

An Exhaustive Criterion for the Non-Existence of Invariant Circles for Area-Preserving Twist Maps

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Abstract. We present a necessary and sufficient condition for the non-existence of rotational invariant circles for area-preserving twist maps of the cylinder or annulus based on the “cone-crossing” and “killends” criteria of MacKay and Percival (1985). Given a number of technical restrictions on the implementation of these criteria, this condition leads to a proof of MacKay and Percival’s Finite Computation Conjecture.

1. Introduction

Much interest has recently been focused on criteria for the non-existence of rotational invariant circles for area-preserving twist maps of the cylinder or annulus [(Chirikov, 1979; Greene, 1979; Mather, 1982, 1984; Newman and Percival, 1983; Aubry, 1983; Katok, 1983; Boyland and Hall, 1985), etc.]. MacKay and Percival (1985) unify a number of related criteria and present a practical algorithm which can be rigorously implemented on a digital computer. Herman (1983) uses essentially the same criterion in his construction of $C^{2-\epsilon}$ counterexamples to the Moser Twist Theorem whilst Mather (1984) and Aubry (1983) apply this criterion to the standard map:

$$\begin{aligned}r' &= r - (k/2\pi) \sin 2\pi\theta, \\ \theta' &= \theta + r'\end{aligned}$$

to show that there are no rotational invariant circles for respectively $|k| > 4/3$ and $|k| > \beta$, where $\beta \simeq 1.23\dots$ is the root of some transcendental equation. Aubry and Mather’s work was done analytically, but by implementing the criterion on a computer MacKay and Percival are able to obtain a considerably better rigorous bound: there are no rotational invariant circles for $|k| \geq 63/64 = 0.984375$. This is

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probably close to optimal since the numerical work of Greene (1979) (and MacKay, 1982, 1983) strongly suggests that there exists a $k_c \simeq 0.971635\dots$ such that there are no rotational invariant circles for $|k| > k_c$, but there is at least one such circle for $|k| \leq k_c$.

This result of MacKay and Percival is considerably better than that obtainable by any other known non-existence criterion. Thus for instance consider the simplest such possible criterion: if there is an orbit which “crosses” the annulus $\mathbb{T}^1 \times [a, b] \subset \mathbb{T}^1 \times \mathbb{R}$ [i.e. there exists $x \in \mathbb{T}^1 \times (-\infty, a)$ such that $f^k(x) \in \mathbb{T}^1 \times (-\infty, b)$ for some $k \in \mathbb{Z}$] then there can be no rotational invariant circle contained in the annulus $\mathbb{T}^1 \times [a, b]$. According to MacKay et al. (1985) and Chirikov (1979) the mean time to cross the “golden cantorus” for the standard map at $k=1.0$ is of the order of 10^6 . To prove non-existence of rotational invariant circles at this value of k we would have to rigorously find such an orbit which is hardly feasible. It thus seems that this simple criterion is unlikely to succeed for even $k=1.0$. By comparison, the longest orbit segment considered by MacKay and Percival had length 96. Their program took less than 90 min of CPU time on a NAS 9080 computer, which is not a particularly fast machine by today’s standards. If more computer time was available their criterion would undoubtedly give even sharper results. Similarly Leage and MacKay (1986) conclude that in practice Boyland and Hall’s “badly ordered periodic orbits” criterion (Boyland and Hall, 1985) is not very efficient.

It is then natural to ask whether this criterion is exhaustive, that is, given a point through which no rotational invariant circles pass, will the criterion succeed in proving so. This is essentially the same question as the following conjecture made by MacKay and Percival (1985).

Finite Computation Conjecture given a compact region through which no rotational invariant circles pass, then the algorithm will prove so using a finite amount of computation.

The purpose of this paper is to prove a theorem which implies this conjecture as long as one places a number of natural restrictions on the way in which the criterion is implemented (e.g. on the accuracy and order of computation etc.). These restrictions and the detailed proof of MacKay and Percival’s conjecture can be found in Stark (1986). Briefly this theorem can be described as follows: the MacKay and Percival non-existence criterion is based on Birkhoff’s theorem, which states that every rotational invariant circle of an area-preserving twist map is the graph of a Lipschitz function. Their algorithm proceeds by obtaining a sequence K_n of increasingly sharp local Lipschitz constants which any rotational invariant circle would have to satisfy and then testing whether or not there are any curves at all (not necessarily invariant) which satisfy these Lipschitz constants. If not, then clearly there are no rotational invariant circles either. The theorem we prove (4.4) is the converse of this:

Theorem. *Given an area-preserving twist map with zero net flux, then if x is a point such that for all $n \in \mathbb{N}$ there is a curve through x satisfying K_n , then there is an invariant circle through x .*

2. The Cone-Crossing Criterion

2.1. *Definition.* An *area-preserving twist map* of the cylinder $M = \mathbb{T}^1 \times \mathbb{R}$ is an area, orientation and end preserving C^1 diffeomorphism $f: M \rightarrow M$ which satisfies $\partial\theta'/\partial y \geq K > 0$, where $f(\theta, y) = (\theta', y')$. One often also considers twist maps of the annulus $\mathbb{T}^1 \times [0, 1]$. Everything below remains valid in this case as long as appropriate care is taken at the boundaries of the annulus.

We often prefer to work in the universal cover \mathbb{R}^2 of $\mathbb{T}^1 \times \mathbb{R}$, and so define a *lift* to \mathbb{R}^2 of f to be a diffeomorphism $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\pi \circ F = f \circ \pi$, where π is the covering map $\pi(t, y) = ((t \bmod 1), y)$. Note that if $R: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the translation $R(t, y) = (t - 1, y)$ then F commutes with R , and if F' is another lift of F , then $F' = F \circ R^k$ for some $k \in \mathbb{Z}$.

