Commun. Math. Phys. 114, 537–548 (1988)

Communications in Mathematical Physics

The Equilibrium States of the Spin-Boson Model

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Abstract. The temperature states of the spin-boson model consisting of a two-level atom in a Bose field are studied. It is proved that for all temperatures there exists a unique solution, hence there is no spontaneous reflection symmetry breaking.

1. Introduction

Spin-boson models are very popular in solid state physics, quantum chemistry as well as in quantum tunneling. A fairly good introduction to the physics can be found in [1].

Here we are particularly interested in the following model:

$$H = \int dk\varepsilon(k)a_k^+ a_k + \sigma_3 \int dk\lambda(k)(a_k^+ + a_k) + \mu\sigma_1$$

describing a two-level atom in a boson field. On the basis of physical arguments one assumes that the following conditions are satisfied:

$$\int dk\lambda(k)^2 < \infty, \quad \int dk \frac{\lambda(k)^2}{\varepsilon(k)} < \infty; \quad \varepsilon(k) \simeq |k|.$$

In this work we study the temperature states of this model in a rigorous way. The ground state problem will be kept for a future occasion. The main aspect we search for is whether or not there is spontaneous symmetry breaking of the reflection symmetry: $\sigma_3 \to -\sigma_3$, $a_k \to -a_k$. Usual techniques for proving the absence of symmetry breaking are not applicable because the group is finite.

As far as we know there exist only a few rigorous results for this model. In [2] one discusses some results on the spectrum of the proposed Hamiltonian, in [3] a thorough analysis is made of a finite mode approximation of the Hamiltonian. In particular the Hartree–Fock solutions are found to show breaking of the symmetry under the condition $\mu < 2\int(\lambda(k)^2/\varepsilon(k))dk$. In [4] the ground state of the model is analyzed. By functional integration techniques it is shown that no

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spontaneous symmetry breaking appears if λ/ε is square integrable (in fact in $k=0$). Otherwise if the coupling λ is large enough there is symmetry breaking.

Our contribution consists in the computation of all temperature states, and hence in actually solving the problem for $T>0$. The method we use consists in considering the term $\mu\sigma_1$ as a perturbation. This point of view was already presented in [2] and [3]. Then we work towards a rigorous formulation of the KMS-equation (equilibrium conditions) for the so-called unperturbed model, and we are able to solve it completely. Finding the solution relies on a detailed study of the representations of the spin-Boson algebra. Then the perturbation theory is applied at the level of the cyclic vector of the unperturbed system.

From the Hartree–Fock computations and the spectral perturbation theory it could be guessed that symmetry breaking occurs for a coupling which is strong enough. Surprisingly enough it turns out that there is never spontaneous breaking of the symmetry, irrespective of the strength of the coupling constant. In particular we prove that for all positive temperatures there is a unique equilibrium state. We work out the model in one dimension. Our analysis is not complete for higher dimensions. There might be a macroscopic occupation of the ground state ($k=0$) for the Boson field, destroying the unicity of the solution in two and more dimensions. However we are not interested in the symmetry breaking due to Bose condensation.

2. The Model

We start with the CCR-algebra $\Delta(\mathscr{H}_0)$ built on a test function space $\mathscr{H}_0$ which is a dense subspace of $L^2(\mathbb{R})$. A reasonable choice is $\mathscr{H}_0=C_0(\mathbb{R})$, the continuous complex functions of compact support on $\mathbb{R}$ vanishing on a neighborhood of zero. As usual we consider the CCR-C^*-algebra $\Delta(\mathscr{H}_0)$ generated by the Weyl operators

$$W(f)=\exp i(a(f)+a^+(f)); \quad f\in\mathscr{H}_0, \tag{1}$$

where $a^+(f)$ and $a(f)$ are the usual Fock creation and annihilation operators, which satisfy the product rule

$$W(f)W(g)=W(f+g)\exp -i\,\mathrm{Im}(f,g). \tag{2}$$

The algebra of observables of the system is then the C^*-algebra

$$\mathscr{B}=\Delta(\mathscr{H}_0)\otimes M_2,$$

the unique tensor product of $\Delta(\mathscr{H}_0)$ and the 2×2 complex matrices.

