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On the Large Order Behavior of Φ_4^4

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Abstract. We continue the rigorous study of the large order behavior of the perturbation series for the φ^4 model in 4 dimensions started in [1]. In this paper we prove a result announced in [1]. We show that the exact radius of convergence of the Borel transform of the renormalized perturbation series for φ_4^4 is greater than or equal to the expected value given by the position of the first "renormalon" [2]. This result holds for any vector $(\boldsymbol{\varphi}^2)^2$ model with N components, and makes use of the "Lipatov bound" of [1]. This result is based on a partial resummation of counterterms similar to the one of [3], but in a phase-space analysis of the renormalized series.

I. Introduction

A) The Renormalon Problem

The large order behavior of the renormalized series for φ_4^4 field theory is expected to be governed by the first "renormalon" singularity in the Borel plane [2]. It happens indeed that this singularity, which should exist only in dimension 4 where the theory is renormalizable, is closer to the origin than the "instanton" singularity which is responsible for the "Lipatov" large order behavior [4] of φ^4 series. Therefore this Lipatov behavior is only expected to hold for the lower dimensions 1, 2, and 3, where the φ^4 theory is superrenormalizable. Although this "Lipatov" behavior in the superrenormalizable domain has now been rigorously established, there is no theorem, up to now, on the existence of a single "renormalon" singularity, except in the trivial case of "infinite component" vector models.

A rigorous study of renormalons is interesting for several reasons. The renormalon is the modern version of the "Landau ghost" [14]. As shown in [2] by Parisi and 't Hooft, for those renormalizable theories which are not asymptotically free (i.e. the one loop β function coefficient is positive), one expects singularities in

★ Permanent address: Mathematics Department, University of British Columbia, Vancouver, B.C., Canada V6T 1Y4 the Borel transform induced by bad ultraviolet behavior at some points on the real positive axis in the Borel plane. These singularities, called renormalons, are an obstruction to resummation of perturbation theory. The connection between the renormalization group flow for non-asymptotically free theories and the appearance of renormalons in the Borel plane has been further investigated in [15, 16].

An understanding of ultraviolet renormalons would shed some light on the problem of triviality of non-asymptotically free theories like φ_4^4 from a perturbative point of view [13], i.e. a direct analysis of the renormalized perturbation expansion. This point of view is different and somewhat complementary to the one of [6], where new correlation inequalities were derived and used to prove that the continuum limit of any φ^4 theory regularized on a four dimensional lattice is a free field under rather mild assumptions.

Another reason for interest in renormalons is the fact that non-abelian gauge theories, although asymptotically free, should also have renormalons, of infrared type. These renormalons should be related to the existence of non-perturbative effects at long distance. An understanding of renormalons is in particular necessary to define correctly the coefficients of the operator product expansion which has been used to parametrize such effects in models like QCD [7].

The fact that the first renormalon singularity of φ_4^4 in the heuristic analysis is the closest one to the origin already suggests that a rigorous proof of its existence should be feasible. The corresponding strategy was explained in [1]; one should combine an upper bound on the instanton effects with a lower bound on the first "renormalon effects." In [1], the first piece of this program was accomplished, namely an upper bound on the piece of the expansion which does not contain renormalons, with the exact value of the Lipatov constant. In this paper, using the results of [1], we do prove that the renormalized Borel transformed series converge and give an analytic function in the optimal expected disk, which is limited by the position of the first expected renormalon. We can also identify many sources of potential singularities when one approaches the corresponding point on the real positive axis. However we have not yet found an organizing principle for these singularities, which would allow us to rule out the very unlikely possibility of a miraculous cancellation between all of them. It is a general feature that when signs are not fixed, lower bounds are much harder to get than upper bounds.

For the $(\boldsymbol{\varphi}^2)^2$ model with N components and N sufficiently large, we do have such an organizing principle, namely the $1/N$ expansion, and it should therefore be possible to prove the existence of the first renormalon. Note that in this case the delicate part of [1], namely getting the exact Lipatov constant in the estimates is no longer necessary, since the instanton singularity, at large N, is arbitrarily farther from the origin on the negative Borel axis than the renormalon on the positive axis. However there is a new technical difficulty. In this paper, to obtain the correct convergence radius in the Borel plane requires only studying the "effective coupling constant flow" in an approximation where one retains only the first non-vanishing coefficient β_2 of the β function. To prove the existence of the first renormalon, even at large N, requires controlling not only the position but also its "strength," and this requires an analysis where one retains both β_2 and β_3 in the β function. The "loglog" correction to the logarithmic decay of the effective constant induced by β_3

is well understood in the asymptotically free case; but in the case of a non-asymptotically free theory it seems more difficult to control. We hope to return to this problem in the future.

In conclusion we consider that the simple question "Is the renormalized series of plain one-component φ_4^4 theory Borel summable" or even "Does it have a non-zero radius of convergence" is still open from the purist point of view, but we hope that this paper is a step towards a convincing negative answer to these two questions.

B) The Results

The perturbation expansion for a connected Schwinger function S_N in the theory with interaction $-g\varphi^4$ is a formal power series in g defined as:

$$S_N = \sum_n (-g)^n a_n^R , \tag{I.1}$$

where a_n^R is the sum of all renormalized Feynman amplitudes for connected graphs with n vertices and N external legs. This requires the choice of a renormalization scheme, which we choose to be the ordinary BPHZ scheme with subtractions at 0 external momenta for the massive theory [8], and the scheme of [9] in the massless case. For simplicity we will assume that we are in the massive case, and we will consider more precisely the large order behavior of two typical quantities, the 6 point function at 0 external momenta and the 2 point function at some external momentum p satisfying $p^2 = \mu^2$, where μ is a fixed energy scale, but of course our results are not limited to this particular case.

Let us sketch our understanding of the large order behavior of φ_4^4. The "Lipatov" analysis would lead to an expected behavior:

$$a_n = n!\, a^n (1+\varepsilon(n))^n , \quad \text{with} \quad \lim_{n\to\infty} \varepsilon(n) = 0 , \tag{I.2}$$

where a, the Lipatov constant, takes the value $a = (3/2\pi^2)$ for an N component model (note the discrepancy with the values of [2], since we do not use an interaction $-g\varphi^4/4!$).

In fact renormalization should disturb the Lipatov analysis in 4 dimensions, so that one should expect a large order behavior [2]:

$$a_n^R = n! \left[\frac{-\beta_2}{2}\right]^n (1+\varepsilon(n))^n , \quad \text{with} \quad \lim_{n\to\infty} \varepsilon(n) = 0 , \tag{I.3}$$

where $\beta_2(N) = (N+8)/(2\pi^2)$ is the one loop coefficient of the beta function for an N component φ_4^4 model. Note that even at $N=1$, $a = \beta_2/3$, so that (I.3) dominates over (I.2). This situation persists for all N, and the ratio a/β_2 tends to 0 as $N \to \infty$.

