Commun. Math. Phys. 116, 365–400 (1988)

Communications in Mathematical Physics

Current Algebras in $d+1$-Dimensions and Determinant Bundles over Infinite-Dimensional Grassmannians*

J. Mickelsson** and S. G. Rajeev***

Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. We extend the methods of Pressley and Segal for constructing cocycle representations of the restricted general linear group in infinite-dimensions to the case of a larger linear group modeled by Schatten classes of rank $1 \leqq p < \infty$. An essential ingredient is the generalization of the determinant line bundle over an infinite-dimensional Grassmannian to the case of an arbitrary Schatten rank, $p \geqq 1$. The results are used to obtain highest weight representations of current algebras (with the operator Schwinger terms) in $d+1$-dimensions when the space dimension d is any odd number.

1. Introduction

In this paper we generalize some results of Pressley and Segal [PS] on the determinant line bundle over infinite-dimensional Grassmannians and on central extensions of infinite-dimensional linear groups. The ultimate aim is to obtain linear representations of current algebras arising in quantum field theory in $3+1$-dimensions. In particular, we want to construct a generalization of the fermionic Fock representation of current algebras in $1+1$-dimensions (including the Schwinger term), adapted to the $3+1$-dimensional case. We have a partial resolution to this problem.

We are able to construct a highest weight representation for the $3+1$-dimensional current algebra, including an explicit realization of the highest weight vector ($=$ vacuum) as a section of the dual Det_2^* of the determinant bundle, Det_2 over a Grassmannian Gr_2, which contains the Grassmannian Gr_1 studied in [PS]

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069

** Permanent address: Department of Mathematics, University of Jyväskylä, Seminaarintatu 15, SF-40100, Jyväskylä Finland

***Address: (after September 1, 1987) Department of Physics, University of Rochester, River Campus Station, Rochester, NY 14627, USA

as a dense subset. In fact, this construction can be generalized without difficulties to current algebra in any odd-dimension. (The even-dimensional case seems to be different and we shall comment briefly on it in Sect. II.)

However, we have not been able to prove the unitarizability of our representation.

Current algebras were introduced in particle physics [A] in the study of strong interactions. The observables of a strongly interacting system (such as the proton) can be thought of as the currents that couple to other forces such as electromagnetism or weak interactions. The hope was that the algebra of these current operators and their representations would provide a theory of strong interactions. But this rather abstract approach fell out of favor when it was realized that Quantum Chromodynamics (QCD) provided a field theoretic description of strong interactions [MP]. However, the current algebra point of view has seen a revival in recent years since it has proved to be too difficult to describe low energy properties of hadrons in terms of QCD. In fact, understanding the meson and baryon physics in terms of QCD is one of the outstanding challenges of particle theory. Meanwhile, current-algebras and effective Lagrangians provide a more direct description of hadrons [B, Tr]. It is also hoped that studying current-algebras and their anomalies (Schwinger terms) as predicted by QCD will provide a way of unraveling the low energy properties of QCD [R].

Consider a Dirac field in $d+1$-dimensions coupled to an external Yang–Mills field A. We can choose space to be a compact d-dimensional spin manifold (such as S^d), and A is then locally a Lie algebra valued one-form. At the first quantized level, where the Dirac field ψ is thought of as a Grassmann number (and not an operator), the currents satisfy the algebra

$$\{J^i(x), J^i(y)\} = iC_k^{ij} J^k(x)\delta(x-y), \tag{1.1}$$

the bracket is the fermionic analogue of a Poisson-bracket (pseudo-Poisson bracket) following from

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y), \tag{1.2}$$

and $J^i(x) = \psi^\dagger \lambda^i \psi(x)$ is the charge density (the time component of the current-density).

If we define

$$J(f) = \int dx f^i(x) J^i(x), \tag{1.3}$$

where $f: S^d \to \underline{g}$ are functions valued in the Lie algebra,

$$[J(f), J(g)] = J([f, g]). \tag{1.4}$$

So the current algebra in this case is just the infinite-dimensional Lie algebra $\mathrm{Map}(S^d; \underline{g})$.

Actually, we have a unitary representation of $\mathrm{Map}(S^d; \underline{g})$ on the Hilbert space of square integrable spinors ["first quantized" representations], given by

$$[J(f), \psi_\alpha(x)] = \lambda_\alpha^{i\beta} f^i(x) \psi_\beta(x), \tag{1.5}$$

λ^i being the representation matrices of $\underline{g}$.

However, this is not the representation of interest in quantum field theory.

There is no vacuum state (highest weight-vector) in this representation. The Dirac Hamiltonian is not bounded below.

So, one constructs the fermionic Fock space, following Dirac [second quantization]. The Dirac field $\psi(x)$ is an operator on this space, providing a representation of the infinite-dimensional Clifford algebra. One then looks for a representation of $\mathrm{Map}(S^d;\underline{g})$ on this Fock space with the Dirac vacuum state as the highest weight-vector. But it is well-known that the operator product $J^i(x)=\psi^\dagger\lambda^i\psi(x)$ is not well-defined due to the ultraviolet divergences of quantum field theory.

If $d=1$, we can define this product by normal ordering. This involves subtracting the vacuum expectation value from $J[f]$. After this subtraction, it is well defined. But the price we pay for this is that we do not obtain a representation of $\mathrm{Map}(S^1,\underline{g})$, but a central extension of it.

Even this will not work for $d>1$. Even after subtracting, the vacuum expectation value of the squares $J[f]^2$ are not well-defined. In the language of renormalization theory, $J[f]$ requires a multiplicative renormalization for $d=3$. This means that there is no meaning to $J[f]$ within the fermionic Fock space. Since this point does not seem to have been appreciated in the literature, we shall show this explicitly in the next section.

What happens is that (even after normal ordering) the operator $J[f]$ creates states of infinite-norm out of the vacuum. One might try to redefine the inner product so that the fermionic states do not form a complete set. Then, we can add to the Fock space new states created out of the vacuum by $J[f]$. There might be a unitary representation on this larger Hilbert space. Note that these new states we have to add are bosonic; they have the same quantum numbers as a two fermion state. However, they have no meaning as a linear combination of two-fermion states. In this sense they are "condensates" of fermion pairs.

We have constructed a linear representation with a highest weight vector, essentially including these bosonic states. But we have not been able to find an invariant inner product.

Instead of a central extension for $d>1$, we find the representation of an Abelian extension of $\mathrm{Map}(S^d;\underline{g})$,

$$[J(f),J(g)]=J([f,g])+c(f,g;A). \tag{1.6}$$

Here, c is the Schwinger term which, for $d>1$, is a function of the gauge field A. c is to be thought of as a linear operator in some Hilbert space [M1, F].

There is a group corresponding to this current algebra, [M2] which is an Abelian extension of $S^dG=\mathrm{Map}(S^d;G)$ by the group $\mathrm{Map}(\mathscr{A},\mathbf{C}^\times)$, where $\mathscr{A}$ is the space of gauge potentials.

The action of gauge transformations in Dirac field, (1.5), defines an embedding of S^dG into the infinite-dimensional general linear group GL_p modeled on Schatten classes of type $I_{2p}(2p\geqq d+1)$ which will be described in Sect. II. Our strategy will be to find a representation of an Abelian extension of GL_p, which will automatically give a representation of an Abelian extension of S^dG. The representation we look for will be a generalization of the wedge representation of GL_1 constructed in [PS]. This was just the representation of fermion bilinears on the fermionic Fock space, as discussed in detail in [BR], for example.

The fermionic Fock space is an infinite-dimensional generalization of the exterior algebra of the one particle Hilbert space H. It is well-known [W] that in the finite-dimensional case, the exterior algebra of a vector space can be thought of as the space of holomorphic sections of a line bundle over the Grassmannian associated to the vector space. This point of view was generalized by [PS] to the infinite-dimensional case. They defined a determinant line bundle over the infinite-dimensional Grassmannian Gr_1, modeled on the Schatten class I_2. This was possible because the determinant involved was that of an operator of type $1+I_1$.

We will instead have to deal with Grassmannian Gr_p modeled on I_{2p}. The Grassmannian Gr_1 of [PS] is a dense subset. However, it will not be possible to define the determinant line bundle as they did, because the operators will be of type $1+I_p$. There is a modified ("renormalized") definition of a determinant for such operators [S] which we use to define a holomorphic line bundle Det_p on Gr_p. We describe this modified determinant in Sect. III. It satisfies most of the properties of the ordinary determinant, except that

$$\det_p AB \neq \det_p A \det_p B.$$

The group GL_p acts holomorphically on Gr_p. But this action does not lift to Det_p. Instead, there is an Abelian extension $\widehat{GL_p}$ by the group $\mathrm{Map}(\mathrm{Gr}_p;\mathbf{C}^\times)$ which does act on Det_p. The pull-back of this extension under the embedding $S^dG \hookrightarrow GL_p$ defines an extension $\mathscr{G}$ of S^dG which is the one we want.

Thus, any linear representation of $\widehat{GL_p}$ automatically gives one for $\mathscr{G}$. In finite-dimensional (as well as for $p=1$, in the infinite-dimensional case) the space of holomorphic sections of the dual line bundle Det* provides a representation of the general linear group. The line bundle Det has no (non-constant) holomorphic sections, but Det* does. This is just the antisymmetric tensor representation (fermionic Fock representation in the infinite-dimensional case with $p=1$).

We might try to find a representation of $\widehat{GL_p}$ on the space of holomorphic section of Det_p^*. Here an important new phenomenon appears for $p>1$ (and hence for $d>1$). The holomorphic structure of Det_p^* is not invariant under the action of $\widehat{GL_p}$. This is related to the failure of the usual method of finding a representation of the current algebra on the fermionic Fock space. We can still think of the fermionic Fock space as the holomorphic sections of Det_p^*. But the action by an element in $\mathscr{G}\subset\widehat{GL_p}$ will take us out of this space, because it produces a non-holomorphic section out of a holomorphic one.

To see this, recall that $\widehat{GL_p}$ is an extension of GL_p by $\mathrm{Map}(\mathrm{Gr}_p;\mathbf{C}^\times)$. The only holomorphic functions on the Grassmannian are constants. So the action of an element of $\mathrm{Map}(\mathrm{Gr}_p;\mathbf{C}^\times)$, on a holomorphic section will in general give a non-holomorphic section.

For the case $d=p=1$, we do not need to worry about this, since already the smaller extension of GL_1 by $\mathbf{C}^\times$ (the constant functions on $\widehat{\mathrm{Gr}_1}$) acts on Det_1.

For $d,p>1$, we can therefore find a representation of $\widehat{GL_p}$ on the space of all sections (not just holomorphic ones) of Det_p^*. There will be a holomorphic section which is a highest weight vector, which represents the vacuum state.

There are similarities between our construction of the action of $\widehat{GL_p}$ on Det_p and the renormalization theory of quantum fields. In the case $p=1$, the Lie algebra extension $\hat{gl}_1$ of the Lie algebra $\underline{gl}_1$ of $\widehat{GL}_1$, corresponding to the group extension $\widehat{GL}_1$ can be described as follows. The Lie algebra $\underline{gl}_1$ consists of operators of the type

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acting in a Hilbert $H = H_+ \oplus H_-$ such that $b: H_- \to H_+$ and $c: H_+ \to H_-$ are Hilbert–Schmidt. The central extension is $\hat{\underline{gl}}_1 = \underline{gl}_1 \oplus \mathbf{C}$ and the commutators in $\underline{gl}_1$ are defined by the Kač–Peterson cocycle

$$\eta_1(X, Y) = \tfrac{1}{8}\operatorname{tr}[[\varepsilon, X], [\varepsilon, Y]]\varepsilon, \tag{1.7}$$

where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By a simple computation,

$$\eta_1(X, Y) = \operatorname{tr}(b(X)c(Y) - b(Y)c(X)). \tag{1.8}$$

Note that the product bc is a trace-class operator, since b and c are Hilbert–Schmidt. In the case $p = 2 (d = 3)$ (1.8) does not make sense, because $b, c \in I_{2_p}$ and $bc \in I_p$ is not of trace-class. The points on Gr_p can be parametrized by idempotent operators F such that the diagonal blocks of $F - \varepsilon$ are in I_p. In particular, the diagonal blocks of $F - \varepsilon$ are Hilbert–Schmidt operators for $p = 2$ and the following formula makes sense in that case:

$$\eta_2 = \tfrac{1}{8}\operatorname{tr}[[\varepsilon, X], [\varepsilon, Y]](\varepsilon - F). \tag{1.9}$$

To check the cocycle property of η_2 one has to take into account that the group GL_2 acts on F; infinitesimally, this is given by the commutator $[X, F]$ for $X \in \underline{gl}_2$. (Strictly speaking, this is true only for the unitary subgroup $U_2 \subset GL_2$. However, one can define a complexified Grassmannian $\mathbf{C}\,\mathrm{Gr}_p$ such that the action of $X \in \underline{g}_2$ is still given by the commutator. The real Grassmannian $\mathrm{Gr}_p \subset \mathbf{C}\,\mathrm{Gr}_p$ is parametrized by Hermitian operators F.) Restricting to the dense subalgebra $\underline{gl}_1 \subset \underline{gl}_2$ (and $F \in \mathrm{Gr}_1$) the difference $\eta_1 - \eta_2$ becomes a trivial two-cocycle. It is a coboundary of the one-cocycle $\alpha = -\frac{1}{16}\operatorname{tr}[X, \varepsilon][F, \varepsilon]$. The form α diverges in the case $p = 2$. Formally, η_1 is a sum of η_2 and of $-\frac{1}{8}\operatorname{tr}[[\varepsilon, X], [\varepsilon, Y]]F$, but the point is that separately these two terms become infinite for $p = 2$ and only the sum makes sense. The latter term can be understood as an infinite charge renormalization in the field theory terminology (the elements of $\underline{gl}_2$ correspond to local charges in $3+1$-dimensional QFT). There is also another renormalization which we shall meet. There is a natural embedding $\mathrm{Gr}_1 \subset \mathrm{Gr}_2$ as a dense subspace. A section of Det_2^* defines thus a section of Det_1^*; the structure of Det_1^* obtained by a restriction from Det_2^* differs from the canonical Det_1^* (of [PS]) in such a way that the sections of Det_1^* are obtained from sections of Det_1^* by multiplying by certain function. When approaching points in $\mathrm{Gr}_2 \setminus \mathrm{Gr}_1$ both the section of Det_1 and the multiplier become infinite but the product converges. We call this the wave function renormalization since the sections of Det_2^* can be thought of as wave functions in the Schrödinger picture of the quantized Dirac field coupled to external Yang–Mills field.

Instead of defining a representation of $\widehat{GL}_p$ on non-holomorphic sections of Det_p^*, we could try to find some line bundle on which $\widehat{GL}_p$ acts preserving the holomorphic structure. Then we could find a representation of $\widehat{GL}_p$ on the holomorphic sections of that line bundle.