A general element of $\mathscr{B}$ is of the form

$$X=\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{pmatrix}; \quad X_{ij}\in\Delta(\mathscr{H}_0)$$

and a state ω of $\mathscr{B}$ can be described by a set $\{\omega_{ij}|i,j=1,2\}$ of linear functionals of $\Delta(H_0)$. A useful notation is the following:

$$\omega=\begin{pmatrix} \omega_{11} & \omega_{21} \\ \omega_{12} & \omega_{22}\end{pmatrix}, \tag{3}$$

then

$$\omega: X \to \omega(X) = \sum_{i,j=1}^{2} \omega_{ij}(X_{ij}).$$

By normalization and positivity ω is a state of $\mathscr{B}$ if and only if the functionals ω_{ij} satisfy

$$\omega_{11}(\mathbb{1}) + \omega_{22}(\mathbb{1}) = 1,\ \omega_{ii}(x^*x) \geqq 0; \quad i = 1,2$$
$$|\omega_{12}(x^*y)|^2 \leqq \omega_{11}(x^*x)\omega_{22}(y^*y),$$
$$\overline{\omega_{12}(x)} = \omega_{21}(x^*)$$

for all $x, y \in \Delta(\mathscr{H}_0)$. For the purpose of the study of the model under consideration we will restrict our attention to the class of states that satisfy the following conditions:

(a) Regularity of the states, i.e. for all $f, g \in \mathscr{H}_0$ the map

$$z \in \mathbb{R} \to \omega_{ij}(W(f + zg))$$

is analytic. This condition implies the existence of correlation functions

$$\omega_{ij}(a^{\#}(f_1) \cdots a^{\#}(f_n)), \quad f_i \in \mathscr{H}_0$$

where $a^{\#}$ stands for a or a^+.
(b) Continuity of the correlation functions: define the Hilbert space $\mathscr{H}$ as the closure of $\mathscr{H}_0$ for the scalar product

$$(f, g)_{\sim} = \int \left(1 + \frac{1}{|k|}\right) \bar{f}(k) g(k) dk.$$

Then we suppose that the states ω_{ii} satisfy

$$|\omega_{ii}(a^+(f_1) \cdots a^+(f_n) a(g_1) \cdots a(g_m))|^2 \leqq C^{n+m} n! m! \prod_{j=1}^{n} \|f_j\|_{\sim}^2 \prod_{j=1}^{m} \|g_j\|_{\sim}^2$$

for $f_j, g_j \in \mathscr{H}_0$.

For any state ω of $\mathscr{B}$ we consider its GNS-triplet $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$. For notational convenience we identify the algebra and the representation ($x = \pi_\omega(x)$ for $x \in \mathscr{B}$) if this is clear from the context, and we denote by $\mathscr{B}''$ the von Neumann algebra generated by $\mathscr{B}$. Because of the continuity conditions (a) and (b), $\mathscr{B}''$ contains the strong limits of $W(f) \otimes \mathbb{1}$, $f \in \mathscr{H}_0$, which again are Weyl operators in that they satisfy the relation (2). This is the meaning of the operators $W(f) \otimes A$ with $f \in \mathscr{H}, A \in M_2$. The same reasoning holds true for creation and annihilation operators

$$a^{\#}(f) \otimes A$$

with $f \in \mathscr{H}$, as operators on $\mathscr{H}_\omega$ affiliated to $\mathscr{B}''$. First we describe the evolution α_t^0 on the von Neumann algebra $\mathscr{B}''$. Denote by $\sigma_i, i = 1, 2, 3$, the Pauli-matrices and define for $\varepsilon(k)=|k|$ the maps:

$$\alpha_t^0(\sigma_1)=\frac{1}{2}\left\{W\left(\frac{2i\lambda}{\varepsilon}(e^{it\varepsilon}-1)\right)(\sigma_1-i\sigma_2)\right.$$

$$\left.+W\left(\frac{2i\lambda}{\varepsilon}(1-e^{it\varepsilon})\right)(\sigma_1+i\sigma_2)\right\}, \tag{4.a}$$

$$\alpha_t^0(\sigma_2)=\frac{1}{2}\left\{W\left(\frac{2i\lambda}{\varepsilon}(e^{it\varepsilon}-1)\right)(\sigma_2+i\sigma_1)\right.$$

$$\left.+W\left(\frac{2i\lambda}{\varepsilon}(1-e^{it\varepsilon})\right)(\sigma_2-i\sigma_1)\right\}, \tag{4.b}$$

$$\alpha_t^0(\sigma_3)=\sigma_3, \tag{4.c}$$

$$\alpha_t^0(W(f))=W(e^{i\varepsilon t}f)\exp\left\{2i\sigma_3\operatorname{Re}\left(\frac{\lambda}{\varepsilon},(e^{it\varepsilon}-1)f\right)\right\},\quad f\in\mathscr{H}. \tag{4.d}$$

(c) Existence of the dynamics: we assume that the dynamics $\{\alpha_t^0|t\in\mathbb{R}\}$ extends to a weakly continuous one-parameter group of *-automorphisms of $\mathscr{B}''$.