The main result of this paper is:

Theorem. *For any number of components N, there exists a function $\varepsilon(n)$ which tends to 0 as n tends to $+\infty$, such that:*

$$|a_n^R| \leq n! \left[\frac{\beta_2(N)}{2}\right]^n (1+\varepsilon(n))^n , \tag{I.4}$$

hence the Borel transform [10] $B(b)=\sum_n(-b)^n a_n^R/n!$ *of the renormalized perturbation series is analytic in a disk of radius* $(2/\beta_2)$ *(which is the optimal expected disk).*

C) Outline of the Proof

In Sect. II we rewrite the renormalized perturbation series, which is a power series in the renormalized coupling constant, as an "effective" power series in an infinite number of "effective coupling constants," one per "momentum slice," which are related to each other through a renormalization group equation, truncated to second order. This rewriting is accomplished by an explicit resummation of some pieces of counterterms in the initial series, in the manner of [3]. In Sect. III, we use inductively the "Lipatov bound" of [1] together with a combinatoric analysis in the style of [8, Appendix C] to extend this "Lipatov bound" to the coefficients of this effective series. Combining this bound with the ultraviolet behavior of the effective coupling constants leads to the proof of the theorem.

II. The Dressing Process

A) The Renormalization Operator $\mathbb{R}$

We write the usual decomposition of the propagator into momentum slices (see e.g. [11]):

$$C(p)=\int_0^\infty d\alpha e^{-\alpha(p^2+m^2)}=\sum_{j\geqq 0} C^j(p), \tag{II.1}$$

$$C^j(p)=\int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha e^{-\alpha(p^2+m^2)}; \qquad C^0(p)=\int_1^\infty d\alpha e^{-\alpha(p^2+m^2)}, \tag{II.2}$$

where $M>1$ is a fixed number which we can choose equal to e, so that $\operatorname{Log} M=1$, which simplifies some of our equations. Moreover the renormalized mass m^2 can be fixed to 1, simply by redefining the unit of mass. Recall that in x space we have the behavior [11]:

$$|C^j(x-y)|\leqq O(1)M^{2j}e^{-(1/2)M^j|x-y|}, \tag{II.3}$$

where $O(1)$ is our generic name for a fixed irrelevant constant. The BPHZ renormalization is performed by Taylor subtractions on the x-space integrand of Feynman amplitudes, according to Part III of [12]:

$$\mathbb{R}=\sum_F \prod_{g\in F}(-T_g), \tag{II.4}$$

where the sum is performed over all forests of proper divergent closed graphs (see [8, 12]) and the T_g operator extracts the beginning of the Taylor expansion of the integrand up to the superficial divergence degree for subgraph g at 0 external momenta. The expression of T_g in x-space is somewhat messy so we refer to [12, p. 27] for its exact definition. The amplitude for a Feynman graph G can then be written as a sum over momentum assignments μ, where μ is a collection of L integers $\mu(l)$, $l=1,\ldots,L$, one for each of the L internal lines of G:

$$I_G^R=(-g)^{n(G)}\sum_\mu\int\prod_v dx_v \mathbb{R}Z_{G,\mu}; \qquad Z_{G,\mu}=\prod_l C^{\mu(l)}(x_l,y_l), \tag{II.5}$$

where v runs over the $n(G)$ vertices of G, and x_l, y_l are the positions of the two ends of the line l. g is the renormalized coupling constant. Next we perform the reorganization of forests which is crucial for analyzing renormalization in [8] and [12]. We define for any forest F and any $g \in F$ two indices $i_g(F, \mu)$ and $e_g(F, \mu)$. Barring subleties, the definition for i and e would be:

$$i_g(F, \mu) = \inf\{\mu(l),\ l \in g/A_g(F)\}, \tag{II.6}$$

$$e_g(F, \mu) = 2nd\max\{\mu(l),\ l \in E(g)B_g(F)\}, \tag{II.7}$$

where $2nd\max\{A\}$ means "second max of A," i.e. $\min_{a \in A}(\max\{A - \{a\}\})$, and as in [8, 3, 12], $A_g(F)$ is the union of the subgraphs of F inside g, and $B_g(F)$ is the smallest graph of F containing g. By convention when $g = G$ is a full graph for the perturbation expansion of the Schwinger function we consider, we put the external index e_G to 0. However in Sect. C) it will be convenient to consider some subgraphs g also as full graphs, and one should remember that this convention does not apply to them; they should keep as external index the index that they have as subgraphs. The true definition of i and e is more complicated, both because it has to be inductive, starting from the largest graphs of F, and because the possible presence of "bipeds" and the necessity to take into account 1 PI and "closed" structures in order not to make a large number of spurious subtractions create painful technicalities. The correct definition that we will use in this paper is the one of [12, p. 48] with a single slight modification: in Eq. (3.3b) the max is taken to be a second max, even in the first line of the equation (the case "g triped or quadruped"). This definition then corresponds exactly to the one in [3], but neither exactly to the one of [8], nor to the one of [9, 12]. This slight change does not alter the basic lemmas which we are going to state.

Let us associate to any forest F two other μ-dependent forests:

- the forest $D_\mu(F)$ of the dangerous graphs of F:

$$D_\mu(F) = \{g \in F:\ i_g(F) > e_g(F)\}, \tag{II.8}$$

- the forest $\Sigma_\mu(F)$ of the safe graphs of F:

$$\Sigma_\mu(F) = \{g \in F:\ i_g(F) \leqq e_g(F)\}. \tag{II.9}$$

Lemma 1. *Σ_μ is a projection: $\Sigma_\mu \circ \Sigma_\mu = \Sigma_\mu$.*

The forests F such that $\Sigma_\mu(F) = F$ are called *compatible* with μ.

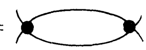
Lemma 2. *Let F be compatible with μ; then there exists a forest $H_\mu(F)$ such that:*
- *$F \cup H_\mu(F)$ is a forest,*
- *$\Sigma_\mu(F') = F \Leftrightarrow F \subset F' \subset F \cup H_\mu(F)$.*

The proof of these two lemmas is the same as in [8]. From Lemmas 1 and 2 we obtain the following reorganization of the $\mathbb{R}$ operator:

$$\sum_\mu \mathbb{R} = \sum_\mu \sum_{\substack{F \\ \Sigma_\mu(F) = F}} \prod_{g \in F} (-T_g) \prod_{g \in H_\mu(F)} (1 - T_g)$$

$$= \sum_F \sum_{\substack{\mu \\ \Sigma_\mu(F) = F}} \prod_{g \in F} (-T_g) \prod_{g \in H_\mu(F)} (1 - T_g). \tag{II.10}$$

B) Resummation of Useless Counterterms

The Taylor operators in (II.10) corresponding to the safe forest F generate pieces of counterterms which we call "useless counterterms" [12], because they do not contribute to the cancellation of ultraviolet divergences, but are forced by the requirement of locality for the total counterterms. It is these pieces of the counterterms which can be combined with the renormalized coupling constant to generate effective constants [3]. We are now in a position to resum the most important useless counterterm, which corresponds to the one loop graph $G_2=$ , with $i_{G_2}(F)<e_{G_2}(F)$ (note the strict inequality). This process generates running coupling constants according to a law governed by the one loop β function. To resum all the useless counterterms, even for the sole graph G_2, would be difficult to control. Therefore we will resum only the pieces truly responsible for the "first renormalon behavior." For this purpose, let us define a projection π on the set of triplets (G,F,μ) made of a graph G, a forest F in G, and a momentum assignment μ compatible with F [i.e. such that $\Sigma_\mu(F)=F$]. (This resummation is inspired by what is called the "first resummation" in [3]; the "second resummation" of [3] will not be necessary here.)

