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On the Asymptotics of Nodes of L^2-Solutions of Schrödinger Equations in Dimensions $\geqq 3$

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Abstract. Let $\Omega_R = \mathbb{R}^n \setminus B_R$, where $n \geqq 3$ and $B_R = \{x \in \mathbb{R}^n : |x| \leqq R\}$. We investigate the asymptotics of real valued solutions $\psi \in L^2(\Omega_R)$ of the Schrödinger equation $(-\Delta + V - E)\psi = 0$, where $E < 0$ and $V(x) \to 0$ for $|x| \to \infty$: Let D denote an unbounded nodal domain of ψ (i.e. a component of $\Omega_R \setminus \{x : \psi(x) = 0\}$), and let $S(r) = \{y \in S^{n-1} : ry \in D\}$ with S^{n-1} the unit sphere in $\mathbb{R}^n$. Under suitable assumptions on V it is shown that for some $\gamma > 0$,

$$\liminf_{r \to \infty} r^\gamma \int_{S(r)} \psi^2 d\sigma \Big/ \int_{S^{n-1}} \psi^2 d\sigma > 0 \text{ and}$$

$$\liminf_{r \to \infty} \ln(\text{Volume}(D \cap B_r))/\ln r \geqq (n+1)/2.$$

Results of this type are already non-trivial for radial problems with ψ satisfying non-radial boundary conditions on $\partial\Omega_R$ or for excited states of the Hydrogen atom if one considers linear combinations of different l-waves.

1. Introduction and Statement of the Results

In [17] we investigated, in collaboration with J. Swetina, the asymptotics of L^2-solutions of Schrödinger equations in exterior domains. For dimension $n = 2$ the asymptotic behaviour of nodal lines has been studied in [16] and in [14] by the first author. In this paper we investigate the asymptotic behaviour of nodal surfaces of such solutions for $n \geqq 3$.

We start by describing the problem in the n-dimensional setting: We consider real valued $W^{2,2}$-solutions $\psi(x)$ of

$$(-\Delta + V - E)\psi = 0 \quad \text{for} \quad x \in \Omega_R,$$
$$\Omega_R = \{x \in \mathbb{R}^n : |x| = r > R\}, \quad R > 0, \quad n \geqq 3. \tag{1.1}$$

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Here the Sobolev space $W^{2,2}(\Omega_R)$ is defined as in [12]. Throughout the paper we assume that

$$E < 0, \tag{1.2}$$

and that $V(x)$ satisfies the following assumptions in Ω_R:

$$V(x) \text{ is real valued and continuous, and} \tag{A.1}$$

$$\lim_{|x|\to\infty} V(x) = 0. \tag{A.2}$$

Equations (1.2), (A.1) and (A.2) imply that we can choose R so that

$$\inf_{x\in\Omega_R} (V(x) - E) > 0. \tag{A.3}$$

These conditions on V imply that $C_0^\infty(\Omega_R)$ is a form core for the quadratic form associated to $-\Delta + V - E$ and its Friedrichs extension is a positive definite selfadjoint operator denoted by H_{Ω_R}-E. This guarantees that given ψ on $\partial\Omega_R$, say $\psi = \varphi$ on $\partial\Omega_R$ with φ continuous in a neighborhood of $\partial\Omega_R$, the corresponding problem (1.1) has a unique solution [12].

We split V so that

$$V(x) = V_1(r) + V_2(x), \tag{A.4}$$

and assume that V_1 and V_2 satisfy the assumptions (A.1–A.3) separately. Furthermore we assume that in Ω_R,

$$V_1 \text{ is continuously differentiable, and that}$$

$$\left|\frac{dV_1(r)}{dr}\right| \leqq cr^{-\varepsilon-1} \quad \text{for some } \varepsilon, \quad c > 0, \tag{A.5}$$

and that

$$|V_2| \leqq c_0 r^{-1-\gamma} \quad \text{for some } c_0 \quad \text{and} \quad \gamma > 0. \tag{A.6}$$

Suppose

$$(-\Delta + V_1 - E)v = 0 \quad \text{in } \Omega_R \tag{1.3}$$

with $0 < v \in L^2(\Omega_R)$ and $v(x) = v(|x|)$. It was shown in [15] that

$$c_- v(r) \leqq \psi_{av}(r) \tag{1.4}$$

and

$$|\psi| \leqq c_+ v(r), \tag{1.5}$$

where

$$\psi_{av}(r) = \left(\int_{S^{n-1}} \psi^2 d\sigma\right)^{1/2}(r),$$

with S^{n-1} the unit sphere in $\mathbb{R}^n$ and $d\sigma$ normalized integration over it.

Upper and lower bounds to ψ_{av} have been subject of many investigations under various conditions on V (see [15, 17, 22, 24] and references therein). In view of (1.4) and (1.5) it is natural to investigate the function

$$F = \frac{\psi}{\psi_{av}}. \tag{1.6}$$

But let us first state our assumptions on V_2. To do so we write $V_2(x) = V_2(ry)$ so that $y = x/r \in S^{n-1}$. We require that

$$r^{1+\alpha} V_2(ry) \in C^{\omega}(S^{n-1})$$
$$\text{uniformly in } r \text{ for } r \geqq \bar{R} > R \text{ for some } \bar{R} > R, \tag{B}$$

where $\alpha > 1/2$ and $C^{\omega}(S^{n-1})$ means real analytic.

In Sect. 2 we shall recall some regularity results for $F(ry)$, which were obtained in [17] and which are the basis of the proofs of the following theorems. These results concern the asymptotic behaviour of F in relation with the asymptotics of its nodal surfaces.

To begin with let us define a nodal domain of ψ. Let $r_0 \geqq R$ and let

$$\mathcal{N}_{r_0} = \{x \in \Omega_{r_0} : \psi = 0\}.$$

A component D_{r_0} of the set $\Omega_{r_0} \backslash \mathcal{N}_{r_0}$ will be called a nodal domain of ψ in Ω_{r_0}. This definition differs slightly from the usual one because ψ cannot vanish identically on $\partial D_{r_0} \cap \partial \Omega_{r_0}$, for this would imply $\psi \equiv 0$ in D_{r_0} by the positivity of $-\Delta + V - E$ on $C_0^{\infty}(D_{r_0})$ as a quadratic form.

We shall consider only unbounded nodal domains D_{r_0}, and we call for each $r_0 \geqq R$,

$$\mathscr{D}_{r_0} = \{\text{unbounded nodal domains } D_{r_0}\}. \tag{1.7}$$

We also introduce the sets $S(r)$ for $D_{r_0} \in \mathscr{D}_{r_0}$,

$$S(r) = \{y \in S^{n-1} : ry \in D_{r_0}\}. \tag{1.8}$$

Its measure on S^{n-1} will be denoted by

$$|S(r)| = \int_{S(r)} d\sigma.$$

After these definitions we can state our main results.

Theorem 1.1. *Let $\psi \not\equiv 0$ be a real valued L^2-solution to (1.1) and assume that V, V_1 and V_2 satisfy the assumptions A and B and that $E < 0$. Then there is a $\beta > 0$ and a constant $c > 0$ not depending on r_0 such that $\forall D_{r_0} \in \mathscr{D}_{r_0}$,*

$$\left(\int_{S(r)} F^2 d\sigma \right)^{1/2} = \frac{\left(\int_{S(r)} \psi^2 d\sigma \right)^{1/2}}{\psi_{av}} \geqq c r^{-\beta}. \tag{1.9}$$

We shall discuss this result and the following one in the next section and we shall also illustrate them with examples.

Theorem 1.2. *Let ψ be as in Theorem 1.1 and let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, then $\forall r_0 > R$ and $\forall D_{r_0} \in \mathscr{D}_{r_0}$,*

$$\liminf_{r \to \infty} \frac{\ln(\text{Volume}(D_{r_0} \cap B_r))}{\ln r} \geqq \frac{n+1}{2}. \tag{1.10}$$

Remark 1.1. Theorem 1.2 tells us that in an averaged sense the sets $D_{r_0} \cap \partial B_r$ are not too "narrow" near infinity. A lot more was shown for $n = 2$ in [16] and especially in [14]. For instance it was shown there for this case that the limit in (1.10) exists and equals either 3/2 or 2.

The next result is a spectral one.

Theorem 1.3. *Let ψ be as in Theorem 1.1. Fix any $r_0 \geqq R$ and consider a $D_{r_0} \in \mathcal{D}_{r_0}$. Then the selfadjoint operator $H_{D_{r_0}}$, which is the Friedrichs extension of the quadratic form associated with $-\Delta + V$ on $C_0^\infty(D_{r_0})$ satisfies*

$$\text{inf essential spectrum } H_{D_{r_0}} = 0. \tag{1.11}$$

Since this theorem follows quite directly from Theorem 1.1, inequality (1.4) and known results [2, 20] we only sketch its proof.

Proof of Theorem 1.3. Taking into account that

$$\lim_{r \to \infty} \frac{\ln v(r)}{r} = -\sqrt{|E|} \tag{1.12}$$

(this follows from (1.3) by standard estimates, see e.g. [2]) inequality (1.4) together with Theorem 1.1 implies that

$$e^{\beta r}\psi \notin L^2(D_{r_0}) \quad \text{for} \quad \beta > \sqrt{|E|}. \tag{1.13}$$

Now let $D_r = D_{r_0} \backslash B_r$ and denote

$$\Sigma_r \equiv \inf_{\varphi \in C_0^\infty(D_r)} \frac{\int (|\nabla \varphi|^2 + V|\varphi|^2)dx}{\int |\varphi|^2 dx} \quad \text{and} \quad \Sigma \equiv \lim_{r \to \infty} \Sigma_r.$$

Following the results of Persson [2] and Agmon [22] we have $\Sigma = \inf \sigma_{\text{ess}} H_{D_{r_0}}$ and further we obtain easily that $\Sigma \geqq \inf \sigma_{\text{ess}} H_{\Omega_R}$. Since due to the assumption on V, $\inf \sigma_{\text{ess}} H_{\Omega_R} = 0$, $\Sigma \geqq 0$ results. Now suppose indirectly that $\Sigma > 0$. Then due to the foregoing considerations $\forall \varphi \in C_0^\infty(D_r)$,

$$\int (|\nabla \varphi|^2 + (V - E)|\varphi|^2)dx \left(\int |\varphi|^2 dx\right)^{-1} \geqq \Sigma_r + |E| \geqq a > |E|,$$

$\forall r \geqq \bar{r}$ ($\bar{r}$ large enough) with $a = \Sigma + |E| - \varepsilon(\bar{r})$ and $\varepsilon(\bar{r}) \to 0$ for $\bar{r} \to \infty$. Using this positivity of the quadratic form and the fact that ψ satisfies Eq. (1.1), $\psi \in H^1_{\text{loc}}(D_{r_0}) \cap L^2(D_{r_0})$ and $\psi = 0$ in $\partial D_{r_0} \backslash \partial B_{r_0}$, it follows by the same methods as developed by Agmon [22, p. 19 and 55] that $\psi \exp(\sqrt{a - \delta}\, r) \in L^2(D_{r_0})$, $\forall \delta > 0$ small. Therefore $\psi \exp(\sqrt{|E| + v}\, r) \in L^2(D_{r_0})$ for some $v > 0$, which is a contradiction to (1.13) and hence $\Sigma = 0$. □

In the following section we shall discuss these results in the light of earlier findings [16, 17]. Sections 3 and 4 will be devoted to proofs. Some of the main ingredients of the proofs will involve lower bounds à la Davies [9] together with the derivation and analysis of some non-linear differential inequalities. We will also prove an upper bound to positive solutions of a linear differential equation (Theorem 3.1) which is quite different from the usual ones following from the maximum principle.

2. Discussion of the Results and Previous Results

In this section we first recall some results from [17]. We will then give a "heuristic" explanation of Theorems 1.1 and 1.2 and try to point out why the case $n \geqq 3$ requires new techniques and is in some respect qualitatively different from the 2-dimensional one. We shall also discuss the radial case, i.e. nonradial solutions to (1.1) if $V_2 \equiv 0$.