We can do this by considering as base space $\mathbf{C}\,\mathrm{Gr}_p$, which is roughly speaking "twice as big" as Gr_p(i.e. it is modeled on $I_{2p} \oplus I_{2p}$ rather than I_{2p}). Unlike Gr_p, this space does have non-constant holomorphic functions. In fact, $\mathbf{C}\,\mathrm{Gr}_p$ can be thought of as a complexification of Gr_p (thought of as a real analytic manifold). Holomorphic functions on $\mathbf{C}\,\mathrm{Gr}_p$ are then analytic continuations of real analytic functions of Gr_p. There is a holomorphic line bundle $\mathbf{C}\,\mathrm{Det}_p^*$ over $\mathbf{C}\,\mathrm{Gr}_p$ and it does admit an action of $\widehat{GL}_p$ preserving the holomorphic structure.

$\mathbf{C}\,\mathrm{Gr}_p$ can be thought of as a holomorphic bundle over Gr_p with fiber I_{2p}. Then we can find a holomorphic vector bundle over Gr_p with fiber $\mathrm{Hol}(I_{2p}; \mathbf{C})$ on which $\widehat{GL}_p$ acts preserving the holomorphic structure. This point of view is analogous to the "Bargmann" picture for the bosonic fields, while considering non-holomorphic functions on Gr_p is like the "Schrödinger" picture.

II. Embedding of the Current Group in GL_p

We assume space to be a compact Riemann manifold, X. If periodic boundary conditions at infinity in $\mathbf{R}^d$ are chosen, $X = T^d$. (We will assume this is the case for the moment.) The group of interest to us is the group $\mathrm{Map}(X; G)$ of smooth maps $\{g: X \to G\}$, G being a compact Lie group. This may be thought of as the group associated to the current algebra

$$[J(f), J(g)] = J([f, g]), \tag{2.1}$$

where $f \in \mathrm{Map}(X \to \underline{g})$ is a function valued in the Lie algebra $\underline{g}$. We are interested in finding representations of this group $\mathrm{Map}(X; G)$.

One representation (the "first quantized" representation) is easy to construct. Consider a free fermion field ψ carrying a unitary representation ρ of G. The set of such ψ's form the "first quantized Hilbert space" H. More precisely, $H = L^2(X; V)$ is the space of square integrable functions on X valued in a finite-dimensional complex vector space V. V is the tensor product of the space of spinors on X with the representation space of ρ. Now define a representation

$$[M(f)\psi](x) = \rho(f(x))\psi(x) \tag{2.2}$$

by pointwise multiplication. If $f: X \to G$ is smooth, $M(f): H \to H$ is a continuous unitary operator.

Using the Dirac operator D on X, we can in fact refine this statement somewhat. The Dirac operator is a self-adjoint operator with discrete spectrum. Let H_+ be the space spanned by eigenstates of non-negative eigenvalues and H_- that by eigenstates of negative eigenvalue. Since the set of eigenstates of D is complete, we have an orthogonal decomposition

$$H = H_+ \oplus H_-.$$

We can now decompose matrices in the general linear group of H into 2×2

blocks,

$$g=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} a: & H_+ \to H_+ \\ b: & H_- \to H_+ \\ c: & H_+ \to H_- \\ d: & H_- \to H_- \end{array}. \tag{2.3}$$

It is clear that if $f: X \to G$ is a constant function, it will commute with the Dirac operator D. So it will map H_+ to H_+ and H_- to H_-:

$$M(f)=\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

More generally, if f is a smooth function, we expect that the off-diagonal elements b and c are not "too large." What we need to make this notion precise is a norm on the space of operators in a Hilbert space. If $\dim X = 1$, (i.e. $X = S^1$) it is known that $\operatorname{tr} b^+ b$ and $\operatorname{tr} c^+ c$ are finite for smooth functions f [PS]. More generally, we will show that $\operatorname{tr}(b^+ b)^p$ and $\operatorname{tr}(c^+ c)^p$ are finite for $2p > d$. The proof was already given in [PS] for the case $X = T^d$ and sketched for the general case; for the sake of completeness we shall explain the full proof here. This, and some results in Sect. 4 also appear in Conne's non-commutative geometry [C]. But since we don't need his full machinery we have chosen to develop the theory from scratch.

The Banach space of operators on a Hilbert space with norm

$$\| A \|_{2p} = [\operatorname{tr}(A^+ A)^p]^{1/2p}$$

is called the Schatten ideal I_{2p} [S]. An equivalent norm that is easier to compute in practice is

$$\| A \|_{2p} = \left[\sum_l \| A e_l \|^{2p} \right]^{1/2p},$$

e_l being vectors in H forming an orthonormal basis.

Proposition 2.1. *Let $H = H_+ \oplus H_-$ be the space of Dirac spinors on T^d carrying a finite-dimensional representation ρ of G. Let $M(f)$ be the operator on H representing $f \in \operatorname{Map}(T^d, G)$, a smooth function on T^d as in (2.2) and b, c as in (2.3). Then $\| b \|_{2p}$ and $\| c \|_{2p}$ exist for $2p > d$. The only functions f for which these exist for $2p \leqq d$ are constant.*

Proof. Define $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $H_+ \oplus H_-$. ε may be thought of as the sign of the Dirac operator. Then,

$$[\varepsilon, M(f)] = 2\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix},$$

and it is sufficient to consider

$$\| [\varepsilon, M(f)] \|_{2p}^{2p} = 2^{2p}(\| b \|_{2p}^{2p} + \| c \|_{2p}^{2p}).$$

Let $\phi_k (k \in \mathbb{Z}^d)$ denote the Fourier components of $\phi: L^2(T^d, V)$. Then

$$D\phi_k = \not{k}\phi_k,$$

where $\not{k} = \sum_{i=1}^{d} \alpha_i k_i$ and the α_i's are the Dirac matrices acting on the spinor components of ϕ. Clearly

$$\begin{aligned} \varepsilon\phi_k &= \frac{\not{k}}{|k|}\phi_k, \quad k \neq 0 \\ &= \phi_k, \qquad k = 0. \end{aligned}$$

Also,

$$(M(f)\phi)_k = \sum_q f_{k-q}\phi_q,$$

where

$$f_k = \int e^{-ik\cdot x} dx\, \rho(f(x))$$

is the Fourier coefficient. Then

$$([\varepsilon, M(f)]\phi)_k = \sum_q \left(\frac{\not{k}}{|k|} - \frac{\not{q}}{|q|}\right) f_{k-q}\phi_q,$$

and

$$\|[\varepsilon, M(f)]\|_{2p}^{2p} = \sum_{k,q} \operatorname{tr}\left\{ f_{k-q}\left(\frac{\not{k}}{|k|} - \frac{\not{q}}{|q|}\right)\left(\frac{\not{k}}{|k|} - \frac{\not{q}}{|q|}\right) f_{k-q}^{+} \right\}^p.$$

The trace inside the summation is just a finite-dimensional trace in V. Now, by properties of the Dirac matrices α,

$$\left(\frac{\not{k}}{|k|} - \frac{\not{q}}{|q|}\right)^2 = 2\left(1 - \frac{k\cdot q}{|k||q|}\right).$$

Redefining $k \to k + q$, the right-hand side is equal to

$$\sum_k \operatorname{tr}(f_k^{+} f_k)^p \times \sum_q \left(1 - \frac{(k+q)\cdot q}{|k+q||q|}\right)^p.$$

Consider first, $S_p(k) = \sum_p (1 - (k+q)\cdot q/|k+q||q|)^{2p}$. We are interested in this sum as $q \to \infty$ as any finite number of terms will produce a convergent sum in k. Now

$$1 - \frac{(k+q)\cdot q}{|k+q||q|} \sim -\frac{(q\cdot k)^2}{|q|^4} + \frac{k^2}{2|q|^2} + O\left(\frac{1}{|q|^3}\right).$$

Thus the sum on q behaves as $|q| \to \infty$, like

$$\begin{aligned} S_p(k) &\sim |k|^{2p} \sum_q \left[\frac{1}{|q|^2}\right]^p \sim |k|^{2p} \int d^d q \left(\frac{1}{|q|^2}\right)^p \\ &\sim |k|^{2p} \int^{\infty} d|q|\, |q|^{d-1} |q|^{-2p}. \end{aligned}$$

Here, $\sim$ means *modulo* a finite constant factor. This is convergent if $2p > d$. In this case

$$\| [\varepsilon, M(g)] \|_{2p}^{2p} \sim \sum_k \operatorname{tr}(k^2 f_k^+ f_k)^p.$$

The sum on k is convergent because, for smooth functions, $|\tilde{f}(k)|$ decreases faster than any power of $|k|$ as $|k| \to \infty$.

If $2p \leqq d$, the sum on q diverges unless $k = 0$. But this happens only for constant functions $f(x)$, and these are the only ones of finite-norm in that case. □

Let $H = H_+ \oplus H_-$ be an orthogonal decomposition of a Hilbert space into two infinite-dimensional subspaces. We are led to consider invertible operators $g: H \to H$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $\operatorname{tr}(b^+ b)^p < \infty$ and $\operatorname{tr}(c^+ c)^p < \infty$ (equivalently, such that $[\varepsilon, g] \in I_{2p}$). These form a group which we will denote by GL_p. The dependence on the splitting $H_+ \oplus H_-$ will be suppressed, since it will be obvious from the context which one we mean. As an abstract-group, GL_p is, of course, well-defined independent of these choices. That GL_p is a group (i.e., closed under multiplication and inverse) follows from the fact that I_{2p} is a two-sided ideal in the algebra of bounded operators [S]. In Sect. IV, a brief discussion of the main properties of GL_p is given. GL_p is a Banach Lie group, with topology given by the norm

$$\| a \| + \| b \|_{2p} + \| c \|_{2p} + \| d \|,$$

where

$$\| a \| = \sup_{\| \psi \| = 1} \| a\psi \|.$$

We can now restate Proposition 2.1 as follows

Proposition 2.2. *There is a continuous injective homomorphism*

$$M: \mathrm{Map}(T^d; G) \to GL_p$$

for $2p > d$.

Let us now see how far we can generalize this situation. We would like to replace T^d by an arbitrary compact Riemannian manifold X. In order to have spinors, X must be a spin manifold. The Hilbert space H is now well-defined, once a spin structure on X is chosen. A representation of $\mathrm{Map}(X; G)$ on H can be defined. Given a choice of connection on the spin bundle, we have the Dirac operator on H. This is a self-adjoint operator with finite-dimensional kernel, so the orthogonal decomposition $H = H_+ \oplus H_-$ into non-negative (H_+) and negative (H_-) eigenspaces goes through as before. The operator ε is now a pseudo-differential operator [T] on X, and we can investigate whether $[\varepsilon, M(f)]$ belongs to $I_{2p}(H)$ as before. We know that only the asymptotic behavior of ε and $M(f)$ as the momenta go to infinity is relevant. So we are interested in the short-distance behavior of the integral kernel associated to $[\varepsilon, M(f)]$. Since X is a smooth manifold, locally it resembles T^d and we expect that the results such as 2.1 to continue to be valid. The calculus of pseudo-differential operators provides us with a precise language to prove this.

Before doing that, let us imagine an even more general situation. Let E be a Hermitian vector bundle over X with structure group G. The earlier case corresponds to the trivial case

$$E = X \times V.$$

To this is related the bundle of automorphism of E, Aut E. Aut E is a fiber bundle with fiber G, but is not necessarily the principal fiber bundle associated to E [FU]. The principal bundle may not admit smooth global sections, Aut E does. Smooth sections of Aut E form an infinite-dimensional group $C^\infty(\text{Aut}\, E)$, the group of gauge transformations of E. These are the bundle maps of E that reduce to the identity on the base space X. In the trivial case $C^\infty(\text{Aut}\, E) = \text{Map}(X; G)$.

Let H be the Hilbert space of squares integrable sections of the bundle $E \oplus S$. (S is the spin bundle over X.) Pointwise action gives a representation of $C^\infty(\text{Aut}\, E)$ on H.

Given a connection on E, we can define the Dirac operator on H, and hence the decomposition $H = H_+ \oplus H_-$. We may then ask if there is a homomorphism

$$M: C^\infty(\text{Aut}\, E) \hookrightarrow GL_p.$$

We now establish that this exists.

Let us begin by recalling a few definitions [T]. Let $\Omega \subset R^d$ be an open subset and $C^\infty(\Omega; R^n)$ the space of smooth functions. $S^m(\Omega; R^n)$ is the set of smooth functions.

$$\varphi: \Omega \times R^d \to \text{End}(R^n)$$

such that for multi-indices $\alpha, \beta \in \mathbf{N}^d$ and any compact subset $K \subset \Omega$, there exist constants $C_{K,\alpha,\beta}$ with

$$|D_x^\alpha D_q^\beta \varphi(x, q)| \leqq C_{K,\alpha,\beta}(1 + |q|)^{m-|\alpha|},$$

for $x \in K$. Given φ, define an operator $\hat{\varphi}$ on $C^\infty(\Omega, R^n)$, by

$$(\hat{\varphi} f)(x) = \int \frac{dq}{(2\pi)^d} \varphi(x, q) \tilde{f}(q) e^{iq\cdot x},$$

$\tilde{f}(q)$ being the Fourier transform. $\hat{\varphi}$ extends to a continuous operator on $L^2(\Omega, R^n)$. $\hat{\varphi}$ is called a pseudo-differential operator of order m and φ is its symbol. The space of such operators is called $PS^m(\Omega; R^n)$. What is mostly of interest is the "most singular" part of φ. The principal symbol of $\hat{\varphi}$ is defined as the equivalence class of φ in S^m/S^{m-1}. We denote some representative of this class by $\sigma\hat{\varphi}$ and call it the principal symbol also by a slight abuse of language.

Now let E be a vector bundle over a compact manifold X (of dimension d) with fibers of dimension n. An operator $\hat{\varphi}$ on $C^\infty(E)$ is in $PS^m(E)$ if for any coordinate neighborhood U in X with chart (and trivialization) $\chi: E|_u \to \Omega \times R^n$, the operator $\chi \circ \hat{\varphi} \circ \chi^{-1}$ on $C^\infty(\Omega; R^n)$ is in $PS^m(\Omega; R^n)$. This definition is then invariant under change of coordinates.

Note that the principal symbol of a pseudo-differential operator on E, is a function on the cotangent bundle $T^*(X)$ valued in $\text{End}(R^n)$.

For the massive Dirac operator on X,

$$D_m = \alpha\cdot\nabla + m\beta = D + m\beta.$$

The principal symbol is,

$$\sigma D_m(x,k) = \alpha\cdot k.$$

$(x,k)\in T^*X$ and $\alpha_i(x)$ are the Dirac matrices with the given metric tensor on X. In an orthonormal basis

$$[\beta,\alpha_i]_+ = 0;\quad \beta^2 = 1;\quad [\alpha_i,\alpha_j]_+ = \delta_{ij}.$$

Obviously, the Dirac operator on a vector bundle with connection is of order one $D_m \in PS^1(E)$.