2.2. *Definition.* An *invariant circle* for an area-preserving twist map f is a homeomorphic image of \mathbb{T}^1 which is invariant under f . A *rotational invariant circle* is one which is homotopically non-trivial, i.e. which is homotopic to the generator of the fundamental group of $\mathbb{T}^1 \times \mathbb{R}$. Intuitively such a circle winds once around the cylinder.

MacKay and Percival's cone crossing criterion is based on the following theorem which was proved by Birkhoff in the 1920's. See Herman (1983) for a modern proof.

2.3. **Theorem (Birkhoff).** *Every rotational invariant circle Γ is the graph $\{(\theta, \gamma(\theta)) : \theta \in \mathbb{T}^1\}$ of a Lipschitz function $\gamma: \mathbb{T}^1 \rightarrow \mathbb{R}$.*

Recall that γ is Lipschitz if there exist constants $C^+, C^- \in \mathbb{R}$ such that $\forall \theta \theta' \in \mathbb{T}^1$, (Fig. 1):

$$C^- \leq \frac{\gamma(\theta) - \gamma(\theta')}{\theta - \theta'} \leq C^+.$$

We shall want to generalise this definition to allow C^\pm to vary with position, and hence define:

2.4. *Definition.* A *conefield* K consists of two functions $K^-, K^+ : M \rightarrow \mathbb{R}$ (not necessarily continuous). A function $\gamma: \mathbb{T}^1 \rightarrow \mathbb{R}$ is said to *satisfy* a conefield K if

$$-1/4 \leq K^-(\theta, \gamma(\theta)) \leq K^+(\theta, \gamma(\theta)) \leq 1/4 \quad \forall \theta \in \mathbb{T}^1$$

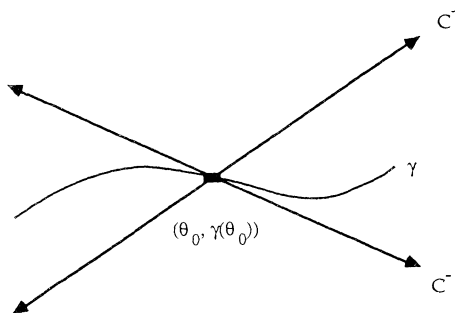


Fig. 1

and $\tan(2\pi K^-)$ and $\tan(2\pi K^+)$ give local Lipschitz constants for γ :

$$\begin{aligned} \tan 2\pi K^-(\theta, \gamma(\theta)) &\leq \liminf_{\theta \rightarrow \theta_0} \frac{\gamma(\theta) - \gamma(\theta_0)}{\theta - \theta_0} \quad \forall \theta \in \mathbb{T}^1, \\ \limsup_{\theta \rightarrow \theta_0} \frac{\gamma(\theta) - \gamma(\theta_0)}{\theta - \theta_0} &\leq \tan 2\pi K^+(\theta, \gamma(\theta)) \quad \forall \theta \in \mathbb{T}^1. \end{aligned}$$

Thus for instance if the graph of γ is a C^1 curve, then the slope of the tangent vector to the curve at x has to lie between $\tan(2\pi K^+(x))$ and $\tan(2\pi K^-(x))$. We shall say that a rotational invariant circle Γ satisfies a cone field K if the function whose graph Γ is satisfies K .

The idea behind the cone-crossing criterion is to obtain a sequence K_m of increasingly sharp cone fields which any invariant circle has to satisfy. Then if for some $m \geq 0$ and $x \in M$ we have $K_m^+(x) < K_m^-(x)$, there can be no rotational invariant circle through the point x .

By Birkhoff's theorem the constant cone field $K^+ = +1/4$, $K^- = -1/4$ given by the upward and downward verticals is satisfied by all rotational invariant circles. We can then obtain further cone fields from this one by using the action of the derivative Df . To define this action observe that we can think of a K^+ and K^- as two sections of the line bundle $M \times \mathbb{R}$ considered as a cover of the unit tangent bundle $M \times \mathbb{T}^1$ [with covering map $p: M \times \mathbb{R} \rightarrow M \times \mathbb{T}^1$ given by $p(x, t) = (x, (t \bmod 1))$]. We take coordinates on $M \times \mathbb{T}^1$ so that $(x, \xi) \in M \times \mathbb{T}^1$ corresponds to the unit tangent vector $(\cos 2\pi\xi, \sin 2\pi\xi)$ at $x \in M$. We then lift these to $M \times \mathbb{R}$, thus $(x, +1/4)$ is the upward vertical at x and $(x, -1/4)$ is the downward vertical at x . Then define:

2.5. Definition. The action of Df on the tangent bundle of M restricts to an action on the unit tangent bundle $M \times \mathbb{T}^1$. We can then lift this to an action on $M \times \mathbb{R}$, and since Df is continuous and f satisfies the twist property, there exists a unique continuous lift (which we continue to denote Df) such that $-1/4 < Df_x(1/4) < 1/4$, $\forall x \in M$. This action on $M \times \mathbb{R}$ then induces an operator on cone fields (which we still denote by Df , Fig. 2) by

$$[Df(K^\pm)](x) = Df_{f^{-1}(x)}[K^\pm(f^{-1}(x))].$$

Similarly we get an action of Df^{-1} by

$$[Df^{-1}(K^\pm)](x) = Df_{f(x)}^{-1}[K^\pm(f(x))].$$

2.6. Definition. Define the cone fields K_m , $m = 0, 1, 2, \dots$ by

$$\begin{aligned} K_0^+(x) &= +1/4 \quad (\text{upward vertical}) \quad \forall x \in M, \\ K_0^-(x) &= -1/4 \quad (\text{downward vertical}) \quad \forall x \in M, \end{aligned}$$

and

$$\begin{aligned} K_m^+ &= Df(K_{m-1}^+) \quad \forall m > 0, \\ K_m^- &= Df^{-1}(K_{m-1}^-) \quad \forall m > 0. \end{aligned} \tag{2.7}$$

