Commun. Math. Phys. 116, 529–538 (1988)

Communications in Mathematical Physics

Non-Integrability of the Truncated Toda Lattice Hamiltonian at Any Order

Haruo Yoshida*

Centre de Physique Théorique, Ecole Polytechnique, F-91128, Palaiseau, France

Abstract. Two sufficient conditions for the non-existence of an additional analytic integral are given for Hamiltonian systems with non-homogeneous polynomial potential of an arbitrary degree. An application is made to the truncated three-particle Toda lattice, which is proved to be non-integrable at any order.

1. Introduction and the Main Result

The so-called three-particle periodic Toda lattice [10] is defined by the Hamiltonian

$$H = (1/2)({p_x}^2 + {p_y}^2) + e^{\sqrt{3}x+y} + e^{-\sqrt{3}x+y} + e^{-2y}, \tag{1.1}$$

and integrable because of the existence of a second analytic integral [4, 6]

$$\Phi = (1/3)p_x({p_x}^2 - 3{p_y}^2) + (p_x - \sqrt{3}p_y)e^{\sqrt{3}x+y} + (p_x + \sqrt{3}p_y)e^{-\sqrt{3}x+y} - 2p_x e^{-2y}. \tag{1.2}$$

We now truncate the Taylor series of exponential functions in (1.1) at a finite order [3], i.e., consider the potential

$$V_N = \sum_{k=1}^{N} \{(\sqrt{3}x + y)^k + (-\sqrt{3}x + y)^k + (-2y)^k\}/k!. \tag{1.3}$$

V_2 is the harmonic oscillator and V_3 is identified with the so-called Hénon–Heiles potential [5],

$$V_3 = (1/2)(x^2 + y^2) + x^2 y - (1/3)y^3, \tag{1.4}$$

after a proper change of scale. In the process of proving the integrability of the Toda lattice [4, 6], the exponential function plays a crucial role. Therefore, by truncating the exponential function to a finite order polynomial, we can no more expect the existence of an additional integral like (1.2). Indeed the main result of this paper is to prove

* Present address: Department of Mathematics, Imperial College of Science and Technology, 180 Queen's Gate, London SW7 2BZ, United Kingdom

Theorem 1.1. *The truncated Toda lattice Hamiltonian with potential V_N cannot have an additional analytic integral $\Phi=$ const. when $N \geqq 3$.*

As a powerful method to prove the non-existence of an additional analytic integral for a given system, Ziglin's theorem [17] was applied to (i) homogeneous potentials [12, 13], (ii) some generalized Toda lattice [16], (iii) some perturbed Kepler potentials [14], and (iv) non-homogeneous polynomial potentials of degree 3 or 4, [7, 8, 9], in addition to Ziglin's original example, i.e., the motion of a rigid body around a fixed point [17].

In the case of systems with homogeneous potential, everything necessary for Ziglin's theorem to be applied (monodromy matrices) is given explicitly. The logic used in [7, 8, 9] to prove the non-integrability of a non-homogeneous potential strongly depends on the speciality of a polynomial potential of degree 3 and 4, and therefore it cannot be applied to arbitrary degree cases like (1.3).

Simple observation shows that when the value of the energy is very large, i.e. $h \to \infty$, the behavior of the system is dominated by the highest degree term of the potential. Also in the limit $h \to 0$, the lowest degree term dominates. Therefore at both limits we can make use of the knowledge on homogeneous potentials entirely. For example, the change of trace of the monodromy matrix as a function of energy [7, 9] can be seen by comparing these two limits if continuity is assumed.

This paper gives, on the basis of Ziglin's theorem, two sufficient conditions for the non-existence of an additional analytic integral for non-homogeneous polynomial potentials, which completely use the previous knowledge on the homogeneous potential case. Section 2 introduces Ziglin's theorem, i.e., necessary assumptions and the statement of the theorem. Known results on homogeneous potentials are reviewed in Sect. 3. The main technical theorems are given as Theorem 4.1 and Theorem 4.2 in Sect. 4. In Sect. 5, the truncated Toda lattice is proved to be non-integrable as direct applications of Theorem 4.1 and Theorem 4.2. There truncations of other integrable generalized Toda lattices are also considered.