Hence the one-parameter group $\{\alpha_t^0|t\in\mathbb{R}\}$ defines an infinitesimal generator $\delta_0:\alpha_t^0=\exp it\delta_0$ such that δ_0 is formally given by:

$$\delta_0=[H_0,\cdot],$$

where H_0 is given by

$$H_0=\int dk\,\varepsilon(k)a_k^+a_k+\sigma_3(a(\lambda)+a^+(\lambda)).$$

We now define the full model using the Dyson expansion: for all $x\in\mathscr{B}''$:

$$\alpha_t(x)=\alpha_t^0(x)+\sum_{n\geq 1}i^n\mu^n\int_{0\leqq s_n\leqq\cdots\leqq s_1\leqq t}\cdots\int ds_1\cdots ds_n$$

$$\cdot[\alpha_{s_n}^0(\sigma_1),[\alpha_{s_{n-1}}^0(\sigma_1),\cdots[\alpha_{s_1}^0(\sigma_1),\alpha_t^0(x)]\cdots] \tag{5}$$

for $t\geqq 0$, and a similar expansion for $t<0$. As the perturbation $[\mu\sigma_1,\cdot]$ is a bounded derivation of $\mathscr{B}''$ the series is uniformly convergent and defines a weak *-continuous group of *-automorphisms of $\mathscr{B}''$. The infinitesimal generator δ of the group is formally given by

$$\delta=[H,\cdot],$$

where

$$H=H_0+\mu\sigma_1.$$

Remark that by formulae (4) and (5) we arrived at a rigorous definition of the dynamics of the model on the appropriate C^*-algebra of observables taking into account the conditions

$$\int\lambda(k)^2dk<\infty,\quad\int\frac{\lambda(k)^2}{\varepsilon(k)}dk<\infty.$$

3. Equilibrium States

We are interested in the equilibrium states at fixed inverse temperature $\beta \geqq 0$ for the full dynamics $\{\alpha_t | t \in \mathbb{R}\}$, defined in (5). The strategy consists in constructing the equilibrium states for the unperturbed dynamics α_t^0, then we use the known stability properties of KMS-states for bounded perturbations to obtain the equilibrium states of the full dynamics.

For any state ω satisfying (a)–(c), we define ω to be a (α^0, β)-KMS-state if for all x, $y \in \mathscr{B}''_{\alpha^0}$, a weakly dense α^0-invariant subalgebra of $\mathscr{B}''$, holds [6]:

$$\omega(x\alpha_{i\beta}^0(y)) = \omega(yx). \tag{6}$$

We prove first that this equilibrium condition has a unique solution for the unperturbed evolution α^0.

Theorem 3.1. *There exists a unique* (α^0, β)-KMS-*state* ω_β^0 *of* $\mathscr{B}$ *satisfying conditions* (a)–(c). *Using the notation (3), it is given by*

$$\omega_\beta^0 = \begin{pmatrix} \frac{1}{2}\omega_+ & 0 \\ 0 & \frac{1}{2}\omega_- \end{pmatrix}, \tag{7}$$

where $\omega_\pm$ *are the states of the* CCR-*algebra* $\Delta(\mathscr{H}_0)$ *given by*:

$$\omega_\pm(W(f)) = \exp\left\{ \pm 2i\,\mathrm{Im}\left(\frac{i\lambda}{\varepsilon}, f\right) - \frac{1}{2}\left(f, \coth\frac{\beta\varepsilon}{2} f\right)\right\}. \tag{8}$$

In order to prove this theorem we proceed in a number of steps:

Lemma 3.2. *If* ω_β^0 *is a* (α^0, β)-KMS-*state, then it is of the form*:

$$\omega_\eta = \begin{pmatrix} \eta\omega_+ & 0 \\ 0 & (1-\eta)\omega_- \end{pmatrix}, \tag{9}$$

where $\omega_\pm$ *are given by (8),* $\eta \in [0, 1]$.

Proof. First we prove that the off-diagonal components of ω_β^0 vanish. Therefore apply (6) with $x = \sigma_1 W(f)$ and $y = \sigma_3$. Using (4.c) one gets:

$$\omega_\beta^0(\sigma_1 W(f)\alpha_{i\beta}^0(\sigma_3)) = \omega_\beta^0(\sigma_3\sigma_1 W(f))$$

and

$$\omega_\beta^0(\sigma_1\sigma_3 W(f)) = \omega_\beta^0(\sigma_3\sigma_1 W(f))$$

or

$$\omega_\beta^0(\sigma_2 W(f)) = 0.$$

Analogously:

$$\omega_\beta^0(\sigma_1 W(f)) = 0.$$

Therefore ω_β^0 is of the form

$$\omega_\beta^0 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix},$$

where

$$\omega_1(W(f)) = \omega_\beta^0(\tfrac{1}{2}(1+\sigma_3)W(f)); \quad f\in\mathscr{H}_0,$$
$$\omega_2(W(f)) = \omega_\beta^0(\tfrac{1}{2}(1-\sigma_3)W(f)); \quad f\in\mathscr{H}_0.$$