Let us first introduce polar coordinates and an atlas on S^{n-1}.

$$x_1 = r\cos\vartheta_1, \quad x_j = r\left(\prod_{i=1}^{j-1} \sin\vartheta_i\right)\cos\vartheta_j, \quad 2 \leqq j \leqq n-2,$$

$$x_{n-1} = r\left(\prod_{i=1}^{n-1} \sin\vartheta_i\right)\cos\varphi, \quad x_n = r\left(\prod_{i=1}^{n-1} \sin\vartheta_i\right)\sin\varphi,$$

with $0 \leqq \vartheta_j \leqq \pi$, $1 \leqq j \leqq n-2$, $-\pi \leqq \varphi \leqq \pi$. For our purpose it will be advantageous to replace these angles ϑ_i by

$$\xi_i = \vartheta_i - \frac{\pi}{2}, \quad 1 \leqq i \leqq n-2, \quad \xi_{n-1} = \varphi.$$

We denote by $\xi = (\xi_1, \ldots, \xi_{n-1})$ a vector in Q, where

$$Q = \left\{\xi \in \mathbb{R}^{n-1} : -\frac{\pi}{2} < \xi_i < \frac{\pi}{2}, 1 \leqq i \leqq n-2, -\pi < \xi_{n-1} < \pi\right\},$$

and define for $\xi \in Q$,

$$\begin{aligned}\Phi^{-1}(\xi) = {} & \cos\left(\xi_1 + \frac{\pi}{2}\right)e_1 + \sum_{j=1}^{n-3}\prod_{i=1}^{j} \sin\left(\xi_i + \frac{\pi}{2}\right)\cos\left(\xi_{j+1} + \frac{\pi}{2}\right)e_{j+1} \\ & + \prod_{i=1}^{n-2} \sin\left(\xi_i + \frac{\pi}{2}\right)(\cos\xi_{n-1}e_{n-1} + \sin\xi_{n-1}e_n),\end{aligned}$$

where the e_j are the canonical basis of $\mathbb{R}^n$, i.e.

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots), \ldots, \quad e_n = (0, \ldots, 0, 1).$$

Let $U = \Phi^{-1}(Q)$, then $\bar{U} = S^{n-1}$, and we obtain from the chart (U, Φ) by rotations the charts (U_i, Φ_i), $i \in I$ for some index set I, so that $\bigcup_{i \in I} U_i = S^{n-1}$. The collection of these charts is our C^ω (real analytic) atlas.

We now restate assumption B for V_2 in terms of these local coordinates.

Assumption B′

There is an $\alpha > 1/2$ such that $r^{1+\alpha}V_2 : \Omega_R \to \mathbb{R}^1$ is continuous and that $\forall R_0 > R$,

$$\{r^{1+\alpha}V_2(r\Phi_i^{-1}), r \geqq R_0, i \in I\}$$

is a uniformly bounded set of $C^\omega(Q)$ functions.

Let now

$$u = \frac{\psi}{v}, \tag{2.1}$$

where v was introduced in (1.3). The main regularity result in [17] was the following

Theorem A. *Suppose that $\psi \in L^2(\Omega_R)$ is real valued, $\psi \not\equiv 0$, and satisfies (1.1). Assume that V, V_1 and V_2 obey the assumptions A and B and that $E < 0$. Then for every $R_0 > R$,*

$$u(ry) \in C^{\omega}(S^{n-1}) \tag{2.2}$$

uniformly in r for $r \geqq R_0$. $A(y) \equiv \lim_{r\to\infty} u(ry)$ exists with

$$A \in C^{\omega}(S^{n-1}) \tag{2.3}$$

and does not vanish identically. Furthermore for every multi-index β there is a $C_{|\beta|} < \infty$ such that $\forall \xi \in Q$, $\forall i \in I$, $\forall r \geqq R_0$,

$$\left| \frac{\partial^{\beta}}{\partial \xi^{\beta}} (u(r\Phi_i^{-1}(\xi)) - A(\Phi_i^{-1}(\xi)) \right| r^a \leqq C_{|\beta|}, \tag{2.4}$$

where $a = \min(1, \alpha)$.

Remark 2.1. In [17], weaker regularity properties of u and A were derived under weaker assumptions on V_2.

Theorem A also implies the following

Corollary 2.1. *If in Theorem A, u is replaced by $F = \psi/\psi_{av}$, then its conclusions remain true.*

This follows from Theorem A, and by noting that

$$F = \frac{\psi}{\psi_{av}} = \frac{\psi}{v} \frac{v}{\psi_{av}}$$

and that v/ψ_{av} has a limit according to Theorem A. In [17] all statements were formulated in terms of u, though F is a more natural function to consider. But u satisfies the following differential equation:

$$-\Delta u - \frac{2(\nabla v)}{v} \nabla u + V_2 u = 0 \tag{2.5}$$

that was used in the proof of Theorem A.

The following consequence of Theorem A will be helpful for the proof and the understanding of Theorem 1.1.

Proposition 2.1. *Let $D_{r_0} \in \mathscr{D}_{r_0}$, then either*

$$\liminf_{r\to\infty} |S(r)| \geqq c > 0 \quad \textit{for some } c$$

or

$$\lim_{r\to\infty} |S(r)| = 0. \tag{2.6}$$

Proof of Proposition 2.1. Let $\mathscr{A} = \{y \in S^{n-1} : A(y) = 0\}$, where A was defined in Theorem A and let $\mathscr{A}_{\varepsilon} = \{y \in S^{n-1} : |A(y)| < \varepsilon\}$. Clearly $|\mathscr{A}| = 0$ and $\lim_{\varepsilon\downarrow 0} |\mathscr{A}_{\varepsilon}| = 0$. Now suppose there is a D_{r_0} such that the corresponding $S(r)$ satisfies $\limsup_{r\to\infty} |S(r)| \geqq c_1 > 0$ and $\liminf_{r\to\infty} |S(r)| = 0$, then there is a sequence of r_i tending to infinity so that $|S(r_i)| > c_1 - \delta > 0$ and for sufficiently small ε, there must be $y_i \in S(r_i)$ with $y_i \in S^{n-1} \backslash \mathscr{A}_\varepsilon$. Now take an accumulation point y_∞. But then $|A(y_\infty)| \geqq \varepsilon$ and the continuity of A together with (2.4) implies a contradiction to $\liminf_{r\to\infty} |S(r)| = 0$. □

We give now a heuristic "explanation" of Theorems 1.1 and 1.2 with the help of an asymptotic expansion of u, that was obtained in [17].

Since $A \in C^\omega(S^{n-1})$ it can be expanded locally in a Taylor series. We write for simplicity $A(\xi)$ and $u(r, \xi)$ instead of $A(\Phi^{-1}(\xi))$, respectively $u(r\Phi^{-1}(\xi))$, and expand A near $\xi = 0$ in a Taylor series, so that

$$A(\xi) = \sum_{m=M}^{M_1} P_m(\xi) + O(|\xi|^{M_1+1}), \quad M_1 \geqq M \geqq 0, \tag{2.7}$$

where the P_m are homogeneous polynomials of degree m in the $\xi_1, \ldots, \xi_{n-1}$, i.e.

$$P_m = \sum_{l_1 + l_2 + \cdots l_{n-1} = m} a^{(m)}_{l_1, \ldots, l_{n-1}} \xi_1^{l_1} \xi_2^{l_2} \cdots \xi_{n-1}^{l_{n-1}}. \tag{2.8}$$

We introduce the following asymptotic regions for $r > \bar{r}$ large, any $\kappa > 0$,

$$D^{(\kappa)}_\beta = \{x = r\Phi^{-1}(\xi) \in \Omega_{\bar{r}} : |\xi| < \kappa r^{-\beta}\}, \quad \beta \in (0, 1/2].$$

The following theorem describes how the asymptotic behaviour of u (and hence of F) in these regions is related to P_M, the first nonvanishing term in (2.8).

Theorem B [17]. *Suppose A is given (2.7) and P_M by (2.8). Then in $D^{(\kappa)}_\beta$ for some $\varepsilon > 0$,*

$$u(r, \xi) = (2\sqrt{|E|}r)^{-M/2} \sum_{l_1 + l_2 + \cdots l_{n-1} = M} a^{(M)}_{l_1 \cdots l_{n-1}} \prod_{j=1}^{n-1} H_{l_j}(b\sqrt{r}\xi_j) \cdot (1 + O(r^{-\varepsilon})) + O(r^{-\beta M - \min(a - 1/2, \beta)}), \tag{2.9}$$

where $b = (|E|/4)^{1/4}$ and the H_l are the usual Hermite polynomials. Here $a = \min(1, \alpha)$.

Remark 2.2 Obviously, up to a multiplicative constant (2.9) holds also for F.

Now let $z = (z_1 \cdots z_{n-1}) \in \mathbb{R}^{n-1}$. Theorem B [17] implies that for $|z|$ finite

$$\lim_{r\to\infty} r^{M/2} u\left(r, \frac{z}{b\sqrt{r}}\right) = \left(\frac{1}{2b}\right)^M \sum_{l_1 + l_2 + \cdots l_{n-1} = M} a^{(M)}_{l_1 \cdots l_{n-1}} \prod_{j=1}^{n-1} H_{l_j}(z_j) := \mathscr{H}_M(z). \tag{2.10}$$

We also observe that

$$\left(-\sum_{i=1}^{n-1} \frac{\partial^2}{\partial z_i^2} + |z|^2\right) e^{-|z|^2/2} \mathscr{H}_M(z) = (2M + n - 1) e^{-|z|^2/2} \mathscr{H}_M(z), \tag{2.11}$$

so that $e^{-|z|^2/2}\mathscr{H}_M(z)$ is an eigenfunction of the $(n-1)$-dimensional isotropic oscillator. Equations (2.9) and (2.10) show a specific $\sqrt{r}$-scaling. Let us consider a bounded nodal domain of $\mathscr{H}_M(z)$, i.e. a bounded set in $\mathbb{R}^{n-1}$ which is a component of $\mathbb{R}^{n-1} \backslash \{z \in \mathbb{R}^{n-1} : \mathscr{H}_M(z) = 0\}$. Then this would correspond to the limit of some $S(r)$, so that $|S(r)| \sim r^{-(n-1)/2}$ (see the proof of Corollary 2.2) and this would imply Theorems 1.1 and 1.2 for this case. For $n = 2$ such an argument can be made rigorous because then A depends only on one angular variable, has only isolated zeros of finite multiplicity, and consequently in (2.9) and (2.10) only a single Hermite polynomial turns up. This made it possible to give a characterization of the asymptotics of nodal lines in [14]. There it was shown, roughly speaking, that near infinity nodal lines look either like branches of parabolas or like straight lines.

For $n \geqq 3$ the situation is more complicated and the arguments which work for $n=2$ do not apply in general: $\mathscr{A}$, i.e. the zero set of A will be usually an $n-2$-dimensional object. It turns out that more accurate asymptotic expansions for u than (2.9) are necessary. But even with such improvements of (2.9) uncontrollable cancellations can occur making such an approach inappropriate. However, for many cases one can make explicit statements on nodal domains using Theorem B, and Theorem 1.1 or 1.2, respectively. For instance,

Corollary 2.2 *Suppose that* $y_0 \in S^{n-1}, A(y_0)=0$ *and that for some* $\varepsilon>0$, $A(y)>0$ *for* $0<|y-y_0|<\varepsilon$. *Then there exist nodal domains* $D_{r_0} \in \mathscr{D}_{r_0}$ *with the following properties*:

(i) $|y(r)-y_0| \to 0$ *for* $r \to \infty$ *for* $y(r) \in S(r), S(r)$ *corresponding to* D_{r_0}.
(ii)

$$\lim_{r \to \infty} \frac{\ln(\mathrm{Volume}(D_{r_0} \cap B_r))}{\ln r} = \frac{n+1}{2}. \tag{2.12}$$

Proof of Corollary 2.2. Without loss we assume $y_0 = e_{n-1}$. We write again $A(\xi)$ instead of $A(\Phi^{-1}(\xi))$, so that $A(0)=0$, and $A(\xi)>0$ for $0<|\xi|<\varepsilon'$ for some $\varepsilon'>0$. Now in a neighborhood of $\xi=0$, $A(\xi)=P_M(\xi)+O(|\xi|^{M+1})$, according to (2.7). So $P_M(\xi)$ must satisfy $P_M(\xi)>0$ for $|\xi|>0$ and M must be $\geqq 2$. The function $\mathscr{H}_M(z)$ defined in (2.10) and the polynomial P_M are connected by the following relation proved in [17, Lemma 3.2]:

$$\sum_{k=0} \frac{1}{k!}\left(-\sum_{i=1}^{n-1} \frac{\partial^2}{\partial z_i^2}\right)^k P_M(z) = \mathscr{H}_M(z/2).$$

Hence there is a $\rho>0$ such that $\mathscr{H}_M(z/2)>0$ for $|z| \geqq \rho$ and the same is true for $\mathscr{H}_M(z)$ by scaling. This implies that the set $\{z : \mathscr{H}_M(z)<0\}$ is contained in a bounded region in $\mathbb{R}^{n-1}$. But $\exp(-|z|^2/2)\mathscr{H}_M(z)$ was shown to be the eigenfunction of the $(n-1)$-dimensional isotropic harmonic oscilator corresponding to an excited eigenvalue and must change sign. This together with (2.10) implies that there exist D_{r_0} with the property (i) of Corollary 2.2, and the corresponding $S(r)$ satisfy $|S(r)|r^{(n-1)/2} \leqq \mathrm{const}$, so that Volume $(D_{r_0} \cap B_r) \leqq cr^{(n+1)/2}$. Combining this result with Theorem 1.2 we obtain (2.12). □

With such considerations many specific cases can be discussed. For instance it can be easily shown that if $n=3$ and $A=\sin^2 \xi_2$, then there is a D_{r_0} such that the limit in (2.12) exists and equals 5/2.