2. Ziglin's Theorem: General

Consider the Hamiltonian system

$$H = (1/2)(p_x{}^2 + p_y{}^2) + V(x, y), \tag{2.1}$$

with an algebraic potential $V(x, y)$, and assume that there exists a straight-line solution of the form

$$x = c_1 \phi(t), \quad y = c_2 \phi(t), \tag{2.2}$$

with constants c_1 and c_2. By an orthogonal transformation of coordinates, it is always possible that the straight-line solution (2.2) is re-expressed as

$$x = 0, \quad y = \phi(t). \tag{2.3}$$

Then $\phi(t)$ satisfies the differential equation

$$d^2\phi/dt^2 + V_{,y}(0, \phi) = 0, \tag{2.4}$$

($V_{,y} = \partial V/\partial y$, etc.), with the integral

$$(1/2)(d\phi/dt)^2 + V(0, \phi) = h. \tag{2.5}$$

Therefore the function $\phi(t)$ is determined by the inverse function of

$$t = \int dw/\sqrt{P(w)} \tag{2.6}$$

with

$$P(w) = 2\{h - V(0, w)\}. \tag{2.7}$$

We now consider the normal variational equation (NVE). The variational equations for $\xi_1 = \delta x$ and $\xi_2 = \delta y$ decouple and by NVE (with respect to $x = 0$) we mean the equation for ξ_1, i.e.

$$d^2\xi_1/dt^2 + V_{,xx}(0, \phi(t))\xi_1 = 0. \tag{2.8}$$

A Riemann surface Γ is defined by the function

$$z = \sqrt{P(w)}, \tag{2.9}$$

with $P(w)$ given by (2.7). There are branch points at $w = w_1, w_2, w_3, \ldots$, the roots of $P(w) = 0$. Take an arbitrary closed circuit γ on the Riemann surface Γ. After this closed circuit, the value of t in (2.6) is increased by

$$T = \oint_\gamma dw/\sqrt{P(w)}. \tag{2.10}$$

If this value is not zero, T in (2.10) is considered a period of the function $\phi(t)$. For each closed circuit γ on Γ, we can associate a 2×2 matrix $g(\gamma)$, called the monodromy matrix, with the evolution of the fundamental set of solution of the NVE (2.8) along the circuit γ. The set of all monodromy matrices, which share a common base point of the closed circuit form a group G, called the monodromy group of the NVE.

We now define a monodromy matrix g to be non-resonant when the eigenvalues $(\rho, 1/\rho)$ are not roots of unity. Then we have the following necessary condition for the existence of an additional analytic integral:

Theorem 2.1 [*Ziglin, 17*]. *Suppose (2.1) has an additional integral $\Phi =$ const. which is holomorphic at least in the neighborhood of the given straight-line solution $x = p_x = 0$. Further, suppose there exists a non-resonant monodromy matrix g_1 in G. Then any monodromy matrix g_2 in G must have the property, that either* (i) *g_1 and g_2 commute, or* (ii) trace $g_2 = 0$.

A completely self-contained proof is given in [13] for the homogeneous potential case. This statement can be rephrased as a sufficient condition for the non-existence of an integral, i.e.,

Theorem 2.2. *If there exist two monodromy matrices g_1, g_2 in G which enjoy one of the following properties, then (2.1) cannot have an additional analytic integral.*

(i) *g_1, g_2 both non-resonant and $[g_1, g_2] \neq 0$ (non-commuting).*
(ii) *g_1: non-resonant, g_2:*trace *$g_2 \neq 0$, and $[g_1, g_2] \neq 0$.*
(iii) *g_1: non-resonant, g_2:*trace *$g_2 = \pm 2$ but non-diagonalizable.*

Condition (i) is useful for the non-integrability proof of homogeneous potentials (Sect. 3), while conditions (ii) and (iii) are used for non-homogeneous potentials in general.

