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# The Evaluation Map in Field Theory, Sigma-Models and Strings—II

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**Abstract.** In this paper, we examine specifically the rôle of the evaluation map in sigma-models and strings. We discuss the difference between sigma-models and field theory, as far as anomaly cancellation is concerned. The introduction of the Wess–Zumino terms in different sigma-models is considered. Anomalies in string theory are discussed, with special attention to the conformal anomalies and to the sigma-model anomalies for the imbedded (or immersed) world-sheet of the string. Conformal anomalies in two dimensions are connected to holomorphic and gravitational anomalies. In order to have the cancellation of the sigma-model anomalies of the string, certain topological conditions must be satisfied by the ambient manifold. The rôle of the evaluation map in the calculations of global anomalies is also discussed, both for field theories and for sigma-models. In particular global anomalies are connected with the differential characters of Cheeger and Simons. We show that the absence of global anomalies in sigma-models is guaranteed by the absence of torsion in suitable homology groups of the target space.

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## Introduction

In [1] we used the evaluation map to study *local* anomalies in field theory. In this paper, we extend the analysis to sigma-models and to string theory. We study also the problem of global anomalies, which appear to have a relevant rôle in sigma-models and string theories, more than in field theories.

Compared to field theory, the new fact about sigma-models is the modification of the universality requirement, which is replaced by the weaker condition that some counterterms be allowed by the specific topology of the target space (see below). This renders sigma-models more flexible than field theory as far as anomaly cancellation is concerned. As we will show in Sect. 1, under the condition that the relevant invariant polynomial is in the kernel of the Weil homomorphism, it is possible to exploit the mathematical properties of suitable loop spaces in order to construct explicitly the counterterm which kills the anomaly in question. This counterterm is a generalization of the Wess–Zumino term.

In sigma-models, anomaly cancellation translates into topological restrictions on the target space. These new features of sigma-models (compared to gauge field theories) arise naturally from the fact that a gauge theory can be regarded as a limiting case of a sigma-model, when the target space approximates the classifying space. This flexibility leads to important consequences when we reexamine the problem of anomaly cancellation for theories derived from superstrings.

An effective field theory of a superstring theory can be studied by considering the fields corresponding to the zero modes of the superstring as background fields in the presence of the propagating superstring. This leads to sigma-model type theories, in which conformal invariance is the origin for the dynamics of the background fields. We can therefore study the problem of chiral anomaly cancellation, already studied in Sect. 8 of [1], in the "category" of sigma-models. This allows us to implement the Green–Schwarz mechanism in a different way than the one considered in [1], that is by means of generalized Wess–Zumino terms. It is at this point that global anomalies, come into play. Indeed even after securing the vanishing of all local anomalies, the fermion determinants and generalized Wess–Zumino terms that define the theory may not constitute a globally invariant object, due to the presence of global anomalies.

Our approach to global anomalies both in field theory and sigma-models assumes as a starting point the results of perturbative calculations, which allow us to define (for example in the case of fermions coupled to a gauge potential) a functional integral in a neighborhood of a given connection. Hence we discuss the conditions which allow us to extend it, in an invariant way, to the whole space of connections.

There are two types of obstructions, namely local anomalies and global anomalies. The latter can be represented by means of differential characters and are related to the integral cohomology of the orbit space (for field theories) or of the target space (for sigma-models). The absence of global anomalies is guaranteed by suitable topological conditions. These conditions become very important for sigma-models, in which case the absence of global (as well as local) anomalies becomes a selective criterion for allowable target spaces.

The work is organized as follows:

Section 1 is dedicated to sigma-models and their anomalies. In this framework, gauge theories can be seen as sigma-models with the classifying space as target. The conditions for the cancellation of local anomalies in sigma-models are connected with the structure of the Weil homomorphism of the target space. Anomalies can be cancelled if the Weil homomorphism applied to the relevant invariant polynomials gives zero. If this is the case, the cancellation is performed by introducing a generalized Wess–Zumino term, which is a functional defined on the space of the paths over the fields of the sigma model, i.e. over the space of maps from the space-time to the target.

A gauge theory can be seen as the limit of sigma models when the Weil homomorphism tends to an isomorphism, i.e. when the target becomes the classifying space of the structure group.

The Wess–Zumino term introduced by Witten [2] is also discussed briefly; it is a special case of our generalized Wess–Zumino terms.

In Sect. 2 we consider the field-theory anomalies of the string. We interpret the conformal anomalies as a special case of the holomorphic anomalies of the bundle of complex frames. A similar interpretation is not possible for higher dimensional (complex) manifolds. We show that in string theory, holomorphic and gravitational anomalies have the same origin, i.e. the fourth cohomology group of the classifying space $BU(1)$.

In Sect. 3 we discuss the sigma-model anomalies of a string imbedded in a non-flat higher dimensional "ambient" manifold. We have both gauge-sigma-model anomalies and gravitational-sigma-model anomalies. For the cancellation of both (including the Lorentz anomalies), a specific requirement on the structure of the Weil homomorphism of the ambient manifold is made.

In Sect. 4 we discuss global anomalies both for field theory and for sigma-models and their relation with local anomalies.

We discuss consistency conditions for finite gauge transformations or diffeomorphisms, and their relation to the cohomology of the corresponding groups. We then introduce the differential characters of Cheeger and Simons and their relations to global anomalies and we study the conditions under which global anomalies are absent. We discuss both global anomalies for field theory and for sigma-models. In field theory, the source of global anomalies is the torsion part of the cohomology group of the orbit space, with integer coefficients. In sigma-models, the relevant object is the torsion part of $H^{n+2}(T, Z)$ where $n$ is the dimension of the space-time manifold and $T$ is the target space. We discuss also the effects of global anomalies on the generalized Wess–Zumino terms, defined in Sect. 1. By requiring the absence of global sigma-model anomalies, we introduce a further topological constraint, which selects the admissible target spaces.

Section 5 and Appendix I are devoted respectively to a few comments and to the discussion of the evaluation map for the diffeomorphisms of a compact manifold. Short comments on covariant anomalies and on the cancellation of global anomalies are added, respectively, as Appendix II and as Appendix III.

For the basic definitions and notations, we refer to [1] and specifically to Sect. 1.

## 1. Sigma-Models, Generalized Wess–Zumino Terms and Path-Spaces

We have seen in paper [1] that non-trivial local anomalies in field theory are obtained by pulling back cohomology classes of classifying spaces via suitably defined "evaluation maps." Here we want to consider the case of sigma-model anomalies. Our goal is to show that (non-trivial) sigma-model anomalies are obtained in a way which is completely analogous to the way field-theory-anomalies are obtained, the main difference being the replacement of the classifying space with a given "target space."

To be more specific, we assume that the space-time $M$ is a compact connected oriented $n$-dimensional Riemannian manifold and that the target space $T$ is a connected Riemannian manifold. We denote by $\mathrm{Map}(M, T)$ the space of $C^\infty$-maps from $M$ to $T$; this will be (a subset of) the space of fields (or dynamical variables) for our theory. The space of fields contains moreover chiral fermions interacting with a pulled-back connection.

We want specifically to consider the evaluation map:

$$\mathrm{ev}: M \times \mathrm{Map}(M, T) \to T. \tag{1.1}$$

Generally we assume $\dim T > \dim M + 1$. Let $G$ be a compact Lie group and let $P(T, G)$ be a principal $G$-bundle over $T$, with connection $\xi$, whose curvature will be denoted by $F_\xi$. We have then the diagram [3]:

$$\begin{array}{ccc} \mathrm{ev}^* P & \xrightarrow{\overline{\mathrm{ev}}} & P \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M \times \mathrm{Map}(M, T) & \xrightarrow{\mathrm{ev}} & T. \end{array} \tag{1.2}$$

Here $\text{ev}^* P$ is, as usual, the induced $G$-bundle over $M \times \text{Map}(M, T)$ and $\overline{\text{ev}}$ is the relevant canonical bundle map.

For example, $P$ can be the bundle of orthogonal (spin) frames and in this case $\xi$ can be its Levi–Civita connection or $P$ can be the principal bundle associated to an Hermitian vector bundle and in this case $\xi$ can be its Hermitian connection [4].

We want now to consider forms on $T$ depending on $\xi$ given by expressions such as $Q(F_\xi, \ldots, F_\xi)$, where $Q$ is an ad-invariant polynomial on Lie $G$ and $F_\xi$ is the curvature of $\xi$. By pulling back the relevant Chern-transgressions via $\overline{\text{ev}}^*$, one obtains forms on $\text{ev}^* P$ which generate the sigma-model anomalies.

To be more definite, let us consider a fixed map $f_0 \in \text{Map}(M, T)$, the induced bundle $f_0^* P$ and an ad-invariant polynomial $Q$ with $(n/2 + 1)$-entries. Let moreover $\mathscr{G}_{f_0}$ denote the group $\text{Aut}_v f_0^* P$, namely the group of vertical automorphisms of $f_0^* P$ (see [1], Sect. 1), and let $\bar{f}_0 : f_0^* P \to P$ be the canonical bundle homomorphism induced by $f_0$.

Then the combination of maps:

$$f_0^* P \times \mathscr{G}_{f_0} \xrightarrow{\text{ev}_{f_0}} f_0^* P \xrightarrow{\bar{f}_0} P \tag{1.3}$$

gives the form:

$$(\bar{f}_0 \circ \text{ev}_{f_0})^* TQ(\xi) \tag{1.4}$$

whose $(n, 1)$-component is by definition a local sigma-model anomaly. Here $\text{ev}_{f_0}$ is the evaluation map for $f_0^* P$ defined as in [1, Eq. (2.1)] and $TQ(\xi)$ is given by

$$TQ(\xi) \equiv \left(\frac{n}{2} + 1\right) \int_0^1 dt Q(\xi, F_t(\xi), \ldots, F_t(\xi)),$$

where $F_t(\xi) \equiv t d\xi + (t^2/2)[\xi, \xi]$.

By using a background connection (see Sect. 3 of [1]), we can write instead of (1.4), the $(n+1)$-form on $M \times \text{Aut}_v f_0^* P$ given by

$$W_Q((\bar{f}_0 \circ \text{ev}_{f_0})^* \xi, A_0), \tag{1.5}$$

where the background connection $A_0$ is the extension to $f_0^* P \times \text{Aut}_v f_0^* P$ of a fixed connection on $f_0^* P$, while $W_Q$ is defined as in Sect. 1 of [1][1]; namely, for any two connections $A, A'$ on any principal $G$-bundle and for any ad-invariant polynomial $Q$ on Lie $G$ with $k$-entries, we set

$$W_Q(A, A') \equiv k \int_0^1 dt Q(A - A', \mathscr{F}_t, \ldots, \mathscr{F}_t),$$

where $\mathscr{F}_t$ denotes the curvature of the connection $(1-t)A' + tA$.

In order to understand better the geometrical meaning of (1.4), we recall that any element of $\text{Aut}_v \text{ev}^* P$ induces in a natural way an element of $\text{Aut}_v f^* P$,

1 We usually denote by the same symbol $A_0$ the given connection of $f_0^* P$ and its trivial extension to $f_0^* P \times \text{Aut}_v f_0^* P$ or to any bundle $f_0^* P \times X \to M \times X$ for any manifold $X$. Moreover we assume from now on that $A_0$ is given by $\tilde{f}^* \xi$ and $\tilde{f} \in \text{Hom}(f_0^* P, P)$, where $\tilde{f}$ is required to cover a map from $M$ to $T$ which is homotopic to the given map $f_0$

$\forall f \in \mathrm{Map}(M, T)$. So we can consider the following evaluation map:

$$\mathrm{ev}^* P \times \mathrm{Aut}_v \mathrm{ev}^* P \xrightarrow{\mathrm{ev}^0} \mathrm{ev}^* P \tag{1.6}$$

and the diagram:

$$\begin{array}{ccccc} \mathrm{ev}^* P \times \mathrm{Aut}_v \mathrm{ev}^* P & \xrightarrow{\mathrm{ev}^0} & \mathrm{ev}^* P & \xrightarrow{\overline{\mathrm{ev}}} & P \\ & & \downarrow \pi & & \downarrow \pi \\ & & M \times \mathrm{Map}(M, T) & \xrightarrow{\mathrm{ev}} & T. \end{array} \tag{1.7}$$

The form on $\mathrm{ev}^* P \times \mathrm{Aut}_v \mathrm{ev}^* P$ given by

$$(\overline{\mathrm{ev}} \circ \mathrm{ev}^0)^* TQ(\xi) \tag{1.8}$$

is such that its restriction to $(\pi \circ \mathrm{ev}^0)^{-1}(M \times \{f_0\})$ is given by (1.4).

Hence, by taking into account Sect. 2, 3, 6 and 7 of [1], we can establish the following correspondences[2]:

| | Gauge theory over $M$ | Sigma-model over $M$ |
|---|---|---|
| Target | $BG$ | $T$ |
| Principal $G$-bundle over the target | $EG(BG, G)$ | $P(T, G)$ |
| Bundle induced by a given map $f_0$: $M \to$ target | $P_0 \equiv f_0^* EG$ | $f_0^* P$ |
| Gauge group | $\mathcal{G} \equiv \mathrm{Aut}_v P_0$ (*or* $\mathcal{G} \equiv \mathrm{Aut}_v^m P_0$) | $\mathcal{G}_{f_0} \equiv \mathrm{Aut}_v f_0^* P$ |
| Space of maps | $\mathrm{Map}(M, BG)_{f_0}$ $\left(\text{or } \dfrac{\mathcal{A}}{\mathcal{G}}\right)$ | $\mathrm{Map}(M, T)$ |

2 In the following table, $\mathcal{A}$ denotes the space of connections on $P_0$ and $\mathrm{Map}(M, BG)_{f_0}$ denotes the space of maps which induce a bundle equivalent to $P_0$, $\mathrm{Aut}_v^m P_0$ denotes the group of vertical automorphisms which leave the fiber over $m \in M$ fixed

| | Gauge theory over $M$ | Sigma-model over $M$ |
|---|---|---|
| Bundle induced from the principal bundle over the target by the evaluation map | $\dfrac{P_0 \times \mathrm{Hom}(P_0, EG)}{\mathscr{G}} \downarrow M \times \mathrm{Map}(M, BG)_{f_0}$ $\left(\text{or } \dfrac{P_0 \times \mathscr{A}}{\mathscr{G}} \to M \times \dfrac{\mathscr{A}}{\mathscr{G}}\right)$ | $\mathrm{ev}^* P \downarrow M \times \mathrm{Map}(M, T)$ |
| The anomaly with the background connection is the $(n, 1)$-component of an $(n+1)$-form over: | $M \times \mathscr{G}$ | $M \times \mathscr{G}_{f_0}$ |

So in many respects we can consider gauge theories as sigma-models with the classifying spaces as targets.

As a remark on the above table, notice that, in gauge theories, all the maps in $\mathrm{Map}(M, BG)_{f_0}$ are by definition homotopic to the given map $f_0$. Also, in sigma-models, we have to consider (as we will do later on) only the space of maps from $M$ to the target which are homotopic to a given map $f_0$. But all the above considerations about sigma-model anomalies do not depend on the choice of $f_0$, as all the considerations about gauge anomalies made in [1] did not depend on the isomorphism class of the principal bundle (for instance they did not depend on the instanton number).

As is now clear, a crucial difference between sigma-models and gauge theories is that in sigma-models the target space is not the classifying space. This difference will allow us to cancel anomalies which are not cancelled in guage theories. The basic fact in this respect is that the Weil homomorphism is an isomorphism for the classifying bundle $EG(BG, G)$ (remember that $G$ is supposed to be a compact Lie group), while it need not to be even a monomorphism for a generic bundle $P(T, G)$.

In fact the most interesting situation in a sigma-model is precisely when the Weil homomorphism of the target space, restricted to some ad-invariant polynomials, *is zero*.