We now illustrate our findings with the classical example of quantum mechanics, namely the Hydrogen atom. Its Hamiltonian is in suitable units

$$H = -\Delta - \frac{2}{r}$$

on $L^2(\mathbb{R}^3)$. An eigenfunction ψ_n satisfies (here n labels the energies not counting degeneracies)

$$H\psi_n = E_n\psi_n, \quad n = 1, 2, \ldots, \quad E_n = -\frac{1}{n^2}.$$

The eigenvalues E_n are n^2-fold degenerate and a general (real valued) eigenfunction reads (see any textbook in quantum mechanics)

$$\psi_n = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} c_{l,m}^{(n)} f_l^{(n)}(r)\, Y_l^{(m)}(y), \quad y \in S^{n-1}. \tag{2.13}$$

Here the $Y_l^{(m)}$ are the usual surface harmonics, i.e. the restriction of the homogeneous harmonic polynomials in $\mathbb{R}^3$ to S^2. They satisfy $L^2 Y_l^{(m)} = l(l+1) Y_l^{(m)}$, $m \in [-l, l]$, where $-L^2$ is the Laplace Beltrami operator on S^2. The $f_l^{(n)}(r)$ satisfy, on $(0, \infty)$, the ordinary differential equation

$$\left(-\frac{d^2}{dr^2} - \frac{2}{r} + \frac{l(l+1)}{r^2} - E_n\right) r f_l^{(n)}(r) = 0.$$

These $f_l^{(n)}$ show the same asymptotics

$$f_l^{(n)}(r) \sim r^{1/\sqrt{|E_n|} - 1} e^{-\sqrt{|E_n|}r}.$$

Now if $r > 2/|E_n|$, $f_0^{(n)}$ does not change sign, and we can investigate

$$u = \frac{\psi_n}{f_0^{(n)}} \quad \text{and} \quad A(y) = \lim_{r \to \infty} u(ry).$$

Since the restriction of any polynomial in $\mathbb{R}^3$ to S^2 can be expressed by a finite linear combination of $Y_l^{(m)}$ [23] one can construct explicit examples with it. For instance, one can construct an A as in Proposition 2.1 and so forth. Since highly excited Hydrogen atoms have been prepared recently in laboratory (there are numerous experimental and theoretical papers in the last few years for instance in Phys. Rev. Lett.) our findings might be of some interest for the understanding and analysis of these states. It would be also interesting to give a more detailed analysis of these hydrogenic wave functions, starting from the present results using Theorem B.

For other radial problems say on $L^2(\mathbb{R}^3)$ we will not usually encounter these phenomena since then the eigenvalues do not show this l-degeneracy. The eigenfunction will look like $\psi_l^{(n)} = f_l^{(n)}(r) \sum_{m=-l}^{l} Y_l^{(m)}(y)$ and the corresponding nodal domains will show that the left-hand side of (2.12) equals 3. But with exterior problems as the exterior Helmholtz equation $(-\Delta + \kappa^2)\psi = 0$ for $r > R$ we find all the complexity as for the hydrogenic case.

Our considerations following Theorem B lead us to the following

Conjecture. Pick a $D_{r_0} \in \mathscr{D}_{r_0}$, then

$$\lim_{r \to \infty} \frac{\ln |S(r)|}{\ln r} \in \left[-\frac{n-1}{2}, 0\right].$$

This would imply that the limit in Theorem 2.2 exists and is in the interval $[n+1/2, n]$.

Finally a few words on the literature. Most of the relevant work on the asymptotics of wave functions has been cited in Sect. 1 or in [2, 22]. We should mention the recent results of Herbst [13] and Froese and Herbst [11] which show the surprising complexity of the asymptotics of solutions of Schrödinger equations in cones.

We should also try to relate our work with the literature on nodal properties of solutions of elliptic partial differential equations. Closest in spirit are perhaps the local results of Bers [5], Cheng [8] and Caffarelli and Friedmann [6]. But we are not aware of any investigations on the asymptotics of nodes. It might be also possible to investigate generic properties of the asymptotics of nodal surfaces in the spirit of the results of Albert [3] and Uhlenbeck [25]. We hope to investigate such questions in future work.

3. Proof of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.1. We consider a $D_{r_0} \in \mathscr{D}_{r_0}$. By Proposition 2.1 the corresponding $S(r)$ satisfies $\lim_{r\to\infty} |S(r)| \to 0$ or $\liminf_{r\to\infty} |S(r)| > 0$. But if $|S(r)|$ does not tend to zero Theorem A and the arguments in the proof of Proposition 2.1 imply that

$$\int_{S(r)} u^2 d\sigma \geqq c \tag{3.1}$$

for some $c>0$ and large r. The same holds for $\int_{S(r)} F^2 d\sigma$ due to (1.5), and there is nothing to prove. Therefore, in the following we assume $\lim_{r\to\infty} |S(r)| = 0$.

Let $-L^2$ be the Laplace Beltrami operator on S^{n-1} and let

$$\lambda^2(S(r)) = \inf_{\varphi \in C_0^\infty(S(r))} \frac{\int |L\varphi|^2 d\sigma}{\int |\varphi|^2 d\sigma}. \tag{3.2}$$

For simplicity we shall write $\lambda^2(r)$ instead of $\lambda^2(S(r))$. Thus, for each r, $\lambda^2(r)$ is just the lowest eigenvalue of the selfadjoint operator $L^2(r)$, which is the Friedrichs extension of the corresponding quadratic form with form core $C_0^\infty(S(r))$. $|S(r)| \to 0$ implies, for instance by the Faber Krahn inequality [4, 7], that

$$\lim_{r\to\infty} \lambda^2(r) = \infty. \tag{3.3}$$

The following two lemmas will enable us to reduce the proof of Theorem 1.1 to the investigation of various one dimensional relations.

Lemma 3.1. *Let*

$$\psi_0(r) = \left(\int_{S(r)} \psi^2 d\sigma \right)^{1/2} (r), \tag{3.4}$$

and let

$$U(r) = \inf_{y \in S(r)} V_2(ry),$$

then in the distributional sense for $r > R$,

$$\left(-\Delta + V_1 - E + U + \frac{\lambda^2(r)}{r^2} \right) \psi_0(r) \leqq 0. \tag{3.5}$$

Remark 3.1. Lemma 3.1 will be proved in the next section. It certainly holds under less restrictive conditions on V. We note that with $\tilde{\psi}_0 = r^{(n-1)/2}\psi_0$, (3.5) can be transformed so that

$$\left(-\frac{d^2}{dr^2} + \frac{(n-1)(n-3)}{4r^2} + V_1 - E + U + \frac{\lambda^2(r)}{r^2} \right) \tilde{\psi}_0 \leqq 0, \tag{3.6}$$

and from this we infer that for $r > R$,

$$\frac{d}{dr} \tilde{\psi}_0 \leqq 0, \tag{3.7}$$

and

$$\frac{d^2}{dr^2} \tilde{\psi}_0 \geqq 0 \quad \text{almost everywhere.} \tag{3.8}$$

Consider now $A(y) = \lim_{r \to \infty} u(ry)$ as defined in (2.3). If $y_0 \in \mathscr{A}$, the zero set of A, then y_0 can only be a zero of finite order, say $M(y_0)$, due to the real analyticity of A. Let

$$M = \max_{y_0 \in \mathscr{A}} M(y_0). \tag{3.9}$$

Lemma 3.2. *Let u be as defined in (2.1). Then $\forall D_{r_0} \in \mathscr{D}_{r_0}$ and the corresponding $S(r)$'s, there is $\gamma \geqq 1$ depending only on M such that*

$$\left(\int_{S(r)} u^2 d\sigma \right)^{1/2} (r) = \frac{\psi_0}{v(r)} \geqq c_0 \lambda(r)^{-2\gamma} \tag{3.10}$$

for some $c_0 > 0$.

Remark 3.2. The proof of (3.10) is rather involved and will be given in Sect. 4. The actual value of γ will show up in this proof but we do not need it.

It will be advantageous to work not with the function ψ_0/v but with

$$w(r) = \left(\frac{\psi_0}{v_1} \right)(r), \tag{3.11}$$

where v_1 is a radial positive L^2-function, such that

$$(-\Delta + V_1 - E + U) v_1(r) = 0. \tag{3.12}$$

We are going to show that $w \geqq cr^{-\gamma}$, and from the following it will be clear that this immediately implies Theorem 1.1. We collect first some properties of w and we assume always $|S(r)| \to 0$ for $r \to \infty$.

Proposition 3.1. *There are positive constants c_1, c_2, c_3, so that*

$$w \geqq c_1 \lambda(r)^{-2\gamma}, \tag{3.13}$$

$$\left(-\frac{d^2}{dr^2}+c_2\frac{d}{dr}+\frac{\lambda^2(r)}{r^2}\right)w \leqq 0, \tag{3.14}$$

and

$$\left(-\frac{d^2}{dr^2}+c_2\frac{d}{dr}+\frac{c_3}{r^2}w^{-1/\gamma}\right)w \leqq 0 \tag{3.15}$$

for $r > R$. (3.14) and (3.15) hold in the distributional sense and w is continuously differentiable and with $'$ denoting d/dr

$$w' \leqq 0, \quad r > R. \tag{3.16}$$

Proof of Proposition 3.1. (3.15) follows from (3.13) and (3.14). To show (3.13) we have to show that for some $c', c'' < \infty$,

$$0 < c' \leqq \frac{v_1}{v} \leqq c'' \tag{3.17}$$

for $r > R$. First we observe that $\tilde{v} = r^{(n-1)/2}v$ and $\tilde{v}_1 = r^{(n-1)/2}v_1$ satisfy

$$\left(-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+V_1-E\right)\tilde{v}=0, \tag{3.18}$$

and

$$\left(-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+V_1-E+U\right)\tilde{v}_1=0. \tag{3.19}$$

Since by assumption $B, U \in L^1(R,\infty)$, Theorem 3.1 of [15] implies (3.17). Next we combine (3.6) and (3.19) so that (since $w\tilde{v}_1 = \tilde{\psi}_0$)

$$\left(-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+V_1-E+U+\frac{\lambda^2(r)}{r^2}\right)w\tilde{v}_1$$
$$= -\tilde{v}_1 w'' - 2\tilde{v}_1' w' + \frac{\lambda^2(r)}{r^2}w\tilde{v}_1 \leqq 0$$

in the distributional sense. Now $\tilde{v}_1' < 0$, and from the maximum principle [21] we infer that also $w' < 0$. Hence, dividing the last inequality by $\tilde{v}_1$ and noting that $-\tilde{v}_1'/\tilde{v}_1 \leqq c_2$ for some constant (see [16]) we obtain (3.14) and the proof is complete. □

Lemma 3.3.