3. Review of Known Results on Two Dimensional Homogeneous Potentials

When the potential $V(x, y)$ is homogeneous (degree k), we have always straight-line solutions (2.2). Indeed, suppose $\mathbf{c} = (c_1, c_2)$ is a solution of the algebraic equation

$$\mathbf{c} = \operatorname{grad} V(\mathbf{c}), \tag{3.1}$$

and suppose $\phi(t)$ satisfies the non-linear differential equation

$$d^2\phi/dt^2 + \phi^{k-1} = 0. \tag{3.2}$$

Then expression (2.2) is known to be a solution of the equations of motion. The NVE (2.8) has the expression [13]

$$d^2\xi/dt^2 + \lambda_k \phi(t)^{k-2}\xi = 0, \tag{3.3}$$

where ($\Delta V = V_{,xx} + V_{,yy}$, the Laplacian of V)

$$\lambda_k = \Delta V(c_1, c_2) - (k-1) \tag{3.4}$$

was called the Integrability Coefficient (IC) [13].

In homogeneous potentials there exists a scaling transformation which changes the value of the energy of a given straight-line solution to $h = 1/k$, provided the original energy is not zero. Then in (2.7),

$$P(w) = (2/k)(1 - w^k), \tag{3.5}$$

and the branch points of the Riemann surface Γ defined by $z = \sqrt{P(w)}$ are located at

$$w = 1, \omega, \omega^2, \omega^3, \ldots, \omega^{|k|-1}, \tag{3.6}$$

with $\omega = e^{2\pi i/|k|}$.

For an arbitrary integer k, we can define two fundamental closed circuits γ_1 and γ_2 as follows; γ_1: a counter-clockwise circuit which encircles two branch points $w = 1$ and $w = \omega$, and will be denoted by $(1 \to \omega \to 1)$. γ_2:$(1 \to \omega^{|k|-1} \to 1)$. A common base point w_0 is taken on the real w-axis. Then with the help of the Gauss hypergeometric equation, we know the explicit expression of two fundamental monodromy matrices $g(\gamma_1)$ and $g(\gamma_2)$ in the form [13]

$$g(\gamma_1) = \begin{bmatrix} 1 + \Omega AB, & B(2 + \Omega AB) \\ A(\Omega - 1 - \Omega AB), & 1 + (\Omega - 2)AB - \Omega(AB)^2 \end{bmatrix},$$

$$g(\gamma_2) = \begin{bmatrix} 1 + (2\Omega - 1)AB - \Omega(AB)^2, & \Omega B(2 - AB) \\ A(1 - 1/\Omega - AB), & 1 - AB \end{bmatrix}, \tag{3.7}$$

where

$$A = 1 - \Omega^{-1} e^{-2\pi i a}, \quad B = 1 - \Omega^{-1} e^{-2\pi i b}$$

and

$$\Omega = e^{2\pi i/k}, \quad a + b = 1/2 - 1/k, \quad ab = -\lambda_k/2k. \tag{3.8}$$

Indeed, since the NVE (3.3) is transformed to the Gauss equation by the change of the independent variable $z=\{\phi(t)\}^k$, $g(\gamma_1)$ and $g(\gamma_2)$ are obtained as products of basic monodromy matrices M_0, M_1 of the Gauss equation i.e.,

$$g(\gamma_1)=M_1M_0M_1M_0^{-1}, \quad g(\gamma_2)=M_0^{-1}M_1M_0M_1. \tag{3.9}$$

For the details, see [13].