This is the situation we will examine after discussing some mathematical preliminaries.

We consider first $\mathscr{P}_{f_0}(\mathrm{Map}(M, T))$, i.e. the space of paths of $\mathrm{Map}(M, T)$, based at $f_0$ and the relevant space of loops $\Omega_{f_0}(\mathrm{Map}(M,T))$,[3,4]. We denote by $J$ the inclusion map of $\Omega_{f_0}(\mathrm{Map}(M,T))$ into $\mathscr{P}_{f_0}(\mathrm{Map}(M,T))$ and by $\pi_1$ the map which associates to each element of $\mathscr{P}_{f_0}(\mathrm{Map}(M,T))$ the element of $\mathrm{Map}(M,T)_{f_0}$ given by the endpoint. Here $\mathrm{Map}(M,T)_{f_0}$ is the path connected component of $f_0\in \mathrm{Map}(M,T)$.

We have then the principal fibration

$$\begin{array}{ccc} \Omega_{f_0}(\mathrm{Map}(M,T) & \overset{J}{\hookrightarrow} & \mathscr{P}_{f_0}(\mathrm{Map}(M,T)) \\ & & \downarrow{\scriptstyle \pi_1} \\ & & \mathrm{Map}(M,T)_{f_0}. \end{array} \tag{1.9}$$

The space $\mathscr{P}_{f_0}(\mathrm{Map}(M,T))$ is contractible and so (1.9) plays the rôle of a "universal bundle" for $\Omega_{f_0}(\mathrm{Map}(M,T))$, even though (1.9) is not a principal fibre bundle.

Next we consider any element $p\in\mathscr{P}_{f_0}(\mathrm{Map}(M,T))$. This can be regarded as a map $p:M\times I\to T$ and so we have the relevant induced bundle $p^*P$ over $M\times I$. If we denote by $p_t$ the restriction of $p$ to $M\times\{t\}$, then $p_0^*P$ is identical to $f_0^*P$, while $p_1^*P$ is equivalent (isomorphic) to $f_0^*P$ [6].

The isomorphism between $f_0^*P$ and $p_1^*P$ is not a canonical one; it depends at least on the path. We will in fact construct such an isomorphism $\tau_\xi(p)$ induced by a connection $\xi$ on $P$ as follows.

Consider on the bundle $p^*P$, the connection $\bar p^*\xi$, where $\bar p$ is the canonical covering of $p$. Next, $\forall x\in M$ take the (trivial) path in $M\times I$ given by $t\mapsto(x,t)$ and lift this path horizontally to a path with initial point $u\in\pi^{-1}(x)\subset f_0^*P$. The endpoint of this lifted path is an element of $p_1^*P$ and the horizontal lift gives us the required isomorphism $\tau_\xi(p):f_0^*P\to p_1^*P$.

We have thus defined a map (determined by $\xi$):

$$\begin{aligned} \tau_0:\mathscr{P}_{f_0}(\mathrm{Map}(M,T)) &\to \mathrm{Hom}(f_0^*P,P) \\ p \qquad &\mapsto \bar p_1\circ\tau_\xi(p), \end{aligned} \tag{1.10}$$

where $\bar p_1$ denotes as usual the canonical bundle homomorphism induced by $p_1$. Obviously $\tau_0(p)$ is a homomorphism which covers the map $\pi_1(p)\equiv p_1$.

Now let $\mathscr{A}_{f_0^*P}$ denote the space of connections on $f_0^*P$. We are also able to define the following maps:

$$\begin{aligned} \tau_1:\mathscr{P}_{f_0}(\mathrm{Map}(M,T)) &\to \mathscr{A}_{f_0^*P} \\ p \qquad &\mapsto \tau_\xi(p)^*\bar p_1^*\xi, \end{aligned} \tag{1.11}$$

$$\begin{aligned} \tau_2:\Omega_{f_0}(\mathrm{Map}(M,T)) &\to \mathrm{Aut}_v f_0^*P \\ l \qquad &\mapsto \tau_2(l)\equiv\tau_\xi(l), \end{aligned} \tag{1.12}$$

3 We always assume our paths and loops to be smooth. For the differentiable structure on path- and loop spaces, see [5]

4 In order to distinguish between the symbol $\Omega$ used for loop spaces and the same symbol used for differential forms, we will always write explicitly the base point for loops at the lower right of the symbol, as in $\Omega_{f_0}$, and the order of forms at the upper right of the symbol, as in $\Omega^n$

where, in the above expression, any loop is considered as a path with initial point equal to the endpoint. The image of the map $\tau_0$ can be referred to as the space of induced bundle homomorphisms, while the image of the map $\tau_1$ is, by definition, the space of connections induced from the principal bundle (with connection) $P$. It is also easy to see that the image of the map $\tau_2$ is a subgroup of $\mathrm{Aut}_v f_0^* P$ which will be referred to as the group of induced gauge transformations. Moreover the group $\mathrm{Im}(\tau_2)$ acts freely on $\mathrm{Im}(\tau_0)$ and we can consider the principal bundle

$$\mathrm{Im}(\tau_0) \to \frac{\mathrm{Im}(\tau_0)}{\mathrm{Im}(\tau_2)} \approx \mathrm{Map}(M, T)_{f_0}. \tag{1.13}$$

Roughly speaking, one should expect $\mathrm{Im}(\tau_2)$ to be bigger the "less reducible" the connection $\xi$ is.

Let us now come back to sigma-models and let us consider an ad-invariant (possibly irreducible) polynomial $Q$ with $(n/2+1)$-entries. We assume specifically that

$$Q(F_\xi, \ldots, F_\xi) = dH, \tag{1.14}$$

where $F_\xi$ is the curvature of $\xi$ and $H$ is an $(n+1)$-form on $T$.

Now let $\pi_1$ be used ambiguously also for $\mathrm{id} \times \pi_1 : M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \to M \times \mathrm{Map}(M, T)_{f_0}$. The induced bundle

$$\pi_1^* \mathrm{ev}^* P \to M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$$

is isomorphic to $f_0^* P \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$; an isomorphism in terms of $\xi$ is given by sending $(u, p) \in f_0^* P \times \mathscr{P}_{f_0} \mathrm{Map}(M, T)$ to the element of $\pi_1^* \mathrm{ev}^* P$ represented by $(\pi(u), p, \tau_\xi(p) u) \in M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \times \mathrm{ev}^* P$.

Henceforth we will identify these two bundles. Thus we have the diagram

$$\begin{array}{ccccc} f_0^* P \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \approx \pi_1^* \mathrm{ev}^* P & \xrightarrow{\bar{\pi}_1} & \mathrm{ev}^* P & \xrightarrow{\overline{\mathrm{ev}}} & P \\ \downarrow & & \downarrow & & \downarrow \\ M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) & \xrightarrow{\pi_1} & M \times \mathrm{Map}(M, T)_{f_0} & \xrightarrow{\mathrm{ev}} & T, \end{array} \tag{1.15}$$

where $\bar{\pi}_1$ is the combination of the canonical bundle homomorphism induced by $\pi_1$ with the isomorphism $f_0^* P \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \to \pi_1^* \mathrm{ev}^* P$.

On the bundle $f_0^* P \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ we can consider two connections, namely a background connection $A_0$ obtained by extending trivially a fixed connection on the bundle $f_0^* P$ (e.g. $\bar{f}_0^* \xi$) and the connection $\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi$.

The $(n+1)$-form $W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0)$ is basic, i.e. it can be considered as a form on $M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$. Moreover we pull it back via the map $J$ defined in (1.9)[5], thus obtaining a form $J^* W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0)$ on $M \times \Omega_{f_0}(\mathrm{Map}(M, T))$.

5 Here again we use ambiguously the symbol $J$ also to denote the map: $M \times \Omega_{f_0}(\mathrm{Map}(M, T)) \to M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ which more properly should be denoted by $\mathrm{id} \times J$

Now from the map $\tau_2$ (1.13), we obtain in turn the map:

$$\mathrm{id} \times \tau_2 : M \times \Omega_{f_0}(\mathrm{Map}(M, T)) \to M \times \mathrm{Aut}_v f_0^* P, \tag{1.16}$$

which yields a homomorphism:

$$(\mathrm{id} \times \tau_2)^* : H^*(M \times \mathrm{Aut}_v f_0^* P) \to H^*(M \times \Omega_{f_0}(\mathrm{Map}(M, T))). \tag{1.17}$$

The form $\mathrm{J}^* W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0) \in \Omega^{n+1}(M \times \Omega_{f_0}(\mathrm{Map}(M, T)))$ is the image under the pullback of the map (1.16) of the form in $\Omega^{n+1}(M \times \mathrm{Aut}_v f_0^* P)$ given by (1.5).

If $H$ satisfies (1.14), then we have

$$d(W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0) - \pi_1^* \mathrm{ev}^* H) = 0, \tag{1.18}$$

where $d$ is the exterior derivative on $M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$.

To prove (1.18), it is enough to notice that

$$d W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0) = \pi_1^* \mathrm{ev}^* Q(F_\xi, \ldots, F_\xi)$$

due to the fact that $Q(F_{A_0}, \ldots, F_{A_0})$ is zero for dimensional reasons.

We have now:

**Theorem (1.19).** *There exists a form* $\beta \in \Omega^n(M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ *such that* $d\beta = W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0) - \pi_1^* \mathrm{ev}^* H$.

*Proof*. The space $\mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ is contractible and so the theorem follows from the Künneth theorem. In order to construct $\beta$, we can now consider the map $(i \circ r): M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \to M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ given by the combination of the retraction (projection) $r: M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \to M$ with the inclusion $i: M \to M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$.

The map $i \circ r$ is homotopic to the identity, so there exists a homotopy operator $\mathscr{Q}$ with[6]:

$$1^* - (i \circ r)^* = \mathscr{Q} d + d \mathscr{Q}.$$

Since $i^*$ gives zero when applied to $(n+1)$-forms, we have $W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0) - \pi_1^* \mathrm{ev}^* H = d\beta$, where $\beta$ is given by:

$$\beta = \mathscr{Q}(W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0) - \pi_1^* \mathrm{ev}^* H).$$

In order to give a more explicit expression of $\beta(p)$, for each $p \in \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$, we consider the path $p^t$ given by $p^t(s) \equiv p(st)$. By assigning to each $t \in I$ the element $\tau_0(p^t) \in \mathrm{Hom}(f_0^* P, P)$ we obtain a map $\tau_0^p: I \times f_0^* P \to P$, and hence we have:

$$\beta(p) = \int_0^1 W_Q(\tau_0^{p*} \xi, A_0) - \int_0^1 p^* H \quad \square \tag{1.20}$$

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6 More precisely $\mathscr{Q}$ is given by $\int_0^1 \mathscr{H}^*$, where $\mathscr{H}$ is the map defined as follows:

$$\mathscr{H}: M \times I \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)) \to M \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T)),$$
$$(x, p, t) \mapsto (x, p^t)$$

where $p^t(x, s) \equiv p(x, st)$, $\forall x \in M$. Namely the homotopy $\mathscr{H}$ "shrinks" the paths

By restricting to $M \times \Omega_{f_0}(\mathrm{Map}(M, T))$ the form considered in Theorem (1.19), we obtain

$$dJ^*\beta = J^* W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0).$$

Notice that $J^*\pi_1^* \mathrm{ev}^* H$ is zero for dimensional reasons.

Theorem (1.19) tells us that the image under (1.17) of the cohomology class represented by (1.5) is zero.

We now define the form $B$ given by

$$B = (n, 0) - \text{component } of\ \beta. \tag{1.21}$$

The restriction $J^*B$ of $B$ to $M \times \Omega_{f_0}(\mathrm{Map}(M, T))$ satisfies the following equation:

$$\delta J^* B = (\mathrm{id} \times \tau_2)^*(Anomaly) + \text{exact}, \tag{1.22}$$

where $\delta$ is the exterior derivative on $\Omega_{f_0}(\mathrm{Map}(M, T))$, by "Anomaly" we denote the $(n, 1)$-component of the form (1.5) and the last term is exact as a form on $M$.

Hence the local anomaly corresponding to an ad-invariant polynomial $Q$ is cancelled provided that the Weil homomorphism of the target space applied to $Q$ gives zero. The price we have to pay, is that we have to introduce in the effective action an n-form $B$ on $M$, depending on the space of paths over $\mathrm{Map}(M, T)_{f_0}$. The term $B$ will be referred to as a (generalized) Wess–Zumino term.

The situation may become considerably simpler when $Q$ is a reducible polynomial (with $(n/2 + 1)$-entries). Let us assume, for instance, that $Q = Q_1 Q_2$ and that

$$Q_1(F_\xi, \ldots, F_\xi) = dH_1 \quad \text{and} \quad Q_2(F_\xi, \ldots, F_\xi) = dH_2.$$

In this case, the Weil homomorphism of the bundle $f_0^* P$ applied to $Q_1$ and $Q_2$ gives zero and so we have also $Q_2(F_{A_0}, \ldots, F_{A_0}) = dH_0$, for a suitable form $H_0$. Hence we can write (1.5) as:

$$\begin{aligned} W_Q((\bar{f}_0 \circ \mathrm{ev}_{f_0})^* \xi, A_0) = {} & W_{Q_1}((\bar{f}_0 \circ \mathrm{ev}_{f_0})^* \xi, A_0) \wedge dH_0 \\ & + W_{Q_2}((\bar{f}_0 \circ \mathrm{ev}_{f_0})^* \xi, A_0) \wedge df_0^* H_1 + \text{exact}, \end{aligned} \tag{1.23}$$

so the anomaly is cancelled (see [1], Eq. (5.17)).

We would now like to make a few more comments on the relation between gauge theories and sigma-models.

In the special case when the target space is given by $BG$, we can consider the bundle $P_0 \equiv f_0^* EG$ and we know [7] that

$$\mathrm{Aut}_v P_0 \hookrightarrow \mathrm{Hom}(P_0, EG) \to \mathrm{Map}(M, BG)_{f_0} \tag{1.24}$$

is a universal bundle for $\mathrm{Aut}_v P_0$[7]. Hence there exists a weak homotopy equivalence

$$\mathrm{Aut}_v P_0 \approx \Omega_{f_0}(\mathrm{Map}(M, BG)).$$

We can then say that when $T$ approaches $BG$ then (in homotopy theory)

7 More precisely if $BG$ is an $(n + h + 1)$-classifying space, then it can be proved (by obstruction theory) that $\mathrm{Hom}(P, EG)$ is $h$-classifying for $\mathrm{Aut}_v P_0$

$\Omega_{f_0}(\mathrm{Map}(M,T))$ and $\mathscr{P}_{f_0}(\mathrm{Map}(M,T))$ approach respectively $\mathrm{Aut}_v P_0$ and $\mathrm{Hom}(P_0, EG)$.

In this sense sigma-models can be considered as an approximation to a gauge theory as the target approximates the classifying space.

Hence sigma-models are potentially more anomaly-free the more the Weil homomorphism of the target is trivial. We can also say that a gauge theory is the limit of sigma-models when the Weil homomorphism tends to an isomorphism. More precisely, approximating the classifying space $BG$ with a target $T$ means considering a sequence of maps as follows:

$$M \times \mathrm{Map}(M,T) \to T \to BG.$$

Since we are interested in taking the limit of the $T$'s when the kernel of the Weil homomorphism tends to zero, we are only interested in considering the cohomology classes of $T$ which are obtained through pullback via the classifying map: $T \to BG$. This explains why the possible non-trivial cohomology classes of the target space which do not belong to the image of the Weil homomorphism do not really matter for sigma models: in the limit they will disappear. This is also consistent with the index theorem approach to the computation of the coefficients of anomalies, since only "universal" classes can be taken into account.