$$\liminf_{r\to\infty}\frac{\lambda^2(r)}{r} < \infty \tag{3.20}$$

Remark 3.3. By Proposition 3.1 this means that there is a sequence $\{r_i\}$ tending to infinity so that

$$w(r_i) \geqq c_4 r_i^{-\gamma}. \tag{3.21}$$

Proof of Lemma 3.3. Suppose that (3.20) does not hold, that means for any m there is an R_m so that for $r > R_m$,

$$\frac{\lambda^2(r)}{r} \geqq m. \tag{3.22}$$

We shall derive a contradiction to (3.22). Equations (3.22) and (3.14) imply

$$-w'' + c_2 w' + \frac{m}{r} w \leqq 0$$

in the distributional sense for $r > R_m$. Since $w \to 0$ a function $h_m > 0$ with

$$-h_m'' + c_2 h_m' + \frac{m}{r} h_m \geqq 0, \tag{3.23}$$

and $h_m(R_m) \geqq w(R_m)$ will satisfy $h_m \geqq w$ for $r > R_m$. Pick $h_m = c_m r^{-k}$, then we have (left-hand side of (3.23)) $= -k(k+1)R_m^{-1} - c_2 k + m > 0$ if $k < m/c_2$ and R_m large enough. Therefore for any m there is a c_m such that for $r > R_m$,

$$w < c_m r^{-m}. \tag{3.24}$$

Using (3.15) this implies (the m was arbitrary) that for every finite l,

$$-w'' + c_2 w' + c_l r^l w \leqq 0 \tag{3.25}$$

for $r > R_l$. We can use again the maximum principle to infer that for any $j > 0$ and some c_j,

$$w \leqq c_j e^{-r^j} \tag{3.26}$$

for $r > R_j$ sufficiently large. Now this implies, via (3.13), that

$$\frac{\lambda^2}{r^2} \geqq \frac{w^{-1/\gamma}}{r^2} = \tilde{\psi}_0^{-1/\gamma} r^{-2} \tilde{v}_1^{1/\gamma} \geqq c_\delta \tilde{\psi}_0^{-1/\gamma} e^{-\delta r}$$

for r sufficiently large, say $r > R_\delta > R_l$, and $\delta < \sqrt{-E}/\gamma$, where we used that $\tilde{v}_1 \leqq c_\varepsilon \exp[-(\sqrt{-E} - \varepsilon)r]$ for $\varepsilon > 0$ and suitable c_ε. Since $V_1 - E + ((n-1)(n-3)/4r^2) + U$ is bounded away from zero, we see that for any α with $2\alpha < 1/\gamma$,

$$V_1 - E + \frac{(n-1)(n-3)}{4r^2} + U + \frac{\lambda^2(r)}{r^2} \geqq \tilde{\psi}_0^{-2\alpha},$$

and hence from (3.6)

$$\left(-\frac{d^2}{dr^2} + \tilde{\psi}_0^{-2\alpha}\right)\tilde{\psi}_0 \leqq 0 \tag{3.27}$$

follows for $r > R_\delta$ sufficiently large.

We now show that (3.27) implies that $\tilde{\psi}_0 \equiv 0$, but this cannot be true because of unique continuation and because D_{r_0} is unbounded. Hence (3.22) cannot hold and the Lemma is proven. So let

$$\Phi = \exp\left(-\int_{R_\delta}^{r} \tilde{\psi}_0^{-\alpha} dx\right),$$

then

$$-\Phi'' + \tilde{\psi}_0^{-2\alpha}\Phi = -\alpha\tilde{\psi}_0^{-\alpha-1}\tilde{\psi}_0'\Phi \geqq 0, \tag{3.28}$$

since $\tilde{\psi}_0' \leqq 0$. If $\tilde{\psi}_0 \not\equiv 0$ then there is a $c > 0$ such that $\Phi \geqq c\tilde{\psi}_0$ for $r > R_\delta$ by the maximum principle. Let $\mu = \int_{R_\delta}^{r} \tilde{\psi}_0^{-\alpha} dx$, then

$$(\mu')^{-1/\alpha}=\tilde{\psi}_0,$$

so that for some $C>0$,

$$\mu' e^{-\alpha\mu}=-\frac{1}{\alpha}\frac{d}{dr}e^{-\alpha\mu}\geqq C>0. \tag{3.28}$$

Integration leads to

$$\frac{1}{\alpha}[\exp(-\alpha\mu(R_\delta))-\exp(-\alpha\mu(r))]\geqq C(r-R_\delta).$$

The left-hand side of this inequality remains bounded, but the right-hand side tends to infinity for $r\to\infty$. Hence $\tilde{\psi}_0$ must vanish identically and the proof of Lemma 3.3 is complete. □

Taking into account (3.17) and (1.5) it becomes clear that Theorem 1.1 will follow from

Lemma 3.4. *Let γ be given according to Lemma 3.2. Then for some $c>0$ and r large*

$$w(r)\geqq cr^{-\gamma}. \tag{3.29}$$

Proof of Lemma 3.4. Let

$$g_d(r)=dr^{-\gamma}, \tag{3.30}$$

then for sufficiently small positive d, say d_0,

$$\left(-\frac{d^2}{dr^2}+c_2\frac{d}{dr}+\frac{c_3}{r^2}g_{d_0}^{-1/\gamma}\right)g_{d_0}\geqq 0. \tag{3.31}$$

Now suppose that (3.29) does not hold, then there is a sequence $\{r_j\}$ tending to infinity so that

$$w(r_j)<d_0 r_j^{-\gamma}. \tag{3.32}$$

We pick $d_0<c_4$, where c_4 was defined in (3.21). Combining (3.15) and (3.31) we obtain

$$-(w-g_{d_0})''+c_2(w-g_{d_0})'+\frac{c_3}{r^2}(w^{1-1/\gamma}-g_{d_0}^{1-1/\gamma})\leqq 0. \tag{3.33}$$

Due to (3.21) we know that $w(r)\leqq g_{d_0}(r)$ cannot hold for all $r\geqq\bar{r}$, where $\bar{r}$ is large, which together with (3.32) implies that $w-g_{d_0}$ must have a positive maximum for some $\tilde{r}>\bar{r}$. But this cannot be true because of the maximum principle [21]. So there is no sequence $\{r_j\}$ such that (3.32) holds and the Lemma is proven. □

3.2 Proof of Theorem 1.2. We start with a one dimensional estimate, which might be of independent interest.

Theorem 3.1. *Let $\tilde{V}$ be continuous, real valued, and let*

$$\lim_{r\to\infty}\tilde{V}(r)=0. \tag{3.34}$$

Suppose $E<0$, and that for $r>R>0$, $\tilde{V}-E>0$. Let for $r>R$

$$-h''+(\tilde{V}-E)h=0 \tag{3.35}$$

with $0 < h \in L^2(R, \infty)$. *Suppose*

$$W(x) \geqq 0,$$

and uniformly locally integrable so that

$$\sup_{r \geqq R + \frac{1}{2}\varepsilon} \int_{r-\varepsilon/2}^{r+\varepsilon/2} W(x)dx = m(\varepsilon) < \infty \quad \text{for} \quad \varepsilon > 0. \tag{3.36}$$

Let $0 < f \in L^2(R, \infty)$ *be a distributional solution so that*

$$\left(-\frac{d^2}{dr^2} + \tilde{V} - E + W \right) f = 0, \tag{3.37}$$

and suppose that $f(R) = h(R)$. *Then there exists a constant* $C > 0$ *depending on the* L^1*-properties of* W *and on* h *so that for* $r > R + 2$,

$$g(r) = \frac{f(r)}{h(r)} \leqq \exp\left(-C \int_{R+1/2}^{r} W dx \right). \tag{3.38}$$

C may be chosen for instance

$$C = \frac{1}{2} \inf_{\varepsilon \in [1/2, 1]} \frac{\varepsilon c_\varepsilon}{2 + \frac{\varepsilon}{2} c_\varepsilon m\left(\frac{\varepsilon}{2}\right)} \quad \text{with} \quad 0 < c_\varepsilon = \inf_{r \geqq R} \frac{h^2(r+\varepsilon)}{h^2(r)}. \tag{3.39}$$

Remark 3.4. It seems difficult to optimize this constant. We note [22] that our conditions on $\tilde{V}$ and W imply that f is absolutely continuous.

Proof of Theorem 3.1. h is monotonically decreasing and therefore we have, for $r \geqq R$,

$$1 > \frac{h^2(r+\varepsilon)}{h^2(r)} \geqq c_\varepsilon > 0,$$

since $h'/h \geqq -c_1$ for $r > R$ and some c_1 depending on E and $\tilde{V}$ (see [16]). g satisfies

$$g(r) = g(R) - \int_R^r \int_x^\infty \frac{h^2(y)}{h^2(x)} W(y) g(y) dy dx. \tag{3.40}$$

Equation (3.40) makes sense since $h' \leqq 0$, $g \leqq 1$ (by standard comparison arguments [21]) and due to (3.36). Note that (3.40) implies that $g(r)$ decreases. For $\varepsilon > 0$ we have

$$\begin{aligned} g(R+\varepsilon) &\leqq g(R) - \int_r^{R+\varepsilon} \int_x^{R+\varepsilon} \frac{h^2(R+\varepsilon)}{h^2(R)} g(R+\varepsilon) W(y) dy dx \\ &\leqq g(R) - c_\varepsilon g(R+\varepsilon) \int_R^{R+\varepsilon} \int_x^{R+\varepsilon} W(y) dy dx \\ &= g(R) - c_\varepsilon g(R+\varepsilon) \int_R^{R+\varepsilon} (x - R) W(x) dx \\ &\leqq g(R) - \frac{\varepsilon}{2} c_\varepsilon g(R+\varepsilon) \int_{R+\varepsilon/2}^{R+\varepsilon} W(x) dx, \end{aligned}$$

so that

$$g(R+\varepsilon)\leqq g(R)\left(1+\frac{\varepsilon}{2}c_\varepsilon\int_{R+\varepsilon/2}^{R+\varepsilon}W(x)dx\right)^{-1}. \tag{3.41}$$

Iteration of (3.41) gives for $k=1,2,\ldots,$

$$\frac{g(R+k\varepsilon)}{g(R)}\leqq\prod_{j=0}^{k-1}\left(1+\frac{\varepsilon}{2}c_\varepsilon\int_{R+(j+1/2)\varepsilon}^{R+(j+1)\varepsilon}W(x)dx\right)^{-1}.$$

We take logarithms so that

$$\ln\frac{g(R+k\varepsilon)}{g(R)}\leqq-\sum_{j=0}^{k-1}\ln\left(1+\frac{\varepsilon}{2}c_\varepsilon\int_{R+(j+1/2)\varepsilon}^{R+(j+1)\varepsilon}W(x)dx\right). \tag{3.42}$$

To get rid of the logarithms on the right-hand side of (3.42) we use that, for $y>0$ [19, p. 273]

$$\ln(1+y)\geqq\frac{2y}{2+y}, \tag{3.43}$$

and that by (3.36)

$$\int_{R+(j+1/2)\varepsilon}^{R+(j+1)\varepsilon}W(x)dx\leqq m\left(\frac{\varepsilon}{2}\right), \tag{3.44}$$

so that

$$\ln\frac{g(R+k\varepsilon)}{g(R)}\leqq-\frac{\varepsilon c_\varepsilon}{2+\frac{\varepsilon}{2}m\left(\frac{\varepsilon}{2}\right)c_\varepsilon}\sum_{j=0}^{k-1}\int_{R+(j+1/2)\varepsilon}^{R+(j+1)\varepsilon}W(x)dx. \tag{3.45}$$

To replace the sum in (3.45) by a single integral we use that, for $k\geqq 2$,

$$\ln\frac{g(R+k\varepsilon)}{g(R)}\leqq\ln\frac{g(R+(k-1/2)\varepsilon)}{g(R+\varepsilon/2)}\leqq-\frac{\varepsilon c_\varepsilon}{2+\frac{\varepsilon}{2}c_\varepsilon m\left(\frac{\varepsilon}{2}\right)}\sum_{j=0}^{k-2}\int_{R+(j+1)\varepsilon}^{R+(j+3/2)\varepsilon}W(x)dx, \tag{3.46}$$

so that we obtain by combining (3.45) and (3.46),

$$\ln\frac{g(R+k\varepsilon)}{g(R)}\leqq-\frac{1}{2}\frac{\varepsilon c_\varepsilon}{2+\frac{\varepsilon}{2}c_\varepsilon m\left(\frac{\varepsilon}{2}\right)}\int_{R+\varepsilon/2}^{R+k\varepsilon}W(x)dx. \tag{3.47}$$

Now let $r\geqq R+2$ and pick $\varepsilon\in[1/2,1]$ so that $r-R=k\varepsilon$. The proof of Theorem 3.1 is complete by noting that

$$C:=\frac{1}{2}\inf_{\varepsilon\in[1/2,1]}\frac{\varepsilon c_\varepsilon}{2+\frac{\varepsilon}{2}c_\varepsilon m\left(\frac{\varepsilon}{2}\right)}>0,$$

so that (3.38) and (3.39) is implied by the above. □

Remark 3.5. Theorem 3.1 is quite different from the usual results obtainable by subsolution estimates. We first tried to obtain a bound like (3.38) by looking for a supersolution to (3.37). But if W does not show any explicit form, so if only (3.36) is known, this seems to be difficult. Note that if $W \in L^1(R, \infty)$, then there are lower bounds [15] showing that (3.38) is optimal (except for the constant). If W tends to infinity so that $m(\varepsilon) = \infty$, then WKB arguments show that bounds like (3.38) will not describe the asymptotics correctly.