π^0 is defined by reducing to a single vertex, in G, every subgraph g of F which is isomorphic to G_2 and such that $i_g(F)<e_g(F)$. This reduction gives the graph G', a forest F' and an assignment μ' (obtained by reducing in the obvious way F and μ to G'). Clearly F' is compatible with μ', and we put $\pi^0(G,F,\mu)=(G',F',\mu')$. Iterating this process until stationary, we can define the projection π as the limit of $(\pi^0)^k$ as $k\to\infty$ [the limit is obtained for any fixed G for some finite $k\leqq n(G)$]. The renormalized perturbation expansion can be viewed as an infinite formal sum over the triplets (G,F,μ):

$$\sum_{(G,F,\mu)} \int \prod_{v\in G} dx_v(-g)^{n(G)} Z_{(G,F,\mu)}\,, \tag{II.11}$$

where $Z_{(G,F,\mu)}$ is defined as:

$$Z_{(G,F,\mu)}=\prod_{g\in F}(-T_g)\prod_{g\in H_\mu(F)}(1-T_g)Z_{G,\mu}\,. \tag{II.12}$$

We define the "fundamental triplets" as the (G,F,μ) such that:

$$\pi(G,F,\mu)=(G,F,\mu)\,. \tag{II.13}$$

We now organize perturbation theory as a sum over fundamental triplets:

$$\sum_{(G,F,\mu)} = \sum_{\substack{(G,F,\mu)\\ \text{fundamental}}} \sum_{\substack{(G',F',\mu')\\ \pi(G',F',\mu')=(G,F,\mu)}}\,. \tag{II.14}$$

The second sum in the right-hand side can be explicitly performed, since it will be shown to correspond to a dressing of the coupling constants of G. Indeed:

$$Z_{(G,F,\mu)}=Z_{\pi(G,F,\mu)}\prod_{\substack{h\in F\\ h \text{ reduced by } \pi}}(-Z_{h/F,\mu})\,. \tag{II.15}$$

In this formula h/F is the graph obtained by reducing to a single vertex the subgraphs of F in h. This new type of vertex is called a "reduction vertex." Remark that from the definition of π, h/F is always isomorphic to G_2. The integrand $Z_{h/F,\mu}$ is the result of the factorization due to the Taylor operator $(-T_h)$ which has to be there from the definition of π. Finally the lines in h/F have momentum assignments fixed by μ.

For a given fundamental triplet (G, F, μ) and for any vertex $v \in G$, let us define

$$e_v(F,\mu) = 2nd\max\{\mu(l),\ l \text{ hooked to } v \text{ and } l \in B_v(F)\}.$$

Performing the second sum [over (G', F', μ')] in Eq. (II.14) we can rewrite the renormalized perturbation expansion as:

$$\begin{aligned}&\sum_{(G,F,\mu)} \int \prod_{v\in G} dx_v(-g)^{n(G)} Z_{(G,F,\mu)}\\ &= \sum_{\substack{(G,F,\mu)\\ \text{fundamental}}} \int \prod_{v\in G} dx_v Z_{(G,F,\mu)} \prod_{v\in G} (-g_v^{\text{eff}}(g)),\end{aligned} \tag{II.16}$$

where $g_v^{\text{eff}}(g)$ depends only on $e_v(F,\mu)$ and on the renormalized coupling constant g. We write:

$$g_v^{\text{eff}}(g) = g^{\text{eff}}(e_v(F,\mu), g). \tag{II.17}$$

In the next section we will study more precisely this effective coupling.

C) The Effective Coupling Constant

Clearly the effective constant $g_v^{\text{eff}}(g)$ is the result of the sum over all subtriplets which the operation π reduces precisely to the vertex v. Therefore it is useful to define "parquet triplets" as triplets (G,F,μ) such that $\pi(G,F,\mu)$ is a single vertex. Recall that this vertex has an index $e_v \equiv i$ (for short) and that the assignments on G/F, which has to be isomorphic to G_2, satisfies the last constraint $i_G(F) < e_v = i$. Therefore:

$$g^{\text{eff}}(i,g) = \sum_{\substack{(G,F,\mu)\\ \text{parquet}}} g^{n(G)} \int \prod_{v\in G} dx_v \prod_{h\in F} (Z_{h/F,\mu}). \tag{II.18}$$

One might worry about the convergence of the sum in (II.18). But from our restrictive definition of the π projection, this sum is in fact a polynomial in g of order at most 2^i, hence perfectly well defined. Since h/F in the formula above is isomorphic to G_2, we need only to define the following integrand, which corresponds to the graph G_2 with momentum assignments j_1 and j_2 on its two internal lines 1 and 2, at 0 external momenta:

$$Z(j_1, j_2) = \beta \int d^4x C^{j_1}(0,x) C^{j_2}(x,0), \tag{II.19}$$

where β is a numerical coefficient corresponding to the symmetry factor associated to G_2. Furthermore we can define:

$$Y(i) = \sum_{\min\{j_1,j_2\}=i} Z(j_1,j_2). \tag{II.20}$$

Lemma 3. *The sum which defines $Y(i)$ is absolutely convergent. Moreover there exist constants K and K' such that:*

$$|Y(i) - \beta_2| \leqq K' \exp(-Ki), \tag{II.21}$$

where β_2 is the one loop coefficient of the β function defined in Sect. I.

The proof follows easily from the α-representation of the propagator C^j (II.2).

Reexpressing the compatibility condition between F and μ in the case where F is a parquet forest we can rewrite Eq. (II.18) as:

$$g^{\mathrm{eff}}(i,g)=\sum_{\substack{(G,F)\\ \text{parquet}}} g^{n(G)} \sum_{j_1(h),\, j_2(h)} \prod_{h\in F} Z(j_1(h), j_2(h)), \tag{II.22}$$

where the sum over the momentum assignments $j_1(h)$, $j_2(h)$ of the two lines of h/F is constrained by

$$\min(j_1(h), j_2(h)) < \min(j_1(h'), j_2(h')) \quad \text{if} \quad h \supset h'. \tag{II.23}$$

This is essentially because in a parquet triplet each reduced subgraph has exactly two internal lines and two external lines internal in the next subgraph of the forest, hence the second max coincides in this case with the min (see Fig. 1). Let us call $c_n(i)$ the coefficient of order n of $g^{\mathrm{eff}}(i,g)$:

$$g^{\mathrm{eff}}(i,g)=\sum_{n\geqq 1} g^n c_n(i). \tag{II.24}$$

Since parquet triplets have a tree structure with coordination number 2 (see Fig. 2) the coefficients c_n satisfy the simple recursion relation:

$$c_n(i)=\sum_{n_1+n_2=n}\sum_{j<i} c_{n_1}(j)c_{n_2}(j)Y(j) \tag{II.25}$$

with initial conditions $c_0(j)=0$, $c_1(j)=1$ $\forall j$. This recursion relation leads to:

$$g^{\mathrm{eff}}(i,g)=g+\sum_{j<i} Y(j)(g^{\mathrm{eff}}(j,g))^2, \tag{II.26}$$

hence to:

$$g^{\mathrm{eff}}(i,g)=g^{\mathrm{eff}}(i-1,g)+Y(i-1)(g^{\mathrm{eff}}(i-1,g))^2. \tag{II.27}$$

This is the usual form of the renormalization group recursion equation for the coupling constant, with the initial condition $g^{\mathrm{eff}}(0,g)=g$.