Define the automorphism groups $\{\alpha_t^\pm | t\in\mathbb{R}\}$ of $\Delta(\mathscr{H}_0)$ by

$$\alpha_t^\pm(W(f)) = W(e^{i\varepsilon t}f)\exp\left\{\pm 2i\,\mathrm{Re}\left(\frac{\lambda}{\varepsilon},(e^{i\varepsilon t}-1)f\right)\right\}.$$

From (4.c) one has

$$\tfrac{1}{2}(1\pm\sigma_3)\alpha_t^\pm(W(f)) = \tfrac{1}{2}(1\pm\sigma_3)\alpha_t^0(W(f)).$$

Remark that for all $f,g\in\mathscr{H}_0$,

$$\begin{aligned}\omega_1(W(f)\alpha_{i\beta}^+W(g)) &= \omega_\beta^0(W(f)\tfrac{1}{2}(1+\sigma_3)\alpha_{i\beta}^+(W(g)))\\ &= \omega_\beta^0(W(f)\tfrac{1}{2}(1+\sigma_3)\alpha_{i\beta}^0(W(g)))\\ &= \omega_\beta^0(W(g)W(f)\tfrac{1}{2}(1+\sigma_3))\\ &= \omega_1(W(g)W(f)),\end{aligned}$$

and similarly ω_2 is a (α^-,β)-KMS-state up to normalization.

The automorphisms $\alpha^\pm$ are up to a displacement the free Bose-gas automorphisms. Under our general conditions λ^2/ε and $\lambda^2\in L^1(\mathbb{R})$, (a)–(c), they yield the unique KMS-states $\omega^\pm$ given by (8) [5]. Hence there exists a $\eta\in[0,1]$ such that

$$\omega_1 = \eta\omega_+, \quad \omega_2 = (1-\eta)\omega_-,$$

where $\eta = \omega_1(\mathbb{1})$. ■

The lemma shows that all solutions of (6) are of the type given by (9) with $\eta\in[0,1]$. We have to prove that there exists only one solution, namely corresponding to the value $\eta=\frac{1}{2}$. Therefore define the reflection symmetry automorphism τ on $\mathscr{B}$ by the following relations:

$$\begin{gathered}\tau(\sigma_1)=\sigma_1, \quad \tau(\sigma_2) = -\sigma_2, \quad \tau(\sigma_3) = -\sigma_3,\\ \tau(W(f)) = W(-f), \quad f\in\mathscr{H},\end{gathered} \tag{10}$$

and remark that $\omega_\eta\circ\tau = \omega_{1-\eta}$.

The state ω_η with $\eta=\frac{1}{2}$ is then precisely the unique τ-invariant state in (9).

Now the strategy to finish the proof of Theorem 3.1 consists in proving that any (α^0,β)-KMS-state is τ-invariant. In order to obtain this statement we prove that under the general conditions on the model any state of the class (9) is a factor state. This will be a consequence of a more general study of the representations induced by states of that type, which we present in the appendix (Sect. 4). Then we remark that τ is an implementable automorphism, commuting with the time evolution, implying that any solution of (6) is τ-invariant which by the argument of above implies $\eta=\frac{1}{2}$.

Lemma 3.3. *Under the conditions*

$$\int dk \frac{\lambda(k)^2}{\varepsilon(k)} < \infty, \quad \int dk \lambda(k)^2 < \infty,$$

the states ω_η *of* $\mathscr{B} = \Delta(\mathscr{H}_0) \otimes M_2$

$$\begin{pmatrix} \eta\omega_+ & 0 \\ 0 & (1-\eta)\omega_- \end{pmatrix}$$

with $\eta \in [0,1]$ *given in* (9) *are factor states.*

Proof. If $\eta = 0$ or 1 one gets the states ω_+ and ω_- which are well known to be factor states [7].

From Proposition 4.3, ω_+ and ω_- are quasi-equivalent states of $\Delta(\mathscr{H}_0)$ because

$$\left|\left(\frac{\lambda}{\varepsilon}, f\right)\right|^2 = \left|\left(\frac{\lambda}{\varepsilon}\tanh^{1/2}\left(\frac{\beta\varepsilon}{2}\right), \coth^{1/2}\left(\frac{\beta\varepsilon}{2}\right) f\right)\right|^2$$
$$\leqq \left(f, \coth\left(\frac{\beta\varepsilon}{2}\right) f\right)\left(\frac{\lambda}{\varepsilon}, \tanh\left(\frac{\beta\varepsilon}{2}\right)\frac{\lambda}{\varepsilon}\right)$$

and $\lambda^2\varepsilon^{-2}\tanh(\frac{1}{2}\beta\varepsilon) \in L^1(\mathbb{R})$ because $\int dk\lambda(k)^2\varepsilon(k)^{-1} < \infty$, fulfilling the conditions of Proposition 4.3 with $A = \coth(\frac{1}{2}\beta\varepsilon)$. Hence there exists a non-zero intertwining unitary operator between ω_+ and ω_-.