With the help of Theorem 3.1 we shall prove the following Lemma, from which Theorem 1.2 will follow easily.

Lemma 3.5. *Pick $\bar{R}$ large enough, then*

$$\liminf_{r \to \infty} r^{-1} |\{x \in (\bar{R}, r) : x \ln x \geqq \lambda^2(x)\}| > 0, \tag{3.48}$$

where $|\{\cdot\}|$ means measure of $\{\cdot\}$.

Proof of Lemma 3.5. In order to apply Theorem 3.1 we make the following identifications:

$$\tilde{V} = V_1 + \frac{(n-1)(n-3)}{4r^2} + U, \tag{3.49}$$

so that

$$h = \tilde{v}_1(r).$$

For W we pick

$$W = \min\left(\frac{\ln r}{r}, \frac{\lambda^2(r)}{r^2}\right). \tag{3.50}$$

Hence a positive L^2-solution f, so that

$$\left(-\frac{d^2}{dr^2} + \tilde{V} - E + W\right) f = 0 \tag{3.51}$$

will be an upper bound to $\tilde{\psi}_0$ by the maximum principle (see (3.6))

$$\frac{f(r)}{f(\bar{R})} \geqq \frac{\tilde{\psi}_0(r)}{\tilde{\psi}_0(\bar{R})} \tag{3.52}$$

for $r \geqq \bar{R}$. Now by Theorem 3.1,

$$g = \frac{f(r)}{\tilde{v}_1(r)} \leqq \exp\left(-C \int_{R+1/2}^{r} W dx\right), \tag{3.53}$$

and since with our choice $W \to 0$ we have from (3.39) that for given small δ there is an R_δ so that $m(\varepsilon) \leqq \delta\varepsilon$ and $C \geqq (1/8 - \delta)c_\varepsilon$. According to Theorem 1.1 (formulated in terms of w defined in (3.11)) we conclude via (3.52) and (3.53) that

$$c_1 r^{-\beta} \leqq w(r) \leqq c_2 g \leqq c_2 \exp\left[-(\tfrac{1}{8} - \delta)c_\varepsilon \int_{R_\delta + 1/2}^{r} W dx\right] \tag{3.54}$$

for positive constants c_1 and c_2. Now let

$$M_1(r) = \left\{x \in (R_\delta + \tfrac{1}{2}, r) : \frac{\ln x}{x} < \frac{\lambda^2(x)}{x^2}\right\}.$$

Equation (3.54) implies that for large r and some constant c_3

$$\begin{aligned} c_3 + \beta \ln r &\geqq (\tfrac{1}{8} - \delta) c_\varepsilon \int_{R_\delta + 1/2}^{r} W dx \\ &\geqq (\tfrac{1}{8} - \delta) c_\varepsilon \int_{M_1(r)} \frac{\ln x}{x} dx \\ &\geqq (\tfrac{1}{8} - \delta) \frac{c_\varepsilon}{2} [(\ln r)^2 - (\ln(r - |M_1(r)|))^2]. \end{aligned} \tag{3.55}$$

Since $R_\delta > 1$, $\ln(r - |M_1(r)|) > 0$, and we have

$$\frac{2\beta}{(\frac{1}{8} - \delta) c_\varepsilon} \ln r + c_4 \geqq \ln \frac{r}{r - |M_1(r)|} (\ln r + \ln(r - |M_1(r)|)).$$

Hence again for large r

$$\frac{r}{r - |M_1(r)|} \leqq c_5,$$

and this implies that for some $c_6 < 1$ and large r, $|M_1(r)| \leqq c_6 r$. Let $M_2(r) = (R_\delta + 1/2, r) \backslash M_1(r)$, then $r^{-1} |M_2(r)|$ is bounded away from zero for $r \to \infty$, verifying Lemma 3.5. □

We can now prove Theorem 1.2. According to the Faber–Krahn inequality [4, 7] we have for some $c_7 > 0$

$$\lambda^2(r) \geqq c_7 |S(r)|^{-2/(n-1)},$$

and therefore

$$|S(r)| \geqq c_8 \lambda(r)^{-(n-1)}.$$

From this inequality and from Lemma 3.5 we obtain for r large

$$\begin{aligned} \text{Volume}(D_{r_0} \cap B_r) &\geqq c_9 \int_{R_\delta + 1/2}^{r} |S(x)| x^{n-1} dx \\ &\geqq c_9 \int_{M_2(r)} x^{(n-1)/2} (\ln x)^{-(n-1)/2} dx \\ &\geqq c_9 \int_{R_\delta + 1/2}^{c_6 r} x^{(n-1)/2} (\ln x)^{-(n-1)/2} dx \\ &\geqq c_{10} r^{(n+1)/2} (\ln r)^{-(n-1)/2}, \end{aligned}$$

and therefrom

$$\ln \text{Volume}(D_{r_0} \cap B_r) \geqq \frac{(n+1)}{2} \ln r + \ln c_{10} - \frac{(n-1)}{2} \ln \ln r,$$

proving Theorem 1.2.

4. Proof of Lemmas 3.1 and 3.2

4.1. Proof of Lemma 3.1. Instead of (3.5) we shall verify inequality (3.6). We reformulate Eq. (1.1) in terms of $\tilde{\psi} = r^{(n-1)/2} \psi$, multiply the equation obtained from the left by $\tilde{\psi}$ and integrate over $S(r)$. This implies immediately that

$$-\int_{S(r)} \tilde{\psi}\frac{\partial^2}{\partial r^2}\tilde{\psi}\,d\sigma + \left(V_1 + U - E + r^{-2}\lambda^2(r) + \frac{(n-1)(n-3)}{4}r^{-2}\right)\tilde{\psi}_0^2 \leqq 0. \quad (4.1)$$

Let $\tilde{\psi}_D$ denote the restriction of $\tilde{\psi}$ to the nodal domain D_{r_0}. Since $\tilde{\psi}\in C^1(\Omega_R)$, $\tilde{\psi}=0$ in $\partial D_{r_0}\setminus\partial B_{r_0}$ and $|\partial D_{r_0}|=0$, it follows that $(\partial/\partial r)\tilde{\psi}_D$ is continuous a.e. in $\Omega_{\bar{R}}$ and $(\partial/\partial r)\tilde{\psi}_D\in L^\infty_{\rm loc}(\Omega_{\bar{R}})$ for $\bar{R}>r_0$. Therefrom we conclude that $\forall\varphi\in C_0^\infty((\bar{R},\infty))$,

$$\begin{aligned}
-\int_{\bar{R}}^{\infty}\varphi(r)\int_{S(r)}\tilde{\psi}\frac{\partial^2}{\partial r^2}\tilde{\psi}\,d\sigma\,dr &= -\int_{S^{n-1}}\int_{\bar{R}}^{\infty}\varphi(r)\tilde{\psi}_D\frac{\partial^2}{\partial r^2}\tilde{\psi}\,dr\,d\sigma\\
&= \int_{S^{n-1}}\int_{\bar{R}}^{\infty}\left(\frac{\partial}{\partial r}\tilde{\psi}\right)\frac{\partial}{\partial r}(\varphi\tilde{\psi}_D)\,dr\,d\sigma\\
&= \int_{S^{n-1}}\int_{\bar{R}}^{\infty}\left(\frac{\partial}{\partial r}\tilde{\psi}_D\right)\frac{\partial}{\partial r}(\varphi\tilde{\psi}_D)\,dr\,d\sigma. \quad (4.2)
\end{aligned}$$

Now note that $\tilde{\psi}'_0\in L^\infty_{\rm loc}((\bar{R},\infty))$ ($'$ denoting d/dr).

Next suppose we can show that $\forall\varphi\in C_0^\infty((\bar{R},\infty))$, $\varphi\geqq 0$

$$\int_{\bar{R}}^{\infty}\int_{S^{n-1}}\left(\frac{\partial}{\partial r}\tilde{\psi}_D\right)\frac{\partial}{\partial r}(\varphi\tilde{\psi}_D)\,d\sigma\,dr \geqq \int_{\bar{R}}^{\infty}(\varphi\tilde{\psi}_0)'\tilde{\psi}'_0\,dr, \quad (4.3)$$

then (4.2) and (4.3) imply that in the distribution sense

$$-\tilde{\psi}_0\tilde{\psi}''_0 \leqq -\int_{S(r)}\tilde{\psi}\frac{\partial^2}{\partial r^2}\tilde{\psi}\,d\sigma, \quad (4.4)$$

which together with (4.1) yields inequality (3.6).

Hence we are left to verify (4.3): Given $f\in C^2(\Omega_{\bar{R}})$, real valued, it follows easily by partial integration and application of Cauchy–Schwarz's inequality that $\forall\varphi\in C_0^\infty((\bar{R},\infty))$, $\varphi\geqq 0$

$$\int_{\bar{R}}^{\infty}\int_{S^{n-1}}\left(\frac{\partial}{\partial r}f\right)\frac{\partial}{\partial r}(\varphi f)\,d\sigma\,dr \geqq \int_{\bar{R}}^{\infty}(\varphi f_0)'f'_0\,dr$$
$$\text{with}\quad f_0 \equiv \left(\int_{S^{n-1}} f^2\,d\sigma\right)^{1/2}. \quad (4.5)$$

Finally by regularization we shall proceed from (4.5) to (4.3): We first note that $\tilde{\psi}_D\in W^{1,2}_{\rm loc}(\Omega_{\bar{R}})$ (see e.g. [1]). Now let f_ε with $\varepsilon>0$ denote a mollification of $\tilde{\psi}_D$ defined according to [1] e.g. If $\Omega'\subset\subset\Omega_{\bar{R}}$, then f_ε converges for $\varepsilon\to 0$ to $\tilde{\psi}_D$ in $W^{1,2}(\Omega')$, and since $\tilde{\psi}_D$ is continuous, $f_\varepsilon\to\tilde{\psi}_D$ for $\varepsilon\to 0$ pointwise uniformly in Ω'. Denoting $f_{\varepsilon,0}=\left(\int_{S^{n-1}} f_\varepsilon^2\,d\sigma\right)^{1/2}$, we conclude further that $f_{\varepsilon,0}\to\tilde{\psi}_0$ for $\varepsilon\to 0$ pointwise uniformly in r and in $W^{1,2}_{\rm loc}((\bar{R},\infty))$. Therefrom it is straightforward to show that $\forall\varphi\in C_0^\infty((\bar{R},\infty))$,

$$\int_{\bar{R}}^{\infty}\int_{S^{n-1}}\left|\left(\frac{\partial}{\partial r}f_\varepsilon\right)\frac{\partial}{\partial r}(\varphi f_\varepsilon)-\left(\frac{\partial}{\partial r}\tilde{\psi}_D\right)\frac{\partial}{\partial r}(\varphi\tilde{\psi}_D)\right|d\sigma\,dr$$

$$\leqq c(\bar{R})\left(\left\|\frac{\partial}{\partial r}(f_\varepsilon - \tilde{\psi}_D)\right\|\cdot\left\|\frac{\partial}{\partial r}(\varphi f_\varepsilon)\right\| + \left\|\frac{\partial}{\partial r}\tilde{\psi}_D\right\|\left(\left\|\varphi\frac{\partial}{\partial r}(f_\varepsilon - \tilde{\psi}_D)\right\| + \|\varphi'(f_\varepsilon - \tilde{\psi}_D)\|\right)\right)\to 0$$

for $\varepsilon\to 0$ with some $c(\bar{R})>0$, and where $\|\cdot\|$ denotes the norm in $L^2(\Omega_R)$. Similarly we have

$$\int_{\bar{R}}^{\infty} |(\varphi f_{\varepsilon,0})' f'_{\varepsilon,0} - (\varphi\tilde{\psi}_0)'\tilde{\psi}'_0|\,dr \leqq \|(\varphi f_{\varepsilon,0})'\|\cdot\|(f_{\varepsilon,0}-\tilde{\psi}_0)'\| + \|\tilde{\psi}'_0\|(\|\varphi(f_{\varepsilon,0}-\tilde{\psi}_0)'\| + \|\varphi'(f_{\varepsilon,0}-\tilde{\psi}_0)\|)\to 0$$

for $\varepsilon\to 0$ with $\|\cdot\|$ denoting the norm in $L^2((\bar{R},\infty))$.