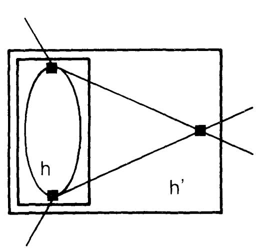


Fig. 1. Parquet triple

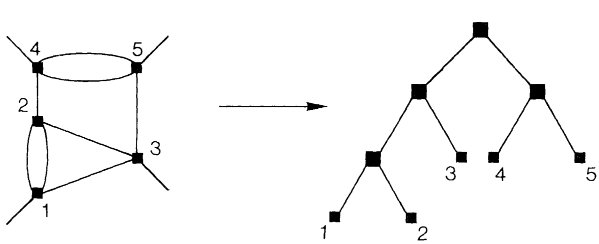


Fig. 2. Parquet graphs; ■ ordinary vertices, ■ reduction vertices

D) The Borel Transform

We use as definition of the Borel transform of a power series $\sum_{n\geqq 1} a_n g^n$ the power series

$$\sum_{n\geqq 1} a_n \frac{b_{n-1}}{(n-1)!}\,. \tag{II.28}$$

With this definition we have the usual correspondence between functions and their Borel transforms and the usual theorems on Borel summability [10] and the "Borel convolution" formula takes the simple form:

$$B[fg](b) = \int_0^b B[f](b')B[g](b-b')db'\,, \tag{II.29}$$

where $B[f]$ means the Borel transform of f.

Let us write $B^{\mathrm{eff}}(i,b)$ for $B[g^{\mathrm{eff}}(i,g)]$. Then (II.27) becomes:

$$B^{\mathrm{eff}}(i,b) = B^{\mathrm{eff}}(i-1,b) + Y(i-1)\int_0^b B^{\mathrm{eff}}(i-1,b')B^{\mathrm{eff}}(i-1,b-b')db' \tag{II.30}$$

with $B^{\mathrm{eff}}(0,b)=1$, from which follows the obvious bound

$$|B^{\mathrm{eff}}(i,b)| \leqq e^{\sum_{j<i} Y(j)|b|}\,. \tag{II.31}$$

A last lemma will give a global bound for the Borel transform of the product of all factors $g^{\mathrm{eff}}(i,g)$ in a fundamental contribution with n vertices. We define $i_m(\mu)$ (or i_m for short) as the maximal momentum assignment appearing in μ: $i_m(\mu) \equiv \sup_{l\in G} \mu(l)$. Then we have [recalling that $n(G)$, or simply n, is the number of vertices of G]:

Lemma 4. *We have, for any fundamental (G,F,μ):*

$$|B[DR_{(G,F,\mu)}](b)| \leqq \frac{|b|^{n-1}}{(n-1)!}\exp\left[\sum_{0\leqq j<i_m(\mu)} Y(j)|b|\right], \tag{II.32}$$

where

$$DR_{(G,F,\mu)} = \prod_{v\in G} [g^{\mathrm{eff}}(e_v(F,\mu),g)]\,. \tag{II.33}$$

Proof. To get this estimate, one has just to remark that $g^{\mathrm{eff}}(i,g)$ is a polynomial in g with positive coefficients and that each coefficient of $g^{\mathrm{eff}}(k,g)$ is bounded by the coefficient of same order of $g^{\mathrm{eff}}(k',g)$ if $k\leqq k'$. Then (II.32) follows simply from (II.31) and the Borel convolution rule (II.29).

Finally combining (II.21) and (II.32) we obtain:

$$|B[DR_{(G,F,\mu)}](b)| \leqq \frac{|b|^{n-1}}{(n-1)!} e^{|b|(\beta_2 i_m(\mu)+c)}\,, \tag{II.34}$$

where c is a numerical constant.

III. The Bounds

In [1] the following "Lipatov bound" was proved for the piece of the perturbation expansion which does not contain "useless counterterms":

Theorem III.1. *There exists a function $\varepsilon(n)$ such that $\varepsilon(n)\to 0$ as $n\to\infty$, and such that:*

$$\sum_{\substack{(G,F,\mu)\\ n(G)=n,\ F=\emptyset}} \int \prod_{v\in G} dx_v |Z_{(G,F,\mu)}| \leqq n!\, K_L^n[1+\varepsilon(n)]^n, \tag{III.1}$$

where K_L is the Lipatov constant of Sect. I. Furthermore there exist constants C and $\varepsilon>0$ such that:

$$1+\varepsilon(n)\leqq \exp\{C(\log n)^{-\varepsilon}\}. \tag{III.2}$$

Of course ε in (III.2) *can be replaced by any number $\varepsilon'<\varepsilon$, which means that ε can be taken as small as desired in* (III.2)*; in particular we will assume later that $\varepsilon<1/2$.*

This estimate was shown in [1] to hold for a fixed Schwinger function at fixed external momenta. The main theorem was expressed by inequality (III.1) but the detailed proof shows that (III.2) [which obviously adds information to (III.1)] also holds.

In fact we also need the following corollary of the results of [1]:

Corollary III.1. *For any $\eta>0$, under the same conditions there exists a positive function $\varepsilon(n)$ and constants C and ε (depending on η) such that $1+\varepsilon(n)$ satisfies the bound* (III.2), *and such that:*

$$\sum_{\substack{(G,F,\mu)\\ n(G)=n,\ F=\emptyset}} \int \prod_{v\in G} dx_v |Z_{(G,F,\mu)}| e^{(2-\eta)i_m(\mu)} \leqq n!\, K_L^n[1+\varepsilon(n)]^n. \tag{III.3}$$

Proof. The proof in [1] of the theorem above uses only the fact that if $F=0$ there is an exponential decay in the space of momentum assignments μ, but does not use the coefficient of this decay, which only affects the function $\varepsilon(n)$ [more precisely the coefficients C and ε in (III.2)]. But by power counting the exponential decay in momentum assignment space is at least 2 [12], which is the worst superficial degree of convergence after renormalization, corresponding to the convergence degree of 6 point functions and of renormalized 2 and 4 point functions. Therefore after multiplying by $e^{(2-\eta)i_m(\mu)}$ we still have exponential decay in momentum space (of strength at least η) and the analysis of [1] does apply, leading to the stated Corollary.

The case of 4 and 2 point functions is controlled by the following bounds:

Corollary III.2. *For a 4 point Schwinger function:*

$$\sum_{\substack{(G,F,\mu)\\ n(G)=n,\, F=\emptyset\\ k\leqq i_G(F,\mu)\leqq e}} \int \prod_{v\in G} dx_v |Z^0_{(G,F,\mu)}| e^{(2-\eta)[i_m(\mu)-i_G(F,\mu)]}$$

$$\leqq (e-k+1)n!\, K_L^n[1+\varepsilon(n)]^n, \tag{III.4}$$

where Z^0 is similar to Z, except that the $(1-T_G)$ operation in the definition of Z is replaced by T_G (hence Z^0 is the counterterm associated to G). Remark that now we have only exponential decay between the maximal (i_m) and minimal (i_G) assignments in

G, hence the corresponding exponential increase in (III.4) *has to be restricted to* $i_m - i_G$.