A final comment concerns the topology of $M$ or the topology of the induced bundle $f_0^* P \to M$. As in gauge theories, the above topologies do not play any rôle in the cancellation of local anomalies. Let us assume, for instance, that we have two ad-invariant irreducible polynomials $Q_1$ and $Q_2$ with $k$ and $k'$ entries respectively. Furthermore we assume that $k_1 + k_2 = n/2 + 1$ and that both $Q_1$ and $Q_2$ are in the kernel of the Weil homomorphism of the induced bundle $f_0^* P \to M$.

Then we know from Sect. 5 of [1], that the anomaly relevant to the product $Q_1 Q_2$ corresponds to the zero element of the first cohomology class of the gauge group. But if both polynomials $Q_1$ and $Q_2$ do not belong to the kernel of the Weil homomorphism of the target space, then one can always find another $n$-dimensional manifold $M'$ and a map $f': M' \to T$, such that the anomaly corresponding to the polynomial $Q_1 Q_2$ represents a non-trivial element of $H^1(\mathscr{G})$ or of $H^1(\Omega_{f'}(\mathrm{Map}(M', T)))$.

In other words, in sigma-models as well as in gauge theories, universality plays an essential rôle; that is, for any given target $T$, the results have to be reasonably independent of a "choice" of the manifold $M$. Anomalies for both sigma-models and field theory are "universal objects," as far as the manifold $M$ is concerned.

We have seen that, in order to cancel sigma-model anomalies, we have to introduce in the effective action (generalized) Wess–Zumino terms, namely terms depending on the paths over the space of maps from $M$ to $T$. This procedure is the only one which can exploit the characteristics of sigma-models and give an extra possibility of cancelling anomalies, with respect to ordinary gauge theories.

It is also well known that Wess–Zumino terms have to be introduced in some sigma-models, for purposes other than the cancellation of anomalies. For example, in the two-dimensional non-abelian bosonization model studied by Witten [2, 8], such a term is needed in order to reproduce the features of the original fermionic theory. We want now to show that also these Wess–Zumino terms can be obtained as a particular case of our construction (see also [9, 10]).

By generalizing the sigma-model considered by Witten [2], we choose the target itself to be a compact semisimple Lie group $G$ with the trivial principal $G$-bundle $G \times G \to G$. The connection on such a principal bundle will be chosen to be the flat connection given by the pullback of the Maurer–Cartan form $\theta$, via the product $G \times G \to G$. At this point we are equipped to carry on the discussion of the (generalized) Wess–Zumino term, along the lines discussed before.

In this respect, we notice that, if in Theorem (1.19) and formulae (1.10)–(1.12) we choose the bundle $P$ over the target $T$ to be a trivial bundle with a connection $\xi$ with zero holonomy, then the image of the map $\tau_2$ defined in (1.12) is just the identity on $\mathrm{Aut}_v f_0^* P$. Moreover we can choose the form $H$ in (1.14) to be zero. In this situation, the bundle $\mathrm{ev}^* P$ (see diagram (1.15)) is isomorphic to the bundle $f_0^* P \times \mathrm{Map}(M, T)_{f_0}$ and the form $W_Q(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \xi, A_0)$ is the pull-back, via $\pi_1$, of a form on $M \times \mathrm{Map}(M, T)_{f_0}$. Accordingly its restriction to $M \times \Omega_{f_0}(\mathrm{Map}(M, T))$ is zero and also the restriction of the (generalized) Wess–Zumino term to $M \times \Omega_{f_0}(\mathrm{Map}(M, T))$ is zero. This is exactly the situation for Witten's sigma-model. In this sigma-model the notation can be further simplified by pulling back to $G$ the non-trivial cohomology classes of $G \times G$, via the inclusion on the first factor. By doing so, we are led to consider the non-trivial cohomology class of $G$ represented by the form

$$TQ(\theta) \equiv \left(\frac{n}{2} + 1\right) \int_0^1 dt Q\left(\theta, \frac{t^2 - t}{2}[\theta, \theta], \ldots, \frac{t^2 - t}{2}[\theta, \theta]\right),$$

where $Q$ is an ad-invariant irreducible polynomial on $\mathrm{Lie}\, G$ with $(n/2 + 1)$-entries (see Sect. 5 of [1]). Let us assume from now on that $TQ(\theta)$ is in the image of the canonical map $i: H^{n+1}(G, \mathbf{Z}) \to H^{n+1}(G, \mathbf{R})$ [11].

Let $\mathrm{Map}^m(M, G)$ be the space of functions which map $m \in M$ (i.e. the "point at infinity") into the identity $e \in G$[8]. For any given $f_0 \in \mathrm{Map}^m(M, G)$ we denote by $\mathrm{Map}^m(M, G)_{f_0}$ the space of maps in $\mathrm{Map}^m(M, G)$ homotopic to $f_0$. We can then consider the evaluation map.

$$\mathrm{ev}: M \times \mathrm{Map}^m(M, G)_{f_0} \to G, \tag{1.25}$$

and the space of paths $\mathscr{P}_{f_0} \mathrm{Map}^m(M, G)$.

We have also a natural homeomorphism

$$\mathscr{P}_{f_0}(\mathrm{Map}^m(M, G)) \xrightarrow{\sim} \mathrm{Map}^m(M, \mathscr{P}_e G),$$

where, as before, the symbol $\mathrm{Map}^m$ denotes the space of pointed maps. In fact, to any path $p \in \mathscr{P}_{f_0}(\mathrm{Map}^m(M, G))$ regarded as a map $p: M \times I \to G$, we can associate the map $\tilde{p}: M \to \mathrm{Map}(I, G)$ given by $\tilde{p}(x)(t) = (f_0(x))^{-1} p(x, t)$.

8 We find it convenient to consider the space of pointed maps in order to discuss better the characteristics of the Wess–Zumino term

So we have the following commutative diagram:

$$\begin{array}{ccc} M \times \mathscr{P}_{f_0}(\mathrm{Map}^m(M,G)) \approx M \times \mathrm{Map}^m(M,\mathscr{P}_e G) & \xrightarrow{\mathrm{ev}} & \mathscr{P}_e G \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ M \times \mathrm{Map}^m(M,G)_{f_0} \xrightarrow{\widetilde{\mathrm{ev}}} M \times G & \xrightarrow{f_0^{-1}} & G, \end{array} \tag{1.26}$$

where $f_0^{-1}: M \times G \to G$ is the function which assigns to $(x,g)$ the group element $f_0^{-1}(x)g$ and $\widetilde{\mathrm{ev}}$ is the combination of the identity map on $M$ with the evaluation map on $M \times \mathrm{Map}^m(M,G)_{f_0}$.

It is now obvious that by pulling back $TQ(\theta)$ via the combination of maps given by $f_0^{-1} \circ \widetilde{\mathrm{ev}}$ we obtain an $(n+1)$-form on $M \times \mathrm{Map}^m(M,G)_{f_0}$ whose $(n,1)$-component satisfies a consistency condition and plays the rôle of an anomaly.

Diagram (1.26) tells us that the relevant Wess–Zumino term, which is a form on $M \times \mathscr{P}_{f_0}(\mathrm{Map}^m(M,G))$, is obtained by considering the pull back, via ev* of $\pi_1^* TQ(\theta)$ which is an exact form on $\mathscr{P}_e G$. Namely there exists an $n$-form $\gamma$ on $\mathscr{P}_e G$ such that $\pi_1^* TQ(\theta) = d\gamma$. The $(n,0)$-component of $\mathrm{ev}^*\gamma$ is by definition our (generalized) Wess–Zumino term and coincides with Witten's Wess–Zumino term.

In order to give an explicit expression of $\mathrm{ev}^*\gamma$ (see (1.20)) we consider the "double" evaluation map [12]:

$$\mathrm{Ev}: M \times I \times \mathrm{Map}^m(M,\mathscr{P}_e G) \to G; \tag{1.27}$$

the Wess–Zumino term is then given by $\int_{M\times I} \mathrm{Ev}^* TQ(\theta)$.

We want now to address the problem of the "admissibility" of the Wess–Zumino term, namely we want to ask ourselves whether the functional $\exp\left(2\pi i \int_{M\times I} \mathrm{Ev}^* TQ(\theta)\right)$ descends to a functional on the space of the dynamical variables of the theory, namely on the space $\mathrm{Map}^m(M,G)_{f_0}$. This is the same as requiring that the Wess–Zumino term is an integer when restricted to the space of loops; namely we want now to ascertain whether the functional on the loop space, given by:

$$\int_{M\times S^1} \mathrm{Ev}^* TQ(\theta): \quad \mathrm{Map}^m(M,\Omega_e G) \to \mathbf{R} \tag{1.28}$$

is an integer or not. Here Ev is regarded as a map,

$$\mathrm{Ev}: M \times S^1 \times \mathrm{Map}^m(M,\Omega_e G) \to G.$$

If $\mathrm{Map}^m(M,\Omega_e G)$ is connected, i.e., if all its elements are homotopic to the map which assigns to each $x \in M$ the "zero"-loop in $G$, then for all $l \in \mathrm{Map}^m(M,\Omega_e G)$ we see that $l^* TQ(\theta)$ is exact on $M \times S^1$, and so (1.28) is zero. If, on the contrary, $\mathrm{Map}^m(M,\Omega_e G)$ is not connected, then (1.28) needs not to be zero.

Now $\pi_0(\mathrm{Map}^m(M,\Omega_e G))$ is isomorphic to $[M \wedge S^1, G]_*$, where $M \wedge S^1$ denotes the smash product of $M$ and $S^1$ and $[M \wedge S^1, G]_*$ denotes the homotopy classes of pointed maps from $M \wedge S^1$ to $G$[9].

9 Notice that $\pi_0(\mathrm{Map}*(M,\Omega_e G))$ is equal to $\pi_1(\mathrm{Map}^m(M,G))$ and so it is always an abelian group, since $\mathrm{Map}^m(M,G)$ is a group

Hence, if $c$ denotes the fundamental cycle of $M \times S^1$ and $p$ denotes the projection $p: M \times S^1 \to M \wedge S^1$, then we have a group homomorphism [13, Proposition 7.38]:

$$\begin{aligned} \pi_0(\mathrm{Map}^m(M, \Omega_e G)) \approx [M \wedge S^1, G]_* &\to H_{n+1}(G, \mathbf{Z}) \\ [\psi] \qquad &\mapsto [\psi_*(p_* c)]. \end{aligned} \tag{1.29}$$

In conclusion, when the form $TQ(\theta)$ has been properly normalized, $\int_{M \times S^1} l^* TQ(\theta)$ is an integer for any loop $l \in \mathrm{Map}^m(M, \Omega_e G)$[10], namely the functional

$$\exp\left(2\pi i \int_{M \times I} \mathrm{Ev}^* \, TQ(\theta)\right): \quad \mathscr{P}_{f_0}(\mathrm{Map}^m(M, G)) \to U(1) \approx \mathbf{R}/\mathbf{Z},$$

depends only on the end-points of paths; i.e. it descends to a functional defined on $\mathrm{Map}^m(M, G)$ with values in $\mathbf{R}/\mathbf{Z}$. It follows that the normalized coupling constant of the Wess–Zumino term can only be an integer.

The corresponding conditions one should require for a generic sigma-model have to do with the requirement that the form $J^* B$ considered in (1.21) be such that $\exp(2\pi i J^* B)$ descends to a functional on the group of the induced gauge transformations. In other words we want to be able to interpret, modulo integers, the cancellation condition (1.22) as an equation on the group of the (induced) gauge transformations. and not simply as an equation on the loop space $\Omega_{f_0}(\mathrm{Map}(M, T))$. Moreover we have to make sure that the final effective action is a functional over the true degrees of freedom of the theory. We will discuss this problem in Sect. 4.

A final remark concerns gravitational anomalies. These anomalies also admit a "sigma-model" interpretation in the following sense.

Consider $M$ itself as the target space, with the frame bundle $LM$ over it, and consider $\mathrm{Diff}\, M$ instead of $\mathrm{Map}(M, T)$ in (1.2). The diagram (1.2) would then become:

$$\begin{array}{ccc} \mathrm{ev}^* LM & \xrightarrow{\overline{\mathrm{ev}}} & LM \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M \times \mathrm{Diff}\, M & \xrightarrow{\mathrm{ev}} & M. \end{array} \tag{1.30}$$

Since $\mathrm{Diff}\, M$ lifts to $\mathrm{Aut}\, LM$, then $\mathrm{ev}^* LM$ is canonically isomorphic to $LM \times \mathrm{Diff}\, M$ and we can identify these two bundles. If $Q$ is an ad-invariant polynomial over $GL(n, \mathbf{R})$ with $(n/2 + 1)$-entries[11], and if $A$ is a Levi–Civita connection for a metric $g \in \mathscr{M}$, then $\overline{\mathrm{ev}}^* \, TQ(A)$ can be considered as a form over $LM \times \mathrm{Diff}\, M$ whose $(n, 1)$-component is the gravitational anomaly.

We can also choose a fixed linear connection $A_0$ on $LM$ and consider the

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10 If $M = S^n$ (as in [8]), then $[S^1 \wedge M, G]_* \approx [S^{n+1}, G]_*$. In this case (1.29) is in fact the Hurewicz homomorphism

$$\pi_{n+1}(G) \to H_{n+1}(G, Z)$$

11 We assume that $n \equiv 2 (\mathrm{mod}\, 4)$ so the gravitational anomaly is not necessarily trivial

$(n+1)$-form over $M \times \operatorname{Diff} M$, given by:

$$W_Q(\overline{\mathrm{ev}^* A}, A_0). \tag{1.31}$$

The $(n, 1)$-component of (1.31) is the expression of the gravitational anomaly with the background connection.

## 2. Strings and Conformal Anomalies

Let $S$ be a two-dimensional compact orientable manifold imbedded in an $n$-dimensional Riemannian manifold $M$ with metric $\Gamma$; $S$ is to be identified with the world-sheet of the string.

We consider two metrics on $S$:

a) an "intrinsic" metric $g$,

b) the induced metric $\gamma$.

If $h: S \to M$ is the given imbedding, then the induced metric is defined in the following way:

$$\gamma_x(X_1, X_2) = \Gamma_{h(x)}(h_* X_1, h_* X_2) \quad x \in S \quad X_1, X_2 \in T_x S. \tag{2.1}$$

The action functional of the bosonic Polyakov string is defined as:

$$S \equiv \int_S \mathrm{dvol}_g \sum_{a,b} g^{ab} \gamma_{ab}, \quad \text{where} \quad g^{ab} = (g)^{-1}_{ab}. \tag{2.2}$$

To give a coordinate free description of the action functional, we proceed as follows.

First we recall that for any imbedding $h$, the map $h_*: TS \to TM$ can be thought of as a section of the bundle $T^*S \otimes h^* TM$. So we can apply to $h_*$ the $*$-operator with respect to the intrinsic metric $g$, yielding a new section of $T^*S \otimes h^* TM$ which is again a map $*(h_*): TS \mapsto TM$ with $*(h_*)(\pi^{-1}(x)) \subset \pi^{-1}(h(x))$ $\forall x \in S$. Here $\pi$ denotes both the projection $TM \to M$ and the projection $TS \to S$.