Let us define

$$\varepsilon_m = \mathrm{D}_m[D^2 + m^2]^{-1/2}.$$

We are considering the massive Dirac operator to avoid headaches with zero modes of D. Since what is relevant is the behavior at short distances, this does not matter [C].

Clearly, $\varepsilon_m^2 = 1$. Now we note that ε_m is also a pseudo-differential operator.

Proposition 2.3. *ε_m is a pseudo-differential operator of order zero with principal symbol*

$$\sigma\varepsilon_m(x,k) = \frac{\alpha\cdot k}{(k^2 + m^2)^{1/2}}.$$

Proof. If C is a contour surrounding the spectrum of D,

$$\varepsilon_m = \int_C dz\, f(z)\frac{1}{z - D}.$$

Within a trivialization, the connection on $S\otimes E$ is a smooth one-form. Then

$$D = \frac{1}{i}\alpha\cdot\frac{\partial}{\partial x} + \alpha\cdot A = D_0 + \alpha\cdot A.$$

Now $(1/z - D)\in(PS^{-1}(E))$ may be written as

$$\frac{1}{z - D} = \frac{1}{z - D_0} - \frac{1}{z - D_0}\alpha\cdot A\frac{1}{z - D}\begin{cases} z\notin \mathrm{Spec}(D_0) \\ z\notin \mathrm{Spec}(D)\end{cases}.$$

The second term is a pseudo-differential operator of order -2. So the principal symbol of $1/z - D$ is

$$\left(\sigma\frac{1}{z - D}\right)(x,k) = \frac{1}{z - \alpha\cdot k}.$$

The result for $\sigma\varepsilon_m$ then follows upon multiplication by f and integrating over the contour C. □

Proposition 2.4. *$Y = [\varepsilon_m, M(f)]$ is in $PS^{-1}(E)$ with principal symbol*

$$Y(x,q)=\frac{1}{i}\frac{1}{(q^2+m^2)^{1/2}}\left[\delta_{ij}-\frac{q_i\cdot q_j}{q^2+m^2}\right]\alpha_i\partial_j\rho(f(x)).$$

Proof. Within a coordinate neighborhood, we can write

$$\varepsilon_m M(f)\psi(x)\sim\int\frac{dk}{(2\pi)^d}\frac{e^{+ik\cdot x}\alpha\cdot k}{(k^2+m^2)^{1/2}}\int\rho(f(y))\psi(y)e^{-ik\cdot y}dy,$$

where $\sim$ denotes equality modulo terms of lower order and

$$M(f)\varepsilon_m\psi(x)\sim\rho(f(x))\int\frac{dk}{(2\pi)^d}e^{ik\cdot x}\frac{\alpha\cdot k}{(k^2+m^2)^{1/2}}\tilde{\psi}(k),$$

$\tilde{\psi}$ being the Fourier transform, so

$$Y\psi(x)\sim\int\frac{dk}{(2\pi)^d}e^{ik\cdot x}\frac{\alpha\cdot k}{(k^2+m^2)^{1/2}}\tilde{f}(k-q)\tilde{\psi}(q)\frac{dq}{(2\pi)^d}$$

$$-\int\frac{dk}{(2\pi)^d}e^{i(k+q)\cdot x}\frac{\alpha\cdot k}{(k^2+m^2)^{1/2}}\tilde{f}(q)\tilde{\psi}(k)\frac{dq}{(2\pi)^d},$$

i.e.

$$Y\psi(x)\sim\int\frac{dk}{(2\pi)^d}\frac{dq}{(2\pi)^d}e^{ik\cdot x}\left[\frac{\alpha\cdot(k+q)}{[(k+q)^2+m^2]^{1/2}}-\frac{\alpha\cdot q}{(q^2+m^2)^{1/2}}\right]\tilde{f}(k)e^{iq\cdot x}\tilde{\psi}(q).$$

Now,

$$\int\frac{dk}{(2\pi)^d}e^{ik\cdot x}\left[\frac{\alpha\cdot(k+q)}{[(k+q)^2+m^2]^{1/2}}-\frac{\alpha\cdot q}{(q^2+m^2)^{1/2}}\right]\tilde{f}(k)$$

$$=\int\frac{dk}{(2\pi)^d}\alpha\cdot k\tilde{f}(k)e^{ik\cdot x}\frac{1}{(q^2+m^2)^{1/2}}$$

$$+\int\frac{dk}{(2\pi)^d}e^{ik\cdot x}\tilde{f}(k)\frac{\alpha\cdot q}{(q^2+m^2)^{1/2}}\left[1+\frac{2q\cdot k}{q^2+m^2}\right]^{-1/2}+\mathcal{O}\left(\frac{1}{|q|^2}\right)$$

$$=\frac{1}{i}\alpha\cdot\partial\rho(f(x))\frac{1}{(q^2+m^2)^{1/2}}-\frac{1}{i}\frac{\alpha\cdot q}{(q^2+m^2)^{1/2}}\frac{q\cdot\partial}{(q^2+m^2)^{1/2}}\rho(f(x))+\mathcal{O}\left(\frac{1}{|q|^2}\right).$$

Thus,

$$Yf(x)\sim\int\frac{dq}{(2\pi)^d}e^{iq\cdot x}\frac{1}{i(q^2+m^2)^{1/2}}\left[\alpha\cdot\partial-\frac{\alpha\cdot q}{q^2+m^2}q\cdot\partial\right]\rho(f(x))\tilde{\psi}(q)$$

$$=\int\frac{dq}{(2\pi)^d}\frac{e^{iq\cdot x}}{i}\frac{\tilde{\psi}(q)}{(q^2+m^2)^{1/2}}\left[\delta_{ij}-\frac{q_iq_j}{q^2+m^2}\right]\alpha_i\partial_j\rho(f(x)).\quad\square$$

Let us now find the relation between the order of a pseudo-differential operator and the Schatten class to which it belongs.

Proposition 2.5. *If $\phi\in PS^m(E)$, then $\phi\in I_{2p}(L^2(E))$ for*

$$p>-\frac{d}{2m}.$$

d being the dimensionality of the base manifold of the vector bundle E.

Proof. Let $U \subset X$ be a coordinate neighborhood, mapped into $\Omega \subset R$. On $L^2(E|_U)$ we have the expression

$$\hat{\varphi}\psi(x) = \int \varphi(x,k) e^{ik\cdot x} \tilde{\psi}(k) \frac{dk}{(2\pi)^d}.$$

The compact manifold X can be covered by a finite number of coordinate neighborhoods $X = \bigcup_\alpha U_\alpha$. To such a covering is associated a partition of unity, i.e. there are functions $f_\alpha : U_\alpha \to R$ with $\sum_\alpha f_\alpha = 1$, and $\operatorname{Supp} f_\alpha$ is contained in a compact subset of U_α. If $\psi \in L^2(E)$, the maps

$$\psi \mapsto \psi_\alpha = f_\alpha \psi$$

are projections $L^2(E) \to L^2(E|_{U_\alpha})$. Since

$$\psi = \sum_\alpha \psi_\alpha,$$

we have an injection

$$L^2(E) \to \bigoplus_\alpha L^2(E|_{U_\alpha}).$$

Now we can consider the projections $\hat{\varphi}_\alpha : L^2(E|_{U_\alpha}) \to L^2(E|_{U_\alpha})$. It is clearly sufficient to estimate $\|\hat{\varphi}\|_{2p}$ on each subspace. By definition,

$$|\varphi_\alpha(x,k)| < C_\alpha(1+|k|)^m.$$

The plane wave states form a complete set on $L^2(E|_{U_\alpha})$ so

$$\|\varphi_\alpha\|_{2p}^{2p} = \int \frac{d^d q}{(2\pi)^d} \|\hat{\varphi}_\alpha e^{iq\cdot x}\|^{2p} \leqq \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \int_U dx |\varphi_\alpha(x,k)|^{2p} \delta(k-q)$$
$$\leqq C_\alpha \int \frac{d^d k}{(2\pi)^d} (1+|k|)^{2pm},$$

C_α being a constant depending on α. The integral over k is convergent if

$$d + 2pm < 0. \quad \square$$

Since $Y \in PS^{-1}(E)$, we see immediately that

Corollary 2.6. $[\varepsilon_m, M(f)] \in I_{2p}(L^2(E))$ *for* $p > d/2$.
By combining the above results we have,

Proposition 2.7. *Let E be a Hermitian vector bundle with connection over a compact Riemannian spin manifold X. Let H be the space* $L^2(E \otimes S)$ *of square integrable sections, were S is a spin bundle of X. Then H admits an orthogonal decomposition* $H = H_+ \oplus H_-$ *into non-negative and negative eigenspaces of the Dirac operator. There is a continuous embedding of the group of gauge transformations* $C^\infty(\operatorname{Aut} E)$ *into unitary operators in* GL_p *for* $p > \dim X/2$.

We have used Dirac spinors rather than Weyl spinors for simplicity. The embedding using Weyl spinors is more fundamental and will be discussed briefly in Sect. VI.

We will be mostly interested in the case of odd d. In the even-dimensional case, a further refinement is possible. Consider for simplicity again $X = T^d$, and define ε as the sign of the massless Dirac operator. (The result again, generalizes to any Hermitian vector bundle with connection over an even-dimensional spin manifold.) Then there is an operator Γ (chirality, γ_5 in the case $d=4$) that anticommutes with D and has square one. So,

$$\Gamma^2 = 1; \quad \Gamma\varepsilon = -\varepsilon\Gamma.$$

Γ acts on the spin indices alone and not on the representation indices of ρ. Therefore,

$$[\Gamma, M(f)] = 0; \quad f \in \mathrm{Map}(T^d; G).$$

It is convenient to choose a decomposition into positive and negative chirality, $H = H_1 \oplus H_2$,

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$M(f) = \begin{bmatrix} M_1(f) & 0 \\ 0 & M_2(f) \end{bmatrix}.$$

But we already know that

$$[M(f), \varepsilon] \in I_{2p}; \quad p > \frac{(\dim X)}{2}.$$

This means simply that

$$M_1(f) - M_2(f) \in I_{2p}.$$

So we are led to consider a Hilbert space H with two anticommuting orthogonal decompositions $H = H_1 \oplus H_2$ and $H = H_+ \oplus H_-$ given by $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define $GL^{(2p)} \subset GL(H_1) \times GL(H_2)$ by $(g_1, g_2) \in GL^{(2p)}$ if $g_1 - g_2 \in I_{2p}$. Then we have

Proposition 2.8. *For even d, there is a continuous homomorphism* $\mathrm{Map}(T^d, G) \hookrightarrow GL^{(2p)}$ *for* $p > d/2$.

$GL^{(2p)}$ is a subgroup of GL_p, but it is of a different homotopy type. In fact, consider $GL^{2p} = (1 + I_{2p}) \cap GL$.

Proposition 2.9. *$GL^{(2p)}$ is contractible to GL^{2p}.*

Proof. GL^{2p} can be thought of as the subgroup of $GL^{(2p)}$ of the form $\{(1, h)\}$. But we may write

$$(g_1, g_2) = (g_1, g_1)(1, g_1^{-1} g_2),$$

so we have a fiber bundle

$$\begin{array}{c} GL(H_1) \to GL^{(2p)} \\ \qquad\quad \downarrow \quad . \\ \qquad\quad GL^{2p} \end{array}$$

The fiber is contractible, being the general linear group of an infinite-dimensional Hilbert space [K]. So $GL^{(2p)}$ is contractible to GL^{2p}. □

It is known [PS] that GL^{2p} and GL_p are related to Fredholm theory on even- and odd-dimensional manifolds, respectively. We have a more complete realization of this idea, but the relevant groups seem to be $GL^{(2p)}$ and GL_p. $GL^{(2p)}$ is connected, but $\pi_1(GL^{(2p)}) = \mathbf{Z}$. Its second cohomology vanishes. It is interesting to construct its universal covering group and representations of this extension of $GL^{(2p)}$.

Groups such as GL_p, GL^p and $GL^{(2p)}$ play an important role in the non-commutative differential geometry of Connes [C].

We conclude by explaining why the standard methods of quantum field theory fail to produce a highest weight unitary representation of U_p for $p > 1$. (U_p is the unitary subgroup of GL_p.) We can restrict to the Lie algebra $\underline{u_p}$ to see this point.

Let us recall how such a representation can be found for $p = 1$. (See [BR] for example). Let $H = H_+ \oplus H_-$ be the one-particle Hilbert space and u_k a basis for H_+, and v_k for H_-. Any element of $\underline{u_p}$ can be written as

$$\underline{g} = \sum_{k,k'} (\Phi_{kk'} u_k \otimes u_{k'}^+ + \Lambda_{kk'} u_k \otimes v_{k'}^+ + \Lambda'_{kk'} v_k \otimes u_{k'}^+ + \Psi_{kk'} v_k \otimes v_k^+),$$

where

$$\bar{\Phi}_{k'k} = -\Phi_{kk'}, \quad \bar{\Psi}_{kk'} = -\Psi_{k'k}$$

and

$$\bar{\Lambda}_{kk'} = -\Lambda'_{k'k}; \quad \Lambda \in I_{2p}.$$

As is well-known, this representation of $\underline{u_p}$ is not a highest weight representation. (There is no vacuum state, since the Hamiltonian is not bounded below). We now define the Fermionic Fock space following Dirac.

Introduce operators A_k, B_k corresponding to u_k and v_k, respectively.

$$\begin{aligned} &[A_k^+, A_{k'}]_+ = \delta_{kk'}, \\ &[B_k^+, B_{k'}]_+ = \delta_{kk'}, \\ &[A_k, A_{k'}]_+ = [B_k, B_{k'}]_+ = [A_k, B_k]_+ \text{ etc.} = 0. \end{aligned}$$

A representation for this Clifford algebra is found by starting with a vector $|0\rangle$ ("vacuum") satisfying

$$A_k|0\rangle = 0, \quad B_k^+|0\rangle = 0.$$

This says that the annihilation operator for positive energy and the creation operator for negative energy vanish on the vacuum state (i.e., the vacuum state has neither "particles" nor "holes").

Now consider the space of finite linear combinations of

$$A_{k_1}^+ \cdots A_{k_a}^+ B_{l_1} \cdots B_{l_b}|0\rangle.$$

We can declare these to be orthonormal to get an inner product and then complete this normed vector space to get a Hilbert space, the fermionic Fock space.

If $H_\pm$ were finite-dimensional,

$$r(\underline{g}) = \sum_{k,k'} (\Phi_{kk'} A_k^+ A_{k'} + \Lambda_{kk'} A_k^+ B_{k'} + \Lambda'_{kk'} B_k^+ A_{k'} + \Psi_{kk'} B_k^+ B_{k'})$$

would produce a representation of the Lie algebra. But in the infinite-dimensional case, this does not make sense because the infinite sum of operators does not converge. For example $\langle 0|r(\underline{g})|0\rangle$ is divergent in general: $\langle 0|r(\underline{g})|0\rangle = \operatorname{tr} \Psi$.