As ω_+ and ω_- are factor states it follows now from Proposition 4.2 that also for $\eta \in (0,1)$ the states (11) are factor states. ■

Lemma 3.4. *Denote again by* $\mathscr{B}''$ *the von Neumann algebra of the state* ω_η *of* $\mathscr{B}$, $\eta \in [0,1]$. *Under the conditions* $\int dk\lambda(k)^2\varepsilon(k)^{-1} < \infty$ *and* $\int dk\lambda(k)^2 < \infty$ *the automorphism* τ *of* $\mathscr{B}$ *defined in* (10), *extends to a unitary implementable automorphism of* $\mathscr{B}''$.

Proof. Let $(\mathscr{H}_+, \pi_+, \Omega_+)$ denote the GNS-representation of ω_+. By the conditions on λ we can extract from Proposition 4.3 and the proof of Lemma 3.3 the existence of a unitary operator $U \in \mathscr{B}(\mathscr{H}_+)$, which satisfies

$$\omega_-(x) = \langle U\Omega_+ | \pi_+(x) U\Omega_+ \rangle, \quad x \in \Delta(\mathscr{H}_0).$$

Consider now the mapping

$$V: \pi(W(f))\Omega_+ \to \pi_+(W(-f))U\Omega_+, \quad f \in \mathscr{H}_0,$$

then

$$\begin{aligned} &\langle \pi_+(W(-f_1))U\Omega_+ | \pi_+(W(-f_2))U\Omega_+ \rangle \\ &= \langle U\Omega_+ | \pi_+(W(f_1)W(-f_2))U\Omega_+ \rangle \\ &= \omega_-(W(f_1)W(-f_2)) \\ &= \omega_+(W(-f_1)W(f_2)) \\ &= \langle \pi_+(W(f_1))\Omega_+ | \pi_+(W(f_2))\Omega_+ \rangle . \end{aligned}$$

Therefore V extends to an isometry of $\mathscr{H}_+$ and as $W(f) \to W(-f)$ defines an automorphism of $\Delta(\mathscr{H}_0)$ it is easy to check that V is unitary. Furthermore

$$\pi_+(W(-f)) = V\pi_+(W(f))V^*, \quad f\in\mathscr{H}_0.$$

Therefore by Proposition 4.2 and using $\pi_+ = \pi_-$ and $\mathscr{H}_+ = \mathscr{H}_-$, one concludes that

$$\mathscr{B}'' = \left\{ \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_4 & 0 & a_3 \\ a_3 & 0 & a_4 & 0 \\ 0 & a_2 & 0 & a_1 \end{pmatrix} \middle| a_i \in \pi_+(\Delta(\mathscr{H}_0))'' \right\}.$$

Now it is easily checked that the automorphism τ is implemented by the unitary

$$Q = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & V \end{pmatrix} \pi_\eta(\sigma_1),$$

where π_η is the representation induced by ω_η. ∎

Combining the arguments of above we now obtain:

Proof of Theorem 3.1. By Lemma 3.4 the automorphism τ of (10) extends to an implementable automorphism of the von Neumann algebra $\mathscr{B}''$ of ω_η and we again denote this automorphism by τ. Clearly $\omega_\eta \circ \tau = \omega_{1-\eta}$.

From the formulas (4.a)–(4.d) it follows furthermore that $\alpha_t^0 \circ \tau = \tau \circ \alpha_t^0, t\in\mathbb{R}$. Let now $\omega_\beta^0 = \omega_\eta$ be an (α_t^0, β)-KMS-state (Lemma 3.2), then also $\omega_{1-\eta}$ has to be (α_t^0, β)-KMS and $\omega_{1-\eta}$ is a normal state of $\mathscr{B}''$ by the arguments of above. As by Lemma 3.3 $\mathscr{B}''$ is a factor we conclude by [6, Prop. 5.3.29] that $\omega_\eta = \omega_{1-\eta}$ or equivalently that $\eta = \frac{1}{2}$. This proves that there exists a unique (α_t^0, β)-KMS-state of $\mathscr{B}$ explicitly given by formula (7). ∎

The equilibrium states of the full model can now be computed by a perturbation of the equlibrium states of the solvable model $\{\alpha_t^0 | t\in\mathbb{R}\}$. In order to do this we need the perturbation technique on von Neumann algebras developed in the context of stability theory for KMS-states.