For two point functions (which, say in a 6 point function, will appear only as subgraphs) it is convenient to include in the corresponding estimate one of the external legs of G, and to get:

Corollary III.3. *For a 2 point Schwinger function plus one of its external legs:*

$$\sum_{\substack{(G,F,\mu)\\ n(G)=n,\,F=\emptyset\\ k\le i_G(F,\mu)\le e}} \int \prod_{v\in G} dx_v |Z^0_{(G,F,\mu)}| e^{(2-\eta)[i_m(\mu)-i_G(F,\mu)]}$$
$$\le n!\,K_L^n[1+\varepsilon(n)]^n\,, \qquad \text{(III.5a)}$$

$$\sum_{\substack{(G,F,\mu)\\ n(G)=n,\,F=\emptyset\\ k\le i_G(F,\mu)\le e}} \int \prod_{v\in G} dx_v |Z^1_{(G,F,\mu)}| e^{(2-\eta)[i_m(\mu)-i_G(F,\mu)]}$$
$$\le n!\,(e-k+1)\,K_L^n[1+\varepsilon(n)]^n\,, \qquad \text{(III.5b)}$$

where Z^0 *and* Z^1 *are respectively the mass and wave function counterterms associated to G. Recall that by convention in* (III.5b) *one of the two external legs of G has been included in the estimate, to bear the derivative associated to the Taylor operator for the wave function counterterm* [12].

These last identities are easy generalizations of Corollary III.1, just taking into account the divergence of the global counterterms which are logarithmically divergent for the coupling constant and wave function renormalizations and quadratically divergent for the mass renormalization; notice that by (II.7) the quadratic divergence of a two point subgraph is always compensated by the quadratic *convergence* of any of its two external legs.

A last remark will be useful. The bounds (III.1) and (III.3)–(III.5) are expressed for unlabeled graphs; if the summation was made over "labeled graphs" with distinguished vertices (which is better for correct graph counting) there would be simply an $(n!)^2$ instead of $n!$ in the right-hand side of the bounds. It is this form of the bounds that we will use below to establish (III.15).

We want now to bound the total n^{th} order of effective perturbation theory in (II.16), i.e. we want a bound on the sum over fundamental triplets (G, F, μ) which is not restricted to the triplets with an empty F. Our target is to prove:

Theorem III.2. *For any* $\eta' > 0$ *there exists a function* $\delta(n)$ *(depending on* η'*) such that* $\delta(n) \to 0$ *as* $n \to \infty$, *and such that:*

$$\sum_{\substack{(G,F,\mu)\,\text{fundamental}\\ n(G)=n}} \int \prod_{v\in G} dx_v |Z_{(G,F,\mu)}| e^{(2-\eta')\,i_m(\mu)} \le n!\,K_L^n[1+\delta(n)]^n\,. \qquad \text{(III.6)}$$

Moreover the details of the proof will give the bound:

$$1+\delta(n) \le \exp\{+C(\log\log\log n)^{-\varepsilon}\} \qquad \text{(III.7)}$$

for some constants $C > 1$ and $\varepsilon > 0$ (depending only on η'). Of course this bound is not optimal and the log log log has no physical significance. In what follows we always assume implicitly in formulas like (III.7) that n is larger than $\exp[\exp[\exp 1]]$ so that $\log\log\log n > 1$; indeed for n bounded there is no problem at all.

In order not to obscure the argument too much, we treat first a simplified case in which the sum is restricted to graphs G which have no two point subgraphs. The subgraphs of F (all of which have 4 external legs) can then be classified into four classes:

– *Class 1*. These are the subgraphs $g \in F$ which have $n(g/F) \geqq 3$.

– *Class 2*. These are the subgraphs $g \in F$ which have $n(g/F) = 2$ (hence g/F is isomorphic to G_2) and have $i_g(F,\mu) = e_g(F,\mu)$.

– *Class 3*. These are the subgraphs $g \in F$ which have $n(g/F) = 2$ (hence g/F is isomorphic to G_2), have $i_g(F,\mu) < e_g(F,\mu)$, and are such that g/F has exactly one reduction vertex [for the definition of this type of vertex, see after (II.15)].

– *Class 4*. These are the subgraphs $g \in F$ which have $n(g/F) = 2$ (hence g/F is isomorphic to G_2), have $i_g(F,\mu) < e_g(F,\mu)$, and are such that g/F has exactly two reduction vertices.

Notice that this exhausts all possibilities because of our resummation rule. For a given (G, F, μ) let us call p_1, p_2, p_3, and p_4 the number of subgraphs of classes 1, 2, 3, and 4, and $p = p_1 + p_2 + p_3 + p_4$ the total number of subgraphs in F. First we remark that:

$$p_4 \leqq p_1 + p_2 - 1, \quad \text{hence} \quad p_4 \leqq p/2. \tag{III.8}$$

This is because a class 4 subgraph must contain two disjoint subgraphs of F of class 1 and 2.

As in Appendix C of [8], we separate F into layers L_i, $1 \leqq i \leqq \lambda(G) \leqq p$:

$$L_{\lambda(G)} = \{g \in F \mid B_g(F) = G\}, \tag{III.9}$$

$$L_i = \{g \in F \mid B_g(F) \in L_{i+1}\}. \tag{III.10}$$

We also define $h_i \geqq 1$ as the number of elements in the layer L_i and

$$k_i = \sum_{1 \leqq i' \leqq i} h_{i'}. \tag{III.11}$$

Next we order arbitrarily the disjoint subgraphs in each layer, and we label them in the following way:

$$L_i = \{g_j, j = k_{i-1} + 1, \ldots, k_i\}. \tag{III.12}$$

[There are of course $\pi_i(h_i!)$ possible different labelings.] By a slight abuse of notations we will say that j is of class 1, 2, 3 or 4 if g_j is respectively of class 1, 2, 3 or 4. We fix which j's are of class 1, 2, 3 or 4, paying a factor 4^p in our estimates. We also define $n_j \geqq 2$ as the number of vertices of the reduced graph g_j/F and $s_j \geqq 0$ as the number of reduction vertices in g_j/F. By convention we write $G = g_{p+1}$. Then the following relations must be satisfied:

$$k_{\lambda(G)} = p; \qquad \sum_{1 \leqq i' \leqq p+1} n_j = n + p; \qquad \sum_{k_{i-1} < j \leqq k_i} s_j = h_{i-1}. \tag{III.13}$$

To obtain the desired estimate (III.6) we use Corollary III.2 inductively, starting from the minimal graphs of F, and inserting them into larger and larger ones to build the whole forest. Let $r_j = n_j - s_j$ be the number of ordinary vertices in g_j/F. As in [8], Appendix C, the binomial coefficients

$$P_j = \begin{bmatrix} n - \Sigma_{j'=1,j-1} r_{j'} \\ r_j \end{bmatrix}; \qquad Q_j = \begin{bmatrix} h_{i-1} - \Sigma_{j'=k_{i-1}+1,j-1} s_{j'} \\ s_j \end{bmatrix} \tag{III.14}$$

count the choices respectively of ordinary and reduction vertices in g_j/F. For each j, a factor (4!) takes into account the identification between the 4 external legs of g_j and the 4 lines attached to the reduction vertex corresponding to it. When the ways of inserting the g_j/F's have been chosen, it remains to bound the amplitudes of the inserted objects, which is done using Corollary III.2 (and Corollary III.1 for the last graph G/F), since the inserted pieces do not contain useless counterterms any more. However we apply these corollaries with $\eta = \eta'/2$; hence we preserve a factor $\exp\{-\eta i_m(\mu)\}$ for later use. $\Big($To see that this is possible one should notice that $i_m(G/F,\mu) + \sum_{h\in F} [i_m(h/F,\mu) - i_{h/F}(F,\mu)\} \geqq i_m(G,\mu) \equiv i_m(\mu).\Big)$ Therefore we get a bound which is the sum over the choices of p, λ, $\{k_i\}$, $\{n_j\}$, $\{s_j\}$ satisfying the constraints (III.13) of:

$$\sum_{i_m(\mu)} \frac{(4.4!)^p}{n!} \prod_{i=1}^{\lambda(G)} \left[\frac{1}{h_i!} \prod_{j=k_{i-1}+1}^{k_i} P_j Q_j m_j [1+\varepsilon(n_j)]^{n_j} K_L^{n_j} (n_j!)^2 \right]$$
$$\times K_L^{n_{p+1}} (n_{p+1}!)^2 e^{-\eta i_m(\mu)} . \tag{III.15}$$

Remark the similarity of this formula with (C.19) in [8]. Here the factors $1/n!$ and the $(n_j!)^2$ take into account the fact that we establish (III.15) by an analysis at the level of *labeled* graphs. Furthermore $(4)^p$ takes into account the choices of the class of g_j, $\pi_i(h_i!)^{-1}$ restores correct counting by dividing out the multiple counting due to the arbitrary labeling of the forest, and the integer m_j corresponds to the factor $(e-k)$ in the bound (III.4) when Corollary III.2 is applied to $G = g_j/F$.