If $p = 1$, this can be avoided by the process of normal ordering. Define

$$\bar{r}(\underline{g}) = \sum_{kk'} (\Phi_{kk'} A_k^+ A_{k'} + \Lambda_{kk'} A_k^+ B_{k'} + \Lambda'_{kk'} B_k^+ A_{k'} - \Psi_{kk'} B_{k'} B_{k'}^+).$$

Then, $\langle 0|\bar{r}(\underline{g})|0\rangle = 0$. Also, $\| \bar{r}(\underline{g})|0\rangle \|^2 = 2 \operatorname{tr} \Lambda^+ \Lambda < \infty$ since $\Lambda \in I_2$. In fact, we can show in this case that $\bar{r}(\underline{g})$ acting on any state produces a vector of finite length. $\bar{r}$ does not provide a representation of $\underline{u}_p$ but rather of its central extension. This extension is determined by the Kač-Peterson cocycle [KP, BR].

But this will fail if $p > 1$. After normal ordering, the vacuum expectation value is well-defined

$$\langle 0|\bar{r}(\underline{g})|0\rangle = 0.$$

However, now $\| \bar{r}(\underline{g})|0\rangle \|^2 = 2 \operatorname{tr} \Lambda^+ \Lambda$ is not convergent in general. If we consider the embedding of $\operatorname{Map}(X;\underline{g})$ in $\underline{u}_p$, for $\dim X > 1$, the only maps for which this converges are the constants (i.e. global transformations). There is no representation of the algebra $\operatorname{Map}(X,\underline{g})$ in the fermionic Fock space. We will find a representation of its Abelian extension on a larger vector space.

This kind of divergence has physical consequences. For example, anomalies arise from precisely such divergences of quantum field theory. However, the divergences we find here persist even if the anomalies cancel, and are related to the renormalization of the composite operator $\psi^+ \lambda^i \psi(x)$.

III. Properties of Generalized Determinants

The ordinary determinants is defined only for linear operators of the type $1 + A$, where A is a trace-class operator. However, there is a generalization $\det_p$ for ecah integer $1 \leqq p < \infty$ such that $\det_p(1 + A)$ exists for $A \in I_p$ and shares some of the basic properties of the ordinary determinant; an account of these properties together with references to the original papers can be found in [S]. Here we shall give the definition of $\det_p$ and list some of its properties for the convenience of the readers.

For each bounded linear operator A let

$$R_p(A) = -1 + (1 + A) \exp\left[\sum_{j=1}^{p-1} (-1)^j \frac{A^j}{j}\right] \tag{3.1}$$

for any $p \in \mathbf{N}^+$. By expanding $R_p(A)$ as a Taylor series of the powers A^n, one sees that the first non-vanishing term is of order p. Thus, in particular $R_p(A) \in I_1$ if $A \in I_p$. It follows that

$$\det_p(1 + A) = \det(1 + R_p(A)) \tag{3.2}$$

exists for any $A \in I_p$. Since R_p is analytic and det is continuous, and $\det_p$ is a continuous function of A (in the I_p topology).

Note that

$$\begin{aligned}\log \det_p(1 + A) &= \log \det(1 + R_p(A)) \\ &= \operatorname{tr} \log(1 + R_p(A)) = \operatorname{tr}\left[\log(1 + A) + \sum_{j=1}^{p-1} (-1)^j \frac{A^j}{j}\right] \\ &= \operatorname{tr}\left((-1)^p \frac{A^p}{p} + (-1)^{p+1} \frac{A^{p+1}}{p+1} + \cdots\right).\end{aligned} \tag{3.3}$$

Thus $\log \det_p(1 + A)$ can be thought of as a regularization of $\det(1 + A)$, where the first $p - 1$ terms have been subtracted in the expansion of $\log(1 + A)$. The following proposition has been proven in [S], p. 107.

Proposition 3.1. *Let $A \in I_p$. Then*

(a) $1 + A$ *is invertible* iff $\det_p(1 + A) \neq 0$.
(b) *If $A \in I_{p-1}$, then*

$$\det_p(1 + A) = \det_{p-1}(1 + A) \cdot \exp\left[(-1)^{p-1} \operatorname{tr} \frac{A^{p-1}}{p-1}\right].$$

In particular from (b) *it follows that* $\det_2(1 + A) = \det(1 + A) \cdot e^{-\operatorname{tr} A}$ *for* $A \in I_1$: $\det_1(1 + A) = \det(1 + A)$ *by the definition (3.1)–(3.2) above.*

Proposition 3.2. *For each $p \in \mathbf{N}^+$ there is a symmetric polynomial $\gamma_p(A, B)$ of two variables $A, B \in 1 + I_p$ such that*

$$\det_p AB = \det_p A \cdot \det_p B \cdot e^{\gamma_p(A,B)}.$$

Proof. This is clear for $p = 1$, since $\det_1 A = \det A$; $\gamma_1 \equiv 0$. We prove the equation by induction on p. Suppose $A, B \in 1 + I_{p-1}$. Then

$$\begin{aligned}\det_p AB &= \det_{p-1} AB \cdot \exp\left[(-1)^{p-1} \operatorname{tr} \frac{(AB - 1)^{p-1}}{p-1}\right] \\ &= \det_{p-1} A \cdot \det_{p-1} B \cdot \exp\left[\gamma_{p-1}(A, B) + (-1)^{p-1} \operatorname{tr} \frac{(AB - 1)^{p-1}}{p-1}\right] \\ &= \det_p A \cdot \det_p B \cdot \exp\left[\gamma_{p-1}(A, B) + (-1)^{p-1} \operatorname{tr} \frac{(AB - 1)^{p-1}}{p-1}\right. \\ &\quad \left. - (-1)^{p-1} \operatorname{tr} \frac{(A - 1)^{p-1}}{p_1} - (-1)^{p-1} \operatorname{tr} \frac{(B - 1)^{p-1}}{p-1}\right],\end{aligned}$$

by the induction hypothesis. Denoting the expression in the square brackets by $\gamma_p(A, B)$ we have proven the claim for the index p in the case $A, B \in 1 + I_{p-1}$. Using the continuity of $\det_p$ this same relation must hold for any pair $A, B \in 1 + I_p$ ($I_{p-1} \subset I_p$ is dense). □

In the case $p=2$, we have

$$\det_2 AB = \det_2 A \cdot \det_2 B \cdot e^{-\operatorname{tr}(A-1)(B-1)}. \tag{3.4}$$

Proposition 3.3. *Define* $\omega_p(A,B)=\det_p B\cdot e^{\gamma_p(A,B)}$. *Then* $\omega_p(A,BC)=\omega_p(AB,C)\cdot\omega_p(A,B)$ *for all* $A,B,C\in 1+I_p$.

Proof. If A is invertible we may write

$$\omega_p(A,B)=\frac{\det_p AB}{\det_p A}, \tag{3.5}$$

and thus for invertible A and B the claim is trivially true. However, both sides of the equation to be proven are continuous functions of the variables A,B,C. Since the space of invertible linear operators is dense in $1+I_p$, the equation holds for all $A,B,C\in 1+I_p$. □

IV. The Determinant bundle and the Abelian Extension of GL_p

Let H be a complex separable Hilbert space with an orthogonal decomposition $H=H_+\oplus H_-$ to a pair of closed infinite-dimensional subspaces. In this section we shall study in more detail the properties of the group GL_p acting in H, and associated homogeneous spaces and line bundles. The index p is an arbitrary positive integer. In addition, we define GL_0 to consist of invertible bounded operators

$$g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

such that the blocks b and c are finite rank operators; we define GL_∞ to consist of operators with b and c compact. Then we have the inclusions

$$GL_0\subset GL_1\subset GL_2\subset\cdots\subset GL_\infty. \tag{4.1}$$

Each GL_p is dense in $GL_{p'}$ for $p\leqq p'$, with respect to the topology of $GL_{p'}$. Let $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_p$. Using the fact that g is invertible and b,c are compact, it follows that the diagonal blocks a and d are Fredholm operators. Now $g_t=\begin{pmatrix} a & td \\ tc & d\end{pmatrix}$ is a Fredholm operator for all $0\leqq t\leqq 1, g_1=g$ is invertible and $g_0=\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}$. Thus, index $g_0=0$ and therefore index $a=-$ index d. The group GL_p can be split into disconnected components labeled by $n=$ index a,

$$GL_p=\bigcup_{n\in\mathbf{Z}} GL_p^{(n)}.$$

In this and the following section we shall denote shortly by GL_p the connected component $GL_p^{(0)}$. In Sect. VI we shall make some remarks about the full group.

There is another infinite sequence of linear groups, closely related to (4.1). We denote $GL^p=GL(H_+)\cap(1+I_p)$, where $p\in\mathbf{N}\cup\{\infty\}$; $I_0=\{\text{finite rank operators}\}$ and $I_\infty=\{\text{compact operators}\}$. Then

$$GL^0 \subset GL^1 \subset GL^2 \subset \cdots \subset GL^\infty. \tag{4.2}$$

The group $GL(H)$ is contractible (when $\dim H = \infty$), [K]. However, GL^p and GL_p have non-trivial topologies [P]. To understand the relation between (4.1) and (4.2) it is useful to define the group

$$\mathscr{E}_p = \{(g, q) | g \in GL_p, \quad q \in GL(H_+), \quad aq^{-1} - 1 \in I_p\} \subset GL_p \times GL(H_+),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The group multiplication is $(g_1, q_1)(g_2, q_2) = (g_1 g_2, q_1 q_2)$. The topology is not the product of space topology, but the topology given by the norm

$$\|(g, q)\| = \|a\| + \|d\| + \|b\|_{2p} + \|c\|_{2p} + \|a - q\|_p. \tag{4.3}$$

The group GL^p acts from the right on $\mathscr{E}_p$ by $(g, q)\cdot t = (g, qt)$. The quotient $\mathscr{E}_p/GL^p$ is GL_p. Thus $\mathscr{E}_p$ can be viewed as a principal GL^p bundle over GL_p. As shown in [PS], the group $\mathscr{E}_p$ is contractible. From this follows

$$\pi_i(GL_p) \simeq \pi_{i-1}(GL^p) \tag{4.4}$$

for the homotopy groups. The homotopy properties do not depend on the index p: all the spaces GL_p are homotopy equivalent for $0 \leqq p \leqq \infty$, [PS, P].

We denote by B_p the subgroup of GL_p consisting of operators of the type $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and by Gr_p the homogeneous space GL_p/B_p. The points on the *Grassmannian* Gr_p can be thought of as infinite-dimensional planes $W \subset H$. (Gr_p means here the connected component, corresponding to the block a of g having Fredholm index zero.) To each $gB_p \in GL_p/B_p$ we associate with plane $W = g \cdot H_+$. Let

$$\mathrm{pr}_\pm : W \to H_\pm$$

be the orthogonal projections. Using the fact that the off-diagonal blocks of g are in I_{2p}, it follows that pr_- is in the class I_{2p}; because the diagonal blocks of g are Fredholm, the projection pr_+ is a Fredholm operator. Using the fact that B_p is contractible and the homotopy equivalence $GL_p \approx GL_{p'}$, we see that $\mathrm{Gr}_p \approx \mathrm{Gr}_{p'}$ for all $p, p' \geqq 0$. More important than the homotopy in our discussion will be the cohomology of these spaces. In particular, the group extensions we shall construct are related to the Chern class $c_1 \in H^2(\mathrm{Gr}_p, \mathbf{Z})$. The Chern classes of Gr_p were recently derived in [Q]. Actually we shall not use his results; instead, one can derive a form for $c_1(\mathrm{Gr}_p)$ from the group extension $\widehat{GL}_p$ of GL_p below.

We define the *Stiefel manifold* $St_p = \mathscr{E}_p/B_p$, where the action of $k = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \in B_p$ on $\mathscr{E}_p$ is given by

$$(g, q)\cdot k = (gk, q\alpha).$$

Let $\{e_1, e_2, \ldots,\}$ be an orthogonal basis of H_+ and $\{e_0, e_{-1}, e_{-2} \cdots\}$ a basis of H_-. Let $\{w_1, w_2, \ldots,\} = w$ be a basis of $W \in \mathrm{Gr}_p$. We can write

$$\mathrm{pr} + w_i = \sum_{j=1}^{\infty} (w_+)_{ji} e_j.$$

We say that the basis w is *admissible* if $w_+ \in 1 + I_p$. Every $W \in \mathrm{Gr}_p$ has an admissible basis: Let $W = g \cdot H_+$ with $g \in GL_p$. Since we are working in the connected component of the identity in GL_p, index $a = 0$. It follows that $a = g + t$, where g is invertible and t is of finite rank. Now $aq^{-1} \in 1 + I_p$ and $w_i = gq^{-1} e_i$ is an admissible basis. Any two admissible bases in a given W are connected by a basis transformation of type $1 + I_p$. The mapping $(g, q) \mapsto \{gq^{-1} e_i\}$ defines a $1-1$ correspondence between St_p and the set of all admissible basis for all W.

The space St_p is a principal GL^p bundle over Gr_p. The bundle projection is $\{w_i\} \mapsto$ the plane spanned by the vectors w_i. The right action of $t \in GL^p$ is just the basis transformation $w_i' = \Sigma w_j t_{ij}$. It is sometimes convenient to write a basis w as a column vector $\begin{pmatrix} w_+ \\ w_- \end{pmatrix}$, $w_\pm = \mathrm{pr}_\pm w$. Then $w_+ \in 1 + I_p$ and $w_- \in I_{2p}$. The topology of St_p is defined by the metric

$$d(w, w') = \| w_+ - w'_+ \|_p + \| w_- - w'_- \|_{2p}.$$

The right action of GL^p on St_p is written shortly as $\begin{pmatrix} w_+ \\ w_- \end{pmatrix} \mapsto \begin{pmatrix} w_+ t \\ w_- t \end{pmatrix}$.

Next, we define a right action of GL^p on $St_p \times \mathbf{C}$ by

$$(w, \lambda) \cdot t = (wt, \lambda \omega_p(w_+, t)^{-1}). \tag{4.5}$$

This action is clearly free and we can define

$$\mathrm{Det}_p = (St_p \times \mathbf{C})/GL^p. \tag{4.6}$$

As a quotient of two complex spaces, Det_p is also a complex manifold. Furthermore, Det_p is a holomorphic line bundle over the Grassmannian Gr_p. The projection is given by $[(w, \lambda)] \mapsto$ the plane spanned by $\{w_1, w_2, \ldots\}$. (In general, we denote by $[x]$ the equivalence class represented by an element x.)