2.7. Lemma. Any rotational invariant circle satisfies K_m for all $m \geq 0$.

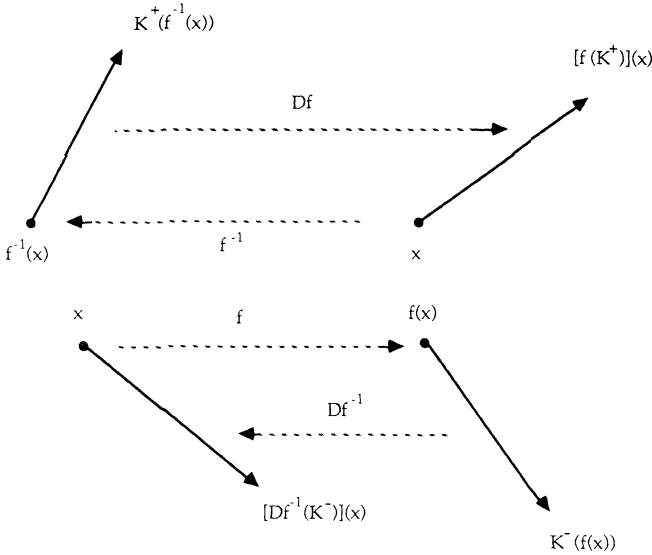


Fig. 2

Proof. By induction. By Birkhoff's theorem all rotational invariant circles satisfy K_0 . Now suppose that Γ satisfies K_{m-1} for some $m > 0$. If for some $x \in \Gamma$, $K_m^+(x)$ is not a Lipschitz constant for Γ , then by definition $K_{m-1}^+(f^{-1}(x))$ is not a Lipschitz constant for $f^{-1}\Gamma$ at $f^{-1}(x)$. But Γ is f -invariant and hence $f^{-1}\Gamma = \Gamma$. Thus $K_{m-1}^+(f^{-1}(x))$ is not a Lipschitz constant for Γ at $f^{-1}(x)$ which contradicts the hypothesis that Γ satisfies K_{m-1} . Similarly if $K_m^-(x)$ is not a Lipschitz constant for Γ at x then $K_{m-1}^-(f(x))$ is not a Lipschitz constant for Γ at $f(x)$. Hence Γ satisfies K_m .

2.8. *Cone-Crossing Criterion.* If for some point $x \in M$ and some $m \in \mathbb{N}$ we have $K_m^+(x) < K_m^-(x)$, then there is no rotational invariant circle through x . We shall say that the ‘‘cones have crossed at x ’’ when this occurs.

2.9. *Remark.* By the twist condition we have $K_1^+(x) < 1/4$ and $K_1^-(x) > -1/4$, $\forall x \in M$. Since the actions of Df and Df^{-1} on slopes are order preserving we get $K_m^+(x) < K_{m-1}^+(x)$ and $K_m^-(x) > K_{m-1}^-(x)$, $\forall m \in \mathbb{N}$, $\forall x \in M$ (Fig. 3). Thus the action of Df and Df^{-1} gives us a sequence of increasingly sharp cone-fields.

2.10. *Remark.* All of the above concepts lift in a straightforward manner to the universal cover \mathbb{R}^2 . Thus rotational invariant circles lift to graphs of periodic Lipschitz functions $\gamma: \mathbb{R} \rightarrow \mathbb{R}$, $\gamma(t+1) = \gamma(t)$, $\forall t \in \mathbb{R}$. Cone-fields lift to functions $K^-, K^+: \mathbb{R}^2 \rightarrow \mathbb{R}$ and we get an action of DF and DF^{-1} on cone-fields. Abusing notation, we shall continue to refer to the lift of the cone-fields K_m as K_m .

2.11. *Remark.* In practice it is not feasible to perform the iteration (2.7) exactly, especially if one is using a digital computer. However one can obviously replace (2.7) by

$$\begin{aligned}
 K_m^+ &= \text{u.b. } Df(K_{m-1}^+) \quad \forall m > 0, \\
 K_m^- &= \text{l.b. } Df^{-1}(K_{m-1}^-) \quad \forall m > 0,
 \end{aligned}
 \tag{2.12}$$

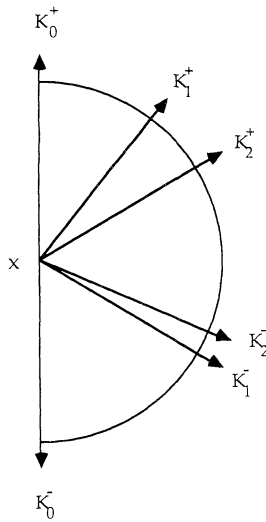


Fig. 3

where u.b. is any upper bound and l.b. is any lower bound. If we restrict the K_m to some class of functions which are representable in a finite manner (e.g. piece-wise constant functions) then we can easily perform (2.12) rigorously on a digital computer.