Let $\mathrm{Imb}(S, M)$ denote the space of all imbeddings[12]. We can now consider a 2-form $\mathscr{L}(h, g)$ on $S$, depending on $h \in \mathrm{Imb}(S, M)$ and on the intrinsic metric $g$, given

12 $\mathrm{Imb}(S, M)$ can be given the structure of a manifold compatible with the Whitney $C^\infty$-topology [5]. The tangent space at $h \in \mathrm{Imb}(S, M)$ is given by the sections of the bundle $h^* TM$, or equivalently, by the elements $f_h \in \mathrm{Map}(S, TM)$ which satisfy the following condition

$$f_h(x) \in \pi^{-1}(h(x)).$$

Consider now

$$\langle f_h, f'_h \rangle (x) \equiv \Gamma(f_h(x), f'_h(x)) \quad x \in S \quad f_h, f'_h \in T_h \mathrm{Imb}(S, M),$$

and define

$$(f_h | f'_h) \equiv \int_S \langle f_h, f'_h \rangle \, \mathrm{dvol}_{h^* \Gamma};$$

$$(f_h | f'_h)_g \equiv \int_S \langle f_h, f'_h \rangle \, \mathrm{dvol}_g.$$

It is clear that both $(f_h | f'_h)$ and $(f_h | f'_h)$ are (weak) Riemannian metrics on $\mathrm{Imb}(S, M)$

by:

$$\mathscr{L}_x(h,g)(X_1,X_2)=\Gamma_{h(x)}(h_*(X_1),*(h_*)(X_2)) - \Gamma_{h(x)}(h_*(X_2),*(h_*)(X_1)) \quad x\in S, X_1,X_2\in T_xS. \tag{2.3}$$

We have also the corresponding action functional:

$$\mathscr{S}(h,g)=\int_S \mathscr{L}(h,g). \tag{2.4}$$

Formula (2.4) gives the coordinate-free expression of (2.2).

So the Polyakov action is invariant under:

1) Diff $M$;
   in fact $\forall\psi\in\mathrm{Diff}\,M$, the imbedding $h$ is transformed into $\psi\circ h$, but also the metric $\Gamma$ is transformed into $\psi^{-1*}\Gamma$. Hence the induced metric $\gamma$ is Diff $M$-invariant.
2) Diff $S$;
   $\forall\psi\in\mathrm{Diff}\,S$, the imbedding $h$ is transformed into $h\circ\psi$ and the metric $g$ is transformed into $\psi^*g$. But the imbedding $h\circ\psi$ induces the metric $\psi^*\gamma$ so (2.2) is Diff $S$-invariant.
3) Conformal (Weyl) rescalings of $g$;
   let $\sigma:S\mapsto\mathbf{R}$ and let $g$ be transformed into $e^\sigma g$. Then $\mathrm{dvol}_g\, g^{ab}$ is invariant, or, equivalently the $*$-operator in (2.3) is conformally invariant since it is applied to 1-forms on a 2-dimensional manifold.
4) $\mathrm{Aut}_v\, CS$;
   this is the group of transformations leaving the complex structure of $S$ unchanged (see below). They are a combination of conformal rescalings and Lorentz transformations.

Corresponding to these "classical" invariances, there are possible anomalies of the string. In this section we first consider conformal and holomorphic anomalies, which correspond to the last two invariance groups described above.

The world-sheet of the string, being an orientable 2-dimensional manifold, can be given a complex structure, i.e. we can define $\forall x\in S$ a linear mapping $J_x: T_xS\mapsto T_xS$ with $J_x^2=-1$. Obviously $J_x$ is assumed to depend smoothly on $x$. A complex structure defines, in a natural way, an orientation [4]. If $J_0$ is the matrix $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, then a complex frame [4, Chap. IX] is defined as a frame $u_x:\mathbf{R}^2\mapsto T_xS$ which satisfies the following condition

$$u_x\circ J_0=J_x\circ u_x, \tag{2.5}$$

i.e. a complex frame at $x$ is given by $(x;X,J_xX)$ for any $X\in T_xM$. The principal bundle $CS$ of complex frames is a reduced bundle (with structure group $\mathbf{C}^*$) of the frame bundle $LS$.

A metric $g$ is said to be compatible with the complex structure $J$ if the following condition is satisfied:

$$g(J_xX_1,J_xX_2)=g(X_1,X_2) \quad \forall X_1,X_2\in T_xS. \tag{2.6}$$

Hence if $(x;X,JX)$ is a complex frame at $x\in S$, then $g(X,JX)=0$; that is a bundle of complex frames is in reality the bundle of oriented frames which are orthogonal and of fixed length at each point. If $g'$ is another metric compatible with the complex structure $J$, then $g$ and $g'$ are conformally related (i.e. $g' = e^{\sigma} g$ with $\sigma: S \mapsto R$) and vice versa. The bundle $O_g S$ is a subbundle of $CS$ with structure group $SO(2) \cong U(1)$.

Let us now consider the space $\mathscr{A}_{CS}^{\text{metric}}$ of connections on $CS$ which are reducible to a connection on $O_g S$ for some compatible $g$. The group of gauge transformations of $CS$ acts on $\mathscr{A}_{CS}^{\text{metric}}$. Since $\mathbf{C}^*$ is an abelian group, $\mathrm{Aut}_v\, CS$ is the group $\mathrm{Map}(S, \mathbf{C}^*)$, which is the direct product of $\mathrm{Map}(S, \mathbf{R}^+)$ and $\mathrm{Map}(S, U(1)) \cong \mathrm{Map}(S, SO(2))$. So $\psi \in \mathrm{Map}(S, SO(2))$ transforms the space of connections reducible to $O_g S$ into itself, while $e^{\sigma} \in \mathrm{Map}(S, \mathbf{R}^+)$ transforms the space of connections reducible to connections on $O_g S$ into the space of connections reducible to connections on $O_{e^{\sigma} g} S$.

In order to prove this, it is enough to note that, if $A$ is a connection reducible to a connection on $O_g S$ and $A'$ is the gauge transform of $A$ by $e^{\sigma} \in \mathrm{Map}(S, \mathbf{R}^+)$, then we have:

$$\nabla_A g = 0 \Rightarrow \nabla_{A'} e^{\sigma} g = 0,$$

where $\nabla_A$ is the covariant derivative on $S^2 TM$ induced by the connection $A$ on $LM$.

If we identify $\mathbf{C}^*$ with the group of constant maps from $S$ to $\mathbf{C}^*$, then $\mathrm{Aut}_v\, CS/\mathbf{C}^*$ acts freely on the space of connections on $CS$ and in particular on $\mathscr{A}_{CS}^{\text{metric}}$.

Let $\mathscr{G}$ denote any subgroup of $\mathrm{Aut}_v\, CS/\mathbf{C}^*$. We can then consider the following diagram (analogous to diagrams (7.5) and (7.14) of [1]):

$$\begin{array}{ccc} CS \times \mathscr{A}_{CS}^{\text{metric}} & \xrightarrow{\hat{\text{ev}}} & E\mathbf{C}^* = EU(1) \times \mathbf{R}^+ \\ \downarrow & & \downarrow \text{id} \\ \dfrac{CS \times \mathscr{A}_{CS}^{\text{metric}}}{\mathscr{G}} & \xrightarrow{\text{Ev}} & E\mathbf{C}^* \\ \downarrow \pi & & \downarrow \pi \\ S \times \dfrac{\mathscr{A}_{CS}^{\text{metric}}}{\mathscr{G}} & \xrightarrow{\text{Ev}} & B\mathbf{C}^* = BU(1). \end{array} \tag{2.7}$$

Diagram (2.7) shows that, in two dimensions, the holomorphic anomalies, defined as the gauge anomalies of the complex frame bundle, are generated by the cohomology classes of $BU(1)$. In turn the cohomology algebra of $BU(1)$ is generated by $F_{\xi}$ (the curvature of a universal connection $\xi$ for $EU(1)$)[13].

Since we are in two dimensions, we are interested in the 4-form $F_{\xi} \wedge F_{\xi}$ whose (Chern) transgression is given by $\xi \wedge F_{\xi}$.

If $\mathscr{G}$ is in particular the group isomorphic to $\mathscr{G} = (\mathrm{Map}(S, \mathbf{R}^+)/\mathbf{R}^+)$ and if $f: CS \mapsto E\mathbf{C}^*$ is a bundle homomorphism, then the connection $A \equiv f^* \xi$ is a metric connection (in particular it can be a Levi Civita connection). If $\Sigma \in (\mathrm{aut}_v\, CS/\mathbf{C})$ is the vector field on $CS$, corresponding to $\sigma \in (\mathrm{Map}(S, \mathbf{R})/\mathbf{R}) = \mathrm{Lie}(\mathrm{Map}(S, \mathbf{R}^+)/R^+)$, then the relevant anomaly is given by the following map:

$$\Sigma \mapsto i_{\Sigma} A \wedge F_A = \sigma F_A. \tag{2.8}$$

13 In the following, $\xi$ will be considered either as a connection on $EU(1)$ or as a connection on $EC^*$, reducible to a connection on $EU(1)$

If we observe that under the isomorphism $U(1)\to SO(2)$, the curvature $F_A$ becomes, up to a factor $\sqrt{-1}$, the Euler density, then we see that the right-hand side of (2.8) coincides, up to a factor $\sqrt{-1}$, with the expression of the local conformal anomaly in two dimensions as defined according to [14].

So we can interpret the conformal anomaly in two dimensions as the holomorphic anomaly restricted to vector fields $\Sigma$ defined as above.

Notice that we could not consider in (2.7) only Levi–Civita connections instead of metric connections since, if $A_g$ is the Levi Civita connection of $g$, then the gauge transform of $A_g$ by $e^\sigma$ is not the Levi Civita connection of $e^\sigma g$. For instance, in holomorphic coordinates, where $d=\partial+\bar{\partial}$, the gauge transform of $A_g$ is $A_g+d\sigma$, while the Levi–Civita connection of $e^\sigma g$ is $A_g+\partial\sigma$. The above observation has some implications concerning the possibility of cancelling conformal anomalies as it will be shown in the following.

As $A$ is a $\mathbf{C}^*$-valued connection form, we can write $A=\mathrm{Re}\,A+i\,\mathrm{Im}\,A$. But $\mathrm{Re}\,A$ is an exact form since $A\in\mathscr{A}_{CS}^{\text{metric}}$. Now $i_\Sigma A=i_\Sigma \mathrm{Re}\,A$ and so we could mistakenly conclude that the conformal anomaly (2.8) is always trivial, since we have:

$$i_\Sigma A\wedge F_A=i_\Sigma d(\beta\wedge F_A)=L_\Sigma(\beta\wedge F_A)-d(i_\Sigma\beta\wedge F_A), \tag{2.9}$$

provided that $d\beta=\mathrm{Re}\,A$. Here $L_\Sigma$ is the Lie derivative along $\Sigma$.

The impossibility of considering the conformal anomaly as a trivial one arises from the fact that we cannot identify $L_\Sigma\beta$ as the "true" variation of $\beta$ along $\Sigma$. In fact, when $A$ is a Levi–Civita connection then $\beta$ is a function of the metric $g$ [4] and its "variation along $\Sigma$" should really be the variation obtained by conformally rescaling the metric $g$ [11, 15][14].

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14 We would have similar problems for a possible "gauge" interpretation of some conformal anomalies in the case of a manifold $M$ of any even dimension $n$.

One could consider the bundle $\mathrm{Orth}\,M$ of oriented frames which are orthogonal and of fixed length at each point. This is a principal bundle with structure group $\mathbf{R}^+\times SO(n)$.

Obviously one could also consider the gauge transformations defined by the elements of $\mathrm{Map}(M,\mathbf{R}^+)$ and construct a diagram like (2.7), by replacing $S, CS, \mathbf{C}^*$ and $U(1)$ respectively with $M, \mathrm{Orth}\,M, \mathbf{R}^+\times SO(n)$ and $SO(n)$. In $n$-dimensions a class of conformal anomalies is given by [14, 16]:

$$(*)\qquad \sigma\mapsto\sigma P(F,\dots,F),$$

where $F$ is the curvature of a metric or Levi Civita connection and $P$ is an ad-invariant polynomial of $SO(n)$ with $n/2$-entries. Other conformal anomalies are constructed with the Weyl tensor.

Interpreting the above anomaly $(*)$ as a gauge anomaly would mean considering the form

$$\mathrm{Tr}\,A\wedge P(F,\dots,F)$$

which is certainly exact,

So the conformal anomaly would be "trivial" and we would have the same situation as in two dimensions. The difference between two and any other even dimension is the following. If $n=\dim M\neq 2$, then there exists no universal $(n+1)$-form $\chi$ (depending on the space of connections on $\mathrm{Orth}\,M$ which are reducible to connection on $O_gM$) which satisfies the following requirements:

(a) $\chi$ is closed and it is not the differential of any other universal form;

(b) The anomaly $(*)$ can be represented as the map $\Sigma\to i_\Sigma\chi$, where $\Sigma$, in turn, is a vector field corresponding to $\sigma\in\mathrm{Map}(M,R)$.

Hence it seems that, if $n\neq 2$, then the *coefficient* of the anomalies $(*)$ cannot be computed with the family's index theorem

We have also to point out that, since the group $\mathrm{Map}(S, \mathbf{R}^+)$ is a contractible space, it does not make any sense to look for the topological significance of conformal anomalies, namely it does not make sense to search for anomalies representing non-trivial 1-cohomology classes of $\mathrm{Map}(S, \mathbf{R}^+)$, since such non-trivial classes do not exist. Also it does not make any sense to look for topological arguments which could justify the use of the index theorem in the computation of conformal anomalies for any dimension of the space time. The possibility of such a use in two dimensions seems to be purely "coincidental" (at least in the present framework), due to the relation between conformal and holomorphic anomalies. But the locality requirement applies perfectly well to conformal anomalies. Locality here means that anomalies may not be cancelled by adding counterterms which are not forms depending on the fields of the theory. But obviously they can be cancelled if the coefficient in front of them is zero. This is what happens to conformal anomalies in the critical dimension [17].

Gravitational anomalies can be considered by pulling back cohomology classes of $BU(1)$ via the map Ev in the following diagram:

$$\begin{array}{ccc} \dfrac{\mathcal{O}_S \ltimes \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1} S} & \xrightarrow{\mathrm{Ev}} & EU(1) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ \dfrac{S \ltimes \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1} S} & \xrightarrow{\underline{\mathrm{Ev}}} & BU(1), \end{array} \tag{2.10}$$

where $\mathcal{O}_S \ltimes \mathscr{A}^{\mathrm{metric}}$ denotes the bundle of orthonormal frames of $S$ paired with the corresponding metric connections for metrics on $S$.

When the world-sheet of the string is imbedded in a flat background, then the coeffcients of both the gravitational and the holomorphic anomaly have been computed [17]. In the Polyakov string, the contributions from the left and the right sectors to the gravitational anomaly annihilate each other. In contrast, the coefficient of the conformal anomaly vanishes only in the critical dimension. In the heterotic string and in the superstring case, both the conformal and the gravitational anomaly vanish only in the critical dimension [17].

Finally, let us point out that two complex structures $J$ and $J'$ are said to be equivalent if the relevant complex frame bundles $C_J S$ and $C_{J'} S$ are such that

$$C_{J'} S = l(\psi)^{-1}(C_J S), \tag{2.11}$$

where $\psi$ is a diffeomorphism and $l(\psi)$ is its lift in $LS$.

If the genus $g$ of $S$ is $\geqq 2$, then there are only a finite number of diffeomorphisms $\psi$ (no more than $84\,(g-1)$ [18]) such that

$$C_J S = l(\psi)^{-1} C_J S \tag{2.12}$$

for any complex structure $J$.

So for $g \geqq 2$, the action of Diff $S$ on the space of complex structures is essentially free and the quotient space is called the moduli space. From now on we will assume $g \geqq 2$.

