in (1.2) and it is easily seen that (Φ^i, A) satisfy the field equations in (1.3). Thus the metric contains a solution to the equations in (1.3).

Remark. In dimension 4 the metric has the form $\Phi d\bar{x} \cdot d\bar{x} + \Phi^{-1}(d\tau + A)^2$, where (Φ, A) is a monopole in $\mathbb{R}^3$. The metrics of Gibbons and Hawking are all of this form. Indeed, in dimension 4, a hyper-Kähler metric is just a self-dual Einstein metric with vanishing cosmological constant.

4. The Solution Induced by the Twistor Space

We have described a hyper-Kähler manifold with $\mathbb{R}^n$ acting on it. If we exponentiate our description from before we get the set-up with a torus action. Thus the twistor space becomes a principal $(\mathbb{C}^*)^n$ bundle and we let $E = L_1 \oplus \cdots \oplus L_n$ be the associated vector bundle on T trivial on sections and with transition matrix

$$g_{01} = \mathrm{diag}\left(\exp\frac{\partial H}{\partial \eta^1}, \ldots, \exp\frac{\partial H}{\partial \eta^n} \right). \tag{4.1}$$

From Chapter 2 we know that such a vector bundle determines a T^n-solution (Φ^i, A) to the field equations: We get a flat connection ∇_π on all of the special $2n$-spaces π. To describe ∇_π we seek a trivialization of E on $\mathbb{CP}_x$: From (3.2) and (4.1) we see that $((0, \ldots, 0, \exp \xi^i, 0, \ldots, 0), (0, \ldots, 0, \exp \tilde{\xi}^i, 0, \ldots, 0))$, $i = 1, \ldots, n$ give such a trivialization. Also, we trivialize $\tilde{E}$ on $\mathbb{R}^3 \otimes \mathbb{R}^n$ by the n sections

$$s^i : \mathbb{R}^3 \otimes \mathbb{R}^n \to \tilde{E}, \tag{4.2}$$

where s^i at x is the section of E on $\mathbb{CP}_x$ given by $((0, \ldots, 0, \exp \xi^i, 0, \ldots, 0), (0, \ldots, 0, \exp \tilde{\xi}^i, 0, \ldots, 0))$. Now, suppose we had functions f^i on $\mathbb{R}^3 \otimes \mathbb{R}^n$ such that $f^i s^i$ satisfied $\psi(f^i s^i) = (0, 0, 1, 0, \ldots, 0)$

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{\psi}"] & E_{(\zeta_0, \eta_0^j)} \\ \pi(\zeta_0, \eta_0^j) \arrow[u, "f^i s^i"] \arrow[ur] & \end{tikzcd}$$

then $\nabla_\pi(f^i s^i) = 0$. We have

$$\psi(f^i s^i) = (0, \ldots, f^i(x) \exp \xi^i(\zeta_0), 0, \ldots, 0), \tag{4.3}$$

so

$$f^i = \exp(-\xi^i(\zeta_0)).$$

Then, from (3.3) we get

$$0 = \nabla_\pi(f^i s^i) = -d\xi^i(\zeta_0) \cdot \exp(-\xi^i) \cdot s^i + \exp(-\xi^i) \theta_\pi^i s^i,$$

$i = 1, \ldots, n$, where $\theta_\pi = (\theta_\pi^1, \ldots, \theta_\pi^n)$ is the connection form of ∇_π in the frame $(s^1, \ldots, s^n)$. Thus

$$\theta_\pi^i = d\xi^i(\zeta_0) = \sum_{n=0} db_n^i \zeta_0^n. \tag{4.4}$$

Consider the connection form of ∇ with respect to the frame $(s^1, \ldots, s^n)$

$$A^i = \sum_j (f^{ij} dz^j + g^{ij} dx^j + h^{ij} d\bar{z}^j). \tag{4.5}$$

From (2.1) and (2.3) we see that on a null space we have

$$A^i = \sum_j (h^{ij} - 2g^{ij}\zeta_0 - f^{ij}\zeta_0^2) d\bar{z}^j, \tag{4.6}$$

which coincide with

$$\theta_\pi^i = \sum_j \{(b_0^i)_{\bar{z}^j} + (-2(b_0^i)_{x^j} + (b_1^i)_{\bar{z}^j})\zeta_0 + (-2(b_1^i)_{x^j} - (b_0^i)_{z^j} + (b_2^i)_{\bar{z}^j})\zeta_0^2\} d\bar{z}^j. \tag{4.7}$$