3. Supplementary Criteria

3.1. *Remark.* As MacKay and Percival (1985) show, the cones K_m^\pm will never cross on an orbit of minimal action. On the other hand the work of Mather (1982) and Aubry and LeDaeron (1983) [see also MacKay and Stark (1985) for a survey] shows that typically there exist many orbits with minimal action which do not lie on rotational invariant circles. Thus just because the cones do not cross at a point does not imply that there is a rotational invariant circle through that point. In other words cone-crossing on its own is not an exhaustive criterion for the non-existence of invariant circles. MacKay and Percival thus introduce a supplementary criterion which they call “killends.” This is based on the observation that if for a given $x \in M$ all the curves through x that satisfy some K_m are forced to pass through a region where $K_m^+ < K_m^-$, then there can be no rotational invariant circle through x . The “end” of the curve is “killed” even though at the starting point x of the curve one might still have $K_m^+(x) \geq K_m^-(x)$. More precisely define:

3.2. *Definition.* Given $x \in M$, choose $z \in \mathbb{R}^2$ such that $\pi(z) = x$, where π is the covering map $\pi(t, y) = ((t \bmod 1), y)$. For $m > 0$ define:

$$L_z^*(K_m) = \{ \gamma : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } z \in \text{graph } \gamma, \gamma \text{ satisfies } K_m \},$$

$$L_z(K_m) = \{ \gamma \in L_z^* \text{ such that } \gamma(t+1) = \gamma(t), \forall t \in \mathbb{R} \}.$$

Note that γ is not required to be f -invariant. The set $L_z(K_m)$ thus consists of those closed curves γ through z that satisfy the cone field K_m . Observe that if $\gamma \in L_z(K_m)$ then $f^{-i}(\gamma)$ and $f^i(\gamma)$ will satisfy K_0 for all $1 \leq i \leq m$. We can thus characterise

$L_z(K_m)$ as the set of all closed curves γ through z such that for all $1 \leq i \leq m$, $f^{-i}(\gamma)$ and $f^i(\gamma)$ are the graphs of a Lipschitz function. Also note that the definition of $L_z(K_m)$ is independent of the choice of z such that $\pi(z) = x$, and the definition of $L_z^*(K_m)$ is independent of z up to translation by $R(t, y) = (t - 1, y)$. Note that $L_z(K_m) \subset L_z(K_{m-1})$.

3.3. “Killends.” If for some $m > 0$, either $L_z^*(K_m) = \emptyset$ or $L_z(K_m) = \emptyset$, then there can be no rotational invariant circles through x . The MacKay and Percival implementation of “killends” tests for $L_z^*(K_m) = \emptyset$, however it takes very little extra work to test for $L_z(K_m) = \emptyset$, and one might as well do this instead. In fact by considering images of the vertical under F^n , one can show that $L_z(K_m) = \emptyset, \forall m > 0$, if and only if $L_z^*(K_m) = \emptyset, \forall m > 0$ (Stark, 1986), and hence the two versions are equivalent. In this paper we shall only consider the condition $L_z(K_m) = \emptyset$, since this makes the mathematical treatment considerably simpler.

3.4. **Lemma.** $L_z(K_m)$ is bounded $\forall m > 0$, that is if $z = (t_0, y_0)$ there exists a constant B such that $|\gamma(t) - y_0| \leq B, \forall t \in \mathbb{R}, \forall \gamma \in L_z(K_m)$.

Proof. Since $L_z(K_m) \subset L_z(K_{m-1})$ it is enough to show that $L_z(K_1)$ is bounded. Let V be the vertical through $F^{-1}(z) = (t_{-1}, y_{-1})$, i.e. $V = \{(t, y) \in \mathbb{R}^2 \text{ such that } t = t_{-1}\}$. Then since F is a twist map, $F(V)$ is the graph of a continuous function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$. By the definition of K_1^+ , if $\gamma \in L_z(K_m)$ then, $\gamma(t) \geq \Psi(t), \forall t \leq t_0$ and $\gamma(t) \leq \Psi(t), \forall t \geq t_0$. So let

$$B = \sup_{t \in [t_0 - 1, t_0 + 1]} |\Psi(t) - y_0|.$$

3.5. *Remark.* As a consequence of this lemma we only need to evaluate K_m on a bounded region in order to prove the non-existence of a rotational invariant circle through a given point.

Another test for non-existence is also required:

3.6. *Definition.* Let $U \subset \mathbb{T}^1 \times \mathbb{R}$ be an open set homeomorphic to $\mathbb{T}^1 \times \mathbb{R}$ such that $\mathbb{T}^1 \times (-\infty, a] \subset U \subset \mathbb{T}^1 \times (-\infty, b]$ for some $a, b \in \mathbb{R}$. Then the *net flux* or *Calabi Invariant* of a twist map f is $[\text{area}(fU \setminus U) - \text{area}(U \setminus fU)]$. Since f is area-preserving this is independent of the choice of U . It can be thought of as the flux of area across a circle $\{y = \text{const}\}$. Clearly if f has a rotational invariant circle then this net flux must be zero, and hence we get:

3.7. *Net Flux Test.* If the net flux of f is non-zero then f can have no rotational invariant circles.