For any complex structure, $C_J S$ there is a unique Levi Civita connection (for a compatible metric) $A^J_{-1}$ such that its scalar curvature is $-1$. If $C_{J'}S = l(\psi)^{-1}(C_J S)$, then $A^{J'}_{-1} = l(\psi)^* A^J_{-1}$. We can consider the space $\mathscr{A}^{-1}$ of all connections $A^J_{-1}$, for any complex structure $J$. Moreover we can define, analogously to definitions (7.18)–(7.21) in [1], the bundle $\mathscr{C}_S \times \mathscr{A}^{-1}$ of all complex frames paired with the corresponding connection in $\mathscr{A}^{-1}$. We have then the following diagram:

$$\begin{array}{ccc} \dfrac{\mathscr{C}_S \times \mathscr{A}^{-1}}{\operatorname{Diff} S} & \xrightarrow{\mathrm{Ev}} & E\mathbf{C}^* \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ \dfrac{S \times \mathscr{A}^{-1}}{\operatorname{Diff} S} & \xrightarrow{\underline{\mathrm{Ev}}} & B\mathbf{C}^* = BU(1). \end{array} \tag{2.13}$$

Here $(S \times \mathscr{A}^{-1})/\operatorname{Diff} S$ is a fiber bundle over the moduli space (or the Teichmüller space if we limit ourselves to considering only diffeomorphisms which are connected to the identity)[15]. Diagram (2.13) allows us to consider the anomaly over the moduli (Teichmüller) space. Such an anomaly is generated by the two form on $\mathscr{A}^{-1}/\operatorname{Diff} S$ obtained by fiber integrating the pull back of $F_\xi \wedge F_\xi \in \Omega^4(BU(1))$.

Notice that since we are interested in local anomalies, the fact that the Teichmüller space is contractible does not imply automatically the cancellation of the relevant anomaly. The anomaly determined by diagram (2.13) can be thought of as obstructions to the definition of a local diff-invariant expression for "$\det \bar{\partial}$" (see also [19]).

It is clear, by comparing diagrams (2.7), (2.10) and (2.13), that the cancellation[16] of the conformal anomaly implies the cancellation of the local anomaly over the moduli (Teichmüller) spaces [20], since both anomalies are generated by the same form $F_\xi \wedge F_\xi$ on $BU(1)$. In conclusion, all the anomalies of the string (except the sigma-model ones) are generated by the same form on $BU(1)$.

As a final remark, we recall that the homomorphism (2.13) can be defined only when we are given a connection $\eta'$ on $(\mathscr{C}_S \times \mathscr{A}^{-1})/\operatorname{Diff} S$. This connection $\eta'$ is in turn determined by a connection $\omega$ on the bundle $\mathscr{A}^{-1} \to (\mathscr{A}^{-1}/\operatorname{Diff} S)$ (see [1], Sect. 7). Now on $\mathscr{A}^{-1}$ there is a natural $\operatorname{Diff} S$-invariant metric, namely the Weil–Petersson metric [21, 22, 23], so we can choose for $\omega$ the connection such that the "horizontal" vectors in $T\mathscr{A}^{-1}$ are, by definition, the ones which are orthogonal to the vertical ones with respect to this metric.

## 3. Sigma-Model Anomalies of the String

In this section the world-sheet $S$ of the string is supposed to be a compact connected Riemann surface imbedded or immersed in an $n$-dimensional compact Riemannian

15 One could also consider an analogous diagram for $\mathscr{C}_S \times \mathscr{A}^{\mathrm{metric}}$, where metric connections (for compatible metrics) are paired with the corresponding complex frame bundles

16 Here by cancellation we mean obviously that the coefficient of the anomaly is zero and not that the anomaly is cancelled by adding local counterterms

manifold $M$, which will be generally assumed to be connected, compact and without boundary. The space of all the imbeddings of $S$ in $M$ is denoted by the symbol$(S, M)$, while the space of all immersions will be denoted by the symbol $\mathrm{Imm}(S,M)$[17]. Obviously we have $\mathrm{Imm}(S,M) \supset \mathrm{Imb}(S,M)$.

We want to discuss the sigma-model anomalies of the string[18].

A few preliminary remarks concerning the mathematical structure of the imbedded or immersed world-sheet of the string are in order:

1) In sigma-models, the relevant space of maps is the space of all maps from the manifold to the target space, while, for the string, we want to consider only the imbeddings or the immersions. One of our aims is in fact to "compare" the "action" of the two difffeomorphism groups: $\mathrm{Diff}\, M$ and $\mathrm{Diff}\, S$.

   In the following we will discuss first the imbedded world-sheet of the string and later we will consider briefly the immersed one. We have to recall, anyway that if the dimension of $M$ is large enough (e.g. $\dim M \geqq 5$), then any map from $S$ to $M$ is homotopic to an imbedding, see [24, 25].
2) For any $h_0 \in \mathrm{Imb}(S,M)$ and for any $\psi \in \mathrm{Diff}\, M$, the composite $\psi \circ h_0$ is also an element of $\mathrm{Imb}(S,M)$ not necessarily distinct from $h_0$. Moreover if we are given $h_0 \in \mathrm{Imb}(S,M)$, then we can define $\psi, \psi' \in \mathrm{Diff}\, M$ to be equivalent when $\psi \circ h_0 = \psi' \circ h_0$. We denote by $\mathrm{Diff}\, M/(h_0)$ the space of such equivalence classes, and by $\mathrm{Diff}_0\, M/(h_0)$ the space of equivalence classes of elements of $\mathrm{Diff}_0\, M$, which is by definition the connected component of the identity. In the following $h_0$ will be an arbitrary, fixed imbedding.
3) We have to consider, at least in principle, three different symmetry groups for the imbedded world-sheet of the string: $\mathrm{Diff}\, S$, $\mathrm{Diff}\, M$ and $\mathrm{Diff}(M, h_0(S))$. The last group is defined as the subgroup of $\mathrm{Diff}\, M$ which maps $h_0(S)$ into itself.

   Any element in $\mathrm{Diff}(M, h_0(S))$ induces a diffeomorphism of $S$, so there is a group homomorphism:

$$\rho_{h_0}: \mathrm{Diff}(M, h_0(S)) \to \mathrm{Diff}\, S, \tag{3.1}$$

   where $\psi \in \mathrm{Diff}(M, h_0(S)) \Rightarrow \psi \circ h_0 = h_0 \circ \rho_{h_0}(\psi)$. The map $\rho_{h_0}$ is surjective on $\mathrm{Diff}_0\, S$, i.e. on the connected component of the identity of $\mathrm{Diff}\, S$.

   For reasons that will be discussed later, we consider also the space $\mathrm{Diff}(M, h_0(S))^l$ of equivalence classes of elements of $\mathrm{Diff}(M, h_0(S))$, where two such elements $\psi, \psi'$ are said to be equivalent if

$$\rho_{h_0}(\psi) = \rho_{h_0}(\psi') \quad \text{and} \quad l(\psi)|_{\pi^{-1}(h_0(S))} = l(\psi')|_{\pi^{-1}(h_0(S))};$$

   here $\pi: LM \to M$ is the projection and $l$ is the natural lift for $\mathrm{Diff}\, M$.

   Notice that $\mathrm{Diff}(M, h_0(S))^l$ is a group when the product of two equivalence classes $[\psi]$ and $[\psi']$ is defined as the equivalence class $[\psi \circ \psi']$. Indeed it is the

17 We recall that an immersion $h_0: S \to M$ is, by definition, a $C^\infty$ map such that $h_{0*}: T_x S \to T_{h_0(x)} M$ is injective $\forall x \in S$. The immersion $h_0$ is, by definition, an imbedding if it is injective

18 All the results of this section are valid also for the case of a $k$-dimensional manifold $S$ imbedded in an $n$-dimensional manifold $M$, with $n > k$, provided that the condition $\dim M \geqq 5$ considered below is replaced by the condition $\dim M \geqq 2k + 1$

quotient of $\mathrm{Diff}(M, h_0(S))$ by the normal subgroup of those $\psi$ such that $l(\psi)\bar{h}_0 = \bar{h}_0$, where $\bar{h}_0 : h_0^* LM \to LM$ is canonically induced by $h_0$. The group $\mathrm{Diff}(M, h_0(S))^l$ can also be identified with the subgroup of $\mathrm{Aut}\, h_0^* LM$, given by:

$$\{\psi \mid \psi \in \mathrm{Aut}\, h_0^* LM \quad \text{such that there exists} \quad \varphi \in \mathrm{Diff}\, M, \quad \text{with} \quad \bar{h}_0 \circ \psi = l(\varphi) \circ \bar{h}_0\}$$

where $l$ is the natural lift in $LM$.

We can also consider the group $\mathrm{Diff}_0(M, h_0(S))$, that is the connected component of the identity of $\mathrm{Diff}(M, h_0(S))$ and the group $\mathrm{Diff}_0(M, h_0(S))^l$ which is given by the equivalence classes of elements of $\mathrm{Diff}_0(M, h_0(S))$.

4) If $h \in \mathrm{Imb}(S, M)$ is homotopic to $h_0$, then there exists a diffeomorphism $\psi \in \mathrm{Diff}_0\, M$ such that $\psi \circ h_0 = h$ [24, 25][19]. So any $h \in \mathrm{Imb}(S, M)$ which is homotopic to $h_0$ can be identified with an element of $\mathrm{Diff}_0\, M/(h_0)$. In other words, we can consider the imbedding-degrees-of-freedom of the string as equivalence classes of diffeomorphisms of the target space M.

We denote by $\mathrm{Imb}(S, M)_{h_0}$ the set of all imbeddings homotopic to $h_0$.

Notice that the above arguments do not depend on the choice of $h_0$ in a given homotopy class of imbeddings. Obviously it can also happen, that there is only one homotopy class in $\mathrm{Imb}(S, M)$. This is the case, for instance, when $\pi_1(M) = \pi_2(M) = 0$.

In order to discuss the anomalies of a (closed super) string propagating in a background corresponding to its zero mass sector [26, 27], we consider a Lagrangian which includes, besides the density of Eq. (2.2), also a term $h_0^* B$, where $h_0 \in \mathrm{Imb}(S, M)$ and $B$ is a 2-form on $M$ (e.g. the fundamental 2-form, if $M$ is a complex manifold). The relevant fermionic part of the string action will contain typically two terms $\langle \phi, \not{\partial}_{\bar{h}_0^* A} \phi \rangle$ and $\langle \lambda, \not{\partial}_{\bar{h}_0^* \omega} \lambda \rangle$, where $\omega$ and $A$ are respectively connections on Spin $M$ and $P$, a (gauge) principal bundle. Moreover, $\lambda$ is assumed to be a section of the bundle $S^{\pm} \otimes h_0^* TM$ (corresponding to the Ramond, Neveu, Schwarz formulation), where $S^{\pm}$ is the spinor bundle over $S$ with positive (negative) chirality and $\phi$ is supposed to be a section of a bundle $S^{\mp} \otimes h_0^* V$, where $V$ is a vector bundle associated to $P$. The fields $\phi$ and $\lambda$ have opposite chirality.

In this section, we will concentrate on the chiral anomalies which have not already been discussed in the previous section. We refer to them generically as sigma-model anomalies. Let us discuss first the sigma-model anomalies of the string, which are analogous to the sigma-model anomalies considered in Sect. 1.

If $P$ is a principal bundle over $M$, then we can consider the diagram:

$$\begin{array}{ccccc} h_0^* P \times \mathscr{P}_{h_0} \mathrm{Imb}(S, M) \approx \pi_1^* \mathrm{ev}^* P & \xrightarrow{\bar{\pi}_1} & \mathrm{ev}^* P & \xrightarrow{\overline{\mathrm{ev}}} & P \\ \downarrow & & \downarrow & & \downarrow \\ S \times \mathscr{P}_{h_0} \mathrm{Imb}(S, M) & \xrightarrow{\pi_1} & S \times \mathrm{Imb}(S, M) & \xrightarrow{\mathrm{ev}} & M. \end{array} \tag{3.2}$$

19 This statement is true, provided that $n = \dim M \geqq 5$

Here the formalism is the same as in Sect. 1 (see in particular diagram (1.15)). The anomaly is generated by $W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0)$ (in the formalism with the background connection) (see also [28, 29, 30]) where $K$ is the (normalized) ad-invariant polynomial with 2-entries on the Lie algebra of the structure group, $A$ is a connection on $P$ (recall expressions (1.4), (1.5)) and $A_0$ is a background connection on $h_0^* P$. One can consider also the bundle $O_\Gamma M$ (or $\mathrm{Spin}_\Gamma M$), where $\Gamma$ is a metric on $M$, and study the "Lorentz" sigma-model anomaly generated by $W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \omega, \omega_0)$, where $\omega$ is a Lorentz connection on $M$ and $\omega_0$ is a connection on $h_0^* O_\Gamma M$. When both gauge and Lorentz anomalies are present, we must refer to the bundle $O_\Gamma M + P$ and the relevant anomaly is generated by the form

$$W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0) - W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \omega, \omega_0).$$

We have essential three cases for the sigma-model anomalies of the string.

(1) In the first case, we denote by $F_A$ the curvature of $A$ and we assume that the Weil homomorphism applied to $K$ gives zero. That is, we assume that there exists a basic 3-form $H_A$ such that $K(F_A, F_A) = dH_A$.

In this case the sigma-model anomaly can be cancelled by the (generalized) Wess–Zumino term $B$ which is an element of $\Omega^2(S \times \mathscr{P}_{h_0}(\mathrm{Imb}(S, M)))$.

In particular $\int_S B$ will be a function defined on the path space $\mathscr{P}_{h_0} \mathrm{Imb}(S, M)$.

Analogously to the guage anomaly, the Lorentz anomaly also will be cancelled provided the first Pontrjagin class of $M$ is zero.

(2) As a second case, we assume that $K(F_A, F_A)$ and $K(F_\omega, F_\omega)$ are not necessarily exact but that the following condition is satisfied:

$$K(F_A, F_A) - K(F_\omega, F_\omega) = dH_{A,\omega}. \tag{3.3}$$

That is we assume that the cohomology class of $K(F_A, F_A)$ coincides with the first Pontrjagin class.

In this situation we can consider the bundle $P + O_\Gamma M$ over $M$ and the diagram (see Sect. 1 for the notation):

$$\begin{array}{cccc} h_0^*(P + O_\Gamma M) \times \mathscr{P}_{h_0}(\mathrm{Imb}(S, M)) & & & \\ \approx \pi_1^* \mathrm{ev}^*(P + O_\Gamma M) & \xrightarrow{\bar{\pi}_1} & \mathrm{ev}^*(P + O_\Gamma M) \xrightarrow{\mathrm{ev}} & P + O_\Gamma M \\ \downarrow & & & \downarrow \\ S \times \mathscr{P}_{h_0}(\mathrm{Imb}(S, M)) & \xrightarrow{\pi_1} & S \times \mathrm{Imb}(S, M)_{h_0} \xrightarrow{\mathrm{ev}} & M \end{array} \tag{3.4}$$

Now the form

$$W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0) - W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* \omega, \omega_0) - \pi_1^* \mathrm{ev}^* H_{A,\omega} \tag{3.5}$$

is closed (due to (3.3)) and hence exact, since it is a 3-form on a space which admits as a deformation retract the two dimensional space $S$. Hence we have a generalized Wess–Zumino term which cancels the anomaly. Notice that the minus sign is due to the opposite chiralities.

(3) In the third case we assume that no equation like (3.3) is satisfied. In this case, there is no anomaly cancellation.

We are now in position to make a further comment on the Green–Schwarz mechanism.

We recall that anomaly cancellation for a ten-dimensional field theory can occur only when we require the imbedding of the orthonormal frame bundle into the gauge bundle, as was explained in [1], Sect. 8. Hence, in field theory, it is the imbedding itself which guarantees the validity of an equation analogous to Eq. (3.3).

On the contrary, if we consider the sigma-model anomalies of the string, then Eq. (3.3) permits anomaly cancellation and no imbedding is required.

In other words, the Green–Schwarz mechanism, without imbedding, should be interpreted as a mechanism for the cancellation of sigma-model anomalies of the string, and not for the cancellation of anomalies in a ten-dimensional field theory. In this case, under suitable geometrical conditions, a cancellation scheme is also possible, which is realized through a Chapline–Manton ansatz (see [31]).