Therefore, a simple computation shows that

$$\operatorname{trace} g(\gamma_1)=\operatorname{trace} g(\gamma_2)=E_k(\lambda_k), \tag{3.10}$$

where

$$E_k(\lambda_k)=2\cos(2\pi/k)+4\cos^2[(\pi/2k)\sqrt{\{(k-2)^2+8k\lambda_k\}}], \tag{3.11}$$

and we can confirm that $g(\gamma_1)$ and $g(\gamma_2)$ commute when and only when λ_k has values such that

$$E_k(\lambda_k)=2 \quad \text{or} \quad 2\cos(2\pi/k) \tag{3.12}$$

or $k=\pm 2$. By using condition (i) of Theorem 2.2 we get the following sufficient condition for the non-integrability:

Theorem 3.1. *[13] If* IC λ_k *is in the region* S_k *such that* $E_k(\lambda_k)>2$, *then there cannot exist an additional analytic integral* $\Phi=$ const. *The non-integrable region* S_k *can be computed as follows: (Note that when* $k=0, \pm 2$, *such regions are not defined.)*

(i) $k\geqq 3$: $S_k=\{\lambda<0, 1<\lambda<k-1, k+2<\lambda<3k-2, \ldots, j(j-1)k/2+j<\lambda<j(j+1)k/2-j, \ldots\}$,

(ii) $S_1=\mathbb{R}-\{0,1,3,6,10,\ldots,j(j+1)/2,\ldots\}$,

(iii) $S_{-1}=\mathbb{R}-\{1,0,-2,-5,-9,\ldots,-j(j+1)/2+1,\ldots\}$,

(iv) $k\leqq -3$:

$$S_k=\{\lambda>1, 0>\lambda>-|k|+2, -|k|-1>\lambda>-3|k|+3, -3|k|-2>\lambda >-6|k|+4, \ldots, -j(j-1)|k|/2-(j-1)>\lambda>-j(j+1)|k|/2+(j+1), \ldots\}.$$

In the case when k is an even integer $k=2m>0$, there is another naive way to define two closed circuits on Γ [12]. One is a counter-clockwise circuit encircling two branch points 1 and -1, which define a real period, thus $\gamma_{\text{real}}=(1\to -1\to 1)$. Another one is to define an imaginary period, $\gamma_{\text{imag}}=(\omega\to -\omega\to\omega)$. Here, however, we use the full closed circuit for which the monodromy matrices can be obtained as the square of those given in [12]. Thus

$$g(\gamma_{\text{real}})=\begin{bmatrix} -1, & -BC \\ A, & ABC-1 \end{bmatrix}^2,$$

$$g(\gamma_{\text{imag}})=\begin{bmatrix} -1-\Omega AB, & -B(ABC+\Omega AB+\Omega^{-1}C) \\ \Omega A, & ABC+\Omega AB-1 \end{bmatrix}^2, \tag{3.13}$$

with $C=2\Omega/(\Omega-1)$, and A, B, Ω are given in (3.8). Therefore

$$\operatorname{trace} g(\gamma_{\text{real}})=\operatorname{trace} g(\gamma_{\text{imag}})=F_{2m}(\lambda_{2m}), \tag{3.14}$$

where

$$F_{2m}(\lambda_{2m}) = 4\cos^2[(\pi/2m)\sqrt{\{(m-1)^2 + 4m\lambda_{2m}\}}]/\sin^2(\pi/2m) - 2, \qquad (3.15)$$

and $g(\gamma_{\rm real})$ and $g(\gamma_{\rm imag})$ commute only when $F_{2m}(\lambda_{2m}) = \pm 2$. From these expressions we get exactly the same non-integrability region S_{2m} as in Theorem 3.1.

4. Two Sufficient Conditions for the Non-existence of an Additional Analytic Integral for Non-homogeneous Potentials

When the potential is non-homogeneous, monodromy matrices depend on the value of the energy, i.e. $g(\gamma; h)$, and there exists no analytical method to express them explicitly. However the following consideration tells us that we can compute monodromy matrices in the limits $h \to 0$ and $h \to \infty$. For simplicity, take the case where the potential is composed of two different degree terms, k and K ($k < K$). The equation (2.4) for $\phi(t)$, can be written, after a proper change of scale, in the form

$$d^2\phi/dt^2 + \phi^{k-1} + \phi^{K-1} = 0 \qquad (4.1)$$

with the energy integral

$$(1/2)(d\phi/dt)^2 + (1/k)\phi^k + (1/K)\phi^K = h. \qquad (4.2)$$

In the same scale, the NVE (2.8) has the form

$$d^2\xi/dt^2 + (\lambda_k\phi^{k-2} + \lambda_K\phi^{K-2})\xi = 0. \qquad (4.3)$$