We will need only the following immediate properties of m_j:

– for g_j in class 2, $m_j = 1$, (III.16a)

– for g_j in classes 1 and 4:

$$m_j \leqq i_m(\mu) + 1 , \tag{III.16b}$$

– for g_j in class 3, there is one and only one maximal $g_{j'}$ included in g_j and corresponding to the unique reduction vertex of g_j; then

$$m_j \leqq e_{g_j}(F,\mu) - i_{g_{j'}}(F,\mu) , \tag{III.17}$$

simply by using the fact that both g_j and $g_{j'}$ belong to a forest F compatible with μ.

Let us define S_j as the binomial coefficient:

$$S_j = \begin{bmatrix} n_j \\ s_j \end{bmatrix} \equiv C(n_j; s_j) , \tag{III.18}$$

where for simplicity of notation we write $C(n;p)$ for the binomial coefficient of x^p in the development of $(1+x)^n$. Then, using relations (III.13), (III.15) is reduced (by

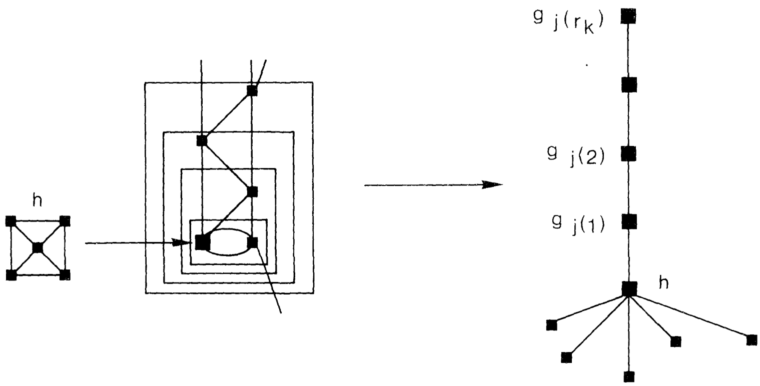


Fig. 3. A chain of type III subgraphs above a particular subgraph h with five vertices

telescopic simplification of factorials) to:

$$K_L^{n+p}(4.4!)^p \prod_{j=1}^{p+1} [S_j m_j n_j![1+\varepsilon(n_j)]^{n_j}]\, e^{-\eta i_m(\mu)}, \tag{III.19}$$

where by convention m_{p+1} is 1 (and we recall that $\eta=\eta'/2$). First let us replace all factors m_j for j of class 2 by 1, and for j of class 1 or 4 by $i_m(\mu)$, using (III.16a, b). Then if we define q as p_1+p_4 we can write:

$$[i_m(\mu)]^q e^{-\eta i_m(\mu)/2} \leqq (2\eta^{-1})^q q!\,. \tag{III.20}$$

For the graphs of class 3, we must be more careful. These graphs are grouped into r chains of r_k graphs, with $1\leqq r_k \leqq p_3$, $\sum_{k=1,r} r_k = p_3$ (see Fig. 3); the k^{th} chain is a maximal subset of subgraphs $g_{j(1)},\dots,g_{j(r_k)}$ of class 3 completely ordered by inclusion, such that $g_{j(l)}$ corresponds to the reduction vertex in $g_{j(l+1)}$. Note that these chains are completely determined by the choices of which g_j are of class 3 and which graphs of layer $i-1$ correspond to the reduction vertices in layer i. Since these choices have been taken into account in (III.15), there is no sum to perform over the r_k, but only a supremum to take over all choices of r_k. Another important fact to notice is that for each such chain, the reduction vertex in $g_{j(1)}$, the smallest graph of the chain, cannot be of class 3, hence must be of class 1, 2 or 4. Therefore we have $r\leqq p_1+p_2+p_4=q+p_2$; hence

$$p_1+p-r/2 \geqq p_1+p_3+(q/2)+(p_2/2)=p_3+q-(1/2)[p_4-(p_1+p_2)]\,.$$

Hence using (III.8) we get:

$$q+p_3\leqq p+p_1-(r/2)\,. \tag{III.21}$$

For such a chain one sees, using (III.17) that (after telescopic simplification):

$$\sum_{l=1}^{r_k} m_l \leqq i_m(\mu), \quad \text{hence} \quad \prod_{l=1}^{r_k} m_l \leqq \left[\frac{i_m(\mu)}{r_k}\right]^{r_k}. \tag{III.22}$$

Using again inequality (III.20) but with q replaced by p_3, we find that since $\sum r_k = p_3$:

$$\left[\prod_k \prod_{l=1}^{r_k} m_l\right] e^{-\eta i_m(\mu)/2} \leqq \frac{[2\eta^{-1}]^{p_3} p_3!}{\prod_k (r_k)^{r_k}} \leqq \frac{[2\eta^{-1}]^{p_3} p_3!}{(p_3/r)^{p_3}}, \tag{III.23}$$

where the last inequality follows from convexity of the function $x\log x$. Combining (III.20) and (III.23) and using trivial bounds we get (remember that $p_3+q\leqq p$):

$$\prod_j m_j e^{-\eta i_m(\mu)} \leqq [2\eta^{-1}]^p q!\, r^{p_3}. \tag{III.24}$$

We will now write a lemma which bounds the product of the factors $[1+\varepsilon(n_j)]^{n_j}$, using the bound (III.2).

Lemma III.1. *Let $\{n_j\}, j=1,\dots,p+1$ be a collection of $p+1$ integers such that $n_j\geqq 2$ and $\sum_j n_j=n+p$. Then for any $C>1$ and ε with $0<\varepsilon<1$, there exists a constant $C'>1$, depending on C and ε, such that (we assume* $\log\log\log n>1$, *and* $\log\log p>1$, *otherwise the bounds are much easier):*

$$\prod_j \exp\{Cn_j(\log n_j)^{-\varepsilon}\} \leqq \exp\{C'[p(\log\log p)^{2\varepsilon}+n(\log\log\log n)^{-\varepsilon}]\}. \tag{III.25}$$

Proof. First we can bound each factor in the left-hand side of (III.25) with $n_j<10$ by a constant, hence all these factors contribute at most $\exp\{C\cdot p\cdot \text{const}\}$, which can be absorbed in the term $\exp\{C'p(\log\log p)^{2\varepsilon}\}$ in the right-hand side of (III.25). Then we take logarithms on both sides of (III.25) and use the concavity, for $x\geqq 10 \geqq e^{1+\varepsilon}$, of the function $x(\log x)^{-\varepsilon}$. This reduces the proof of (III.25) to the proof of:

$$C(n+p)\{\log[(n+p)/(p+1)]\}^{-\varepsilon} \leqq C'[p(\log\log p)^{2\varepsilon}+n(\log\log\log n)^{-\varepsilon}]. \tag{III.26}$$

Since $p+1\leqq n$ we have:

$$C(n+p)\{\log[(n+p)/(p+1)]\}^{-\varepsilon} \leqq 2Cn\{\log(n/p)\}^{-\varepsilon}. \tag{III.27}$$

The proof of (III.26) is then trivial if one distinguishes the two cases $p\leqq n(\log\log n)^{-\varepsilon}$ and $n(\log\log n)^{-\varepsilon}<p\leqq n$. In the latter case one has $\log\log p >(\log\log n)^{1/2}$.