Other sigma-model anomalies of the string need to be considered.

We will now consider new sigma-model anomalies which have not been studied before in the literature. We will call them diffeomorphism-sigma-model anomalies. The first question we have to ask ourselves is whether it makes sense at all to look for $\mathrm{Diff}\,M$-anomalies.

In order to understand better the situation, notice that, if we assume that a diffeomorphism $\psi \in \mathrm{Diff}\,M$ transforms the imbedding $h_0 \in \mathrm{Imb}(S, M)$ into $\psi \circ h_0$ and the Lorentz (spin) connection $\omega$ into $l(\psi)^{-1*}\omega$, then not only the string Lagrangian is invariant under $\mathrm{Diff}\,M$, but all of its terms are completely insensitive to the diffeomorphisms of $M$. This action of $\mathrm{Diff}\,M$ is the only one which leaves the string Lagrangian invariant. The Dirac operator $\partial\!\!\!/_{\tilde{h}_0^*\omega}$ does not transform covariantly under $\mathrm{Diff}\,M$, it is simply left unchanged. So it does not make sense to speak of $\mathrm{Diff}\,M$-sigma-model anomalies of the string, when the full group $\mathrm{Diff}\,M$ is considered.

The situation is different if, on the contrary, only the group $\mathrm{Diff}(M, h_0(S))$ is taken into account. In this case we can assume that, under the action of the above group, the imbedding is kept fixed, while $M$ is transformed. Thus, if we identify $S$ with $h_0(S)$, this group is "perceived" by the imbedded string as the group of diffeomorphisms of the string itself.

The action of $\mathrm{Diff}(M, h_0(S))$ differs from the action of $\mathrm{Diff}\,S$, only when we include fields like the $\lambda$'s considered before, i.e. when we consider sections of bundles associated to $h_0^* LM$ (or $h_0^* LM^+$)[20]. In this case two elements of $\mathrm{Diff}(M, h_0(S))$ will have the same effect on the above fields $\lambda$ if and only if they belong to the same class in $\mathrm{Diff}(M, h_0(S))^l$, i.e. if they coincide on $h_0(S)$ and their natural lifts coincide for all the points of $LM|_{h_0(S)}$.

In this respect, an invariance group we have to consider is $\mathrm{Diff}(M, h_0(S))^l$.

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20 In dealing with spinor fields one should be careful about the possible change of the spin structure of $S$ [32] (see also [33]). If we want to avoid such a change, then we can limit ourselves to considering the group $\rho_{\tilde{h}_0}^{-1}(\mathrm{Diff}_0\,S)$

As opposed to the case of Diff $M$, the action of $\mathrm{Diff}(M, h_0(S))^l$ on the fields $\lambda$ and on the Dirac operator is not trivial. Hence, in principle, if the string action contains the fermionic part described before, then one has to consider also the sigma-model anomalies of the string which come from the action of $\mathrm{Diff}(M, h_0(S))^l$. These anomalies differ from the ordinary sigma-model anomalies since for any chosen $h_0 \in \mathrm{Imb}(S, M)$ they are 1-forms on $\mathrm{Diff}(M, h_0(S))^l \subset \mathrm{Aut}\, h_0^* LM$, while the ordinary sigma-model anomalies are 1-forms on $\mathrm{Aut}_v h_0^* P$, for a given principal bundle $P$.

More generally we will consider the action (and the relevant anomalies) of the full group $\mathrm{Aut}\, h_0^* LM$ under whose action, the Dirac operator $\partial\!\!\!/_{\bar{h}_0^* \omega}$ transforms covariantly. Obviously our main interest concerns the possibility of anomaly cancellation. From this point of view there is nothing "special" in the bundle $LM$, hence we will study the anomalies of the full group of automorphisms of a bundle induced through $h_0$ from a generic bundle $P$ over $M$ with connection $A$.

First we notice that $\mathrm{Diff}_0 S$ acts freely on $\mathrm{Imb}(S, M)_{h_0}$ as follows

$$\begin{aligned} \mathrm{Imb}(S, M)_{h_0} \times \mathrm{Diff}_0 S &\to \mathrm{Imb}(S, M)_{h_0} \\ (h, \psi) &\mapsto h \circ \psi, \end{aligned} \tag{3.6}$$

yielding a principal fibre bundle [5]. Then we can consider the fibration:

$$\begin{array}{ccc} \mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S} & \xrightarrow{\;j\;} & \mathscr{P}_{h_0}(\mathrm{Imb}(S, M)) \\ & & \Big\downarrow \tilde{\pi}_1 \\ & & \dfrac{(\mathrm{Imb}(S, M))_{h_0}}{\mathrm{Diff}_0 S}, \end{array} \tag{3.7}$$

where $\mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S}$ is the space of paths on $\mathrm{Imb}(S, M)$ with starting point $h_0$ and endpoints given by $h_0 \circ \psi$, with $\psi \in \mathrm{Diff}_0 S$[21]. If $\pi_1(p)$ is the endpoint of $p \in \mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S}$ then we set

$$\rho_1(p) \equiv \psi \quad \text{if} \quad \pi_1(p) = h_0 \circ \psi.$$

Let now $P$ be a principal $G$-bundle on $M$ with connection $A$ and let $h_0^* P$ be the bundle on $S$ induced via $h_0$. To any $\psi \in \mathrm{Diff}_0 S$ we can associate the canonical homomorphism

$$\bar{\psi}: \psi^* h_0^* P \to h_0^* P.$$

We are now in a position to define a map

$$\tau_3: \mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S} \to \mathrm{Aut}\, h_0^* P. \tag{3.8}$$

Namely for any $p \in \mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S}$ we set $\tau_3(p) \equiv \overline{\rho_1(p)} \circ \tau_A(p)$ where the homomorphism $\tau_A(p): h_0^* P \to (h_0 \circ \psi)^* P$ is defined as in Sect. 1 (see Eq. (1.10)–(1.12)).

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21 In order to prove that the map $\tilde{\pi}_1$ gives a fibration, it is enough to notice that $\tilde{\pi}_1$ is the combination of the projection map $\pi_1$ of the fibration $\mathscr{P}_{h_0}(\mathrm{Imb}(S, M)) \to \mathrm{Imp}(S, M)_{h_0}$ with the projection map of the principal bundle $\mathrm{Imb}(S, M)_{h_0} \to (\mathrm{Imb}(S, M))_{h_0} / \mathrm{Diff}_0 S$

The map $\tau_3$ is an extension of the map $\tau_2$(1.12), but while the image of $\tau_2$ is a subgroup of $\operatorname{Aut} h_0^* P$, the same is not necessarily true for the image of $\tau_3$.

The evaluation map

$$\mathrm{ev}: S \times \mathrm{Imb}(S, M)_{h_0} \to T \tag{3.9}$$

descends to a map

$$\mathrm{ev}': \frac{S \times \mathrm{Imb}(S, M)_{h_0}}{\mathrm{Diff}_0 S} \to T. \tag{3.10}$$

So we can consider the closed 2 form on

$$\frac{(\mathrm{Imb}(S, M))_{h_0}}{\mathrm{Diff}_0 S}$$

given by

$$\int_S \mathrm{ev}'^* K(F_A, F_A), \tag{3.11}$$

where $F_A$ is the curvature of the connection $A$. The form (3.11) can be lifted to a 2-form on $\mathscr{P}_{h_0}(\mathrm{Imb}(S, M))$ where it becomes exact since $\mathscr{P}_{h_0}(\mathrm{Imb}(S, M))$ is a contractible space. Namely we have

$$\tilde{\pi}_1^* \int_S \mathrm{ev}'^* K(F_A, F_A) = dC \tag{3.12}$$

and, taking into account the fibration (3.7), we can set[22]

$$\tilde{J}^* C = \int_S \tilde{J}^* W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0). \tag{3.13}$$

The above expression is the pullback under the map $\tau_3$ of the full $\operatorname{Aut} h_0^* P$ anomaly. Analogously to Sect. 1 we can easily prove that the exactness of the form $K(F_A, F_A)$ implies the exactness of $\tilde{J}^* C$ as a 1-form on $\mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S}$. More precisely the equation

$$K(F_A, F_A) = dH_A \tag{3.14}$$

implies

$$W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0) = \pi_1^* \mathrm{ev}^* H + d\mathscr{Q}(W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0) - \pi_1^* \mathrm{ev}^* H), \tag{3.15}$$

where the homotopy operator $\mathscr{Q}$ is defined as in Theorem (1.19).

From (3.15) we obtain in turn

$$\int_S \tilde{J}^* W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0) = d \int_S \tilde{J}^* \mathscr{Q}(W_K(\bar{\pi}_1^* \overline{\mathrm{ev}}^* A, A_0) - \pi_1^* \mathrm{ev}^* H), \tag{3.16}$$

where the last $d$ is the exterior derivative on $\mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S}$.

This is a way in which sigma-model anomalies relevant to the full group of automorphisms of an induced principal bundle can be cancelled. As we said before,

22 In order to avoid a cumbersome notation we use ambiguously the symbol $\tilde{J}$ to denote also the map $S \times \mathscr{P}_{h_0}(\mathrm{Imb}(S, M))_{\mathrm{Diff}_0 S} \to S \times \mathscr{P}_{h_0}(\mathrm{Imb}(S, M))$ which should more properly be denoted by the symbol $\mathrm{id} \times \tilde{J}$

the case which is specially relevant for string theory is when the bundle $P$ is the frame bundle over the target $M$.

In conclusion we can say that the cancellation of (local) gravitational-sigma-model anomalies for the string, is guaranteed by the vanishing of the first Pontrjagin class of the ambient manifold $M$.

Alternatively, it is enough to a require a condition like (3.3), if we want to consider simultaneously the gravitational sigma-model anomalies and the anomalies of the full automorphism group of the "gauge" bundle $h_0^* P$.

Finally, we want to comment briefly on a more general situation than the imbedded string. Namely we want to consider a string whose world-sheet $S$ is immersed into the ambient manifold $M$ as opposed to being imbedded.

We denote by $N_{h_0}$ the subset of $S$ defined as follows:

$$N_{h_0} = \{x \in S \mid \exists x' \in S \quad \text{with} \quad x' \neq x \quad \text{and} \quad h_0(x) = h_0(x')\}.$$

The subsets $N_{h_0} \subset S$ and $h_0(N_{h_0}) \subset M$ are closed. In general both $N_{h_0}$ and $h_0(N_{h_0})$ will be the union of closed submanifolds (possibly with boundary) respectively of $S$ and $M$. We are especially interested here in the case when $h_0(N_{h_0})$ and hence [34] also $N_{h_0}$ are given by a discrete set of points. In this situation we can take into account the following groups:

$$\begin{aligned} \mathrm{Diff}(M, h_0(S); h_0(N_{h_0})) = \{\psi \in \mathrm{Diff}\, M, \quad &\text{such that} \quad \psi(h_0(S)) = h_0(S) \\ \text{and} \quad &\psi(h_0(N_{h_0})) = h_0(N_{h_0})\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \mathrm{Diff}(S; N_{h_0}) = \{\psi \in \mathrm{Diff}\, S, \quad &\text{such that} \quad \forall y \in h_0(N_{h_0}) \quad \text{we have:} \\ &x_1, x_2 \in h_0^{-1}(y), x_1 \neq x_2 \Rightarrow h_0 \psi(x_1) = h_0 \psi(x_2)\}. \end{aligned} \tag{3.18}$$

We can discuss the sigma-model anomalies of the immersed string exactly in the same way as we did for the imbedded string, provided that we replace the groups $\mathrm{Diff}(M, h_0(S))$ and $\mathrm{Diff}\, S$ with the groups (3.17) and (3.18) respectively, and that we take "special care" of the elements of (3.17) and (3.18) which "exchange" the multiple points.

## 4. Evaluation Map, Differential Characters and Global Anomalies

### A) *Consistency Conditions, Group Cohomology and Line Bundles Over the Orbit Space*

In order to introduce global anomalies ([35–41]), it is convenient to reformulate the consistency conditions by including also transformations which are not connected with the identity. Let $\mathscr{G}$ be a symmetry group of the theory (e.g. the group of gauge transformations or the group of diffeomorphisms). Perturbative calculations try to produce an effective procedure to calculate a fermion determinant. The latter will be denoted by $\mathscr{Z}(A)$; it is supposed to be a non-vanishing complex function, defined locally on the space of connections, $\mathscr{A}$. Then aim is to construct a globally defined $\mathscr{G}$-invariant functional on $\mathscr{A}$.

Let us represent the action of $\mathscr{G}$ on $\mathscr{Z}(A)$ as follows:

$$\mathscr{U}(\psi)\mathscr{Z}(A) \equiv \rho(A; \psi)\mathscr{Z}(A). \tag{4.1}$$

If the (complex) factor $\rho(A;\psi)$ is non-trivial, then $\mathscr{Z}(A)$ is not invariant and the effective procedure fails; we have an anomaly. The consistency condition for $\rho(A;\psi)$ is found [42] by requiring $\psi \mapsto \mathscr{U}(\psi)$ to be a representation of $\mathscr{G}$, namely by requiring the following equality: $\mathscr{U}(\psi_1)\mathscr{U}(\psi_2) = \mathscr{U}(\psi_1\psi_2)$. This implies the cocycle condition:

$$\rho(A\psi_1;\psi_2)\rho(A;\psi_1)\rho(A;\psi_1\psi_2)^{-1} = 1. \tag{4.2}$$

If moreover we suppose that $\rho(A;\psi)$ has the form

$$\rho(A;\psi) = \exp[2\pi i\alpha(A;\psi)] \tag{4.3}$$

for a real function $\alpha$, then Eq. (4.2) becomes:

$$\alpha(A\psi_1;\psi_2) + \alpha(A;\psi_1) - \alpha(A;\psi_1\psi_2) = 0 \mod \mathbf{Z}. \tag{4.4}$$

In field theory we always have to require that $\rho(A;\psi)$ has the form (4.3). The anomaly corresponding to $\rho(A;\psi)$ is trivial if there exists a function $\sigma$ defined on the space of connections $\mathscr{A}$, such that:

$$\rho(A;\psi) = \sigma(A\psi)\sigma^{-1}(A). \tag{4.5}$$

In this case, by setting $\hat{\mathscr{Z}}(A) \equiv \sigma^{-1}(A)\mathscr{Z}(A)$ we obtain a $\mathscr{G}$-invariant functional. When $\rho(A;\psi)$ is represented by (4.3), the anomaly is trivial if there exists a real function $\theta$ defined on $\mathscr{A}$, with:

$$\alpha(A;\psi) = \theta(A\psi) - \theta(A) \mod \mathbf{Z}. \tag{4.6}$$

The above formulae are the basis of our analysis of global (as well as local) anomlies. Conditions (4.4) and (4.6) are, respectively the cocycle and coboundary conditions for the group cohomology of $\mathscr{G}$ with coefficients in the reduction mod $\mathbf{Z}$ of $C^\infty(\mathscr{A})$ (see [43]).

The usual consistency conditions for local anomalies are easily obtained from Eq. (4.4) by considering $\psi_1$ and $\psi_2$ infinitesimally close to the identity; we are then involved with the Lie algebra cohomology. As pointed out before, we look for a $\mathscr{G}$-invariant fermion determinant. Hence the problem is to see whether one can re-define the functional $\mathscr{Z}(A)$ first to be infinitesimally invariant, and further to be $\mathscr{G}$-invariant (and globally defined on the whole space $\mathscr{A}$). While trying to implement this latter extension, we will come across global anomalies.

In order to understand, from a general point of view, the problem of global anomalies we will discuss briefly the connection between group cohomology and complex line bundles.