The group GL_p acts on the base manifold Gr_p but the action cannot be lifted to the bundle Det_p for $p \geqq 1$. The obstruction comes from the non-triviality of the bundle Det_p. In fact, already the subbundle obtained by restricting the base to $\mathrm{Gr}_1 \subset \mathrm{Gr}_p (p \geqq 1)$ is non-trivial: If $w_+ \in 1 + I_1$, then

$$\begin{aligned} \omega_p^{-1}(w_+, t) &= (\det_p t)^{-1} \cdot e^{-\gamma_p(w_+, t)} = (\det t)^{-1} \cdot e^{-\beta_p(t)} \\ &\quad \cdot e^{-\gamma_p(w_+, t)} = (\det t)^{-1} e^{\beta_p(w_+) - \beta_p(w_+ t)}, \end{aligned}$$

where we have used the fact that $\det_p A = \det A \cdot e^{\beta_p(A)}$ for some polynomial β_p, $A \in 1 + I_1$. Thus the cocycle $w_p(w_+, t)$ is cohomologous (in the group cohomology of Eilenberg–MacLane) to the cocycle given by the inverse of the determinant and therefore the bundle over Gr_1 is equivalent to the non-trivial line bunle Det studied in [PS].

In the case $p = 1$ there is a central extension of GL_1 which acts in Det_1, [PS]. Since $\pi_1(GL_1) = 0$, [P], the various central extensions are classified by elements of $H^2(GL_1, \mathbf{Z}) = \pi_2(GL_1) = \mathbf{Z}$. The generator of $H^2(GL_1)$ can be represented by a constant coefficient two-form which corresponds to the Lie algebra extension determined by the two-cocycle

$$\eta_1(X, Y) = \tfrac{1}{8} \operatorname{tr} \varepsilon [[\varepsilon, X], [\varepsilon, Y]].$$

The general two-cocycle for $\underline{gl}_1$ can be thus written as

$$\lambda\eta_1(X, Y) + s[(X, Y]),$$

where $\lambda \in \mathbf{C}$ and $s:\underline{gl}_1 \to \mathbf{C}$ is any continuous linear form. One can check that for $\lambda \neq 0$ there is no way to choose s in such a way that the sum above is finite for all $X, Y \in \underline{gl}_p$ when $p > 1$: When restricted to the diagonal blocks, s gives a linear form on $\underline{gl}(H_+) \oplus \underline{gl}(H_-)$, continuous in the operator norm topology. The diverging terms in η_1 (for $p > 1$) are due to the off-diagonal blocks of X and Y; so let us assume that X, Y are off-diagonal. Now $[X, Y]$ consists of diagonal blocks, and therefore $s([X, Y])$ is necessarily finite for any p. Thus, the divergence cannot be removed by adding the trivial two-cocycle s. Since any central extension of $\underline{gl}_p(p \geqq 1)$ gives by restriction a central extension of $\underline{gl}_1$, we conclude that $\underline{gl}_p$ (and thus GL_p) does not have any non-trivial central extension for $p > 1$. However, GL_p does have non-trivial Abelian extensions, to be described below.

Lemma 4.1. *There are smooth functions $\alpha(g, q; w)$ on $\mathscr{E}_p \times St_p$ such that*

$$\frac{\alpha(g, q; wt)}{\alpha(g, q; w)} = \frac{\omega_p(w_+, t)}{\omega_p((gwq^{-1})_+, qtq^{-1})} \tag{4.7}$$

for $t \in GL^p$. Let $F = F(w) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ be the linear operator in $H = H_+ \oplus H_-$ such that $F|_W = +1$ and $F|_{W^\perp} = -1$, where W is the plane determined by the basis $w = \{w_i\}$. A general solution of (4.7) is given by

$$\alpha(g, q; w) = f(g, q; W) \frac{\det_p w_+}{\det_p (gwq^{-1})_+} \cdot \frac{\det_p \frac{1}{2}(q^{-1}a(F_{11} + 1) + q^{-1}bF_{21})}{\det_p \frac{1}{2}(F_{11} + 1)}, \tag{4.8}$$

where $f:\mathscr{E}_p \times \mathrm{Gr}_p \to \mathbf{C}^\times$ is an arbitrary smooth function.

Proof. If we can find one solution α of (4.7), then the general solution is clearly obtained by multiplying by a function on $\mathscr{E}_p \times GR_p$. Formally, $\det_p w_+/\det_p(gwq^{-1})_+$ is a solution of (4.7). However, this function has zeroes and singularities. We can regularize it by multiplying by a function on $\mathscr{E}_p \times \mathrm{Gr}_p$. Let

$$h = \begin{pmatrix} w_+ & \alpha \\ w_- & \beta \end{pmatrix}$$

be an invertible operator such that $W = h \cdot H_+$ and $W^\perp = h \cdot H_-$. Denote

$$h^{-1} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}.$$

Then $F = h\varepsilon h^{-1}$, where

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and in particular $F_{11} = w_+x - \alpha u = 2w_+x - 1$ and $F_{21} = w_-x - \beta u = 2w_-x$.

Thus

$$\exp[\gamma_p(xw_+,(gwq^{-1})_+)-\gamma_p(w_+,\tfrac{1}{2}q^{-1}(a(F_{11}+1)+bF_{21}))]$$
$$=\exp[\gamma_p(xw_+,(gwq^{-1})_+)-\gamma_p(w_+,q^{-1}aw_+x+q^{-1}bw_-x)]$$
$$=\frac{\det_p(q^{-1}aw+xw_++q^{-1}bw_-xw_+)}{\det_p(q^{-1}aw_++q^{-1}bw)\det_p(xw_+)}$$
$$\cdot\frac{\det_p w_+\cdot\det_p(q^{-1}aw_+x+q^{-1}bw_-x)}{\det_p(q^{-1}aw_+xw_++q^{-1}bw_-xw_+)}$$
$$=\frac{\det_p w_+}{\det_p(gwq^{-1})_+}\cdot\frac{\det_p(q^{-1}a\cdot\frac{1}{2}(F_{11}+1)+q^{-1}b\cdot\frac{1}{2}F_{21})}{\det_p\frac{1}{2}(F_{11}+1)},$$

where we have used Proposition (3.2) and the symmetry $\det_p AB=\det_p BA$. This shows that the ratio of the determinants in (4.8) is a regular function. □

Let $\pi:St_p\to Gr_p$ denote the canonical projection.

Proposition 4.2. *The formula*

$$(g,q,\mu)\cdot(w,\lambda)=(gwq^{-1},\mu(\pi(w))\lambda\alpha(g,q;w)),$$

where α *is any fixed solution of* (4.7), *defines an action of* $\mathscr{E}_p\times\mathrm{Map}(\mathrm{Gr}_p,\mathbf{C}^\times)$ *on* Det_p; *the multiplication in* $\mathscr{E}_p\times\mathrm{Map}(\mathrm{Gr}_p,\mathbf{C}^\times)$ *is defined by* $(g_1,q_1,\mu_1)(g_2,q_2,\mu_2)=(g_1g_2,q_1q_2,\mu_1(g_{2.}F)\mu_2\alpha(g_1,q_1;g_2wq_2^{-1})\times\alpha(g_2,q_2;w)\alpha(g_1g_2,q_1q_2;w)^{-1})$.

Proof. We have to show that $(g,q,\mu)\cdot(w,\lambda)$ and $(g,q,\mu)\cdot(wt,\lambda\omega_p(w_+,t)^{-1})$ represent the same class in $\mathrm{Det}_p=(St_p\times\mathbf{C})/GL^p$. But

$$(g,q,\mu)\cdot(wt,\lambda\omega_p(w_+,t)^{-1})=(gwtq^{-1},\mu(\pi(w))\lambda\omega_p(w_+,t)^{-1}\alpha(g,q;wt))$$
$$=(gwq^{-1},\mu(\pi(w))\lambda\omega_p(w_+,t)^{-1}\omega_p((gwq^{-1})_+,qtq^{-1})\alpha(g,q;wt)).$$

This represents the class of $(g,q,\alpha)\cdot(w,\lambda)$ iff

$$\omega_p((gwq^{-1})_+,qtq^{-1})\omega_p(w_+,t)^{-1}\alpha(g,q;wt)=\alpha(g,q;w);$$

this is precisely Eq. (4.7). The triple product of α functions is really a function on Gr_p, and not on St_p. To see this one has to replace the basis w by $wt(t\in GL^p)$ and to show that value of the product does not change; but this is an easy consequence of a repeated use of (4.7). □

Theorem 4.3. *There is an Abelian extension of* GL_p *by* $\mathrm{Map}(\mathrm{Gr}_p,\mathbf{C}^\times)$ *which acts on* Det_p. *There extension is*

$$\widehat{GL}_p=(\mathscr{E}_p\times\mathrm{Map}(\mathrm{Gr}_p,\mathbf{C}^\times))/N,$$

where N *is the normal subgroup consisting of elements* $(1,q,\mu_q)$, *where* $\mu_q(w)=\alpha(1,q,w)^{-1}\cdot\omega_p(w_+,q^{-1})^{-1}$, $q\in GL^p$, *and the action on* Det_p *is given by Proposition 4.2.*

Proof. An element $(g,q,\mu)\in\mathscr{E}_p\times\mathrm{Map}(\mathrm{Gr}_p,\mathbf{C}^\times)$ belongs to the kernel of the group action on Det_p iff $g=1$ and $(wq^{-1},\alpha(1,q;w)\mu(\pi(w)))=(w,1)\cdot q^{-1}$. The last relation is equivalent to

$$\alpha(1,q;w)\mu(\pi(w))=\omega_p(w_+,q^{-1})^{-1}.\quad\square$$

We shall study the Abelian extension in the case $p=2$ in more detail. This case corresponds to the physically important problem of obtaining representations of current algebras in $3+1$ space-time dimensions, as we mentioned in the Introduction. For $p=2$ one can adjust the function f in (4.8) in such a way that

$$\alpha = \exp - \operatorname{tr}[(1-q^{-1}a)(w_+ - 1) + q^{-1}b(\tfrac{1}{2}F_{21} - w_-)]. \tag{4.9}$$

For this choice of α the group N consists of elements $(1, q, (\det_2 q^{-1})^{-1})$, since now $\alpha(1, q; w) = \exp - \operatorname{tr}(1-q^{-1})(w_+ - 1) = \det_2 w_+ \cdot \det_2 q^{-1}/\det_2 w_+ q^{-1} = \omega_2(w_+, q^{-1})^{-1} \cdot \det_2 q^{-1}$. We shall compute the local two-cycle corresponding to the extension $\widehat{GL}_2$. Near the unit element $g=1$ we can define the local section $\Gamma : \widehat{GL}_2 \to GL_2$ by

$$\Gamma(g) = (g, a, 1) \bmod N. \tag{4.10}$$

The two-cocycle is defined by

$$\Gamma(g_1)\Gamma(g_2) = \Gamma(g_1 g_2)(1, 1, \xi(g_1, g_2)), \tag{4.11}$$

where $\xi(g_1, g_2) \in \operatorname{Map}(GL_2, \mathbf{C}^\times)$. From Proposition 4.2, we get

$$\Gamma(g_1)\Gamma(g_2) = (g_1 g_2, a_1 a_2, \alpha(g_1, a_1; g_2 w q_2^{-1})\alpha(g_2, a_2; \cdot)\alpha(g_1 g_2, a_1 a_2; \cdot)^{-1}).$$

On the other hand,

$$\begin{aligned}\Gamma(g_1 g_2) &= (g_1 g_2, a(g_1 g_2), 1)\\ &\equiv (g_1 g_2, a_1 a_2, [\det_2(a_2^{-1} a_1^{-1} a(g_1 g_2))]^{-1} \alpha(g_1 g_2, a(g_1 g_2); w a_2^{-1} a_1^{-1} a(g_1 g_2))\\ &\quad \cdot \alpha(g_1 g_2, a_1 a_2; \cdot)^{-1} \alpha(1, a(g_1 g_2)^{-1} a_1 a_2; \cdot)) \bmod N.\end{aligned}$$

Therefore,

$$\begin{aligned}\xi(g_1, g_2) &= \det_2(a_1^{-1} a_2^{-1} a(g_1 g_1)) \cdot \alpha(g_1 g_2, a(g_1 g_2); w a_2^{-1} a_1^{-1} a(g_1 g_2))^{-1}\\ &\quad \cdot \alpha(1, a(g_1 g_2)^{-1} a_1 a_2; w) \alpha(g_1, a_1; g_2 w q_2^{-1}) \alpha(g_1 g_2, a_1 a_2; w).\end{aligned} \tag{4.12}$$

In particular, if g_1 and g_2 are of the type

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix},$$

then $\xi(g_1, g_2) = 1$. In general, the expression for ξ is rather complicated but the corresponding cocycle for the Lie algebra commutators is much simpler. The Lie algebra of $\widehat{GL}_p$ is as a vector space equal to $\underline{gl}_p \oplus \operatorname{Map}(\mathrm{Gr}_p, \mathbf{C})$, where $\underline{gl}_p$ is the Lie algebra of GL_p. The commutator in $\underline{gl}_p$ can be written as

$$[(X, \mu), (Y, v)] = ([X, Y], X \cdot v - Y \cdot \mu + \eta(X, Y; \cdot)), \tag{4.13}$$

where η is an antisymmetric bilinear form on $\underline{gl}_p$ taking values in $\operatorname{Map}(\mathrm{Gr}_p, \mathbf{C})$ and the Lie derivative of a function v on Gr_p to the direction of the vector field X (defined by the GL_p action on Gr_p) is denoted by $X \cdot v$. From the Jacobi identity, it follows that η has to satisfy the equation

$$\eta([X, Y], Z) + \eta([Y, Z], X) + \eta([Z, X], Y) - Z \cdot \eta(X, Y) - X \cdot \eta(Y, Z) - Y \cdot \eta(Z, X) = 0. \tag{4.14}$$

Let $\exp tX$ and $\exp tY$ be two one-parameter subgroups in GL_p. Then

$$\frac{\partial^2}{\partial t \partial s} \Gamma(e^{tX})\Gamma(e^{sY})\Gamma(e^{-tX})\Gamma(e^{-sY})|_{t=s=0} = ([X, Y], 0, \eta(X, Y)). \tag{4.15}$$

We do not have a closed formula for η valid for an arbitrary $p \geqq 1$. However, in the case $p = 1$ one gets

$$\eta(X, Y) = \mathrm{Tr}(b_X c_Y - b_Y c_X),$$

which is the Kač–Peterson cocycle, [KP]. For $p = 2$ we have derived for the Lie algebra of the unitary subgroup $U_2 \subset GL_2$ the formula

$$\eta(X, Y) = \tfrac{1}{8} \mathrm{tr}[[\varepsilon, X], [\varepsilon, Y]](\varepsilon - F), \tag{4.16}$$

where X, Y are anti-Hermitian operators (the off-diagonal blocks are in I_4 and the diagonal blocks are bounded operators). One can define a two-cycle for $\underline{gl}_2$ by a complex extension from (4.16); however, this does not correspond to our choice of the group cocycle ξ defined by our choice (4.9) for α.