3.8. *Remark.* On the other hand there are twist maps with non-zero flux for which $K_m^+(x) > K_m^-(x), \forall x \in M, \forall m > 0$, e.g. $y' = y + 1, \theta' = \theta + y$ (example due to R. de la Llave, private communication). For such a map cone crossing and “killends” will never prove that there are no rotational invariant circles. We thus need to test for non-zero flux, this can be done by sufficiently precise rigorous numerical integration. Note that in practice it will usually be obvious whether or not a given twist map has zero net flux (e.g. if f has at least one rotational invariant circle somewhere then it has zero net flux, thus maps of an annulus $\mathbb{T}^1 \times [0, 1]$ always have zero net flux). Furthermore in practical examples (e.g. the standard map) the net flux is usually zero.

4. The Finite Computation Conjecture

Note that all of the criteria described in the previous two sections can be applied to a whole C^1 ball of twist maps at a time. Indeed this happens naturally in a computer implementation since in that case the map is only determined up to a finite accuracy. MacKay and Percival (1985) then make the following conjecture:

4.1. *Finite Computation Conjecture.* Let X be a compact subset of $M \times A(M)$, where $A(M)$ is the space of all C^1 area-preserving twist maps of M , such that for all $(x, f) \in X$, the map f has no rotational invariant circle through $x \in M$. Then with a finite amount of computation the combination of the net flux test, cone-crossing and killends will prove that for all $(x, f) \in X$, the map f has no rotational invariant circle through x .

4.2. *Remark.* If any of the net flux test, cone-crossing or killends succeed for some $x \in M, f \in A(M)$, then they also succeed for some open C^1 ball in $M \times A(M)$ around (x, f) . Hence the above conjecture is equivalent to the algorithm based on these criteria being exhaustive.

4.3. *Remark.* As stated above the conjecture is too vague to admit a mathematical proof. To have some hope of proving it we have to place some reasonable restrictions on the way in which the criteria are implemented: there clearly exist algorithms based on the above criteria which are not exhaustive. A proof of the conjecture will then have two parts:

1) A proof that if f has zero net flux and $L_z(K_m) \neq \emptyset, \forall m > 0$, i.e. for any m there is a closed curve through z that satisfies the cone field K_m , then there exists a rotational invariant circle through z . Thus if z has no rotational invariant circles through it, either f has non-zero net flux or $L_z(K_m) = \emptyset$ for some $m > 0$. Hence an idealised criterion which calculates the cone fields K_m and the net flux exactly will indeed show that there are no invariant circles through z .

2) A proof that the restrictions imposed on a practical implementation [using (2.11) rather than (2.7) to calculate the cone fields] ensure that it is a sufficiently good approximation of the idealised criterion to also be exhaustive. This essentially amounts to showing that we can calculate the cone fields and the net flux to an arbitrarily high precision using rigorous interval arithmetic on a digital computer.

We shall give a proof of 1) below, 2) is rather technical and the details can be found in Stark (1986). Note that the proof gives no upper bound on the amount of computation required, and hence we cannot use this method to prove the existence of invariant circles (which in general is much harder to do than to prove non-existence). It seems highly unlikely that an upper bound could be given except in special cases, since given a point at which the cones will eventually cross, there is no a priori way of estimating how many iterations of the cone field are required for this to happen.

4.4. **Theorem.** *If f has zero net flux and $L_z(K_m) \neq \emptyset, \forall m > 0$, then there exists a rotational invariant circle through z .*

Observe that the converse is trivially true and hence this theorem gives necessary and sufficient conditions for the existence of rotational invariant circles.

4.5. *Definition.* Let R be the translation $R(u, v) = (u - 1, v)$. If $z = (t_0, y_0)$, then for $i \in \mathbb{N}$ define $z_i = (t_i, y_i) = R^{p(i)} F^i(z)$ with $p(i)$ chosen such that $t_0 \leq t_i < t_0 + 1$.

4.6. *Definition.* Let

$$\begin{aligned} L_z &= \bigcap_{m>0} L_z(K_m), \\ L &= \bigcup_{z \in \mathbb{R}^2} L_z, \\ L_{z,n} &= \{\gamma \in L_z \text{ such that } F^j(z) \in \text{graph } \gamma, 0 \leq j \leq n\}, \\ L_{z,\infty} &= \bigcap_{n \geq 0} L_{z,n}. \end{aligned}$$

Clearly if a rotational f -invariant circle through z exists it will belong to $L_{z,\infty}$. Our proof of (4.4) will proceed in three stages:

i) $L_z(K_m) \neq \emptyset, \forall m > 0 \Rightarrow L_{z,0} = L_z \neq \emptyset$.

ii) $\forall n > 0, L_{z,n-1} \neq \emptyset \Rightarrow L_{z,n} \neq \emptyset$.

iii) If $L_{z,n} \neq \emptyset, \forall n \in \mathbb{N}$, then there exists a rotational f -invariant circle in $L_{z,\infty}$.

Let $C_{\text{per}}(\mathbb{R})$ be the Banach space of continuous functions $\varphi: \mathbb{R} \rightarrow \mathbb{R}$, such that $\varphi(t+1) = \varphi(t), \forall t \in \mathbb{R}$, with the sup norm.

4.7. **Lemma.** $L_z(K_m)$ is closed in $C_{\text{per}}(\mathbb{R}), \forall m > 0$.