We now make a change of scale as

$$\phi \to h^{1/K}\phi, \quad t \to h^{(2-K)/(2K)}t. \qquad (4.4)$$

Then Eqs. (4.1), (4.2) and (4.3) are transformed to

$$d^2\phi/dt^2 + \mu\phi^{k-1} + \phi^{K-1} = 0, \qquad (4.5)$$

$$(1/2)(d\phi/dt)^2 + \mu(1/k)\phi^k + (1/K)\phi^K = h, \qquad (4.6)$$

$$d^2\xi/dt^2 + (\mu\lambda_k\phi^{k-2} + \lambda_K\phi^{K-2})\xi = 0, \qquad (4.7)$$

respectively, where

$$\mu = h^{k/K-1}. \qquad (4.8)$$

Therefore in the limit $h \to \infty$, the NVE(4.7) tends to the NVE for the homogeneous potential of degree K, since μ goes to 0. Similarly by the change of scale

$$\phi \to h^{1/k}\phi, \quad t \to h^{(2-k)/(2k)}t, \qquad (4.9)$$

we recover, in the limit $h \to 0$, the NVE for the homogeneous potential of degree k, for which we know the explicit expression of monodromy matrices. Note that the presence of intermediate degree terms does not change this situation at all.

Any non-homogeneous polynomial potential $V(x, y)$ is written as a finite sum of homogeneous parts:

$$V(x, y) = \sum_k V_k(x, y). \qquad (4.10)$$

Assume that $x = 0$ is a straight-line solution, i.e. $V_{,x}(0, y) = 0$. From the lowest part $V_{k\min}$ and highest part $V_{k\max}$ we can compute the $\lambda_{k\min}$ and $\lambda_{k\max}$, the lowest IC and the highest IC. The first sufficient condition for the non-existence of an integral is

Theorem 4.1. *If either $\lambda_{k\min} \in S_{k\min}$ or $\lambda_{k\max} \in S_{k\max}$ holds, then there cannot exist an additional analytic integral.*

This theorem means that if the lowest or the highest part of the potential is shown to be non-integrable, using Theorem 3.1, then the total system (4.10) is also non-integrable.

Proof. Suppose $\lambda_{k\max} \in S_{k\max}$ holds. We define two closed circuits γ_1 and γ_2 on Γ continuously as h varies such that those tend to $(1 \to \omega \to 1)$ and $(1 \to \omega^{k-1} \to 1)$ as defined in Sect. 3 in the limit $h \to \infty$. By the assumption that $\lambda_{k\max} \in S_{k\max}$, there exists an open interval of $h \in (h_0, \infty)$ in which

$$\operatorname{trace} g(\gamma_i; h) > 2 \quad (i = 1 \text{ and } 2) \tag{4.11}$$

hold. This means that in the interval (h_0, ∞), the matrices $g(\gamma_1; h)$ and $g(\gamma_2; h)$ are both non-resonant. Therefore in order to have integrability they must commute identically in the interval (h_0, ∞). However we know already that if $\lambda_{k\max} \in S_{k\max}$, then $g(\gamma_1; \infty)$ and $g(\gamma_2; \infty)$ do not commute by (3.12). The same consequence follows when $\lambda_{k\min} \in S_{k\min}$ is assumed. □

Next we consider the restricted cases when $k_{\min} = 2$, $k_{\max} = 2m$ and

$$V_{k\min} = (a^2 x^2 + b^2 y^2). \tag{4.12}$$

One further crucial assumption is that the real function $V(0, y)$ is convex, i.e., the only real root of $V_{,y}(0, y) = 0$ is $y = 0$. Under this assumption we can define the real period of $\phi(t)$ continuously from $h = 0$ to $h = \infty$. At both limits we know the explicit expression of trace $g(\gamma_{\text{real}}; h)$ and we could confirm, a priori, whether trace $g(\gamma_{\text{real}}; h)$ changes or not as a function of h, since it is a continuous function of $h \in (0, \infty)$. Let λ_{2m} be the highest IC in this case. Then the second sufficient condition is