Remark. Because the action of GL_p on Gr_p is smooth, it is easy to define different topologies on $\mathrm{Map}(\mathrm{Gr}_p, \mathbf{C}^\times)$ which make the extension GL_p a topological group. In fact, in the cases $p = 1, 2$ (and probably for higher p, too) the extension can be defined in such a way that it becomes a Banach Lie group. In the case $p = 1$ we have the known central extension by $\mathbf{C}^\times$, which is a Banach Lie group, [PS]. In the case $p = 2$, we can choose α in such a way, that the extension of GL_2 is by the functions $\lambda \exp \mathrm{tr}\, \xi(F - \varepsilon)$, where

$$\xi \in \begin{pmatrix} I_2 & I_{4/3} \\ I_{4/3} & I_2 \end{pmatrix}, \quad \lambda \in \mathbf{C}^\times$$

acts on Det_2. The parameter space of these functions is $I_2 \times I_2 \times I_{4/3} \times I_{4/3} \times \mathbf{C}^\times$, which is a Banach space with a smooth U_p action (the action is $\xi \mapsto g\xi g^{-1}, g \in U_p$). The choice of α needed is precisely the choice leading to the affine form (4.16) of the infinitesimal two-cocycle.

Next we shall construct a metric on the bundle Det_p. Define a function $l: St_p \to \mathbf{R}_+$ by $l(w) = e^{-1/2\gamma_p(w_+, w_+^\dagger)}$.

Note that

$$\frac{|\det_p w_+|}{|\det_p w_+ w_+^\dagger|^{1/2}} = \frac{|\det_p w_+|}{|\det_p w_+|^{1/2} |\det_p w_+^\dagger|^{1/2}} \cdot e^{-1/2\gamma_p(w_+, w_+^\dagger)} = e^{-1/2\gamma_p(w_+, w_+^\dagger)} = l(w), \tag{4.17}$$

where we have used the property $\det_p A^\dagger = \overline{\det_p A}$ of the generalized determinant; the latter follows by induction from Proposition 3.1(b).

Proposition 4.4. *Let $e \in \mathrm{Det}_p$ be represented by a pair (w, λ), where $w \in St_p$ is a unitary basis and $\lambda \in \mathbf{C}$. Then, $|e| = |\lambda| l(w)$ defines a metric in Det_p which is invariant under the subgroup $\hat{U}_p \subset \widehat{GL}_p$ corresponding to triplets (g, q, λ) such that g and q are unitary and*

$$|\lambda(F)| = \frac{l(w)}{l(gwq^{-1})} \cdot |\alpha(g, q; w)|^{-1}.$$

Proof. Suppose that $e=[(w',\lambda')]$, where w' is another unitary basis. Then $w'=wt$ and $\lambda'=\lambda\omega_p(w,t)^{-1}$ for some unitary $t\in 1+I_p$. The metric is well-defined if $l(w')|\lambda'|=l(w)|\lambda|$; this is equivalent to

$$l(wt)\left|\frac{\det_p w_+}{\det_p w_+ t}\right|=l(w), \tag{4.18}$$

but the latter follows immediately form (4.17).

The condition $|(g,q,\lambda)\cdot e|=|e|$ can be written as

$$l(gwq^{-1})|\alpha(g,g;w)|\,|\lambda(F)|=l(w).$$

The proof is completed by showing that $\mu(g,q;w)=l(w)l(gwq^{-1})^{-1}|\alpha|^{-1}$ is really a function of $F=\pi(w)$ and not of w. Again, using (4.18) we get

$$\frac{\mu(g,q;wt)}{\mu(g,q;w)}=\left|\frac{\det_p w_+ t}{\det_p w_+}\right|\cdot\left|\frac{\det_p(gwq^{-1})_+}{\det_p(gwtq^{-1})_+}\right|\cdot\left|\frac{\alpha(g,q;w)}{\alpha(g,q;wt)}\right|.$$

The right-hand sude is $=1$, by (4.7). □

Suppose there is an invariant measure m on Gr_p. Then we could define a unitary representation of $\hat{U}_p$ in the space of L_2-sections of Det_p^* as follows. A section of Det_p^* is a function $\psi:St_p\to\mathbf{C}$ such that

$$\psi(wt)=\psi(w)\omega_p(w_+,t). \tag{4.19}$$

An inner product for L_2-sections is defined by

$$\langle\psi_1,\psi_2\rangle=\int\bar{\psi}_1\psi_2 l(w)^{-2}dm. \tag{4.20}$$

From (4.19) and (4.20) it follows that $\bar{\psi}_1\psi_2 l^{-2}$ is invariant under the transformation $w\mapsto wt$, thus being really a function on Gr_p. The action of $\hat{U}_p$ on sections is

$$(T(g,q,\lambda)\psi)(w)=\psi(g^{-1}wq)\alpha(g,q;g^{-1}wq)^{-1}\lambda(g^{-1}F)^{-1}.$$

Using the invariance of the measure and the invariance of the metric under $\hat{u}_p$ it is easily seen that the inner product (4.20) is invariant. Quasi-invariant measures have been recently studied by Pickrell [Pi] in the case $p=1$, but we do not know at the moment if his results can be extended to higher values of p; quasi-invariance is really all we need, since in that case the loss of unitarity due to non-invariance of the measure can be compensated by adding a factor $\sqrt{m_g}$ under the integral sign in (4.20), where m_g is the Radon–Nikodym derivative of m with respect to $g\in U_p$.

Even without the inner product, the representation of $\widehat{GL}_p$ in the space of sections of Det_p^* has the important property that there is a *vacuum vector*:

Theorem 4.5. *Suppose* (*for the sake of simplicity*) *that the extension* $\widehat{GL}_p$ *is defined by the choice* $f=1$ *in* (4.8). *Let* $\psi:St_p\to\mathbf{C}$ *be the section of* Det_p^* *defined by* $\psi(w)=\det_p w_+$. *Then*

$$T(g,a,1)\psi=\psi$$

for any $g=\begin{pmatrix}a&0\\c&d\end{pmatrix}$ *in* GL_p.

Proof. It follows directly from the fact that with the above choice $\alpha(g, a; w) \equiv 1$ and that $\det_p a^{-1} w_+ a = \det_p w_+$. In the general case one must consider the subgroup consisting of elements (g, a, μ_g), $\mu_g(F) = f(g, a; F)^{-1}$. □

Remark. In the case $p = 1$, the vector ψ is the highest weight vector in the Fock representation of $\underline{gl}_1$, [PS]. The vector of highest weight (in the mathematical terminology) is the state of lowest energy (in physics language).

V. Holomorphic Bundles and Group Actions

The action of $\widehat{GL}_{2p}$ on Det_p constructed in Sect. IV does not preserve the holomorphic structure of Det_p. This is because the function $\alpha: \mathscr{E}_p \times St_p \to \mathbf{C}^\times$ is not holomorphic.

However, the holomorphic action of a complex Lie group on a line bundle is a useful notion in representation theory. The well-known Borel–Weil theory [W] produces the antisymmetric tensor (wedge) representation of a finite-dimensional general linear group using its action on the Det line bundle over the Grassmannian [PS]. This has been extended to $\widehat{GL}_1$ by Pressley and Segal [PS].

A highest weight-vector in a representation of a Lie algebra is one that is annihilated by the "step-down" operators. This notion makes sense only on a complex Lie algebra. The analogous notion for a group therefore involves a holomorphic representation of a complex Lie group. One defines a higher weight vector as one that spans a one-dimensional representation of a parabolic subgroup. We will be able to construct such a highest weight representation in this section.

We will show that there are no non-trivial holomorphic functions on Gr_p. So it will not be possible to choose the cocycle of $\mathscr{E}$ to be a holomorphic function on Gr_p. However, we will finite another coset-space $\mathbf{C}\,\mathrm{Gr}_p$ (which is a complexification of Gr_p), which does admit non-trivial holomorphic functions. The action of GL_p on $\mathbf{C}\,\mathrm{Gr}_p$ lifts to an action of an Abelian extension $\widetilde{GL}_p$ of GL_p by $\mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p)$ on $\mathbf{C}\,\mathrm{Det}_p$. This action preserves the holomorphic structure on $\mathbf{C}\,\mathrm{Det}_p$.

We will then also construct an infinite-dimensional vector bundle on Gr_p admitting an action of $\widetilde{GL}_p$ that preserves the holomorphic structure. There is a linear representation of $\widetilde{GL}_p$ on the space of holomorphic sections of this bundle.

Let us begin by recalling a similarity between Gr_p and a compact complex manifold [PS].

Proposition 5.1. *Any holomorphic function on* Gr_p *is constant on each connected component.*

Proof. Gr_0 is a dense subset of Gr_p. Any holomorphic function on Gr_p will therefore restrict to one on Gr_0. However, Gr_0 is the inductive limit of finite-dimensional Grassmannians. These are compact complex manifolds, and therefore holomorphic functions are constant on each connected component on them. So any holomorphic function on Gr_0 is constant on each connected component. □

This leads to the result that for $p > 1$, there is no interesting holomorphic solution to the function α of Lemma 4.1. If there were, we would see that the triple product of function α in Proposition 4.2 is a holomorphic function on Gr_p, and therefore constant [on the connected component which is all we are interested in now]. So we would be constructing an extension of GL_p by the space of constant functions, which would be a central extension. For $p > 1$, GL_p has no non-trivial central extensions.

Now consider the space

$$\mathbf{C}\,\mathrm{Gr}_p = GL_p/GL_+ \times GL_-, \tag{5.1}$$

where $GL_+ \times GL_- = GL(H_+) \times GL(H_-)$ is the subgroup of elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

$\mathbf{C}\,\mathrm{Gr}_p$ can be viewed as the space of infinite-dimensional planes (W_1, W_2) which are *transverse* [i.e. $W_1 \cap W_2 = \{0\}$]. $\mathbf{C}\,\mathrm{Gr}_p$ is the complexification of Gr_p. To see this, note that

$$\mathrm{Gr}_p = U_p/U_+ \times U_- \tag{5.2}$$

and that GL_p is the complexification of U_p. Unlike Gr_p, $\mathbf{C}\,\mathrm{Gr}_p$ does admit non-trivial holomorphic functions. Any such function can be viewed as a holomorphic function

$$f : GL_p \to \mathbf{C}$$

satisfying

$$f(gh^{-1}) = f(g), \quad \text{if } h = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} \in GL_+ \times GL_-. \tag{5.3}$$

An example is

$$f(g) = \det_p a\alpha, \tag{5.4}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.5}$$

as usual and

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{5.6}$$

Note that $\det_p a\alpha$ is invariant under $GL_+ \times GL_-$ but not under $\mathscr{B}_p$. So it is a holomorphic function on $\mathbf{C}\,\mathrm{Gr}_p$ and not on Gr_p. We will often talk of functions on coset spaces as functions on the groups in this fashion without further comment.

Define analogously,

$$\mathbf{C}\,St_p = \mathscr{E}_p/GL_+ \times GL_-, \tag{5.7}$$

where the action of $GL_+ \times GL_-$ on $\mathscr{E}_p$ is

$$(g, q)\cdot h = (gh, qa'), \quad h = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} \in GL_+ \times GL_-. \tag{5.8}$$

Furthermore, consider the right action of GL^p on $\mathscr{E}_p$,

$$(g, q)t \mapsto (g, t^{-1}q). \tag{5.9}$$

It commutes with the action of $GL_+ \times GL_-$ and we can verify that it is well-defined free action on $\mathbf{C}\,St_p$. In fact,

$$\mathbf{C}\,\mathrm{Gr}_p = \mathbf{C}\,St_p/GL_p. \tag{5.10}$$

We denote by $\pi:\mathbf{C}\,St_p \to \mathbf{C}\,\mathrm{Gr}_p$ the projection map. Note that in spite of the notation $\mathbf{C}\,St_p$ is not a complexification of St_p. Considered as a GL^p-bundle, only the base has been complexified. Let u denote a point on $\mathbf{C}\,St_p$ and $u \mapsto ut$ the action of GL^p. Define an action on $\mathbf{C}\,St_p \times \mathbf{C}$ by

$$(u, \lambda)\cdot t = (ut, \lambda\omega_p^{-1}(u, t)), \tag{5.11}$$

where the function $\omega_p:\mathbf{C}\,St_p \times GL^p \to \mathbf{C}$ can be thought of as a function

$$\omega_p:\mathscr{E}_p \times GL^p \to \mathbf{C}^\times, \quad (g, q, t) \mapsto \omega_p(g, q, t), \tag{5.12}$$

invariant under the action of

$$h = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} \in GL_+ \times GL_-. \tag{5.13}$$

We put

$$\omega_p(g, q, t) = (\det_p t)^{-1} e^{-\gamma_p(aq^{-1}, t)}. \tag{5.14}$$

where as usual

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A moment's thought will show that this is in fact the same function as in Sect. IV. Any function on St_p can be thought of as a function on $\mathbf{C}\,St_p$.

ω_p is a one-cocycle for GL_p. Therefore we can verify that the action (5.11) is well-defined. Furthermore, the action is free. So we can define the coset space

$$\mathbf{C}\,\mathrm{Det}_p = (\mathbf{C}\,St_p \times \mathbf{C})/GL^p. \tag{5.15}$$

By construction, $\mathbf{C}\,\mathrm{Det}_p$ is a holomorphic line bundle over $\mathbf{C}\,\mathrm{Gr}_p$. We want to lift the action on GL_p on the base to action on $\mathbf{C}\,\mathrm{Det}_p$. As before, we will find an Abelian extension of GL_p that acts on $\mathbf{C}\,\mathrm{Det}_p$.

Let us find an analogue of Lemma 4.1.

Lemma 5.2. *There are holomorphic functions $\beta:\mathscr{E}_p \times \mathbf{C}\,St_p \to \mathbf{C}^\times$ such that*

$$\frac{\beta(g, q; ut)}{\beta(g, q; u)} = \frac{\omega_p(u, t)}{\omega_p((g, q)u, qtq^{-1})} \tag{5.16}$$

for $t \in GL^p$. Let us regard β as a function $\beta:\mathscr{E}_p \times \mathscr{E}_p \to \mathbf{C}^\times$ invariant under $GL_+ \times GL_-$ on the second argument. Then, the general solution to (5.16) is,

$$\beta(g, q; \tilde{g}, \tilde{q}) = \varphi(g, q; \tilde{g}) \times \frac{\det_p \tilde{a}\tilde{q}^{-1}}{\det_p \tilde{a}\tilde{\alpha}} \frac{\det_p(a\tilde{a} + b\tilde{c})\tilde{\alpha}q^{-1}}{\det_p(a\tilde{a} + b\tilde{c})\tilde{q}^{-1}q^{-1}}, \tag{5.17}$$

where $\varphi \in \mathscr{E}_{2p} \times \mathrm{Gr}_{2p} \to \mathbf{C}^\times$ is any holomorphic function and $\tilde{g}^{-1} = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$.