Theorem 3.5. *Under the conditions* $\int dk\lambda^2\varepsilon^{-1} < \infty$ *and* $\int dk\lambda^2 < \infty$ *the full model defined in* (5) *admits for every positive* β *a unique* (α, β)-KMS-*state* ω_β *which satisfies* (a)–(c). *Furthermore* ω_β *is normal with respect to the unique* (α^0, β)-KMS-*state* ω_β^0 *and is given by the following strongly convergent perturbation expansion*:

Let $(\mathscr{H}_0, \pi_0, \Omega_0)$ *be the* GNS *triplet of* ω_β^0, *then*

$$\omega_\beta(x) = \frac{\langle \Omega | \pi_0(x) \Omega \rangle}{\|\Omega\|^2}, \quad x\in\mathscr{B}, \tag{11}$$

where

$$\Omega = \Omega_0 + \sum_{n\geq 1} (-\beta\mu)^n \int_{0\leqq s_n \leqq \cdots \leqq s_1 \leqq 1/2} ds_1 \cdots ds_n \alpha_{i\beta s_n}^0(\sigma_1) \cdots \alpha_{i\beta s_1}^0(\sigma_1)\Omega_0.$$

It follows that ω_β *is* τ-*invariant where* τ *is defined in* (10), *in particular* $\omega_\beta(\sigma_3) = 0$.

Proof. Using [6, Theorem 5.4.4] one constructs a KMS-state ω_β for a perturbed dynamics α (5) from the unperturbed one α^0 (4). The full dynamics α is obtained by adding a bounded operator $\mu\sigma_1$ to the Hamiltonian.

The state ω_β is given by its cyclic vector which is constructed in terms of a series expansion (11). As the unperturbed state ω_β^0 is a factor state, ω_β is the unique (α, β)-KMS-state which is normal with respect to ω_β^0. As ω_β^0 is unique (Theorem 3.1) also ω_β is unique. In particular ω_β is τ-invariant. ■

4. Appendix

A. Consider the C^*-algebra $\mathscr{B} = \mathscr{A} \otimes M_2$, where $\mathscr{A}$ is a C^*-algebra. Hence a general element of $\mathscr{B}$ can be written in the matrix form

$$x \in \mathscr{B} : x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad x_{ij} \in \mathscr{A}.$$

Suppose that ω_1 and ω_2 are states of $\mathscr{A}$, then

$$\omega_\eta : x \in \mathscr{B} \to \omega_\eta(x) = \eta\omega_1(x_{11}) + (1-\eta)\omega_2(x_{22})$$

is a state of $\mathscr{B}$ for all $\eta \in [0, 1]$. Let $(\mathscr{H}_i, \pi_i, \Omega_i), i = 1, 2$ be the GNS-representations of $\mathscr{A}$ defined by the states $\omega_i, i = 1, 2$, then we have:

Proposition 4.1. *For all* $\eta \in (0, 1)$, ω_η *induces the* GNS-*representation* $(\mathscr{H}_\eta, \pi_\eta, \Omega_\eta)$, *where*

$$\mathscr{H}_\eta = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2,$$

$$\Omega_\eta = \begin{pmatrix} \eta^{1/2}\Omega_1 \\ (1-\eta)^{1/2}\Omega_2 \\ 0 \\ 0 \end{pmatrix} \in \mathscr{H}_\eta,$$

$$\pi_\eta(x) = \begin{pmatrix} \pi_1(x_{11}) & 0 & \pi_1(x_{12}) & 0 \\ 0 & \pi_2(x_{22}) & 0 & \pi_2(x_{21}) \\ \pi_1(x_{21}) & 0 & \pi_1(x_{22}) & 0 \\ 0 & \pi_2(x_{12}) & 0 & \pi_2(x_{11}) \end{pmatrix}.$$

Proof. A straightforward computation yields

$$\langle \Omega_\eta | \pi_\eta(x) \Omega_\eta \rangle = \omega(x), \quad x \in \mathscr{B},$$

$$\pi_\eta(x)\pi_\eta(y) = \pi_\eta(xy), \quad x, y \in \mathscr{B},$$

and the cyclicity of Ω_η for π_η follows from $0 < \eta < 1$. ■

Denote by $\mathscr{M}_i = \pi_i(\mathscr{A})''$ the von Neumann algebras defined by the states ω_i of $\mathscr{A}$ for $i = 1, 2$, by $\mathscr{M}_i'$ the commutants of the $\mathscr{M}_i$ and by $\mathscr{T}$ the set of operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, intertwining the representations π_1 and π_2, i.e. if $t \in \mathscr{T}$ then $t \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ such that for all $y \in \mathscr{A}$ holds $t\pi_1(y) = \pi_2(y)t$. In the next proposition we characterize the von Neumann algebra $\mathscr{M}_\eta$ of the state ω_η.

Proposition 4.2. *With the notations of above we have that $\mathcal{M}_\eta'$ is the set of operators in $B(\mathcal{H}_\eta)$:*

$$\begin{pmatrix} y_1' & 0 & 0 & t_1^* \\ 0 & y_2' & t_2 & 0 \\ 0 & t_1^* & y_1' & 0 \\ t_2 & 0 & 0 & y_2' \end{pmatrix},$$

where $y_i' \in \mathcal{M}_i', t_i \in \mathcal{T}\,(i=1,2)$; furthermore, $\mathcal{M}_\eta$ is given by

$$\begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & b_4 & 0 & b_3 \\ a_3 & 0 & a_4 & 0 \\ 0 & b_2 & 0 & b_1 \end{pmatrix},$$

where $a_i \in \mathcal{M}_1, b_i \in \mathcal{M}_2, t a_i = b_i t$, $a_i t^* = t^* b_i$ for $i=1,2,3,4$ and $t \in \mathcal{T}$.