Theorem 4.2. *If λ_{2m} is not a value such that $F_{2m}(\lambda_{2m}) = \{2, 0, -2, 2\cos(2\pi a/b)\}$, then the system cannot have an additional analytic integral.*

Proof. We define two closed circuits γ_{real} and γ_{imag} on Γ, so that γ_{real} encircles only two real roots of $P(w) = 0$ in (2.7) for all h, and γ_{imag} tends to $(\omega \to -\omega \to \omega)$, i.e., γ_{imag} of Sect. 3, in the limit $h \to \infty$. Then we know

$$\operatorname{trace} g(\gamma_{\text{real}}; 0) = 2\cos(2\pi a/b) \tag{4.13}$$

and

$$\operatorname{trace} g(\gamma_{\text{real}}; \infty) = \operatorname{trace} g(\gamma_{\text{imag}}; \infty) = F_{2m}(\lambda_{2m}). \tag{4.14}$$

Since trace $g(\gamma_{\text{real}}; h)$ is a continuous analytic function of h in the whole interval $h \in (0, \infty)$, it follows that trace $g(\gamma_{\text{real}}; h)$ actually changes as h varies if $F_{2m}(\lambda_{2m}) \neq 2\cos(2\pi a/b)$. Suppose $F_{2m}(\lambda_{2m}) \neq 2\cos(2\pi a/b)$ holds. Then in a dense subset of $h \in (0, \infty)$, $g(\gamma_{\text{real}}; h)$ becomes non-resonant. Therefore for the integrability, it is necessary that either

$$[g(\gamma_{\mathrm{real}}; h), g(\gamma_{\mathrm{imag}}; h)] = 0 \tag{4.15}$$

holds, or

$$\mathrm{trace}\, g(\gamma_{\mathrm{imag}}; h) = 0 \tag{4.16}$$

holds in the same dense set of h. Since (4.15) and (4.16) are analytic relations in h, if they hold in some dense set of h, they should hold identically for all $h \in (0, \infty)$. In order that (4.15) holds at $h \to \infty$, it is necessary that $F_{2m}(\lambda_{2m}) = \pm 2$. For (4.16) to hold as $h \to \infty$ we need $F_{2m}(\lambda_{2m}) = 0$. Therefore if $F_{2m}(\lambda_{2m}) \neq \{2, 0, -2, \cos(2\pi a/b)\}$ is assumed, the none-existence of an additional integral follows. □

5. Application to the Truncated Toda Lattice

We now go back to the truncated Toda lattice (1.3). $x = 0$ is a trivial straight-line solution. Then

$$V_N(0, y) = \sum_{j=1}^{N} \{2 + (-2)^j\} y^j / j!. \tag{5.1}$$

The highest IC, λ_N is computed as

$$\lambda_N = 6(N-1)/\{2 + (-2)^N\}. \tag{5.2}$$

It is clear that if N is an odd integer ($N \geqq 3$), then $\lambda_N < 0$, i.e. in the non-integrable region S_N. Thus, by Theorem 4.1, (1.3) is non-integrable if N is an odd integer. Now consider the case, $N = 2m$. We can first confirm that $V(0, y)$ is a convex function of y (see Appendix). For small m's the values of λ_{2m} and $\sigma_{2m} = F_{2m}(\lambda_{2m})$ are

$$\lambda_4 = 1, \quad \lambda_6 = 0.454, \quad \lambda_8 = 0.163, \quad \lambda_{10} = 0.053, \quad \lambda_{12} = 0.016,$$
$$\sigma_4 = 1, \quad \sigma_6 = -1.975, \quad \sigma_8 = -0.543, \quad \sigma_{10} = 1.056, \quad \sigma_{12} = 1.704.$$