Proof. If $L_z(K_m) = \emptyset$, then it is closed. So suppose not and take a sequence $\gamma_i \in L_z(K_m)$, with $\gamma_i \rightarrow \gamma$, uniformly. Note that K_m^+ and K_m^- are continuous as functions $\mathbb{R}^2 \rightarrow \mathbb{R}$, so let $J^+ = \sup\{K_m^+(t, \gamma(t)), t \in [0, 1]\}$ and $J^- = \inf\{K_m^-(t, \gamma(t)), t \in [0, 1]\}$. Recall from (2.8) that $1/4 > J^+$ and $J^- > -1/4$. Let $w_0 = (t_0, \gamma(t_0)) \in \text{graph } \gamma$. Since the γ_i satisfy K_m , we have $K_m^+(x) \geq K_m^-(x), \forall x \in \text{graph } \gamma_i, \forall i \in \mathbb{N}$. Since w_0 is in the closure of such points and K_m^+ and K_m^- are continuous, we have $K_m^+(w_0) \geq K_m^-(w_0)$ and hence $1/4 > J^+ \geq K_m^+(w_0) \geq K_m^-(w_0) \geq J^- > -1/4$. Now given an $\varepsilon > 0$ sufficiently small (such that $\varepsilon < 1/4 - J^+$ and $\varepsilon < J^- + 1/4$), $\exists \delta > 0$ such that $\forall w \in [t_0 - \delta, t_0 + \delta] \times [\gamma(t_0) - \delta, \gamma(t_0) + \delta]$, we have $|K_m^\pm(w) - K_m^\pm(w_0)| < \varepsilon$. Then on $[t_0 - \delta, t_0 + \delta]$ the γ_i for $i \geq N$, some $N \in \mathbb{N}$, have uniform Lipschitz constants $C^+ = [\tan 2\pi(K_m^+(w_0) + \varepsilon)]$ and $C^- = [\tan 2\pi(K_m^-(w_0) - \varepsilon)]$, i.e. $C^- \leq [\gamma_i(t) - \gamma_i(t')]/(t - t') \leq C^+, \forall t, t' \in [t_0 - \delta, t_0 + \delta]$. Hence so has γ . But this holds for all sufficiently small $\varepsilon > 0$ and hence γ satisfies K_m at w_0 .

4.8. **Lemma.** $L_z(K_m)$ is uniformly Lipschitz, i.e. there is a constant $J = J(z, m), 0 \leq J < \infty$ such that $|\gamma(t) - \gamma(t')| \leq J|t - t'|, \forall \gamma \in L_z(K_m), \forall t, t' \in \mathbb{R}$. In fact since $L_z(K_m) \subset L_z(K_{m-1})$, we can take $J(z, m)$ independent of m .

Proof. If $L_z(K_m) = \emptyset$, then the lemma is trivially true. Suppose not. By (3.4), let $A = \mathbb{T}^1 \times [-B, B]$ be such that $\text{graph } \gamma \subset A, \forall \gamma \in L_z(K_m)$. Note that K_m^\pm are continuous as functions $\mathbb{R}^2 \rightarrow \mathbb{R}$, so let $J^+ = \sup\{K_m^+(u), u \in A\}$ and $J^- = \inf\{K_m^-(u), u \in A\}$. By (2.8), $J^+ < 1/4$ and $J^- > -1/4$. If $\gamma \in L_z(K_m)$, then $K_m^+(x) \geq K_m^-(x), \forall x \in \text{graph } \gamma$. Hence $J^+ \geq J^-$. Let $J = \max\{|\tan 2\pi J^+|, |\tan 2\pi J^-|\}$. Then $0 \leq J < \infty$ and $|\gamma(t) - \gamma(t')| < J|t - t'|, \forall \gamma \in L_z(K_m), \forall t, t' \in \mathbb{R}$.

4.9. **Corollary.** $L_z(K_m) \neq \emptyset, \forall m > 0 \Rightarrow L_z \neq \emptyset$.

Proof. Let $\gamma_i, i > 0$, be a sequence with $\gamma_i \in L_z(K_i)$. Since $L_z(K_m) \subset L_z(K_{m-1})$, $\gamma_i \in L_z(K_j), \forall 1 \leq j \leq i$. In particular $\{\gamma_i, i > 0\} \subset L_z(K_1)$. By (3.4) and (4.8) $L_z(K_1)$ is pointwise bounded and equicontinuous. Thus by Ascoli's Theorem (Rudin, 1973), γ_i has a uniformly convergent subsequence. Let γ be the limit of this subsequence. By (4.7), $\gamma \in L_z(K_m), \forall m > 0$, as required.

4.10. *Definition.* Since the action of DF, DF^{-1} on angles is order preserving, if $\gamma \in L$, then $F(\text{graph } \gamma)$ must also satisfy $K_m, \forall m > 0$, and hence be in L . Thus $F(\text{graph } \gamma)$ is the graph of a Lipschitz function which we denote $F\gamma$. In other words $F\gamma$ is defined by the graph transform:

$$\text{graph}(F\gamma) = F(\text{graph } \gamma).$$

4.11. *Remark.* If f has zero net flux then γ and $F\gamma$ must intersect. More specifically, either $\gamma = F\gamma$ (and so γ represents a rotational invariant circle) or $\exists t, t' \in [0, 1]$ such that $\gamma(t) < (F\gamma)(t)$ and $\gamma(t') > (F\gamma)(t')$. If $L_{z,n} \neq \emptyset$, we shall use F to construct a $\gamma \in (L_{z,n} \cap L_{F(z),n}) \subset L_{z,n+1}$.