Thus, we get

$$h^{ij} = (b_0^i)_{\bar{z}^j}, \quad 2g^{ij} = 2(b_0^i)_{x^j} - (b_1^i)_{\bar{z}^j}, \quad f^{ij} = (b_0^i)_{z^j} + 2(b_1^i)_{x^j} - (b_2^i)_{\bar{z}^j}, \tag{4.8}$$

and from (3.4), (3.6), (3.8), and (3.15) it follows that

$$(b_0^i)_{x^j} = \tfrac{1}{2}F_{x^i x^j}, \quad (b_0^i)_{z^j} = \tfrac{1}{2}F_{x^i z^j}, \quad (b_0^i)_{\bar{z}^j} = \tfrac{1}{2}F_{x^i \bar{z}^j}, \tag{4.9}$$
$$(b_1^i)_{x^j} = -F_{x^i z^j}, \quad (b_1^i)_{\bar{z}^j} = -F_{z^i \bar{z}^j}, \quad (b_2^i)_{\bar{z}^j} = -F_{x^i z^j}.$$

(For example: $b_2^i = -1/2\pi i \int_\Gamma \zeta^{-3}(\partial H/\partial \eta^i)d\zeta$, so $(b_2^i)_{\bar{z}^j} = 1/2\pi i \int \zeta^{-1}(\partial^2 H/\partial\eta^j \partial\eta^i)d\zeta = -F_{x^i z^j}$). This gives

$$A^i = -\frac{1}{2}\sum_j \{F_{x^i z^j} dz^j - F_{x^i \bar{z}^j} d\bar{z}^j\}. \tag{4.10}$$

Next, to find the Higgs fields Φ^j we note that on the null planes we have (2.7). Thus from (2.8) we get

$$A^i - \theta_\pi^i = \frac{i}{2}\sum_j \Phi^{ij}(\zeta_0^{-1} dz^j + \zeta_0 d\bar{z}^j), \tag{4.11}$$

and this gives

$$\Phi^{ij} = iF_{x^i x^j}. \tag{4.12}$$

Compare (4.10) and (4.12) with (3.19), and we have proved that the solution coming from the twistor space is equal to the solution contained in the metric (up to a scalar multiple).

Remark. The situation in dimension 4 of having an Einstein metric given in terms of a monopole has been considered earlier in a different setting [8, 13].

5. Sheaf Theoretical Considerations

The cohomology group $H^1(T, \mathcal{O}(-2))$ corresponds to solutions to the linear system of differential equations in (3.7): If $[f(\zeta, \eta^i)d\zeta] \in H^1(T, \mathcal{O}(-2))$, then the function

$$F(x^j, z^j, \bar{z}^j) = \frac{1}{2\pi i}\int_\Gamma f(\zeta, z^j - x^j\zeta - \bar{z}^j\zeta^2)d\zeta$$

satisfy $F_{x^i x^j} + F_{z^i \bar{z}^j} = 0$ [2, 4, 8, 12]. These equations become the Laplacian on the 3-space obtained by choosing a vector in the $\mathbb{R}^n$ factor of $\mathbb{R}^3 \otimes \mathbb{R}^n$. Furthermore the group $H^1(T, \mathcal{O})$ corresponds to the holomorphic line bundles trivial on sections of $T \to \mathbb{CP}^1$: We have a short exact sequence,

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0,$$

and we know that $H^1(T, \mathcal{O}^*)$ consists of the isomorphism classes of holomorphic line bundles on T. Also, $H^2(T, \mathbb{Z}) \cong H^2(\mathbb{CP}^1, \mathbb{Z}) \cong \mathbb{Z}$. Then, from the long exact sequence on cohomology we get

$$0 \to H^1(T, \mathcal{O}) \xrightarrow{\exp} H^1(T, \mathcal{O}^*) \xrightarrow{\delta} \mathbb{Z} \to .$$