When m is large, $F_{2m}(\lambda)$ approaches the limiting curve

$$F_\infty(\lambda) = 16(\lambda - 1/2)^2 - 2 \tag{5.3}$$

and λ_{2m} is monotonically decreasing to zero. Thus σ_{2m} is in the interval between 0 and 2 and approaches to 2. Since the lowest term of the potential is $6(x^2 + y^2)$, it follows that $2\cos(2\pi a/b) = 2$ in Theorem 4.2. Thus if $N = 2m \geqq 6$, Theorem 4.2 implies the non-integrability of the system. The case $N = 4$ is exceptional in the sense that the highest term yields an integrable potential, i.e. $(x^2 + y^2)^2$. This case was extensively studied in [15], and the main result is summarized as follows. Near $h = 0$, a perturbation expansion of the solution of NVE shows

$$\mathrm{trace}\, g(\gamma_{\mathrm{real}}; h) = 2 + \alpha h^3 + O(h^4), \quad (\alpha \neq 0), \tag{5.4}$$

i.e., $\mathrm{trace}\, g(\gamma_{\mathrm{real}}; h)$ is not a constant function and therefore for a dense set of h, $g(\gamma_{\mathrm{real}}; h)$ is non-resonant. There is another monodromy matrix $g(\gamma_{\mathrm{all}}; h)$, where γ_{all} enclose all roots of $P(w) = 0$ in (2.7). Then it is shown that $g(\gamma_{\mathrm{all}}; h)$ is non-diagonalizable independent of h. Therefore for a dense set of h, V_4 is non-integrable by the condition (iii) of Theorem 2.2. In conclusion the truncated Toda lattice (1.3) is always non-integrable when $N \geqq 3$. This completes the proof of Theorem 1.1.

Apart from the three-particle periodic Toda lattice (1.1), there exist several integrable generalized Toda lattices with two degrees of freedom [1, 2, 11]. The existence of a straight-line solution $x = 0$ is related to the symmetry of the Dynkin diagram which represents the system. The Dynkin diagram of (1.1) is symmetric and labeled by $A_{2,1}$. Other symmetric cases are [1, 11]

$$A_2\colon\ V = e^{\sqrt{3}x+y} + e^{-\sqrt{3}x+y}, \tag{5.5}$$

$$B_{2,2}\colon\ V = e^{x+y} + e^{-x+y} + e^{-2y}, \tag{5.6}$$

$$C_{2,1}\colon\ V = e^{x+y} + e^{-x+y} + e^{-y}. \tag{5.7}$$

As in the case of (1.1) we consider truncated potentials. Then the highest IC are

$$A_2\colon\ \lambda_N = 3(N-1), \tag{5.8}$$

$$B_{2,2}\colon\ \lambda_N = 2(N-1)/\{2+(-2)^N\}, \tag{5.9}$$

$$C_{2,1}\colon\ \lambda_N = 2(N-1)/\{2+(-1)^N\}. \tag{5.10}$$

(i) A_2:

λ_N is in the non-integrable region $N+2 < \lambda_N < 3N-2$ if $N \geqq 3$.

(ii) $B_{2,2}$:

When N is an odd integer ($N \geqq 3$), $\lambda_N < 0$ and is in the non-integrable region. If N is an even integer ($N = 2m \geqq 4$), we could confirm the assumption of Theorem 4.2,

$$F_{2m}(\lambda_{2m}) \neq \{2, 0, -2, 2\cos(2\pi a/b) = -1.768\},$$

thus all cases are non-integrable.

(iii) $C_{2,1}$:

If N is an even integer ($N = 2m \geqq 4$), λ_{2m} is in the non-integrable region, $1 < \lambda_{2m} < 2m-1$. If N is an odd integer ($N \geqq 5$), λ_N is in the non-integrable region $N+2 \lessdot \lambda_N < 3N-2$. Unfortunately, non-integrability of the case $N = 3$ cannot be shown by the present analysis.

These examples illustrate that the existence of integrable systems is indeed a rare phenomenon, as expected.