Using the estimate (III.2), the lemma leads immediately to:

$$\prod_j [1+\varepsilon(n_j)]^{n_j} \leqq \exp\{C'p(\log\log p)^{2\varepsilon}\}\,(1+\delta'(n))^n, \tag{III.28}$$

where $\delta'(n)$ satisfies a bound similar to (III.7).

We are now in a position to combine (III.19) with (III.24) and (III.28). By so doing, the proof of (III.6) is reduced to the proof of:

$$\sum (8\eta^{-1}K_L 4!)^p r^{p_3} q! \prod_{j=1}^{p+1} [S_j n_j!] \exp\{C'p(\log\log p)^{2\varepsilon}\} \leqq n![1+\delta''(n)]^n, \tag{III.29}$$

where $\delta''(n)$ satisfies a bound similar to (III.7). In (III.29) the sum is over choices of $\lambda, p, \{k_i\}, \{n_j\}$, and $\{s_j\}$ satisfying the constraints (III.13) and a supremum should be taken over $p_1, p_2, p_3, p_4, q=p_1+p_4$ and r satisfying the constraints (III.8) and (III.21).

To prove (III.29) we replace first all the n_j and S_j for j of class 2, 3 or 4 by 2 (which is the value of n_j, hence a bound over S_j). Since the labeling of the g_j with j of class 2, 3 or 4 does not play a significant role any longer, we relabel the subgraphs g_j of class 1 with indices $j=1,\dots,p_1$, by simply skipping in the former numbering all the numbers corresponding to subgraphs of class 2, 3 or 4, and we rename $G=g_{p+1}$ as g_{p_1+1}. With these notations the left-hand side of (III.29) is bounded by:

$$\sum (32\eta^{-1}K_L 4!)^p r^{p_3} q! \prod_{j=1}^{p_1+1} [S_j n_j!] \exp\{C'p(\log\log p)^{2\varepsilon}\}. \tag{III.30}$$

In (III.30), the sum over λ and p can be bounded by n^2, hence absorbed in the factor $[1+\delta''(n)]^n$ of (III.29). The sum over the choices of $\{k_i\}$, using (III.13), is bounded by 2^p. Hence there exists a constant $D=64\eta^{-1}K_L 4!$ depending only on η such that (III.30) is bounded by:

$$\sup_{p,q,r,p_1,p_3} \sum_{\{n_j\},\{s_j\}} D^p r^{p_3} q! \prod_{j=1}^{p_1+1} [S_j n_j!] \exp\{C'p(\log\log p)^{2\varepsilon}\}, \tag{III.31}$$

where the constraints over the sup and the sum which arise from (III.8), (III.13), and (III.21) can be rewritten in terms of the integer variables p,q,r,p_1,p_3 and $\{n_j\},\{s_j\}$, $j=1,\dots,p_1+1$, as:

$$\begin{gathered}\sum_j n_j = n-p+2p_1;\; n_j \geqq 3 \quad \text{for any} \quad j;\, 0\leqq s_j \leqq n_j;\, \sum_j s_j \leqq p;\\ p_1 \leqq q \leqq p \leqq n;\; q\leqq p_1+p/2;\; q+p_3 \leqq p+p_1-r/2;\; r\leqq p_3 \leqq p.\end{gathered} \tag{III.32}$$

The only constraint that the reader may find questionable is $n_{p_1+1}\geqq 3$; it is true if we look at a 1PI Schwinger function with 6 external legs or more. Of course this condition is completely unnecessary in fact, and we use it only to slightly simplify the following lemma:

Lemma III.2. *Under the conditions* (III.32) *we have:*

$$\prod_j n_j! \leqq (n-p-p_1)!\, 6^{p_1}. \tag{III.33}$$

Proof. The left-hand side is the product of $n-p+2p_1$ integers. First we can bound p_1 blocks of 3 integers by $(1.2.3)^{p_1}=6^{p_1}$. There remains then a product of $(n-p-p_1)$ integers; by listing them as first the integers in $n_{p_1+1}!$, then these in $4.5\dots n_j$, $j=1,\dots,p_1$, each of them is smaller or equal to the one of corresponding rank in the product $1.2.3\dots(n-p-p_1)$, which proves (III.33) (it is here that we use that $n_{p_1+1}\geqq 3$).

Using (III.33), (III.31) is bounded by:

$$\sup_{p,q,r,p_1,p_3} \sum_{\{n_j\},\{s_j\}} (6D)^p r^{p_3} q!\,(n-p-p_1)! \prod_{j=1}^{p_1+1} S_j \exp\{C'p(\log\log p)^{2\varepsilon}\}. \tag{III.34}$$

The next lemma bounds the factor $r^{p_3}q!\,(n-p-p_1)!$ with the help of constraints (III.32):

Lemma III.3. *Under conditions* (III.32) *we have, for some fixed α and A:*

$$r^{p_3} q!\,(n-p-p_1)! \leqq Ae^p \cdot n!\, e^{-\alpha p \log\log p} \tag{III.35}$$

(α is small, and A is large).

Proof. We distinguish three cases:

– if $p_3 \leqq p/4$, then $q+p_3 \leqq p_1+(3p/4)$; by Stirling's formula $r^{p_3} \leqq e^p p_3!$, and $q!\,p_3!\,(n-p-p_1)! \leqq [n-(p/4)]!$; but $(n-s)! \leqq n!\,(s/2)^{-s/2}$, which for $s=p/4$ gives a much smaller bound than the right-hand side of (III.35).

– if $p_3>p/4$ and $r>p_3(\log p)^{-1/2}$; again by Stirling's formula we bound $r^{p_3}\cdot q!\,(n-p-p_1)!$ by $e^p[n-(r/2)]!$, using (III.21) hence further by $e^p[n-(p/8)(\log p)^{-1/2}]!$, hence further by $e^p n!\,(s/2)^{-s/2}$ with $s=(1/8)p(\log p)^{-1/2}$, which again is bounded for some constants A' and α' by $A'\cdot e^p n!\cdot\exp\{-\alpha'\cdot p(\log p)^{1/2}\}$, if p is large enough, a much smaller bound than the right-hand side of (III.35).

– if $p_3>p/4$ and $r \leqq p_3(\log p)^{-1/2}$ we get the worst bound; we simply bound r^{p_3} by $e^p p_3!\,(\log p)^{-p/8}$; then using (III.32) we can bound $p_3!\,q!\,(n-p-p_1)!$ simply by $n!$, and we get the bound (III.35), with $\alpha=1/8$.