The center $\mathcal{M}_\eta \cap \mathcal{M}_\eta'$ is then

$$\begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix},$$

where $z_i \in \mathcal{M}_i \cap \mathcal{M}_i'$, $(i=1,2)$ and satisfy $t z_1 = z_2 t$ and $z_1 t^* = t^* z_2$ for all $t \in \mathcal{T}$.

Proof. Using the matrix representation of Proposition 4.1 one computes explicitly the commutant $\mathcal{M}_\eta'$ by solving for X the matrix equations

$$\pi_\eta(x) X = X \pi_\eta(x) \quad \text{for all} \quad x \in \mathcal{B}$$

with $X \in \mathcal{B}(\mathcal{H}_\eta)$. The computation of the von Neumann algebra $\mathcal{M}_\eta$ is obtained in a similar way. The center $\mathcal{M}_\eta \cap \mathcal{M}_\eta'$ is then immediately recognized. ■

B. Consider $\mathcal{H}$, a complex Hilbert space with scalar product $(\cdot,\cdot)$, and A a self-adjoint operator on $\mathcal{H}$ such that $A \geqq \mathbb{1}$. Let $\mathcal{H}_0$ be a dense subspace of $\mathcal{H}$ contained in the domain of $A^{1/2}$ and consider the CCR-C^*-algebra $\Delta(\mathcal{H}_0)$. It is well known that for each linear functional χ on $\mathcal{H}_0, \omega_{A,\chi}$ defined by

$$\omega_{A,\chi}(W(f)) = \exp\{-\tfrac{1}{2}(f, Af) + i \operatorname{Im} \chi(f)\}$$

extends to a state of $\Delta(\mathcal{H}_0)$. The GNS-representation of $\omega_{A,\chi}$ is given by:

—the representation space is a subspace of $\mathcal{H}_F \otimes \mathcal{H}_F$, where $\mathcal{H}_F$ is the usual Fock space;

—the representation

$$\pi_{A,\chi}(W(f)) = \exp(i \operatorname{Im} \chi(f)) W\left(\left(\frac{A+\mathbb{1}}{2}\right)^{1/2} f\right) \otimes \overline{W\left(\left(\frac{A-\mathbb{1}}{2}\right)^{1/2} f\right)},$$

where $\bar{\cdot}$ means complex conjugation;

—the cyclic vector: $\Omega_{A,\chi} = \Omega_F \otimes \Omega_F, \Omega_F$ is the Fock vacuum.

It is also well known that $\pi_{A,\chi}$ is a factor representation [7].

Here we are interested in the necessary and sufficient condition in order that two states ω_{A,χ_1} and ω_{A,χ_2} are quasi-equivalent. The case $A=\mathbb{1}$ and $\chi_1=0$ is treated in [8]. In order to formulate the condition we introduce the Hilbert space $\mathscr{H}_A=\mathscr{D}(A^{1/2})$ with norm $\|f\|_A=\|A^{1/2}f\|$.

Proposition 4.3. *The states ω_{A,χ_1} and ω_{A,χ_2} of $\Delta(\mathscr{H}_0)$ are quasi-equivalent if and only if $\chi_1-\chi_2$ extends continuously to $\mathscr{H}_A$. Moreover quasi-equivalence implies unitary equivalence.*

Proof. Suppose first that $\chi_1-\chi_2$ extends continuously to $\mathscr{H}_A$, then there exists a $h\in\mathscr{H}_A$ such that

$$(\chi_1-\chi_2)(f)=(h,f)_A,\quad f\in\mathscr{H}_A. \tag{12}$$

Take the element

$$g=-\frac{A^{1/2}}{2}\left(\frac{2A}{A+\mathbb{1}}\right)^{1/2}h. \tag{13}$$

As $h\in\mathscr{D}(A^{1/2})$ and $(A/A+\mathbb{1})^{1/2}\leqq\mathbb{1}$, one has $g\in\mathscr{H}$, clearly $W(g)\otimes\mathbb{1}$ is a unitary operator on the representation space and

$$\begin{aligned}
&\langle\Omega_{A,\chi_1}|W(-g)\otimes\mathbb{1}\,\pi_{A,\chi_1}(W(f))W(g)\otimes\mathbb{1}\,\Omega_{A,\chi_1}\rangle\\
&=\exp\left\{2i\,\mathrm{Im}\left(g,\left(\frac{A+\mathbb{1}}{2}\right)^{1/2}f\right)\right\}\omega_{A,\chi_1}(W(f))\\
&=\exp\left\{2i\,\mathrm{Im}\left(g,\left(\frac{A+\mathbb{1}}{2}\right)^{1/2}f\right)+i\,\mathrm{Im}(\chi_1-\chi_2)(f)\right\}\omega_{A,\chi_2}(W(f))\\
&=\omega_{A,\chi_2}(W(f)),
\end{aligned}$$

using (12) and (13). Hence ω_{A,χ_1} and ω_{A,χ_2} are quasi-equivalent.