4.12. *Definition.* If $L_{z,n} \neq \emptyset$, define

$$\Gamma_n^+(t) = \sup_{\gamma \in L_{z,n}} \gamma(t), \quad t \in \mathbb{R},$$

$$\Gamma_n^-(t) = \inf_{\gamma \in L_{z,n}} \gamma(t), \quad t \in \mathbb{R}$$

4.13. **Lemma.** *If for some $n \in \mathbb{N}, L_{z,n} \neq \emptyset$, then $\Gamma_n^+, \Gamma_n^- \in L_{z,n}$.*

Proof. By (4.8), let J be a Lipschitz constant for $L_z(K_1)$. Since $L_{z,n} \subset L_z(K_1)$, $|\gamma(t) - \gamma(t')| \leq J|t - t'|, \forall \gamma \in L_{z,n}, \forall t, t' \in \mathbb{R}$. Hence the same holds for Γ_n^\pm . For any $i \in \mathbb{N}$, choose N_i such that $N_i > 2iJ$. For $k = 0, \dots, N_i - 1$, choose $\gamma_{i,k} \in L_{z,n}$ such that $|\Gamma_n^\pm(k/N_i) - \gamma_{i,k}(k/N_i)| < 1/2i$, (Fig. 4). Define γ_i by $\gamma_i(t) = \max$ or $\min \{\gamma_{i,k}(t), k = 0, \dots, N_i - 1\}$ respectively. Note that $\gamma_i \in L_{z,n}$. Then $|\Gamma_n^\pm(t) - \gamma_i(t)| < 1/i, \forall t \in \mathbb{R}, \forall i \in \mathbb{N}$, i.e. $\gamma_i \rightarrow \Gamma_n^\pm$ uniformly and thus by (4.7), $\Gamma_n^\pm \in L_z$. But by definition $F^j(z) \in \text{graph } \Gamma_n^\pm, 0 \leq j \leq n$, and thus $\Gamma_n^\pm \in L_{z,n}$ as required.

4.14. **Lemma.** *Suppose $L_{z,n} \neq \emptyset$ for some $n \in \mathbb{N}$ and $z = (t_0, y_0)$. Then:*

$$(F\Gamma_n^-)(t_0) \leq y_0 \leq (F\Gamma_n^+)(t_0).$$

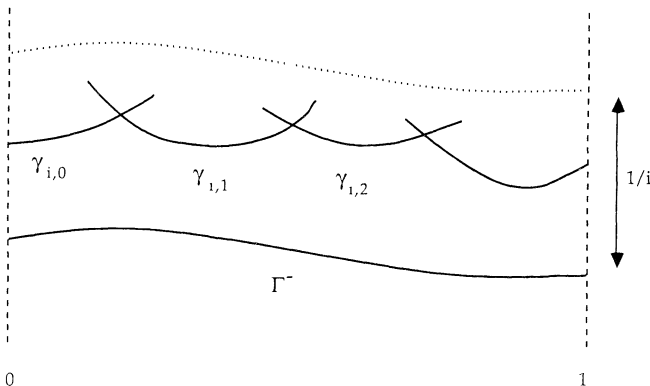


Fig. 4

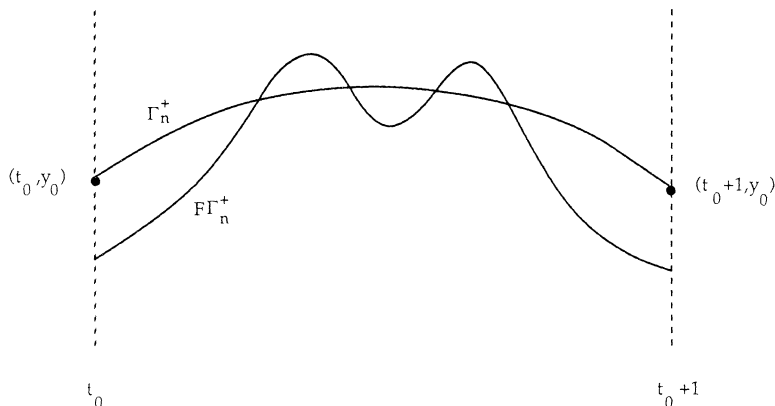


Fig. 5

Proof. We shall prove $y_0 \leq (F\Gamma_n^+)(t_0)$; the other half of the inequality is proved similarly. So suppose $\Gamma_n^+(t_0) = y_0 > (F\Gamma_n^+)(t_0)$, (Fig. 5). By zero net flux $\exists t' \in [t_0, t_0 + 1]$ such that $(F\Gamma_n^+)(t') > \Gamma_n^+(t')$. Define ψ by:

$$\psi(t) = \max \{ \Gamma_n^+(t), (F\Gamma_n^+)(t) \} \quad \forall t \in \mathbb{R}.$$

Then $\psi \in L$. Since $\Gamma_n^+(t_0) > (F\Gamma_n^+)(t_0)$, $\psi(t_0) = y_0$, so $z \in \text{graph } \psi$. Also $F^j(z) \in \text{graph } \Gamma_n^+$, $\forall 0 \leq j \leq n$, and hence $F^j(z) \in \text{graph}(F\Gamma_n^+)$, $\forall 1 \leq j \leq n+1$. Thus $\Gamma_n^+(t_j) = y_j = (F\Gamma_n^+)(t_j)$, $\forall 1 \leq j \leq n$, and so $\psi(t_j) = y_j$, $\forall 1 \leq j \leq n$. Hence $F^j(z) \in \text{graph } \psi$, $\forall 1 \leq j \leq n$. So $\psi \in L_{z,n}$. But $(F\Gamma_n^+)(t') > \Gamma_n^+(t')$, and thus $\psi(t') > \Gamma_n^+(t')$, which contradicts the definition of Γ_n^+ .