Proof. It is obvious that any solution can be multiplied by a non-vanishing function on $\mathbf{C}\,\mathrm{Gr}_p$ to produce another solution. As before

$$\frac{\det_p \tilde{a}\tilde{q}^{-1}}{\det_p(a\tilde{a}+b\tilde{c})\tilde{q}^{-1}q^{-1}}$$

is a formal solution. The denominator has zeros that are cancelled when this formal solution is multiplied by

$$\frac{\det_p(a\tilde{a}+b\tilde{c})\tilde{\alpha}q^{-1}}{\det_p \tilde{a}\tilde{\alpha}},$$

which is a function on the $\mathbf{C}\,\mathrm{Gr}_p$. This "regularizes" the formal solution to produce the claimed result. In fact, we can write (5.17) in a way that shows explicitly that it is a well-defined function on $\mathscr{E}_p \times St_p$:

$$\beta(g,q;\tilde{g},\tilde{q}) = \varphi(g,q;\tilde{g})\exp[-\gamma_p(\tilde{a}\tilde{q}^{-1},\tilde{q}\tilde{\alpha}) + \gamma_p(q^{-1}(a\tilde{a}+b\tilde{c})\tilde{q}^{-1},\tilde{q}\tilde{\alpha})]. \quad (5.18)$$

□

Let $\mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p;\mathbf{C}^\times)$ be the Abelian group of holomorphic functions on $\mathbf{C}\,\mathrm{Gr}_p$.

Proposition 5.3. *The formula*

$$(g,q,v)\cdot(u,\lambda) = ((g,q)u, v(\pi(u))\lambda\beta(g,q;u)), \quad (5.19)$$

where β is any fixed solution of (5.12), defines an action of $\tilde{\mathscr{E}} = \mathscr{E}_p \times \mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p,\mathbf{C}^\times)$ on $\mathbf{C}\,\mathrm{Det}_p$. The multiplication in $\mathscr{E}_p \times \mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p,\mathbf{C}^\times)$ is given by

$$\begin{aligned}(g_1,q_1,v_1)(g_2,g_2,v_2) = (&g_1g_2,q_1q_2,v_1((g_2,q_2)u)\\ &\cdot v_2(u)\beta(g_1,q_1;(g_2,q_2)u)\beta(g_2,q_2;u)\beta(g_1g_2,q_1q_2;u)^{-1}).\end{aligned}$$

Proof. As before we need to show that the action (5.19) of $\mathscr{E}_p \times \mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p;\mathbf{C}^\times)$ on $\mathbf{C}\,St_p \times \mathbf{C}$ maps point equivalent under GL_p to equivalent ones. This, by an analogous calculation, is just the condition (5.17) on β. That the triple product of β's in (5.20) is a function on $\mathscr{E}_p \times \mathbf{C}\,\mathrm{Gr}_p$ (rather than $\mathscr{E}_p \times \mathbf{C}\,St_p$) also follows from a straightforward use of (5.17). □

Proposition 5.4. *There is an Abelian extension $\widetilde{GL}_p$ of GL_p by $\mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p,\mathbf{C}^\times)$ which acts on $\mathbf{C}\,\mathrm{Det}_p$ preserving its holomorphic structure. Hence,*

$$\widetilde{GL}_p = (\mathscr{E}_p \times \mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p,\mathbf{C}^\times))/P,$$

where P is the normal subgroup of elements $(1,q,v_q)$ with $v_q(u) = \alpha(1,q,u)^{-1}\omega_p(u,q^{-1})^{-1}$ and the action on $\mathbf{C}\,\mathrm{Det}_p$ is given as in Proposition 4.2.

Proof. As before, it is enough to show that P is the kernel of the action of $\mathscr{E}_p \times \mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p,\mathbf{C}^\times)$. This is a straightforward calculation. Since all the maps involved are holomorphic, it is obvious that the action leaves the holomorphic structure of $\mathbf{C}\,\mathrm{Det}_p$ invariant. □

Let $\mathbf{C}\,\mathrm{Det}_p^*$ be the dual line bundle of $\mathbf{C}\,\mathrm{Det}_p$. It can also be defined as

$$\mathbf{C}\,\mathrm{Det}_p^* = (\mathbf{C}\,St_p \times \mathbf{C})/GL_p,$$

where the action of GL^p is now

$$(u,\lambda)\cdot t=(ut,\lambda\omega_p(u,t)). \tag{5.21}$$

A holomorphic section ψ of $\mathbf{C}\,\mathrm{Det}_p^*$ can be thought of as a holomorphic function $\psi:\mathscr{E}_p\to\mathbf{C}$ satisfying

$$\psi(\tilde{g},t\tilde{q})=\psi(\tilde{g},\tilde{q})\omega_p(u,t^{-1}),$$

and

$$\psi(\tilde{g},\tilde{q})=\psi(\tilde{g}h,\tilde{q}a') \quad \text{for} \quad h=\begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}\in GL_+\times GL_-.$$

A canonical section is

$$\psi_0(\tilde{g},\tilde{q})=\det_p \tilde{a}\tilde{q}^{-1}. \tag{5.22}$$

We see that this is just the canonical section of Sect. IV, by the natural correspondence of sections of Det_p^* to those of $\mathbf{C}\,\mathrm{Det}_p^*$. If χ is any holomorphic function on $\mathscr{C}\,\mathrm{Gr}_p$,

$$\psi_1(\tilde{g},\tilde{q})=\det_p \tilde{a}\tilde{q}^{-1}\chi(\tilde{g})$$

is also a section of $\mathbf{C}\,\mathrm{Det}_p^*$. Let us hold χ to be a fixed nowhere zero function for the remainder of the paper and regard ψ_1 as a canonical section of $\mathbf{C}\,\mathrm{Det}_p^*$.

Theorem 5.5. *On the space of holomorphic sections* $\mathrm{Hol}(\mathbf{C}\,\mathrm{Det}_p^*)$ *we have a linear representation on* T *of* $\tilde{\mathscr{E}}_p$ *given by*

$$(T(g,q,v)\psi)(\tilde{g},\tilde{q})=\beta^{-1}(g,q;g^{-1}\tilde{g},q^{-1}\tilde{q})v(g^{-1}\tilde{g})\psi(g^{-1}\tilde{g},q^{-1}\tilde{q}).$$

The normal subgroup P of Proposition 5.4 levels all elements of $\mathrm{Hol}(\mathbf{C}\,\mathrm{Det}_p^*)$ *invariant so that this is in fact a representation of* $\widetilde{GL}_p=\tilde{\mathscr{E}}_p/P$. *The kernel of* ψ_1 *is a subgroup* K_p *of* $\widetilde{GL}_p$ *isomorphic to* $B_p^-=\left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right\}$:

$$K_p=\left\{g=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, a, k\right\},$$

where

$$k(\tilde{g})=\frac{\chi(g\tilde{g})}{\chi(\tilde{g})}\varphi(g,q;\tilde{g})\frac{\det_p a\tilde{a}\tilde{\alpha}q^{-1}}{\det_p \tilde{a}\tilde{\alpha}}.$$

Proof. We know how $\tilde{\mathscr{E}}_p$ acts on $\mathbf{C}\,\mathrm{Det}_p$ from Proposition (5.3):

$$(g,q,v)\cdot(u,\lambda)=((g,q)u,v(u)\lambda\beta(g,q;u)).$$

From this, we see that on $\mathbf{C}\,\mathrm{Det}_p^*=(St_p\times\mathbf{C})/GL^p$ we have the action

$$(g,q,v)(u,\lambda)=((g,q)u,v(u)\lambda\beta^{-1}(g,q;u)).$$

A section of $\mathbf{C}\,\mathrm{Det}_p^*$ is a map $\psi:\mathbf{C}\,St_p\to\mathbf{C}$ satisfying the condition described earlier. To see how sections transform, we note that

$$(g,q,v)(u,\psi(u))=((g,q)u,(T(g,q,v)\psi)((g,q)u)),$$

so that

$$(T(g,q,v)\psi)(g\tilde{g},q\tilde{q})=v(\tilde{g})\beta^{-1}(g,q;\tilde{g},\tilde{q})\psi(\tilde{g},\tilde{q}).$$

That P leaves all sections invariant is obvious, since it acts trivially on $\mathbf{C}\,\mathrm{Det}_p$ and hence on $\mathbf{C}\,\mathrm{Det}_p^*$.

If (g,q,v) is to be in the kernel of ψ_1, it must satisfy

$$\psi_1(g\tilde{g},q\tilde{q})=v(\tilde{g})\beta^{-1}(g,q;\tilde{g},\tilde{q})\psi_1(\tilde{g},\tilde{q}),$$

i.e.

$$\chi(g\tilde{g})\det_p(a\tilde{a}+b\tilde{c})\tilde{q}^{-1}=v(\tilde{g})\beta^{-1}(g,q;\tilde{g},\tilde{q})\det\tilde{a}\tilde{q}^{-1}\cdot\chi(\tilde{g}).$$

This means that we must choose $b=0$ and

$$v(\tilde{g})=\frac{\chi(g\tilde{g})}{\chi(\tilde{g})}\beta(g,q;\tilde{g},\tilde{q})\frac{\det_p a\tilde{a}\tilde{q}^{-1}q^{-1}}{\det_p\tilde{a}\tilde{q}^{-1}}=\frac{\chi(g\tilde{g})}{\chi(\tilde{g})}\cdot\varphi(g,q;\tilde{q})\frac{\det_p a\tilde{a}\tilde{\alpha}q^{-1}}{\det_p\tilde{a}\tilde{\alpha}}.$$

Now P will obviously leave ψ_1 invariant. We can represent the elements in $\widetilde{GL}_p=\mathscr{E}_p/P$ corresponding to the kernel of ψ_1 with the choice $q=a$. (For a is invertible if $b=0$.) That K_p is isomorphic to B_p can be verified by a simple computation. □

So ψ_1 can be interpreted as the highest weight vector of the representation T.

For the special case $p=2$ (which is relevant for $3+1$-dimensional current algebras) we can choose the function φ so that the cocycle is an affine function.

Let us digress a little bit to explain what is meant by this. Define $\mathscr{A}_p$ to be the affine space modelled on the vector space of operators in $H_+\oplus H_-$ of the form $\begin{pmatrix} I_p & I_{2p} \\ I_{2p} & I_p \end{pmatrix}$. This carries an affine action of GL_p:

$$A\mapsto gAg^{-1}+[g,\varepsilon]g^{-1}.$$

That $[g,\varepsilon]g^{-1}\in\mathscr{A}_p$ follows from the following explicit computation:

$$A(g)=[g,\varepsilon]g^{-1}=\begin{pmatrix} -b\gamma & -b\delta \\ c\alpha & c\beta \end{pmatrix},$$

with, as usual $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g^{-1}=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

The orbit of $A=0$ is in fact just $\mathbf{C}\,\mathrm{Gr}_p$. So the map

$$g\mapsto A(g)=[g,\varepsilon]g^{-1}$$

gives an affine embedding of $\mathbf{C}\,\mathrm{Gr}_p$ into $\mathscr{A}_p$. (There is of course an analogous embedding of Gr_p into the space of skew-adjoint matrices of type $\begin{pmatrix} I_p & I_{2p} \\ I_{2p} & I_p \end{pmatrix}$ obtained by restricting g to the unitaries.) So to see this, note that $g\to g\begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}$ leaves $A(g)$ invariant.

An affine function (or a polynomial of degree k, more generaly), on $\mathbf{C}\,\mathrm{Gr}_p$ can now be defined as the pull-back under this embedding of an affine function (or a polynomial of degree k) on $\mathscr{A}_p$. An affine function on $\mathbf{C}\,\mathrm{Gr}_p$ has the form $f(\tilde{g}) = \mathrm{tr}\,\xi A(\tilde{g}) + \eta$ for $\xi = \begin{pmatrix} I_q & I_{q'} \\ I_{q'} & I_q \end{pmatrix}$, where $q = p/p-1, q' = 2p/2p-1$ and $\eta \in \mathbf{C}$.

For $p = 2$, one may verify that the choice

$$\beta(g, q; \tilde{g}, \tilde{q}) = \exp[\gamma_2(q^{-1}a, \tilde{a}\tilde{q}^{-1}) - \mathrm{tr}\, q^{-1}b\tilde{c}(\tilde{q}^{-1} - \tilde{\alpha})]$$

solves (5.17). Note that for $g, \tilde{g} \in GL_1$,

$$\beta_1 = \exp[\gamma_2(q^{-1}a, \tilde{a}\tilde{q}^{-1}) - \mathrm{tr}\, q^{-1}b\tilde{c}\tilde{q}^{-1}]$$

is a solution with two-cocycle (for $\mathscr{E}_p$) equal to 1, but this fails to be a continuous function in the GL_2 topology. We "renormalize" by multiplying by $e^{-\mathrm{tr}\, q^{-1}b\tilde{a}\tilde{\alpha}}$ and then β exists even in GL_2 because $\tilde{q}^{-1} - \tilde{\alpha} \in I_2$. Note that the renormalization is by the exponential of an affine function. This means (by a straightforward computation) that the Lie algebra two-cocycle $w(X, Y)$ of $\underline{gl}_2$ will be an affine function. Furthermore, the group two-cocycle of $\tilde{\mathscr{E}}_2$ will be the exponential of an affine function. Therefore, we can define an extension of GL_2 by the multiplicative group of functions on $\mathbf{C}\,\mathrm{Gr}_2$ of the form

$$v(\tilde{g}) = \exp \mathrm{tr}\, \xi A(\tilde{g}) + \eta$$

for

$$\xi \in \begin{pmatrix} I_2 & I_{4/3} \\ I_{4/3} & I_2 \end{pmatrix}, \quad \eta \in \mathbf{C}.$$

Hence $\widetilde{GL}_{2p}$ is now a Banach Lie group modeled on the Banach space

$$B \oplus I_4 \oplus I_4 \oplus B \oplus I_2 \oplus I_{4/3} \oplus I_{4/3} \oplus I_2 \oplus \mathbf{C},$$

B being the space of bounded operators.

We would now like to find a holomorphic vector bundle over Gr_p which carries an action of $\tilde{G}L_p$ that preserves the holomorphic structure. Also, we would like to interpret $\mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p)$ as holomorphic sections of some vector bundle on Gr_p.