Since (4.5) holds with $f=f_\varepsilon$, $\forall\varepsilon>0$, we conclude by the foregoing considerations that for $\varepsilon\to 0$ inequality (4.3) follows, finishing the proof of Lemma 3.1. □

4.2. Proof of Lemma 3.2. According to Proposition 2.1 either $\liminf|S(r)|>0$ or $|S(r)|\to 0$ for $r\to\infty$. However in the first case inequality (3.1) holds, and inequality (3.10) holds in a trivial way. Hence we have to consider the case $\lim_{r\to\infty}|S(r)|=0$. Without loss we assume $u>0$ in D_{r_0}.

The basic ideas which lead to (3.10) can be indicated fairly easily for the 2-dimensional case: Using polar coordinates $x=r\Phi^{-1}(\xi)$ we write $u=u(r,\xi)$, $S(r)\subset S^1$, and we assume that $S(r)$ is simply connected and shrinks to a point $\bar{y}\in S^1$ for $r\to\infty$. Let in polar coordinates $\Phi(S(r))=(\xi^{(1)}(r),\xi^{(2)}(r))\subset\mathbb{R}$ with $\xi^{(j)}(r)\to 0$ for $r\to\infty$ for $j=1,2$. Equation (3.10) in Lemma 3.2 then reads

$$\left(\int_{\xi^{(1)}(r)}^{\xi^{(2)}(r)} u^2(r,\xi)\,d\xi\right)^{1/2} \geqq c_0\lambda(r)^{-2\gamma}.$$

Now by Theorem A, since $A(\xi)=\lim_{r\to\infty}u(r,\xi)$ has a zero of some order $M\geqq 1$ in $\xi=0$, $|(\partial^M/\partial\xi^M)u(r,\xi)|$ is bounded away from zero for large r and small $|\xi|$. By elementary estimates (see Proposition 4.3) this implies

$$\left(\int_{\xi^{(1)}(r)}^{\xi^{(2)}(r)} u^2(r,\xi)\,d\xi\right)^{1/2} \geqq c_M|\xi^{(1)}(r)-\xi^{(2)}(r)|^{M+1/2} = c_M|S(r)|^{M+1/2}.$$

But $\lambda^2(r)=c|S(r)|^{-2}$ (with some $c>0$) for the 2-dimensional case, hence inequality (3.10) follows with $2\gamma=M+1/2$.

Unfortunately for the n-dimensional case various complications arise. In order to make the procedure of the proof more transparent it will be convenient to treat first the special case that the nodal domain D_{r_0} under consideration shrinks into a point for $r\to\infty$, i.e.

$$\exists\bar{y}\in S^{n-1}\quad\text{such that}\quad|y(r)-\bar{y}|\to 0\quad\text{for}\quad r\to\infty\quad\forall y(r)\in S(r). \tag{4.6}$$

Without loss we shall take $\bar{y}=e_{n-1}$. By $B_\varepsilon(e_{n-1})$ we denote the geodesic disc in S^{n-1} with center e_{n-1} and radius ε. Clearly, due to assumption (4.6) we have for $\varepsilon>0$ and $R(\varepsilon)$ large enough, $S(r)\subset B_\varepsilon(e_{n-1})$, $\forall r\geqq R(\varepsilon)$. Using the local coordinates

$\xi=\Phi(y), y\in B_\varepsilon(e_{n-1})$ introduced in Sect. 2, we have $\Phi(e_{n-1})=0$, the Laplace Beltrami operator reads

$$-L^2=\frac{\partial^2}{\partial\xi_1^2}-(n-2)\tan\xi_1\frac{\partial}{\partial\xi_1}+\sum_{j=1}^{n-2}\left(\prod_{i=1}^{j}\cos\xi_i\right)^{-2} \cdot\left(\frac{\partial^2}{\partial\xi_{j+1}^2}-(n-j-2)\tan\xi_{j+1}\frac{\partial}{\partial\xi_{j+1}}\right), \tag{4.7}$$

and $\forall\varphi\in W_0^{1,2}(B_\varepsilon(e_{n-1}))$,

$$\int_{S^{n-1}}|L\varphi|^2d\sigma=\int_{\Phi(B_\varepsilon(e_{n-1}))}\left(\left|\frac{\partial\varphi\circ\Phi^{-1}}{\partial\xi_1}\right|^2+\sum_{j=1}^{n-2}\prod_{i=1}^{j}\cos\xi_i^{-2}\left|\frac{\partial\varphi\circ\Phi^{-1}}{\partial\xi_{j+1}}\right|^2\right) \cdot\prod_{i=1}^{n-2}\cos\xi_i^{n-i-1}d\xi. \tag{4.8}$$

Since $|\xi|<c(\varepsilon)$, $\forall\xi\in\Phi(B_\varepsilon(e_{n-1}))$ with $c(\varepsilon)\to 0$ for $\varepsilon\to 0$ we obtain

$$\lambda^2(r)\geqq(1-c_0(\varepsilon))\inf_{f\in W_0^{1,2}(\Phi(S(r)))}\frac{\int|\nabla f|^2d\xi}{\int|f|^2d\xi} \tag{4.9}$$

$\forall r\geqq R(\varepsilon)$ and some $c_0(\varepsilon)$ small for ε small.

Next we apply a result of Davies [9, 10], namely:

Proposition 4.1. *Let $\mathscr{G}$ be a domain in $\mathbb{R}^{n-1}$, let*

$$d(\xi,e)\equiv\inf\{|t|:\xi+te\notin\mathscr{G}\}\quad\forall\xi\in\mathscr{G}\quad\text{and}\quad\forall e\in S^{n-2},$$

and

$$q(\xi)^{-2}\equiv\int_{S^{n-2}}d(\xi,e)^{-2}d\mu(e) \tag{4.10}$$

with $\mu(e)$ denoting the usual normalized invariant measure on S^{n-2}. Then

$$\inf_{f\in W_0^{1,2}(\mathscr{G})}(\|\nabla f\|_2^2/\|f\|_2^2)\geqq\frac{n-1}{4}\|q\|_\infty^{-2}, \tag{4.11}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm on $\mathscr{G}$.

Taking $\mathscr{G}=\Phi(S(r))$ and applying inequality (4.11) to (4.9) we obtain

Lemma 4.1. *For some $c(n,\varepsilon)>0$ and $\bar{R}$ large enough*

$$\lambda(r)\geqq c(n,\varepsilon)q_\infty(r)^{-1}\quad\forall r\geqq\bar{R}, \tag{4.12}$$

where

$q_\infty(r)\equiv\|q\|_\infty$ and $\|\cdot\|_\infty$ denotes the L^∞-norm on $\Phi(S(r))$.

Lemma 4.1 and the following lemma imply Lemma 3.2 under the assumption (4.6).

Lemma 4.2. *There exist $M,c>0$ such that $\forall r\geqq\bar{R}$ ($\bar{R}$ large enough),*

$$\frac{\psi_0}{v}(r)\geqq cq_\infty(r)^{M+(n-1)/2}. \tag{4.13}$$

To prove Lemma 4.2 we start with

Proposition 4.2. *Let R large enough and $\delta_0, \nu > 0$ small. Then $\forall r \geqq \bar{R}$ there exists $\bar{\xi}(r) \in \Phi(S(r))$ and $I_{\nu,r} \subset S^{n-2}$ with $|I_{\nu,r}| \geqq 1 - \nu^2(1+\delta_0)^2$ such that*

$$\nu^2 q_\infty^2(r) \leqq d^2(\bar{\xi}(r), e) \quad \forall e \in I_{\nu,r}. \tag{4.14}$$

Thereby $d(\xi, e), q_\infty(r)$ are given according to (4.10) and (4.12) and $|I_{\nu,r}| \equiv \int_{I_{\nu,r}} d\mu$.

Proof of Proposition 4.2. Obviously for $\delta_0 > 0$ small, there exists $\bar{\xi}(r) \in \Phi(S(r))$ for $r \geqq \bar{R}$ with $q(\bar{\xi}(r)) \geqq (1+\delta_0)^{-1} q_\infty(r)$. Now let for small $\nu = 0$

$$B_{\nu,r} = \{e \in S^{n-2} \mid d^2(\bar{\xi}(r), e) < \nu^2 q_\infty^2(r)\},$$

then

$$|B_{\nu,r}| \leqq \nu^2 q_\infty(r)^2 \int_{B_{\nu,r}} d^{-2}(\bar{\xi}(r), e) d\mu(e) \leqq \nu^2 q_\infty(r)^2 q^{-2}(\bar{\xi}(r)) \leqq \nu^2(1+\delta_0)^2.$$

Define $I_{\nu,r} = S^{n-2} \backslash B_{\nu,r}$, then $|I_{\nu,r}| \geqq 1 - \nu^2(1+\delta_0)^2$ and $\forall e \in I_{\nu,r}$ (4.14) holds. □

Remark 4.1. Now consider for $r \geqq \bar{R}$ and small $\nu > 0$

$$I'_{\nu,r} \subset I_{\nu,r} \quad \text{with} \quad |I'_{\nu,r}| \geqq c_0(\nu) \quad \text{for some} \quad c_0(\nu) > 0, \tag{4.15}$$

and define

$$\mathscr{C}_{\nu,r} = \{\xi \in \mathbb{R}^{n-1} \mid \xi = \bar{\xi}(r) + te, t \in [0, \nu q_\infty(r)], e \in I'_{\nu,r}\}, \tag{4.16}$$

where $\bar{\xi}(r)$ is given according to Proposition 4.2. Then due to (4.14) $\Phi^{-1}(\mathscr{C}_{\nu,r}) \subset S(r)$ and

$$|S(r)| \geqq c(\varepsilon) \int_{\mathscr{C}_{\nu,r}} d\xi \geqq c(n, \nu, \varepsilon) q_\infty(r)^{n-1} \quad \forall r \geqq \bar{R} \tag{4.17}$$

for some constants $c(\varepsilon)$, $c(n, \nu, \varepsilon) > 0$. Since $|S(r)| \to 0$ for $r \to \infty$, (4.17) implies

$$q_\infty(r) \to 0 \quad \text{for} \quad r \to \infty. \tag{4.18}$$

Now we are ready to prove Lemma 4.2: Due to our assumption (4.6) $\forall y \in S(r)$, $|y(r) - e_{n-1}| \to 0$ for $r \to \infty$ and $A(e_{n-1}) = 0$. Since A is real analytic (due to Theorem A) there exists $M \in \mathbb{N}$ such that

$$A \circ \Phi^{-1}(\xi) = P_M(\xi) + O(|\xi|^{M+1}) \quad \text{for } |\xi| \text{ small}, \tag{4.19}$$

where P_M is a homogeneous polynomial of degree M. Denoting $\xi = \rho e$ with $e = \xi/|\xi|$ it follows that

$$\frac{\partial^M}{\partial \rho^M} A \circ \Phi^{-1}(\rho e) = M! P_M(e) + O(\rho) \quad \text{for } \rho \text{ small}. \tag{4.20}$$

Since $P_M(e) \neq 0$ for a.e. $e \in S^{n-2}$, and since $q_\infty(r) \to 0$ for $r \to \infty$ (due to (4.18)) there exists some $\varepsilon' > 0$ and $J_{\varepsilon'} \subset S^{n-2}$ with $|J_{\varepsilon'}| \geqq 1 - c(\varepsilon') > 0$ such that

$$\operatorname{sgn}\left(\frac{\partial^M}{\partial \rho^M} A \circ \Phi^{-1}(\rho e)\right) = \text{const in } J_{\varepsilon'},$$

$$\left|\frac{\partial^M}{\partial \rho^M} A \circ \Phi^{-1}(\rho e)\right| > \varepsilon' \quad \forall e \in J_{\varepsilon'}. \tag{4.21}$$