If we have a cocycle $\rho(A;\psi)$ satisfying (4.2), then we can construct a (locally trivial) line bundle $\mathscr{L}_\rho$ over $\mathscr{A}/\mathscr{G}$, defined by the following equivalence relation in $\mathscr{A} \times \mathbf{C}$:

$$(A, c) \sim (A\psi, \rho(A,\psi)^{-1}c) \quad \psi\in\mathscr{G}, A\in\mathscr{A}, c\in\mathbf{C}.$$

It is easy to verify that the above equivalence relation is well defined only if Eq. (4.2) is satisfied.

If $\rho(A;\psi)$ is trivial, namely if it satisfies Eq. (4.5), then we can construct a global non-vanishing section of $\mathscr{L}_\rho$, which associates to $[A]\in\mathscr{A}/\mathscr{G}$ the equivalence class in $\mathscr{A}\times\mathbf{C}/\sim$ given by $[(A,\sigma(A)^{-1})]$. Vice versa, if $\mathscr{L}_\rho$ is trivial, then there exists a global section $s$ which we write as $s([A])=[(A,f(A)]$, where $f$ is a function: $\mathscr{A}\to\mathbf{C}^*$. Since $(A\psi,f(A\psi))\sim(A,\rho(A,\psi)f(A\psi))$, it follows that $f(A\psi)=\rho(A,\psi)^{-1}f(A)$. Hence the group cocycle (4.2) defines a line bundle over $\mathscr{A}/\mathscr{G}$ which is trivial if and only if the cocycle itself is trivial.

At this point the problem of global anomalies can be generally stated as follows. Suppose we have a group theoretical 1-cocycle $\rho(A;\psi)$, which is defined only for $\psi$ belonging to the identity connected component $\mathscr{G}_0$ of $\mathscr{G}$. If $\rho(A;\psi)$ is trivial, then, by definition, there exists a function $\sigma:\mathscr{A}\to\mathbf{C}^*$ such that $\rho(A;\psi)=\sigma(A\psi)\sigma(A)^{-1}$, for $A\in\mathscr{A}$, $\psi\in\mathscr{G}_0$. Hence, we can extend trivially the cocycle $\rho(A;\psi)$ to the whole group $\mathscr{G}$, by setting $\bar\rho(A;\psi)=\sigma(A\psi)\sigma^{-1}(A)$ for $A\in\mathscr{A}$ and $\psi\in\mathscr{G}$. The problem of global anomalies can be now formulated as follows: are there also non-trivial extensions of the cocycle $\rho(A;\psi)$ to the whole group $\mathscr{G}$? In other words: are there non-trivial group theoretical 1-cocycles of $\mathscr{G}$, which become trivial when restricted to $\mathscr{G}_0$?

We will enter the details of global anomalies after devoting the next subsection to the illustration of some important technical tools. But before, let us remark that the approach outlined is aimed at emphasizing the connection of global anomalies with perturbative field theory and locality.

B) *Differential Characters* [44]

From the above discussion it is evident that we are looking for objects which satisfy certain equations modulo integers.

Let us now denote the reduction $\mathbf{R}\to\mathbf{R}/\mathbf{Z}$ by a tilde; namely for any real number $r$ and for any real cochain (cohomology class) $\alpha$, their reduction mod $\mathbf{Z}$ will be denoted by $\tilde r$ and $\tilde\alpha$. Hence the consistency condition (4.4) and the triviality condition (4.6) can be written as

$$\tilde\alpha(A\psi_1;\psi_2)+\tilde\alpha(A;\psi_1)-\tilde\alpha(A;\psi_1\psi_2)=0. \tag{4.4$'$}$$

$$\tilde\alpha(A;\psi)=\widetilde{\theta}(A\psi)-\widetilde{\theta}(A). \tag{4.6$'$}$$

Corresponding to the above reduction, we have the Bockstein exact sequence of cohomology groups [11, 13]:

$$\to H^l(M,\mathbf{Z})\xrightarrow{i}H^l(M,\mathbf{R})\xrightarrow{\sim}H^l(M,\mathbf{R}/\mathbf{Z})\xrightarrow{\beta}H^{l+1}(M,\mathbf{Z})\to, \tag{4.7}$$

where $i$ is the canonical map between integral and real cohomology, the tilde denotes the reduction mod $\mathbf{Z}$ and $\beta$ is the Bockstein operator.

Let us now denote by $Z_l$ the group of normalized smooth singular $l$-cycles and by $\partial$ and $\delta$ respectively the boundary and coboundary operators. An $l$-differential character $u$ is a homomorphism $u:Z_l\to\mathbf{R}/\mathbf{Z}$ subject to the following condition:

$$u\circ\partial \text{ is the reduction mod } \mathbf{Z} \text{ of an } (l+1)\text{-differential form.} \tag{4.8}$$

We denote by $\hat H^l(M,\mathbf{R}/\mathbf{Z})$ the space of $l$-differential characters. The relevance of differential characters for some physical applications, has been pointed out first by Cocqueraux [45].

One can always find a real cochain $b$ such that [44][23]

$$\tilde{b} = u. \tag{4.9}$$

Moreover if both $b_1$ and $b_2$ satisfy Eq. (4.9) for the same differential character $u$, then $\tilde{b}_1$ and $\tilde{b}_2$ are cohomologous or, equivalently,

$$b_1 = b_2 + q + \text{exact}, \tag{4.10}$$

where $q$ is an integral $l$-cochain.

From (4.8) and (4.9) we obtain [44]

$$\delta b = \omega - s, \tag{4.11}$$

where $\omega$ is an $(l+1)$-differential form and $s$ is an integral $(l+1)$-cochain. One can show that $\omega$ is closed with integral periods, that $s$ is a cocycle and moreover that

$$i[s] = [\omega],$$

where $i$ is the natural map $H^{l+1}(M,\mathbf{Z}) \to H^{l+1}(M,\mathbf{R})$.

We define now

$$\delta_1(u) \equiv \omega, \quad \delta_2(u) \equiv [s], \tag{4.12}$$

where $\omega$ and $s$ are defined as in (4.11). The definitions (4.12) are valid since they do not depend on the choice of the real cochain $b$ satisfying (4.9).

Cheeger and Simons [44] show also that, denoting by $\Omega_0^*(M)$ the space of closed differential forms with integer periods, we have the following exact sequences:

$$0 \to H^l(M,\mathbf{R}/\mathbf{Z}) \to \hat{H}^l(M,\mathbf{R}/\mathbf{Z}) \xrightarrow{\delta_1} \Omega_0^{l+1}(M) \to 0$$

$$0 \to \frac{\Omega^l(M)}{\Omega_0^l(M)} \to \hat{H}^l(M,\mathbf{R}/\mathbf{Z}) \xrightarrow{\delta_2} H^{l+1}(M,\mathbf{Z}) \to 0. \tag{4.13}$$

Moreover by defining

$$R^l(M,\mathbf{Z}) = \{(\omega,h) \in \Omega_0^l(M) \times H^l(M,\mathbf{Z}) \,|\, i(h) = [\omega]\},$$

we obtain from (4.13) the exact sequence

$$0 \to \frac{H^l(M,\mathbf{R})}{i(H^l(M,\mathbf{Z}))} \to \hat{H}^l(M,\mathbf{R}/\mathbf{Z}) \xrightarrow{(\delta_1,\delta_2)} R^{l+1}(M,\mathbf{Z}) \to 0, \tag{4.14}$$

which shows that whenever we have $H^l(M,\mathbf{R}) = 0$, then $u \in \hat{H}^l(M,\mathbf{R}/\mathbf{Z})$ is determined uniquely by its image in $R^{l+1}(M,\mathbf{Z})$. Moreover we know [44] that if $u \in H^l(M,\mathbf{R}/\mathbf{Z})$, then $\delta_2 u = -\beta u$ and that $\delta_1$ restricted to $\Omega^l(M)/\Omega_0^l(M)$ is equal to the exterior derivative.

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23 A better notation for Eq. (4.9) would be

$$\tilde{b}|_{Zl} = u,$$

but we understand, here and in the future, that any real cochain (or any differential form) reduced mod $\mathbf{Z}$, gives a differential character only when it is restricted to cycles

If $H_l(M,\mathbf{Z})$ has no torsion, then the map $i: H^{l+1}(M,\mathbf{Z}) \to H^{l+1}(M,\mathbf{R})$ is injective and for $(\omega,h) \in R^{l+1}(M,\mathbf{Z})$, the class $h$ is completely determined by $[\omega]$, hence $R^{l+1}(M,\mathbf{Z}) \approx \Omega_0^{l+1}(M)$. If, on the contrary, $H_l(M,\mathbf{Z})$ has torsion, then there is indeterminacy in choosing $h$ given $\omega$. We consider now a $G$-universal bundle $EG \to BG$ with universal connection $\xi$ (whose curvature is $F_\xi$) and an ad-invariant polynomial $Q$ with $k$-entries, such that $Q(F_\xi,\ldots,F_\xi) \in \Omega_0^{2k}(BG)$.

We know that $H^{2k-1}(BG,\mathbf{R}) = 0$; hence given any integral class $\tau \in H^{2k}(BG,\mathbf{Z})$, with $i(\tau) = [Q(F_\xi,\ldots,F_\xi)]$ there exists a unique differential character $u_Q(\tau) \in \hat{H}^{2k-1}$ $(BG,\mathbf{R}/\mathbf{Z})$ such that $\delta_1(u_Q(\tau)) = Q(F_\xi,\ldots,F_\xi)$ and $\delta_2(u_Q(\tau)) = \tau$.

If $H_{2k-1}(BG,\mathbf{Z})$ has no torsion, then $\tau$ and hence $u_Q(\tau)$ is completely determined by $Q$, but torsion would require a choice of $\tau$ (see also the next point $E$ on cobordism).

Differential characters can be pulled back via smooth maps even though, in the general case, homotopic maps do not yield the same differential character. We now associated to any bundle $P(M,G)$ with connection $A$ a bundle morphism

$$\begin{array}{ccc} P & \xrightarrow{f'} & EG \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & BG, \end{array} \tag{4.15}$$

inducing $A$ from the universal connection $\xi$ [46].

It can be proved [44] that the differential character given by $f^* u_Q(\tau)$ is the unique one which satisfies the following conditions:

1) $\delta_1(f^* u_Q(\tau)) = f^* Q(F_\xi,\ldots,F_\xi)$,
2) $\delta_2(f^* u_Q(\tau)) = f^* \tau$,
3) $f^* u_Q(\tau)$ is functorial with respect to bundle morphisms.

If we consider on $P(M,G)$ two connections $A_i, i = 1,2$ and correspondingly two classifying morphisms $(f'_i, f_i), i = 1,2$, then we have the following equalities [44]:

$$\pi^* f_i^* u_Q(\tau) = \widetilde{TQ}(A_i)|_{Z_{2k-1}(P)}, \tag{4.16}$$

$$f_1^* u_Q(\tau) - f_2^* u_Q(\tau) = \tilde{W}_Q(A_1,A_2)|_{Z_{2k-1}(M)}, \tag{4.17}$$

where on the right-hand side of the above equations we consider only the restriction to cycles. We are now equipped to construct, for any $n$-dimensional compact manifold $M$ and for any ad-invariant polynomial with $(n/2+1)$-entries, an object which satisfies the consistency condition (4.4).

As a matter of notation, we denote by the symbol $u[a]$ the evaluation of the differential character $u$ (or the singular cochain $u$) on the cycle $a$.

## C) *Consistency Conditions and Local Anomalies*

Let $P(M,G)$ be, as usual, a principal $G$-bundle over an $n$-dimensional compact manifold $M$ and let $\mathscr{A}$ be the relevant space of connections.

On the principal $G$-bundle $P \times \mathscr{A} \to M \times \mathscr{A}$, we consider the connection $\eta$[24] defined in [1], Eq. (7.11), which descends to a connection $\eta'$ on the bundle $(P \times \mathscr{A})/\mathscr{G} \to (M \times \mathscr{A})/\mathscr{G}$. Consider also a background connection $A_0$ and define, for any path $p: I \to \mathscr{A}$ with $A = p(0)$ and $A' = p(1)$, the functional

$$W_Q^{\{A,A'\}}(p) \equiv \int_{M \times p(I)} W_Q(\eta, A_0).$$

Here, and in the following, we assume that $Q$ is an ad-invariant polynomial with $(n/2+1)$-entries, such that $Q(F_\xi, \ldots, F_\xi) \in \Omega_0^{2k}(BG)$, where $\xi$ is, as before, a universal connection with curvature $F_\xi$.

Let $\mathscr{G}_0$ denote the connected component of the identity of $\mathscr{G}$, let $A \in \mathscr{A}$, $\psi \in \mathscr{G}_0$ and let $p$ be a path lying in the $\mathscr{G}_0$-orbit through $A$, with starting point $A$ and endpoint $A\psi$.

We have the following

**Theorem (4.18).** *Under the above assumptions, the reduction* mod $\mathbf{Z}$ *of the functional* $W_Q^{\{A,A\psi\}}(p)$ *is independent of the choice of the path p if p lies in a $\mathscr{G}$-orbit. Hence it can be simply denoted by* $\tilde{W}_Q^{\{A,A\psi\}}$.

*Proof.* The restriction of the connection $\eta$ to the $\mathscr{G}_0$-orbit through $A$ is simply given by ev* $A$, where ev:$P \times \mathscr{G} \to P$ is the evaluation map (see [1], Sect. 2, 3 and7). Choose a class $\tau \in H^{n+2}(BG, \mathbf{Z})$ and consider the differential character $u_Q(\tau)$. Let $(f'_1, f_1)$ and $(f'_0, f_0)$ be the classifying morphisms relevant to the two connections $A$ and $A_0$. Obviously we have, for dimensional reasons, $f_i^* u_Q(\tau) = 0$, $i = 0, 1$. Given two paths $p_1, p_2: I \to \mathscr{A}$ between $A$ and $A\psi$, we combine them to obtain a loop $l: S^1 \to \mathscr{A}$. The difference of the forms $W_Q^{\{A,A\psi\}}(p_i)$ is then given by $\int_{M \times l(S^1)} W_Q(\eta, A_0)$. Due to (4.17) and to the functoriality of the differential character $f^* u_Q(\tau)$ we have:

$$\tilde{W}_Q(\text{ev}^* A, A_0)[M \times l(S^1)] = (\underline{\text{ev}}^* f_1^* u_Q(\tau) - f_0^* u_Q(\tau))[M \times l(S^1)] = 0,$$

where $\underline{\text{ev}}: M \times \mathscr{G} \to \mathscr{G}$ is the map induced by the evaluation map ev:$P \times \mathscr{G} \to P$. Hence $\tilde{W}_Q^{\{A,A\psi\}}(l) = 0$ and so $\tilde{W}_Q^{\{A,A\psi\}} \equiv \tilde{W}_Q^{\{A,A\psi\}}(p)$ depends only on the connection $A \in \mathscr{A}$ and on the group element $\psi$. □

**Corollary (4.19).** $\tilde{W}_Q^{\{A,A\psi\}}$ *satisfies the consistency condition (4.4)′, namely satisfies the*

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24 Recall that $\eta$ is defined as follows:

$$\eta_{p,A}(X_1, X_2) \equiv A(X_1) + A(\omega(X_2))_p,$$

where $X_1 \in T_p P$, $X_2 \in T_A \mathscr{A}$ and $\omega$ is a connection for the bundle $\mathscr{A} \to \mathscr{A}/\mathscr{G}$.
If $\mathscr{G}$ is the group of diffeomorphisms (strongly fixing a point), then the space of all connections $\mathscr{A}$ is to be replaced by the space of all metric connections $\mathscr{A}^{\text{metric}}$ or by the space of all Levi–Civita connections $\mathscr{A}^{\text{LC}}$ as it has been explained in [1], Sect. 7. If $\mathscr{G}_A$ denotes the $\mathscr{G}$-orbit passing through $A \in \mathscr{A}$, then $\eta$ restricted to $P \times \mathscr{G}_A$ is simply given by ev* $A$, where ev:$P \times \mathscr{G}_A \approx P \times \mathscr{G} \to P$ is the evaluation map (see [1], Sect. 2).
In the string case we can consider the identity component of the diffeomorphism group acting on the space of all complex structures. In this case $\omega$ will be determined by the Weil–Petersson metric (see Sect. 2)

*following equation*:

$$\tilde{W}_Q^{\{A,A\psi_1\}} + \tilde{W}_Q^{\{A\psi_1,A\psi_1\psi_2\}} - \tilde{W}_Q^{\{A,A\psi_1\psi_2\}} = 0.$$

*In other words* $\tilde{W}_Q^{\{A,A\psi\}}$ *is a group theoretical 1-cocycle on* $\mathcal{G}_0$.