As preparation, let I_{2p} be the Abelian contractible subgroup of elements in GL_p of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. For simplicity of notation we may denote this element simply as b. (I_{2p} may be thought of as the additive group of the vector space I_{2p}.) I_{2p} carries an affine action of $\mathscr{B}_p$:

$$\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \cdot b = a'bd'^{-1} - b'd'^{-1}. \tag{5.23}$$

Denote by V the space of holomorphic functions $V = \mathrm{Hol}(I_{2p})$. V carries by pull-back a representation of $\mathscr{B}_{2p}$:

$$\left(\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} v \right)(b) = v(a'^{-1}bd' + a'^{-1}b'). \tag{5.24}$$

Define the holomorphic vector bundle over Gr_p

$$\begin{array}{c} V \to \mathscr{V} \\ \downarrow \\ \mathrm{Gr}_p \end{array}$$

by this action

$$\mathscr{V} = (GL_p \times V)/\mathscr{B}_p.$$

Now we note the following result:

Proposition 5.6. *There is a natural isomorphism between* $\mathrm{Hol}(\mathbf{C}\,\mathrm{Gr}_p)$ *and the space of holomorphic sections of* $\mathscr{V}$, $\mathrm{Hol}(\mathscr{V})$

Proof. A holomorphic function on $\mathbf{C}\,\mathrm{Gr}_p$ can be thought of as a holomorphic function

$$f: GL_p \to \mathbf{C}$$

satisfying

$$f(gh) = f(g) \quad \text{for} \quad h \in GL_+ \times GL_-.$$

$\tilde{f} \in \mathrm{Hol}(\mathscr{V})$ can be identified as a function

$$\tilde{f}: GL_{2p} \times I_{2p} \to \mathbf{C}$$

satisfying

$$\tilde{f}\left(g\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}, \left(a'^{-1} r d'^{-1} + a'^{-1} b' \right) \right) = \tilde{f}(g, r).$$

We may identify these by

$$\tilde{f}(g, r) = f\left(g\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \right). \quad \square$$

Now consider the holomorphic vector bundle on $\mathrm{Gr}_{2p}, \mathscr{V} \otimes \mathrm{Det}_p^*$. Since Det_p^* is a line bundle, the fibers are then isomorphic to V, but this bundle is more "twisted" topologically. By an argument analogous to that for Proposition (5.5), we can show:

Proposition 5.7. *There is a natural isomorphism between the space* $\mathrm{Hol}(\mathbf{C}\,\mathrm{Det}_p^*)$ *and* $\mathrm{Hol}(\mathscr{V} \otimes \mathrm{Det}_p^*)$.

Then we see that $\widetilde{GL}_p$ has a representation on the space of holomorphic sections of $\mathscr{V} \otimes \mathrm{Det}_p^*$ over Gr_p. So one could view this as the analogue of the Det_1^* line bundle over Gr_1.

VI. Extensions of the Full Group GL_p and Extensions of $\mathrm{Map}(S^{2n+1}, G)$

Up to this point, we have constructed the Abelian extensions only for the component of the identity $GL_p^{(0)} \subset GL_p$. We want now to construct the extension for the full group GL_p. We shall generalize the method described in [PS], Sect. VI.6, to our case. Let $\sigma \in GL_p$ be any element such that the Fredholm index of $a(\sigma)$ is equal to one. Denote by $\mathbf{Z}$ the subgroup of GL_p generated by σ. Now

$$GL_p \simeq \mathbf{Z} \times GL_p^{(0)}$$

is a semi-direct product, where the action of σ in $GL_p^{(0)}$ is given by $g \mapsto \sigma g \sigma^{-1}$; $(n_1, g_1)(n_2, g_2) = (n_1 + n_2, g_1 \sigma^{n_1} g_2 \sigma^{-n_1})$. We extend the action to an endomorphism of $\mathscr{E}_p \times \mathrm{Map}(\mathrm{Gr}_p, \mathbf{C}^\times)$ by

$$\sigma(g, q, \lambda) = (\sigma g \sigma^{-1}, q_\sigma, \lambda^\sigma h_\sigma(g, q; g^{-1}F)), \tag{6.1}$$

where $\lambda^\sigma(F) = \lambda(\sigma^{-1}F\sigma)$ and h_σ is a function on $\mathscr{E}_p \times \mathrm{Gr}_p$ taking values in $\mathbf{C}^\times$; the structure of h_σ will be determined below. In order that $\sigma^{-1}F\sigma$ makes sense on the real Grassmannian Gr_p we choose σ to be unitary; for example, $\sigma(e_i) = e_{i+1} \forall i \in \mathbf{Z}$. The map $q \mapsto q_\sigma$ in $GL(H_+)$ is chosen in such a way that $(g, q) \mapsto (\sigma g \sigma^{-1}, q_\sigma)$ as a map $\mathscr{E}_p \to \mathscr{E}_p$ covers the action $g \mapsto \sigma g \sigma^{-1}$ on $GL_p^{(0)}$. For example, with the above choice for σ, one can define

$$q_\sigma = \begin{cases} \sigma q \sigma^{-1} & \text{on} \quad \sigma(H_+) \\ 1 & \text{on} \quad H_+ \ominus \sigma(H_+). \end{cases}$$

The map (6.1) is well-defined even in the quotient $\widehat{GL}_p^{(0)} = (\mathscr{E}_p \times \mathrm{Map}(\mathrm{Gr}_p, \mathbf{C}^\times))/N$. This follows from the fact that $\det_p q_\sigma = \det_p q$. Using the multiplication rule

$$(g_1, q_1, \lambda_1)(g_2, q_2, \lambda_2) = (g_1 g_2, q_1 q_2, \lambda_1(g_2 \cdot F)\lambda_2(F)\Omega_p(g_1, q_1, g_2, q_2; F))$$

on $\mathscr{E}_p \times \mathrm{Map}(\mathrm{Gr}_p, \mathbf{C}^\times)$, where Ω_p is given by Proposition 4.2, the condition that the map σ is an automorphism on the group $\widehat{GL}_p^{(0)}$ can be written as

$$\frac{\Omega_p(g_1, q_1, g_2, q_2; F)}{\Omega_p(\sigma g_1 \sigma^{-1}, q_{1\sigma}, \sigma g_2 \sigma^{-1}, q_{2\sigma}; \sigma F \sigma^{-1})} = h_\sigma(g_1; g_2 F) h_\sigma(g_2; F) h_\sigma(g_1 g_2; F)^{-1}. \tag{6.2}$$

Equation (6.2) says that the quotient on the left should be a coboundary of a 1-chain h_σ of the group $GL_p^{(0)}$ (with respect to the natural action on Gr_p). Thus, (6.2) has a solution h_σ if and only iff Ω_p and the two-cocycle $\Omega_p^{(\sigma)}$ in the denominator represent the same cohomology class. The cohomology classes of the different group extensions are determined by the de Rham cohomology classes obtained by evaluating the corresponding Lie algebra cocycle. We give the proof of the invariance of the cohomology classes in the case $p = 2$ (in the case $p = 1$, $\Omega_p = 1$ and we can take $h_\sigma = 1$); the case $p > 2$ requires more computation but is essentially straightforward. The Lie algebra cocycle $\eta^{(\sigma)}$ obtained from (4.16) through the automorphism σ of GL_2 is

$$\begin{aligned} \eta^{(\sigma)}(X, Y) &= \tfrac{1}{8} \operatorname{tr}[[\varepsilon, \sigma X \sigma^{-1}], [\varepsilon, \sigma Y \sigma^{-1}]](\varepsilon - \sigma F \sigma^{-1}) \\ &= \tfrac{1}{8} \operatorname{tr}[[\sigma^{-1}\varepsilon\sigma, X], [\sigma^{-1}\varepsilon\sigma, Y]](\sigma^{-1}\varepsilon - F). \end{aligned}$$

Thus $\eta^{(\sigma)}$ is obtained from η by substituting $\varepsilon \mapsto \varepsilon_\sigma = \sigma^{-1}\varepsilon\sigma$. The difference $\varepsilon - \varepsilon_\sigma$ is of finite rank. Therefore,

$$\beta(X; F) = \tfrac{1}{16} \operatorname{tr}([X, \varepsilon][F, \varepsilon - \varepsilon_\sigma] + [X, \varepsilon - \varepsilon_\sigma][F, \varepsilon_\sigma]) + \tfrac{1}{2} \operatorname{tr} X(\varepsilon - \varepsilon_\sigma)$$

is well-defined; by a simple computation,

$$\eta - \eta^{(\sigma)} = \delta\beta.$$

The extension of $\widehat{GL}_p$ for the whole group GL_p can now be defined as

$$\widehat{GL}_p = \mathbf{Z} \times \widehat{GL}_p^{(0)},$$

where the action of $\mathbf{Z}$ on $\widehat{GL_p^{(0)}}$ is defined by the action of its generator σ, Eq. (6.1).

As was explained in Sect. II, in the case of a trivial vector bundle, the group of gauge transformation $\mathscr{G} = \mathrm{Map}(X, G)(d = 2n+1$ odd, X a compact manifold of dimension $d)$ can be embedded in GL_p for $p = n+1$. On the other hand, we have a map $\mathscr{G} \to \mathrm{Gr}_p$ given by $g \mapsto g \cdot H_+$. Thus by pull-back, using the map $\mathscr{G} \times \mathscr{G} \to GL_p \times \mathrm{Gr}_p$, we get an extension $\hat{\mathscr{G}}$ of $\mathscr{G}$ by $\mathrm{Map}(\mathscr{G}, \mathbf{C}^\times)$ from the extension GL_p of $\widehat{GL_p}$ by $\mathrm{Map}(\mathrm{Gr}_p, \mathbf{C}^\times)$.

The Lie algebra of $\mathscr{G}$ is $\mathrm{Map}(X, \underline{g})$, where $\underline{g}$ is the Lie algebra of G. Let us compute explicitly the Lie algebra extension in the case $X = T^3 = S^1 \times S^1 \times S^1$, $G = U(N)$ and $p = 2$. We shall use two-cocycle (4.16) for $\underline{gl_2}$. To an element $g \in \mathscr{G}$ there corresponds $F = g\varepsilon g^{-1} \in \mathrm{Gr}_2$, where g is thought of as a multiplication operator in the Hilbert space H of square integrable spinor fields $\psi : T^3 \to \mathbf{C}^2 \otimes \mathbf{C}^N$. Using the Fourier decompositions

$$g = \sum_{p \in \mathbf{Z}^3} g_p e^{ip\cdot\theta}, \quad g^{-1} = \sum_p f_p e^{ip\cdot\theta},$$

one can write the integral kernel of the operator $[g, \varepsilon]$

$$K(p, p') = g_{p-p'} \left(\frac{p\cdot\sigma}{|p|} - \frac{p'\cdot\sigma}{|p'|} \right),$$

where σ_1, σ_2 and are the Pauli matrices acting in $\mathbf{C}^2, \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$. Writing $F - \varepsilon = g\varepsilon g^{-1} - \varepsilon = [g, \varepsilon] g^{-1}$ and inserting to (4.16), we get the following expression for $\eta(X, Y; F)$:

$$\eta = -\frac{1}{8} \sum_{p_i} \mathrm{tr}(X_{p_1-p_2} Y_{p_2-p_3} - Y_{p_1-p_2} X_{p_2-p_3})$$

$$\cdot g_{p_3-p_4} f_{p_4-p_1} \left(\frac{p_1\cdot\sigma}{|p_1|} - \frac{p_2\cdot\sigma}{|p_2|} \right) \left(\frac{p_2\cdot\sigma}{|p_2|} - \frac{p_3\cdot\sigma}{|p_3|} \right) \left(\frac{p_3\cdot\sigma}{|p_3|} - \frac{p_4\cdot\sigma}{|p_4|} \right), \tag{6.3}$$

where the X_p's and Y_p's are the Fourier components of $X, Y; T^3 \to \underline{g}$; the sum is over all the momenta $p_i \in \mathbf{Z}^3$.

The non-local formula (6.3) can be compared with the local expression for the extension of $\mathrm{Map}(X, \underline{g})$ by the $\mathrm{Map}(\mathscr{A}_d, \mathbf{C})$, where $\mathscr{A}_d$ is the space of $\underline{g}$ valued one-forms on X. The different cohomology classes of extensions are integral multiplets of the two-cocycle

$$\eta'(X, Y; A) = \int_{n_d} \mathrm{tr}(XdY + YdX) \wedge P_d(A), \tag{6.4}$$

where $A \in \mathscr{A}_d$ and P_d is a differential form of degree $d-1$ which is a polynomial of A, [M1], [F], [Si]. For example, $P_1 = 1/4\pi$ and $P_2 = (i/12\pi^2) dA$. The former case gives the affine Kač–Moody algebra corresponding to $\underline{g} = \underline{u}(N)$ and the latter is the current algebra of the $3+1$-dimensional field theory. In order to compare with (6.3), we restrict A_3 to the space of gauge potentials $A = g^{-1} dg, g \in \mathscr{G}$ [R]. Computing (6.4) (fór $X = T^3$) in terms of the Fourier components one sees that $\eta \neq \eta'$. However, from the general cohomological classification (see [PS], Sect. VI.10) of extensions follows that η and η' represent the same cohomology class.

Acknowledgements. We wish to thank Daniel Freed for drawing our attention to ref. [S].

References

[A] Adler, S., Dashen, R.: Current algebras. New York: Benjamin 1968
[B] Balachandran, A. P.: Skyrmions. Jackiw, R.: Anomalies and topology. In: High energy physics, 1985. Bowick, M. J., Gürsey, F. (eds.). Singapore: World Scientific 1985
[BR] Bowick, M., Rajeev, S. G.: Anomalies as curvature in complex geometry. MIT preprint CTP#1449 (1987)
[C] Connes, A.: Non-commutative differential geometry, I.H.E.S. Publ. Math. **62**, 41 (1985)
[F] Faddeev, L.: Operator anomaly for the gauss law. Phys. Lett. **145B**, 81 (1984)
[K] Kuiper, N. H.: The homotopy type of the unitary group of Hilbert space. Topology **3**, 19 (1965)
[KP] Kač, V. G., Peterson, D. H.: Lectures on the infinite wedge representation and the MPP hierarchy. Proceedings of the Summer School on Completely Integrable Systems Montreal, Canada, August, 1985
[M1] Mickelsson, J.: Chiral anomalies in even- and odd-dimensions. Commun. Math. Phys. **97**, 361 (1985)
[M2] Mickelsson, J.: Kač–Moody Groups, Topology of the Dirac Determinant Bundle and Fermionization. Commun. Math. Phys. **110**, 173 (1987)
[MP] Marciano, W., Pagels, H.: Quantum chromodynamics. Phys. Rep. **36C**, 137 (1976)
[P] Palais, R. S.: On the homotopy type of certain groups of operators. Topology **3**, 271 (1965)
[Pi] Pickrell, D.: Measures on infinite-dimensional Grassmann manifolds. J. Funct. Anal. (to appear)
[PS] Pressley, A., Segal, G.: Loop groups and their representations. Oxford, UK: Oxford University Press 1986
[Q] Quillen, D.: Superconnection character forms and the cayley transform. preprint (1987)
[R] Rajeev, S. G.: Fermions from Bosons in 3 + 1-dimensions through anomalous commutators. Phys. Rev. **D29**, 2944 (1984)
[S] Simon, B.: Trace ideals and their applications. Cambridge, UK: Cambridge University Press 1979
[Si] Singer, I.: Families of Dirac operators with application to physics. Asterisque 323 (1985)
[T] Taylor, M. E.: Pseudo-differential operators. Princeton, NJ: Princeton University Press 1981
[Tr] Trieman, S. B., Jackiw, R., Zumino, B., Witten, E.: Current algebras and anomalies. Princeton NJ: Princeton University Press 1985
[W] Wallach, N.: Harmonic analysis on homogeneous spaces. New York: Marcel–Dekker 1973
[FU] Freed, D., Uhlenbeck, K.: Instantons and four-manifolds. Berlin, Heidelberg, New York: Springer 1984

Communicated by A. Jaffe

Received June 30, 1987