6. Appendix. Proof that (5.1) is a Convex Function if $N = 2m$

First we show that every even order polynomial obtained by a truncation of the exponential function is convex. Let us define

$$f_n(y) = \sum_{j=1}^{n} y^j/j!. \tag{6.1}$$

Then observe two identities; i.e.,

$$f_{2m}(ay) = f_{2m-1}(ay) + (ay)^{2m}/(2m)! \tag{6.2}$$

and

$$f'_{2m}(ay) = af_{2m-1}(ay). \tag{6.3}$$

The minimum value of polynomial $f_{2m}(ay)$ is taken at y_0, such that $f'_{2m}(ay_0)=0$. Thus

$$f_{2m}(ay) \geqq f_{2m}(ay_0) = f_{2m-1}(ay_0) + (ay_0)^{2m}/(2m)!$$
$$= (ay_0)^{2m}/(2m)! > 0, \tag{6.4}$$

for all real a and y. Since $f''_{2m}(ay) = a^2 f_{2m-2}(ay)$, it also follows that $f''_{2m}(ay) > 0$.

Since $V_{2m}(0,y) = 2f_{2m}(y) + f_{2m}(-2y)$, a sum of two convex functions (both with positive second derivative), $V_{2m}(0,y)$ itself is a convex function.

Acknowledgements. The author thanks A. Ramani and B. Grammaticos for their excellent hospitality during his stay at the Ecole Polytechnique during the academic year 1986–1987.

References

1. Adler, M., van Moerbeke, P.: Kowalewski's asymptotic method, Kac–Moody Lie algebras and regularization. Commun. Math. Phys. **83**, 83 (1982)
2. Bogoyavlensky, O. I.: On perturbations of the periodic Toda lattices. Commun. Math. Phys. **51**, 201 (1976)
3. Contopoulos, G., Polymilis, C.: Approximations of the 3-particle Toda lattice, Physica **24D**, 328 (1987)
4. Flaschka, H.: The Toda lattice II. Existence of integrals. Phys. Rev. **B9**, 1924 (1974)
5. Hénon, M., Heiles, C.: The applicability of the third integral of motion. Astron. J. **69**, 73 (1964)
6. Hénon, M.: Integrals of the Toda lattice. Phys. Rev. **B9**, 1921 (1974)
7. Ito, H.: Non-integrability of Hénon–Heiles system and a theorem of Ziglin. Kodai Math. J. **8**, 120 (1985)
8. Ito, H.: A criterion for non-integrability of Hamiltonian systems with nonhomogeneous potentials. J. Appl. Math. Phys. **38**, 459 (1987)
9. Rod, D. L.: On a theorem of Ziglin in Hamiltonian dynamics, preprint
10. Toda, M.: Theory of nonlinear lattice. Berlin, Heidelberg, New York: Springer 1981
11. Yoshida, H.: Integrability of generalized Toda lattice systems and singularities in the complex t-plane. In: Nonlinear integrable systems-classical theory and quantum theory. Jimbo, M., Miwa, T. (eds.) Singapore: World Science 1983, p. 273
12. Yoshida, H.: Existence of exponentially unstable periodic solutions and the non-integrability of homogeneous Hamiltonian systems. Physica **21D**, 163 (1986)
13. Yoshida, H.: A criterion for the non-existence of an additional integral in Hamiltonian systems with a homogeneous potential. Physica **29D**, 128 (1987)
14. Yoshida, H.: Non-integrability of a class of perturbed Kepler problems. Phys. Lett. **A120**, 388 (1987)
15. Yoshida, H., Ramani, A., Grammaticos, B.: Non-integrability of the fourth-order truncated Toda Hamiltonian. Physica D, (1988) (in press)
16. Yoshida, H., Ramani, A., Grammaticos, B. Hietarinta, J.: On the non-integrability of some generalized Toda lattice. Physica **144A**, 310 (1987)
17. Ziglin. S. L.: Branching of solutions and the nonexistence of first integrals in Hamiltonian mechanics. Funct. Anal. Appl. **16**, 181 (1983), **17**, 6 (1983)

Communicated by A. Jaffe

Received July 29, 1987; in revised form November 19, 1987