Conversely suppose that $\chi_1-\chi_2$ does not extend continuously to $\mathscr{H}_A$, then there exists a sequence $(f_n)_n$ of elements f_n in $\mathscr{H}_0$, such that $\|f_n\|\leqq(1/n)$ and $|\mathrm{Im}(\chi_1-\chi_2)(f_n)|=\pi$, and for every $g\in\mathscr{H}_0$ one has

$$\begin{aligned}
\|\omega_{A,\chi_1}(W(-g)\cdot W(g))-\omega_{A,\chi_2}\|&\geqq|\omega_{A,\chi_1}(W(-g)W(f_n)W(g))-\omega_{A,\chi_2}(W(f_n))|\\
&=|\exp i\,\mathrm{Im}((g,f_n)+\chi_1(f_n))\\
&\quad-\exp i\,\mathrm{Im}\,\chi_2(f_n)|\exp-\tfrac{1}{2}(f_n,Af_n)\\
&\to 2 \textit{ if } n\to\infty.
\end{aligned} \tag{14}$$

Suppose now that there exists a non-zero intertwining operator $T\in\mathscr{B}(h_2,h_1)$ for the representations π_{A,χ_2} and π_{A,χ_1}, such that for all $x\in\Delta(\mathscr{H}_0)$

$$\pi_{A,\chi_1}(x)T=T\pi_{A,\chi_2}(x).$$

It follows that $0<T^*T\in\pi_{A,\chi_2}(\Delta(\mathscr{H}_0))'$ and that a state η of $\Delta(\mathscr{H}_0)$ can be defined by:

$$\eta(x)=\frac{\langle\Omega_{A,\chi_2}|T^*T\pi_{A,\chi_2}(x)\Omega_{A,\chi_2}\rangle}{\langle\Omega_{A,\chi_2}|T^*T\Omega_{A,\chi_2}\rangle}=\frac{\langle T\Omega_{A,\chi_2},\pi_{A,\chi_1}(x)T\Omega_{A,\chi_2}\rangle}{\|T\Omega_{A,\chi_2}\|^2}$$

such that

$$\eta(\cdot) \leqq c\omega_{A,\chi_2}(\cdot) \tag{15}$$

for some $c \in \mathbb{R}^+$. From (14) it follows that for any $\varepsilon > 0$ one can find a selfadjoint x, $\|x\| \leqq 1$, such that

$$\omega_{A,\chi_1}(W(-g)xW(g)) - \omega_{A,\chi_2}(x) \geqq 2 - \varepsilon.$$

Hence

$$\omega_{A,\chi_1}(x) \geqq 1 - \varepsilon; \quad \omega_{A,\chi_2}(x) \leqq -1 + \varepsilon$$

and with (15)

$$0 \leqq \eta(\mathbb{1} + x) \leqq c\omega_{A,\chi_2}(\mathbb{1} + x) = c\varepsilon.$$

Now it follows that

$$\|\omega_{A,\chi_1}(W(-g)\cdot W(g)) - \eta(\cdot)\| = 2. \tag{16}$$

For any two normalized vectors ϕ_1 and ϕ_2 in a Hilbertspace one has in general

$$\|\langle \phi_1 | \cdot \phi_1 \rangle - \langle \phi_2 | \cdot \phi_2 \rangle\| \leqq 2(1 - |\langle \phi_1 | \phi_2 \rangle|^2)^{1/2}.$$

Take now $\phi_1 = \pi_{A,\chi_1}(W(g))\Omega_{A,\chi_1}$ and $\phi_2 = T\Omega_{A,\chi_2}$, then from (16)

$$\langle T\Omega_{A,\chi_2}, \pi_{A,\chi_1}(W(g))\Omega_{A,\chi_1} \rangle = 0$$

As Ω_{A,χ_1} is cyclic for $\pi_{A,\chi_1}(\Delta(\mathscr{H}_0))$ and Ω_{A,χ_2} is separating for $\pi_{A,\chi_2}(\Delta(\mathscr{H}_0))'$, we can conclude that $T = 0$. Therefore ω_{A,χ_1} and ω_{A,χ_2} cannot be quasiequivalent. ■

Acknowledgement. One of us (A.V.) thanks Herbert Spohn for bringing the model to his attention.

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Communicated by H. Araki

Received May 11, 1987; in revised form August 24, 1987