Using (III.35), we see that (III.34) is bounded by:

$$A \sup_p \sum_{\{n_j\},\{s_j\}} D'^p n! \prod_{j=1}^{p_1+1} S_j \exp\{C'p(\log\log p)^{2\varepsilon}-\alpha p\log\log p\}\,, \tag{III.36}$$

where $D'=6eD$ is another constant depending only on η, and the constraints (III.32) can be simplified to the less stringent ones (where j runs from 1 to p_1+1):

$$p_1 \leqq p;\; n_j \geqq 3;\; \sum_j n_j \leqq n+p;\; 0 \leqq s_j \leqq n_j;\; \sum_j s_j \leqq p \leqq n\,. \tag{III.37}$$

Since ε in (III.2), hence in (III.25) and (III.36) can be taken arbitrarily small, and in particular strictly smaller than 1/2, we have for any $p \geqq 1$:

$$A\cdot D'^p \exp\{C'p(\log\log p)^{2\varepsilon}-(1/2)\alpha p\log\log p\} \leqq D''\,, \tag{III.38}$$

where D'' is again a constant depending only on D', C', α, and ε, but not on p. To achieve the proof of (III.29), it remains therefore only to show that under constraints (III.37) we have:

$$D'' \sup_p e^{-(\alpha/2)p\log\log p} \sum_{\{n_j\},\{s_j\}} \prod_{j=1}^{p_1+1} [S_j] \leqq [1+\delta''(n)]^n\,, \tag{III.39}$$

where $\delta''(n)$ behaves as $\delta(n)$ in (III.7).

Firstly since $n_j \geqq 1$, and $\sum_j n_j \leqq n+p$ we can bound by a standard combinatoric argument the choice of the integers $\{n_j\}$ by the binomial coefficient $C(n+p;p_1+1)$. Then we use the following bound on binomial coefficients:

$$C(m;m') \leqq e^{m'}(m/m')^{m'}\,, \tag{III.40}$$

which is essentially given by Stirling's formula. Hence in this case we can bound the choice over the $\{n_j\}$ by (since $p_1 \leqq p$):

$$e^{p+1}[(n+p)/(p+1)]^{p+1} \leqq (2e)^{p+1}\,[n/(p+1)]^{p+1}\,. \tag{III.41}$$

Secondly, let us choose, paying a factor at most 2^{p+1}, which s_j are 0. Recall that the corresponding S_j are 1, hence can be forgotten. Let $p' \leqq p+1$ be the remaining number of s_j with $s_j \geqq 1$, and let us rename them $s_{j'}, j'=1,\ldots,p'$ (we also write $n_{j'}$ for n_j, where j in the old labeling corresponds to j' in the new one). Since $\sum_{j'} s_{j'}=p''$ $\leqq p+1$, the choice of these $\{s_{j'}\}$ is bounded again by a factor 2^{p+1}. Then using (III.40) we bound each $S_{j'}$ by $(e\cdot n_{j'}/s_{j'})^{s_{j'}}$. Using the concavity of the logarithm, we have therefore:

$$\prod_{j'} S_{j'} \leqq e^{p+1} \left[\frac{(n+p)}{p''}\right]^{p''} \leqq (4e)^{p+1} \left[\frac{n}{p+1}\right]^{p+1} . \tag{III.42}$$

Collecting all these factors, the left-hand side of (III.39) is bounded by:

$$D'' \sup_{1 \leqq p+1 \leqq n} e^{-(\alpha/2) p \log\log p} (32 e^2)^{p+1} \left[\frac{n}{p+1}\right]^{2p+2} . \tag{III.43}$$

Again we distinguish two cases:
– if $p+1 < n(\log\log n)^{-1/2}$ we have $(n/p+1)^{2p+2} \leqq \exp\{2c'' n(\log\log n)^{-1/4}\}$ for some constant c'', since $\exp\{(1/x)\log x\}$ is bounded by $\exp\{c'' x^{-1/2}\}$ for $x>0$ [here we apply this relation with $x = n/(p+1)$]. Hence since:

$$D'' \sup_{1 \leqq p+1 \leqq n} e^{-(\alpha/4) p \log\log p} (32 e^2)^{p+1} \leqq D''' \tag{III.44}$$

for some constant D''' depending only on D'' and α, we can bound (III.43) by:

$$D''' e^{2c'' n(\log\log n)^{-1/4}} , \tag{III.45}$$

which is better than the desired form (III.7).
– if $p+1 \geqq n(\log\log n)^{-1/2}$ we have $p \geqq (1/2)\, n(\log\log n)^{-1/2}$, hence under this condition:

$$e^{-(\alpha/4) p \log\log p} \left[\frac{n}{p+1}\right]^{2p+2} \leqq e^{-\alpha' n(\log\log n)^{1/2}} e^{n \log\log\log n} \leqq D'''' \tag{III.46}$$

for some constant D'''' depending only on α. Again using (III.44) we can in this second case bound (III.43) by $D''' \cdot D''''$ hence by a constant (depending finally only on η'). This achieves the proof of Theorem III.2, in the case where no 2-point subgraphs are present in G.

When "bipeds," i.e. two point subgraphs, are present they complicate a little further the argument, but can essentially be considered as class 1 subgraphs, the ones for which the argument above is therefore the easiest. Let us sketch why this is true. The mass counterterms, which are bounded by (III.5a) at first sight seem dangerous because of their quadratic divergence, but when combined with one of their external legs, which is quadratically convergent, they become bounded. (Note that we can always associate to an 1PI biped one of its external lines, so that the same line is not attributed to two different bipeds.) Hence the worst case is the wave-function counterterms, for which the bound (III.5b) gives a factor $(e-k)$ *after* one of the external legs has been combined with the biped. But the block made of a wave function counterterm plus one of its external legs is really a (one particle reducible ...) four point subgraph, which has at least 3 vertices since the only biped with a single vertex is the tadpole, which vanishes in the BPHZ prescription. Hence it can be estimated exactly as a class 1 subgraph in the analysis above, since we never used the fact that class 1 subgraphs were one particle irreducible. This sketchy argument (the reader is invited to fill in the details ...) extends the proof of Theorem III.2 to the general case.

The proof of the Theorem of Sect. I is now easy. Let us take a radius $r < (2/\beta_2)$, and define $\eta' = \beta_2[(2/\beta_2) - r]$. Then by (II.16) and (II.34) the Borel transform of the renormalized series is bounded for $|b| \leqq r$ by

$$\sum_{n} \sum_{\substack{(G,F,\mu)\text{fundamental} \\ n(G)=n}} \int \prod_{v \in G} dx_v |Z_{(G,F,\mu)}| \frac{r^{n-1}}{(n-1)!} e^{(2-\eta') i_m(\mu) + \text{c.r.}},$$

hence, applying Theorem III.2 it is bounded by:

$$\sum_{n} n K_L \left[\frac{2K_L}{\beta_2}\right]^{n-1} e^{2c/\beta_2} [(1+\delta(n)]^n. \tag{III.47}$$

Since $2K_L/\beta_2 \leqq 2/3$ (the value 2/3 corresponding to ordinary φ_4^4 with $N=1$) the Borel transform of the renormalized series is analytic in the disk $|b| \leqq r$, because it is a sum of analytic terms (in fact polynomials in b) and this sum is bounded uniformly for b in this disk by (III.47), which is an absolutely convergent series.

Since this is true for all $r < 2/\beta_2$, this shows that the radius of convergence of the Borel transform of the renormalized perturbation series is at least $(2/\beta_2)$ which is equivalent to the Theorem of Sect. I by Hadamard's formula.

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