4.15. Lemma. Suppose that $L_{z,n} \neq \emptyset$ for some $n \in \mathbb{N}$ and $z = (t_0, y_0)$. Then (see Fig. 6, drawn for the case $n=0$):

$$(F\Gamma_n^+)(t) \geq \Gamma_n^-(t) \quad \forall t \in \mathbb{R},$$

$$(F\Gamma_n^-)(t) \leq \Gamma_n^+(t) \quad \forall t \in \mathbb{R}.$$

Proof. We only prove the first inequality; the proof of the other is similar. Define φ by

$$\varphi(t) = \min \{ \Gamma_n^-(t), (F\Gamma_n^+)(t) \} \quad \forall t \in \mathbb{R}.$$

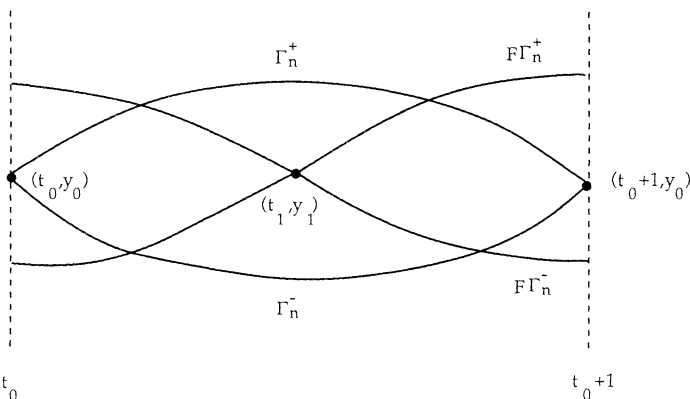


Fig. 6

Then $\varphi \in L$. By Lemma 4.14, $\Gamma_n^-(t_0) = y_0 \leq (F\Gamma_n^+)(t_0)$, hence $\varphi(t_0) = y_0$. So $z \in \text{graph } \varphi$, i.e. $\varphi \in L_z$. As in the proof of Lemma 4.14 we have $F^j(z) \in \text{graph } \varphi$, $\forall 1 \leq j \leq n$. So $\varphi \in L_{z,n}$, which by the definition of Γ_n^- , gives $\varphi(t) \geq \Gamma_n^-(t)$, $\forall t \in \mathbb{R}$. But $(F\Gamma_n^+)(t) \geq \varphi(t)$, $\forall t \in \mathbb{R}$, and hence $(F\Gamma_n^+)(t) \geq \Gamma_n^-(t)$, $\forall t \in \mathbb{R}$, as required.

4.16. Lemma. *If $L_z \neq \emptyset$ then $L_{z,n} \neq \emptyset$, $\forall n \in \mathbb{N}$.*

Proof. By induction $L_{z,0} = L_z \neq \emptyset$. So now suppose that for some $n \in \mathbb{N}$, $L_{z,n} \neq \emptyset$. Define η by (see Fig. 6, drawn for the case $n=0$)

$$\eta(t) = \min \{ \Gamma_n^+(t), (F\Gamma_n^+)(t) \} \quad \forall t \in \mathbb{R}.$$

Then $\eta \in L$. By Lemma 4.14, $\Gamma_n^+(t_0) \leq (F\Gamma_n^+)(t_0)$, hence $\eta(t_0) = y_0$. As in the proof of Lemma 4.14 we have $F^j(z) \in \text{graph } \eta$, $\forall 1 \leq j \leq n$, and so $\eta \in L_{z,n}$. By Lemma 4.15, $(F\Gamma_n^-)(t_{n+1}) \leq \Gamma_n^+(t_{n+1})$. But $(F\Gamma_n^+)(t_{n+1}) = y_{n+1} = (F\Gamma_n^-)(t_{n+1})$, so $(F\Gamma_n^+)(t_{n+1}) \leq \Gamma_n^+(t_{n+1})$. Thus $\eta(t_{n+1}) = y_{n+1}$, and so $F^{n+1}(z) \in \text{graph } \eta$. Hence $\eta \in L_{z,n+1}$ as required.

Proof of Theorem 4.4. By above Γ_n^+ exists $\forall n \in \mathbb{N}$. $\Gamma_n^+ \in L_z$, $\forall n \in \mathbb{N}$, and by (3.4) and (4.8), L_z is pointwise bounded and equicontinuous. Hence by Ascoli's theorem (Rudin, 1974) Γ_n^+ has a uniformly convergent subsequence. Let Γ be the limit of this subsequence. Then $\Gamma \in L_z$ by (4.7). Since $L_{z,n+1} \subset L_{z,n}$ we have $F^n(z) \in \text{graph } \Gamma$, $\forall n \in \mathbb{N}$ and hence $\Gamma \in L_{z,\infty}$. It remains to show that $F\Gamma = \Gamma$. Note that $\Gamma(t) \geq \gamma(t)$, $\forall t \in \mathbb{R}$, $\forall \gamma \in L_{z,\infty}$. Now $(F^{-1}\Gamma) \in L$ and $F^n(x) \in \text{graph}(F^{-1}\Gamma)$, $\forall n \in \mathbb{N}$, hence $(F^{-1}\Gamma) \in L_{z,\infty}$. So $\Gamma(t) \geq (F^{-1}\Gamma)(t)$, $\forall t \in \mathbb{R}$. But F^{-1} has zero net flux, hence if $\exists t' \in \mathbb{R}$ such that $\Gamma(t') > (F^{-1}\Gamma)(t')$ then $\exists t'' \in \mathbb{R}$ such that $\Gamma(t'') < (F^{-1}\Gamma)(t'')$. Hence $\Gamma(t) = (F^{-1}\Gamma)(t)$, $\forall t \in \mathbb{R}$, and so $F\Gamma = \Gamma$ as required.

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