Since the coboundary map δ is the Chern class we see that the image

$$\exp(H^1(T, \mathcal{O})) \subseteq H^1(T, \mathcal{O}^*)$$

consists of the line bundles with vanishing Chern class. Thus if $L \in \exp(H^1(T,\mathcal{O}))$ and $\mathbb{CP}^1$ is a section of T we get

$$\operatorname{degree}(L|_{\mathbb{CP}^1}) = \int_{\mathbb{CP}^1} c_1(L) = 0.$$

Hence from a class $[\partial H/\partial\eta^i(\zeta,\eta^j)]$ in $H^1(T,\mathcal{O})$ we obtain the line bundle L_i with transition function $\exp[\partial H/\partial\eta^i]$—trivial on sections—and the bundle $E = L_1 \oplus \cdots \oplus L_n$ gives the monopole as described above.

Next we shall describe an isomorphism

$$H^1(T,\mathcal{O}) \xrightarrow{\sim} H^1(T,\mathcal{O}(-2)\otimes\mathbb{C}^n).$$

Consider "differentiation along fibres," d_F, defined by the composite map

$$\mathcal{O}_T \xrightarrow{d} \Omega^1_T \xrightarrow{\pi} \Omega^1_F,$$

where Ω^1_M is the sheaf of germs of holomorphic 1-forms on M and F is the fibre of the projection

$$T \xrightarrow{p} \mathbb{CP}^1.$$

Now, it is obvious that $\Omega^1_F \cong p^*\mathcal{O}(-2)\otimes\mathbb{C}^n$. We then have the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^1} \to \mathcal{O}_T \xrightarrow{d_F} p^*\mathcal{O}(-2)\otimes\mathbb{C}^n \to 0,$$

and from the long exact sequence on cohomology we get

$$\to H^1(\mathbb{CP}^1,\mathcal{O}) \to H^1(T,\mathcal{O}) \xrightarrow{d_F} H^1(T,\mathcal{O}(-2)\otimes\mathbb{C}^n) \to H^2(\mathbb{CP}^1,\mathcal{O}) \to .$$

Hence, since $H^1(\mathbb{CP}^1,\mathcal{O}) = 0 = H^2(\mathbb{CP}^1,\mathcal{O})$ we obtain the isomorphism

$$H^1(T,\mathcal{O}) \xrightarrow{d_F} H^1(T,\mathcal{O}(-2)\otimes\mathbb{C}^n).$$

Now, the hyper-Kähler metric is given in terms of the solution to the equations in (3.7),

$$F = \int_\Gamma H\zeta^{-2}d\zeta,$$

represented by $H\zeta^{-2}d\zeta \in H^1(T,\mathcal{O}(-2))$. But the solution $(A,\Phi^1,\ldots,\Phi^n)$ to the field equations in (2.7), represented by $\partial H/\partial\eta^i \in H^1(T,\mathcal{O})$, $i=1,\ldots,n$, also gives solutions Φ^{kl} to the equations in (3.7), and we have seen that

$$\Phi^{kl} = F_{x^k x^l} = \int_\Gamma \frac{\partial^2 H}{\partial\eta^k\partial\eta^l}d\zeta.$$

Hence the solutions $(\Phi^{k1},\ldots,\Phi^{kn})$ are represented by $d_F[\partial H/\partial\eta^k] \in H^1(T,\mathcal{O}(-2)\otimes\mathbb{C}^n)$.

Acknowledgements. The ideas discussed here on the twistorial aspects of $\mathbb{R}^3$ are heavily dependent on lectures given by N. J. Hitchin at the Mathematical Institute, Oxford. H.P. would like to thank the

Department of Mathematics at Rice University for its generous hospitality. Y.S.P. would like to thank the Department of Mathematics and Computer Science at Odense University for its generous hospitality.

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Communicated by S.-T. Yau

Received October 13, 1987; in revised form February 23, 1988