Having in mind Proposition 4.2, where the existence of $I_{\nu,r} \subset S^{n-2}$ with $|I_{\nu,r}| \geqq 1 - \nu^2(1+\delta_0)^2$ and inequality (4.14) was shown, we can choose $\nu > 0$ small such that $|I_{\nu,r} \cap J_{\varepsilon'}| \geqq c_0 > 0$, $\forall r \geqq \bar{R}$ large enough. Hence we can define $\mathscr{C}_{\nu,r}$ as in (4.16) by taking

$$I'_{\nu,r} = I_{\nu,r} \cap J_{\varepsilon'}. \tag{4.22}$$

Now let $\bar{\xi}(r)$ be given according to Proposition 4.2 and define

$$g(r,t) \equiv \int_{I'_{\nu,r}} u(r\Phi^{-1}(\bar{\xi}(r) + te))\,d\mu(e), \tag{4.23}$$

then obviously $\forall r \geqq \bar{R}$

$$g(r,t) \geqq 0 \quad \forall t \in [0, \nu q_\infty(r)]. \tag{4.24}$$

Noting that for some $c(n,\nu) > 0$,

$$|\mathscr{C}_{\nu,r}| \leqq c(n,\nu) q_\infty(r)^{n-1} \quad \forall r \geqq \bar{R}, \tag{4.25}$$

we conclude by Cauchy–Schwarz that

$$\left(\int_{\mathscr{C}_{\nu,r}} u^2(r\Phi^{-1}(\xi))\,d\xi \right)^{1/2} \geqq c(n,\nu) q_\infty(r)^{-(n-1)/2} \int_0^{\nu q_\infty(r)} g(r,t) t^{n-2}\,dt \tag{4.26}$$

for some $c(n,\nu) > 0$. Next we shall bound the right-hand side of (4.26) from below by $(\partial^M/\partial\rho^M) A \circ \Phi^{-1}(\rho e)$, using

Proposition 4.3. *Let $J \subset \mathbb{R}$ be a bounded interval, $M \geqq 1$, $f \in C^M(J)$ and f bounded in $\bar{J}$. Let $\{J_i, 1 \leqq i \leqq 3^M\}$ denote a partition of J in intervals J_i with $|J_i| = |J| 3^{-M}$ and $J = \bigcup_{i=1}^{3M} J_i$, then*

$$\inf_{t \in J} \left| \frac{d^M}{dt^M} f(t) \right| \leqq 3^{\sum_{j=1}^{M} j} |J|^{-M} \sum_{i=1}^{3^M} \inf_{t \in J_i} |f(t)|. \tag{4.27}$$

Proof of Proposition 4.3. For $M = 1$ inequality (4.27) reads

$$\inf_{t \in J} |f'(t)| \leqq \frac{3}{|J|} \sum_{i=1}^{3} \inf_{t \in J_i} |f(t)|,$$

which is easily verified. For arbitrary $M \in \mathbb{N}$ (4.27) can be easily shown by induction. See for instance [19, p. 140] for a related inequality. □

Now we choose a partition $\{J_i, 1 \leqq i \leqq 3^M\}$ of $[0, \nu q_\infty(r)]$ with $|J_i| = \nu q_\infty(r) 3^{-M}$, and conclude via the above proposition that $\forall r \geqq \bar{R}$,

$$\begin{aligned} \int_0^{\nu q_\infty(r)} g(r,t) t^{n-2}\,dt &\geqq \sum_{i=1}^{3^M} \inf_{t \in J_i} g(r,t) \int_{J_i} t^{n-2}\,dt \\ &\geqq c' q_\infty(r)^{n-1} \sum_{i=1}^{3^M} \inf_{t \in J_i} g(r,t) \\ &\geqq c q_\infty(r)^{M+n-1} \inf_{t \in [0, \nu q_\infty(r)]} \left| \frac{\partial^M}{\partial t^M} g(r,t) \right| \end{aligned} \tag{4.28}$$

for some $c = c(M, n, \nu) > 0$. Nothing that

$$\frac{\partial^M}{\partial t^M} g(r,t) = \int_{I'_{\nu,r}} \frac{\partial^M}{\partial t^M} u(r\Phi^{-1}(\bar{\xi}(r) + te))d\mu(e), \tag{4.29}$$

and taking into account Theorem A, we have for some $\delta > 0$ and $c(M) > 0$,

$$\left| \frac{\partial^M}{\partial t^M} (u(r\Phi^{-1}(\bar{\xi}(r) + te)) - A \circ \Phi^{-1}(\bar{\xi}(r) + te)) \right| \leqq c(M) r^{-1/2-\delta} \tag{4.30}$$

$\forall t \in [0, \nu q_\infty(r)]$, $\forall e \in S^{n-2}, \forall r \geqq \bar{R}$ ($\bar{R}$ large enough).

Furthermore, since due to our assumption (4.6) $\Phi^{-1}(\bar{\xi}(r)) \to e_{n-1}$ for $r \to \infty$, $\bar{\xi}(r) \to 0$ for $r \to \infty$ and therefore $\forall t \in [0, \nu q_\infty(r)]$,

$$\left| \frac{\partial^M}{\partial t^M} (A \circ \Phi^{-1}(\bar{\xi}(r) + te) - A \circ \Phi^{-1}(te)) \right| \leqq \delta_1(r) \tag{4.31}$$

with $\delta_1(r) \to 0$ for $r \to \infty$.

Collecting the above findings (4.29), (4.30) and (4.31) we arrive at

$$\frac{\partial^M}{\partial t^M} g(r,t) = \int_{I'_{\nu,r}} \frac{\partial^M}{\partial t^M} (A \circ \Phi^{-1})(te) d\mu(e) + \delta_2(r) \tag{4.32}$$

$\forall t \in [0, \nu q_\infty(r)]$, with $\delta_2(r) \to 0$ for $r \to \infty$.

Now we take into account (4.21) and obtain from (4.32) for some $0 < c < \varepsilon'$, $\forall r \geqq \bar{R}$ with $\bar{R}$ sufficiently large

$$\left| \frac{\partial^M}{\partial t^M} g(r,t) \right| \geqq c \quad \forall t \in [0, \nu q_\infty(r)]. \tag{4.33}$$

Combining (4.26), (4.28) and (4.33) we get

$$\left(\int_{\mathscr{C}_{\nu,r}} u^2(r\Phi^{-1}(\xi))d\xi \right)^{1/2} \geqq c(M,n,\nu,\varepsilon) q_\infty(r)^{M+(n-1)/2} \quad \forall r \geqq \bar{R}. \tag{4.34}$$

Having in mind (4.16), where $\mathscr{C}_{\nu,r}$ is defined, the foregoing obviously implies (4.13). Hence we have proven Lemma 4.2.

Combination of this result with Lemma 4.1 verifies Lemma 3.2 under the assumption (4.6) with $2\gamma = M + (n-1)/2$.

So finally we have to verify Lemma 3.2 for the general case, when $|S(r)| \to 0$ and loosely speaking "D_{r_0} does not shrink into a single point": Let

$$\mathscr{A}_0 = \{ y \in S^{n-1} \mid y = \lim_{r \to \infty} y(r) \quad \text{for some } y(r) \in S(r) \},$$

then due to Theorem A, $A(y) = 0$, $\forall y \in \mathscr{A}_0$ and $|\mathscr{A}_0| = 0$ since A is real analytic.

In order to proceed in an analogous manner as before we need

Proposition 4.4 *Given a geodesic disc B_δ on S^{n-1} with $|B_\delta| = \delta$ and $\delta > 0$ small, then $\forall r \geqq \bar{R}$ ($\bar{R}$ large enough) there exist $\mathscr{R}(r) \in O(n)$ (the rotation group of S^{n-1}) such that for some $c > 0$,*

$$\lambda^2(r) \geqq c\lambda^2(G(r)), \tag{4.35}$$

where $\lambda^2(G(r))$ is defined analogously to (3.2) with $G(r) \equiv B_{\delta,\mathcal{R}(r)} \cap S(r) \neq \phi$. $B_{\delta,\mathcal{R}(r)}$ denoting the geodesic disc obtained from B_δ by the rotation $\mathcal{R}(r)$.

Proof of Proposition 4.4 Applying a result of Lieb [18] it follows that there exist $\mathcal{R}(r) \in O(n)$ such that $\lambda^2(r) > \lambda^2(G(r)) - \lambda^2(B_\delta)$. Since $|S(r)| \to 0$ for $r \to \infty$, $|G(r)| \to 0$ for $r \to \infty$, and we conclude by Faber–Krahn's inequality (see e.g. [4, 7]) that $\lambda^2(G(r)) \to \infty$ for $r \to \infty$. Hence the above inequality immediately implies (4.35). □

Now we consider the geodesic disc $B_\varepsilon(e_{n-1})$, define the local coordinates $\xi = \Phi(y)$ as before and denote $U_0 = \Phi^{-1}(B_\varepsilon(e_{n-1}))$. Then by suitable rotations we obtain from the chart (U_0, Φ), charts (U_i, Φ_i), $1 \leqq i \leqq N$ for some $N = N(\varepsilon) < \infty$ such that

$$\bigcup_{r \geqq \bar{R}} \overline{G(r)} \subset \bigcup_{i=0}^{N} U_i.$$

Thereby $\bar{R}$ is taken large enough, such that $\forall r \geqq \bar{R}$, $\overline{G(r)} \subset U_i$ for some $i \in \{0, 1, \ldots, N\}$ (having in mind that $G(r) = B_{\delta,\mathcal{R}(r)} \cap S(r)$, this is possible provided δ is chosen sufficiently small relatively to ε). So for each i for which $\overline{G(r)} \subset U_i$ we can apply Proposition 4.1 with $\mathcal{G} = \Phi_i(G(r))$. Thereby $d(\xi, e)$, $q(\xi)$ and $q_\infty(r)$ clearly depend on i (which will be suppressed in the notation for simplicity). Further analogous to Proposition 4.2, there exist $\bar{\xi}(r) \in \Phi_i(G(r))$ and $I_{v,r}$ such that (4.14) holds.

Consider now sequences $\{r_m\}$ with $r_m \to \infty$ for $m \to \infty$ with the following properties:

There exist $i \in \{0, 1, \ldots, N\}$ such that $\overline{G(r_m)} \subset U_i \forall m$. Further $\lim_{r \to \infty} \bar{\xi}(r_m) \equiv \bar{\xi}$ exists (where $\bar{\xi}(r_m)$ is defined as before). (4.36)

If we can show that for some $M \in \mathbb{N}$,

$$\liminf_{m \to \infty} \frac{\psi_0}{v}(r_m) \lambda(r_m)^{M + (n-1)/2} > 0$$

for every sequence $\{r_m\}$ with property (4.36), (3.10′)

then Lemma 3.2 is proven with $2\gamma = M + (n-1)/2$: For assume indirectly that $\forall K \in \mathbb{N}$ there is a sequence $R_m^{(K)} \to \infty$ such that

$$\frac{\psi_0}{v}(R_m^{(K)}) \lambda(R_m^{(K)})^{K + (n-1)/2} \to 0 \quad \text{for} \quad m \to \infty,$$

then clearly $\forall K$ we can pick a subsequence $\{\bar{R}_m^{(K)}\}$ of $\{R_m^{(K)}\}$ which has the property (4.36). But due to (3.10′) there exist $M \in \mathbb{N}$ such that $\forall K \in \mathbb{N}$

$$\liminf_{m \to \infty} \frac{\psi_0}{v}(\bar{R}_m^{(K)}) \lambda(\bar{R}_m^{(K)})^{M + (n-1)/2} > 0.$$

Hence for $K = M$ we have a contradiction.

Therefore it suffices to prove inequality (3.10′): So let $r_m \to \infty$ be arbitrary but fixed with property (4.36) and without loss we shall assume that $\overline{G(r_m)} \subset U_0$, $\forall m$. Note that $\bar{\xi}(r_m) \in G(r_m)$, $\forall m$ implies that $\bar{\xi} \in U_0$ and $A \circ \Phi^{-1}(\bar{\xi}) = 0$. It is obvious that in the same way as before by applying Proposition 4.1 with $\mathscr{G} = \Phi(G(r_m))$ the analog to Lemma 4.1 follows, namely,

Lemma 4.1′. *For some* $c(n,\varepsilon) > 0$,

$$\lambda(G(r_m)) \geqq c(n,\varepsilon) q_\infty(r_m)^{-1} \quad \text{for } m \text{ large},$$

where $\{r_m\}$ *is an arbitrary sequence with property (4.36) and* $i=0$ *in (4.36).*

Next we are going to derive the analog to Lemma 4.2:

Lemma 4.2′. *There exist* $M, c > 0$ *such that for large* m

$$\frac{\psi_0}{v}(r_m) \geqq c q_\infty(r_m)^{M+(n-1)/2},$$

where $\{r_m\}$ *is an arbitrary sequence with property (4.36) and* $i=0$ *in (4.36).*

Proof of Lemma 4.2′. Basically we proceed as in the proof of Lemma 4.2. As already noted Proposition 4.2 remains valid with $S(r)$ replaced by $G(r_m)$, so that I_{v,r_m} is given in the same manner as before. Further, in the same way as in Remark 4.1, I'_{v,r_m} and $\mathscr{C}_{v,r_m}$ are defined and we obtain analogously

$$|G(r_m)| \geqq c(n,v) q_\infty(r_m)^{n-1} \quad \text{for large } m. \tag{4.37}$$

Further we shall derive an analog to (4.21), namely:

Proposition 4.5. *Let* $\{r_m\}$ *be a sequence with property (4.36), where* $i=0$ *in (4.36), let* I_{v,r_m} *be given according to Proposition 4.2 and let* M *denote the highest order of the zeros of* A *as in (3.9). Then* $\forall \bar{\xi} \in \Phi(U_0 \cap \mathscr{A}_0)$ *there exist* $\bar{l}(\bar{\xi}) \equiv \bar{l}$ *with* $\bar{l} \in \{1, 2, \ldots, M\}$, *and there exist for* m *large*

$$I'_{v,r_m}(\bar{\xi}) \subset I_{v,r_m} \quad \text{with} \quad |I'_{v,r_m}(\bar{\xi})| \geqq c_0(v) > 0$$

such that

$$\inf_{\bar{\xi} \in \Phi(\mathscr{A}_0 \cap U_0)} \left| \int_{I'_{v,r_m}(\bar{\xi})} \left. \frac{\partial^{\bar{l}}}{\partial t^{\bar{l}}} A \circ \Phi^{-1}(\bar{\xi} + \rho e) \right|_{\rho=0} d\mu(e) \right| \geqq c(v) > 0. \tag{4.38}$$

Proof of Proposition 4.5. Since A is real analytic and $A \circ \Phi^{-1}(\bar{\xi}) = 0$ $\forall \bar{\xi} \in \Phi(U_0 \cap \mathscr{A}_0)$, we have

$$A \circ \Phi^{-1}(\bar{\xi} + \rho e) = \sum_{k=0}^{M} a_{\bar{\xi},k} \rho^k P_{\bar{\xi},k}(e) + O(\rho^{M+1}) \quad \text{for} \quad \rho \to 0$$

with $a_{\bar{\xi},k} \in \mathbb{R}$, and where $P_{\bar{\xi},k}(e)$ is a homogeneous polynomial of degree k. For every homogeneous polynomial $P(e)$ of degree $\leqq M$ define

$$\mathscr{M}_{\pm}(P) = \{e \in S^{n-2} \mid P(e) \gtrless 0\}$$

and

$$\mathscr{M}(P) = \begin{cases} \mathscr{M}_+ & \text{if } |\mathscr{M}_+| > |\mathscr{M}_-| \\ \mathscr{M}_- & \text{otherwise} \end{cases}.$$

Then clearly $|\mathscr{M}(P)| \geqq 1/2$. Without loss we assume that for $\varepsilon > 0$ small enough there exist $\mathscr{M}_\varepsilon(P) \subset \mathscr{M}(P)$ with $|P| > \varepsilon$, $\forall e \in \mathscr{M}_\varepsilon(P)$ and $|\mathscr{M}_\varepsilon(P)| \geqq 1/4$. Since $|I_{v,r_m}| \geqq 1 - v^2$ with v arbitrarily small, it follows that for large m

$$|\mathscr{M}_\varepsilon(P)\cap I_{v,r_m}| \geqq c(v,\varepsilon) > 0, \tag{4.39}$$

and therefore

$$\left| \int_{\mathscr{M}(P)\cap I_{v,r_m}} P(e)d\mu(e) \right| \geqq \varepsilon c(v,\varepsilon) > 0. \tag{4.40}$$

Define for $K \in \mathbb{N}$

$$\mathscr{N}_K = \{\xi \in \Phi(U_0 \cap \mathscr{A}_0) | A \circ \Phi^{-1}(\xi) = 0, \xi \text{ is a zero of order } K\}.$$

It is easily seen that $\mathscr{N}_M$ is a closed set, which implies that

$$\inf_{\xi\in\mathscr{N}_M} |a_{\xi,M}| \equiv k_M > 0. \tag{4.41}$$

Having in mind the foregoing considerations this yields that for large m

$$\inf_{\bar\xi\in\mathscr{N}_M} \left| \int_{I'_{v,r_m}(\bar\xi)} \frac{\partial^M}{\partial\rho^M} A \circ \Phi^{-1}(\bar\xi + \rho e)|_{\rho=0} d\mu(e) \right| \geqq M! k_M c(v,\varepsilon) \tag{4.42}$$

with

$$I'_{v,r_m}(\bar\xi) = \mathscr{M}_\varepsilon(P_{\bar\xi,M}) \cap I_{v,r_m}.$$

Next we show that for $j = 0, 1, \ldots, M$,

$$\inf_{\bar\xi\in\mathscr{N}_{M-j}} \sup(|a_{\bar\xi,M-j}|, |a_{\bar\xi,M-j+1}|, \ldots, |a_{\bar\xi,M}|) \equiv k_{M-j} > 0. \tag{4.43}$$

For $j = 0$ this is given in (4.41).

Assume indirectly that for some $j \geqq 1, k_{M-j} = 0$. Then clearly there exists a converging sequence $\xi_m \in \mathscr{N}_{M-j}$, $m \in \mathbb{N}$ with $|a_{\xi_m,M-i}| \to 0$ for $m \to \infty$ $\forall 0 \leqq i \leqq j$. Let $\lim_{m\to\infty} \xi_m = \bar\xi$, then obviously $\bar\xi$ is a zero of $A \circ \Phi^{-1}$ of order $\geqq M - j$ and therefore

$$\sup(|a_{\bar\xi,M-j}|, \ldots, |a_{\bar\xi,M}|) > 0. \tag{4.44}$$

But since $\forall l \geqq 0$,

$$\left| \frac{\partial^l}{\partial\rho^l} (A \circ \Phi^{-1}(\xi_m + \rho e) - A \circ \Phi^{-1}(\bar\xi + \rho e))|_{\rho=0}) \right| \to 0$$

for $m \to \infty$, it follows that $|a_{\xi_m,M-i} - a_{\bar\xi,M-i}| \to 0$ for $m \to \infty$ $\forall i$, and therefore $a_{\bar\xi,M-i} = 0$ for all i with $0 \leqq i \leqq j$, which is a contradiction to (4.44). Hence (4.43) is verified.

From the foregoing we conclude that $\forall \bar\xi \in \mathscr{N}_{M-j}$ there exist $l(\bar\xi) = \bar l$ with $M - j \leqq \bar l \leqq M$, such that $|a_{\bar\xi,\bar l}| \geqq k_{M-j}$, and for large m

$$\left| \int_{I'_{v,r_m}(\bar\xi)} \frac{\partial^{\bar l}}{\partial\rho^{\bar l}} A \circ \Phi^{-1}(\bar\xi + \rho e)|_{\rho=0} d\mu(e) \right| \geqq \bar l! k_{M-j} c(v,\varepsilon),$$

where $I'_{v,r_m}(\bar\xi) = \mathscr{M}_\varepsilon(P_{\bar\xi,\bar l}) \cap I_{v,r_m}$. Taking the infimum over $\bar\xi \in \mathscr{N}_{M-j}$ and the minimum over $1 \leqq j \leqq M - 1$ we obtain finally inequality (4.38), and $I'_{v,r_m}(\bar\xi)$ shows the desired property due to (4.39). This verifies Proposition 4.5. □

Now we are ready to finish the proof of Lemma 4.2′: We recall that we consider an arbitrary but fixed sequence $r_m \to \infty$ having property (4.36), where $i = 0$ in (4.36) and $|\bar{\xi}(r_m)-\bar{\xi}|\to 0$ for $m\to\infty$. Thereby (compare (4.14)) $v^2 q_\infty^2(r_m)\leqq d^2(\bar{\xi}(r_m),e)$ $\forall e\in I_{v,r_m}$, $\forall m$. According to Proposition 4.5 there exist $\bar{l}(\bar{\xi})=\bar{l}$ and $I'_{v,r_m}(\bar{\xi})$ such that (4.38) holds. Now we define analogously to (4.23)

$$g(r_m,t)=\int_{I'_{v,r_m}(\bar{\xi})} u(r\Phi^{-1}(\bar{\xi}(r_m)+te))d\mu(e), \tag{4.23'}$$

and clearly the analogs to (4.24), (4.25) and (4.26) hold. So we have

$$\left(\int_{\mathscr{C}_{v,r_m}} u^2(r_m\Phi^{-1}(\xi))d\xi\right)^{1/2}\geqq cq_\infty(r)^{-(n-1)/2}\int_0^{vq_\infty(r_m)} g(r_m,t)t^{n-2}dt. \tag{4.26'}$$

Applying Proposition 4.3 to $(\partial^{\bar{l}}/\partial t^{\bar{l}})g(r_m,t)$ we obtain analogously to (4.28),

$$\int_0^{vq_\infty(r_m)} g(r_m,t)t^{n-2}dt\geqq cq_\infty(r_m)^{\bar{l}+n-1}\inf_{t\in[0,vq_\infty(r_m)]}\left|\frac{\partial^{\bar{l}}}{\partial t^{\bar{l}}}g(r_m,t)\right| \tag{4.28'}$$

with $c=c(M,n,v)$. The analogs to (4.29) and (4.30) are evident, and instead of (4.31) we have $\forall t\in[0,vq_\infty(r_m)]$,

$$\left|\frac{\partial^{\bar{l}}}{\partial t^{\bar{l}}}(A\circ\Phi^{-1}(\bar{\xi}(r_m)+te)-A\circ\Phi^{-1}(\bar{\xi}+te))\right|\leqq\delta_1(r_m) \tag{4.31'}$$

with $\delta_1(r_m)\to 0$ for $m\to\infty$. Combining the above considerations we obtain the analog to (4.32),

$$\frac{\partial^{\bar{l}}}{\partial t^{\bar{l}}}g(r_m,t)=\int_{I'_{v,r_m}(\bar{\xi})}\frac{\partial^{\bar{l}}}{\partial t^{\bar{l}}}(A\circ\Phi^{-1})(\bar{\xi}+te)d\mu(e)+\delta_2(r_m) \tag{4.32'}$$

$\forall t\in[0,vq_\infty(r_m)]$ with $\delta_2(r_m)\to 0$ for $m\to\infty$.

Now we take into account (4.38) and obtain for m large enough

$$\left|\frac{\partial^{\bar{l}}}{\partial t^{\bar{l}}}g(r_m,t)\right|\geqq c>0\quad\forall t\in[0,vq_\infty(r_m)]. \tag{4.33'}$$

Combining (4.33′) with (4.28′) and (4.26′), and having in mind the definition of $\mathscr{C}_{v,r_m}$, we get

$$\left(\int_{S(r_m)} u^2 d\sigma\right)^{1/2}\geqq c(M,n,v,\varepsilon)q_\infty(r_m)^{\bar{l}+(n-1)/2}.$$

But $q_\infty(r_m)\to 0$ for $m\to\infty$ and $\bar{l}\leqq M$, which together with the above inequality verifies Lemma 4.2′. □

From the foregoing procedure it becomes clear that for the case $\overline{G(r_m)}\subset U_i,\forall m$ and $i\neq 0$ in (4.36), Lemmas 4.1′ and 4.2′ follow in the same way, whereby $q_\infty(r_m)$ is then given with $\mathscr{G}=\Phi_i(G(r_m))$ (compare (4.10)). Hence Lemmas 4.1′ and 4.2′ hold for any sequence $\{r_m\}$ with property (4.36). Combination of the two lemmas implies that inequality (3.10′) holds and, as already noted, this finishes the proof of Lemma 3.2. □

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