*Remarks.*

1) While $\tilde{W}_Q^{\{A,A\psi\}}$ does not depend on the chosen path lying in the $\mathcal{G}_0$-orbit connecting $A$ and $A\psi$, the object $W_Q^{\{A,A\psi\}}(p)$, which is not reduced mod $\mathbf{Z}$, depends, in general, on $p$.
2) If $p$ is the infinitesimal path in $\mathcal{G}$ represented by the vector field $X \in T_e\mathcal{G}$, then, with a slight abuse of notation, we can consider $W_Q^{\{A,A+L_XA\}}(X)$[25] and show that it is equal to $\int_M j_X W_Q(\mathrm{ev}^* A, A_0)$, namely equal to the integrated local anomaly. Here $j_X$ is defined as in [1], Sect. 3. In other words, by performing the "infinitesimal variation" in the $\mathcal{G}_0$-orbit of $W_Q^{\{A,A\psi\}}(p)$ we obtain the integrated local anomaly.
3) We recall that an (integrated) local anomaly is said to be of topological origin if it represents a non-trivial real 1-cohomology class of $\mathcal{G}$ and of non-topological origin if it represents $0 \in H^1(\mathcal{G}, \mathbf{R})$ (see Sects. 5, 6 of [1]). From the discussion above we can conclude that, if the anomaly corresponding to the ad-invariant polynomial $Q$ is of non-topological origin, then also $W_Q^{\{A,A\psi\}}(p)$ does depend only on the $A \in \mathcal{A}$ and $\psi \in \mathcal{G}_0$ and not on the choice of the path $p$ lying in the $\mathcal{G}_0$-orbit.
4) Summarizing the above remarks, we can say that, since the polynomial $Q$ is such that $w_{BG}(Q)$ is the real image of an integral class of $BG$ (here $w_{BG}$ denotes the Weil homomorphism for the classifying space), then the integrated local anomaly $\int_M j_X W_Q(\mathrm{ev}^* A, A_0)$ represents a real 1-cohomology class of $\mathcal{G}$, which is the image of an element of $H^1(\mathcal{G}, \mathbf{Z})$. In fact the class $\left[\int_M j_X W_Q(\mathrm{ev}^* A, A_0)\right]$ is obtained by antitransgressing (suspending) a class in $H^2(\mathcal{A}/\mathcal{G}, \mathbf{R})$, which is in turn obtained by fiber integrating the pull-back of a class in $H^{n+2}(BG, \mathbf{R})$ (see [1], Sect. 7). Since the latter class is the real image of an integral class, then also the result of the antitransgression is the real image of an integral class.

We would like now to discuss in more detail the relation between local anomalies, on one side, and the consistency (4.4) and triviality (4.6) conditions, on the other side. Here, and in the next two subsections, we will be concerned with field theory, but, in order to prepare the discussion on sigma models, we will put temporarily aside the locality (universality) requirement.

First we remark that we are looking for obstructions to defining a non-vanishing gauge invariant functional. More precisely, in perturbative field theory one tries to define in a neighborhood of $A \in \mathcal{A}$, an effective action, i.e. the logarithm of the

25 A better notation would be given by choosing paths $p_\varepsilon$, such that $p_\varepsilon(t) = A + \varepsilon t L_X A$ and by considering $\lim_{\varepsilon\to 0}(1/\varepsilon) W_Q^{\{A,A+\varepsilon L_X A\}}(p_\varepsilon)$

functional $\mathscr{Z}(A)$ considered at the beginning of the section. The obstructions to extending the functionals $\log \mathscr{Z}(A)$ and $\mathscr{Z}(A)$ to the whole $\mathscr{G}_0$-orbit, in such a way that (4.3) is satisfied, are given by the non-trivial 1-cohomology classes of $\mathscr{G}_0$. In fact, if $\mathscr{Z}(A)$ transforms under the action of $\mathscr{G}$ as in (4.1) and $\alpha(A;\psi)$ is given by $W_Q^{\{A,A\psi\}}(p)$, then we have:

$$\delta_{\mathscr{G}} \log \mathscr{Z}(A)|_{\psi=e}(X) = \frac{\delta_{\mathscr{G}} \mathscr{Z}(A)|_{\psi=e}}{\mathscr{Z}(A)}(X)$$
$$= 2\pi i \delta_{\mathscr{G}} \alpha(A;\psi)|_{\psi=e}(X) = 2\pi i \int_M j_X W_Q(\text{ev}^* A, A_0)|_{\psi=e}, \quad (4.20)$$

where $\delta_{\mathscr{G}}$ is the exterior derivative on $\mathscr{G}$ and $X \in T_e\mathscr{G}$. If the anomaly $\int_M j_{(\cdot)} W_Q$ $(\text{ev}^* A; A_0)$ is of topological origin, namely if it represents a non-trivial element of $H^1(\mathscr{G}, \mathbf{R})$, then one can never find a function $\alpha(A;\psi) \in C^\infty(\mathscr{G}_0)$ satisfying (4.20).

On the contrary, if the anomaly $\int_M j_{(\cdot)} W_Q(\text{ev}^* A; A_0)$ is of non-topological origin, then there exists a real functional $\gamma_A(\psi)$, defined on the $\mathscr{G}_0$-orbit through $A$, such that

$$W_Q^{\{A,A\psi\}}(p) = \gamma_A(\psi) - \gamma_A(e). \quad (4.21)$$

We recall now that integrated anomalies are obtained by considering cohomology classes in $H^2(\mathscr{A}/\mathscr{G}, \mathbf{R})$ and the relevant antitransgression (suspension) map:

$$H^2\left(\frac{\mathscr{A}}{\mathscr{G}}, \mathbf{R}\right) \to H^1(\mathscr{G}, \mathbf{R})$$

(see [1], Sect. 7)[26].

When $\mathscr{A}/\mathscr{G}$ is simply connected, then the antitransgression is an isomorphism. On the contrary if $\pi_1(\mathscr{A}/\mathscr{G}) \neq 0$, then the kernel of the antitransgression is not zero and includes $H^1(\mathscr{A}/\mathscr{G}, \mathbf{R}) \wedge H^1(\mathscr{A}/\mathscr{G}, \mathbf{R})$. In this latter case, when we are given an anomaly representing the trivial element of $H^1(\mathscr{G}, \mathbf{R})$, then we may ask ourselves whether it "comes from the kernel of the antitransgression" or not. Here the antitransgression itself must be considered as a correspondence from closed 2-forms on $\mathscr{A}/\mathscr{G}$ to closed 1-forms on $\mathscr{G}$, as opposed to being considered as a map between cohomology groups. We have now the following:

**Theorem (4.22).** *Let the anomaly $\int_M j_{(\cdot)} W_Q(\text{ev}^* A, A_0)$ represent the trivial element of $H^1(\mathscr{G}, \mathbf{R})$ and let $\alpha(A;\psi)$ be given by $W_Q^{\{A,A\psi\}}(p)$, for any path $p$ joining $A$ and $A\psi$. Then there exists a function $\theta$ defined on $\mathscr{A}$, satisfying the equation $\alpha(A;\psi) = \theta(A\psi) - \theta(A)$ if and only if the above anomaly is obtained by antitransgressing an exact 2-form on $\mathscr{A}/\mathscr{G}$.*

26 Here we are considering the principal bundle $\pi: \mathscr{A} \to \mathscr{A}/\mathscr{G}$, where $\mathscr{G}$ is either the group of gauge transformations leaving a point fixed or the group of diffeomorphisms strongly fixing a point. In the latter case the symbol $\mathscr{A}$ should be properly replaced by $\mathscr{A}^{\text{metric}}$, i.e. the space of all metric linear connections (for any metric on $M$)

*Proof.* The antitransgression associates to the closed 2-form $a$, representing a class in $H^2(\mathscr{A}/\mathscr{G},\mathbf{R})$, the restriction to the orbit of $b\in\Omega^1(\mathscr{A})$ which satisfies $\pi^*a=db$. In our case $b=\int_M W_Q(\eta,A_0)$.

If $[a]=0$, then $a=dc$ on $\mathscr{A}/\mathscr{G}$ and $d(\pi^*c-b)=0$ in $\mathscr{A}$. Hence $b-\pi^*c=d\theta$ for a suitable element $\theta\in C^\infty(\mathscr{A})$. Let $\theta_{\mathscr{G}}$ denote the restriction of $\theta$ to the orbit and $\delta_{\mathscr{G}}$ denote the exterior derivative in $\mathscr{G}$. The 1-form $\delta_{\mathscr{G}}\theta_{\mathscr{G}}$ is obviously equal to the restriction of $b$ to the orbit and hence $\theta$ satisfies the conditions of the theorem. Conversely if there exists a functional $\theta$ satisfying the equation $\theta(A\psi)-\theta(A)=\alpha(A,\psi)$, then the anomaly is given by $\delta_{\mathscr{G}}\theta_{\mathscr{G}}$. Let the closed 2-form $a$ on $\mathscr{A}/\mathscr{G}$ generate the anomaly by antitransgression; as before we can set $\pi^*a=db$. Consider now $\tau\equiv d\theta-b\in\Omega^1(\mathscr{A})$; we can show that it is a basic form. In fact the restriction to $\mathscr{G}$ of $\tau$ is zero and so is the restriction to $\mathscr{G}$ of its derivative $d\tau=\pi^*a$. Hence there exists $\tau'\in\Omega^1(\mathscr{A}/\mathscr{G})$ with $\pi^*\tau'=\tau$ and moreover we have $d\tau'=a$. □

The above theorem tells us that if the anomaly $\int_M j_{(\cdot)}W_Q(\mathrm{ev}^*A,A_0)$ represents $0\in H^1(\mathscr{G},\mathbf{R})$[27], then we must check whether the class in $H^2(\mathscr{A}/\mathscr{G},\mathbf{R})$ represented by the fiber integration (over $M$) of $Q(F'_\eta,\dots,F'_\eta)$, is exact or not, where $\eta'$ is the connection on $(P\times\mathscr{A})/\mathscr{G}$ considered before[28]. If the latter form is exact, then our polynomial $Q$ does not give an obstruction to finding a solution of Eq. (4.6) for the cocycle $W_Q^{\{A,A\psi\}}(p)$ defined on $\mathscr{G}_0$.

We are now able to understand better the relation between local anomalies and group cohomology with coefficients in $\mathbf{R}/\mathbf{Z}$. If a local anomaly, corresponding to an ad-invariant polynomial $Q$, represents a real cohomology class of $\mathscr{G}$ which is in the image of $H^1(\mathscr{G},\mathbf{Z})$, then we are able to construct the relevant group 1-cocycle $\tilde{W}_Q^{\{A,A\psi\}}$ on $\mathscr{G}_0$.

We now assume crucially that $\mathscr{G}$ is connected. In this case, if the given anomaly represents $0\in H^1(\mathscr{G},\mathbf{R})$, then, thanks to the above theorem and to the fact that $\pi_1(\mathscr{A}/\mathscr{G})=0$, there exists a real function $\theta$ on $\mathscr{A}$, whose reduction mod $\mathbf{Z}$ trivializes the group 1-cocycle $\tilde{W}_Q^{\{A,A\psi\}}$. In conclusion local anomalies allow us to define 1-cocycles in the group cohomology; when these anomalies are of non-topological origin, then the relevant 1-cocycles are coboundaries.

The above arguments are valid only when $\mathscr{G}$ is connected. When this is not the case, the situation is completely different. To start with, $\tilde{W}_Q^{\{A,A\psi\}}$ is a group

27 In order for this condition to be satisfied, $Q$ must be the product of two polynomials $Q_1$ and $Q_2$ and moreover we must have $w(Q_1)=w(Q_2)=0$. Here $w$ denotes the Weil-homomorphism for the bundle $P(M,G)$ (see [1], Sect. 5)

28 As an important remark, notice that the triviality of the above class in $H^2(\mathscr{A}/\mathscr{G},\mathbf{R})$ is not automatically guaranteed by the requirement that the corresponding local anomaly represents $0\in H^1(\mathscr{G},\mathbf{R})$. In addition to the conditions which guarantee that the local anomaly is of non-topological origin, some extra topological constraints have to be met: for instance, if we are in gauge theories and if $Q$ is the product of two polynomials $Q_1$ and $Q_2$ with $k_{i=1,2}$, entries $(k_1+k_2=(n/2+1))$, then sufficient conditions which imply $\left[\int_M (Q(F'_\eta,\dots,F'_\eta))\right]=0$ are $w(Q_1)=w(Q_2)=0$ (see the previous footnote) and $H^{2k_i-1}(M,\mathbf{R})=0$ for at least one of the two indices $k_i$

1-cocycle only on the component connected to the identity of $\mathscr{G}$ and not on the whole group $\mathscr{G}$. The case when $\mathscr{G}$ is not connected will be dealt with in the next subsection. Before, let us recall that in this paragraph we have provisionally put aside the "locality" (universality) problem. We have been stressing only the "topological significance" of anomalies in order to see the connection between elements of $H^1(\mathscr{G},\mathbf{R})$ and 1-cocycles of the group cohomology. But we have to remind that locality plays an essential rôle in field theory, hence we will come back to it.

### D) *Differential Characters on $\mathscr{A}/\mathscr{G}$ and Global Anomalies in Field Theory*

Now we start considering the case when $\mathscr{G}$ is not necessarily connected. In this subsection we continue putting aside the locality problem; moreover we assume that, for a given ad-invariant polynomial $Q$, we have $\left[\int_M Q(F'_\eta,\ldots,F'_\eta)\right] = 0\in H^2(\mathscr{A}/\mathscr{G},\mathbf{R})$, namely we assume that the given anomaly is obtained by antitransgressing an exact 2-form on $\mathscr{A}/\mathscr{G}$. So, if there are no obstructions represented by other anomalies, then, taking into account Theorem (4.22), one can define a $\mathscr{G}$-invariant functional given by:

$$\hat{\mathscr{Z}}(A) \equiv e^{-2\pi i\theta(A)}\mathscr{Z}(A). \tag{4.23}$$

To be more precise one can define a non-vanishing functional locally in $\mathscr{A}/\mathscr{G}$. The problem is then to extend this non-vanishing functional to the whole space $\mathscr{A}/\mathscr{G}$, or equivalently to the whole space $\mathscr{A}$ in a $\mathscr{G}$-invariant way.

We can now consider $\hat{\mathscr{Z}}(A)$ as a local non-vanishing section of a complex line bundle over $\mathscr{A}/\mathscr{G}$. If we require $\hat{\mathscr{Z}}(A)$ to be the local expression of a global non-vanishing section, then this means that we are dealing with a trivial line bundle (see subsection *A*). $C^\infty$-line bundles over $\mathscr{A}/\mathscr{G}$ are classified by their Chern class, which is an element of $H^2(\mathscr{A}/\mathscr{G},\mathbf{Z})$. We identify the class represented by

$$\mathscr{F}_0 \equiv \int_M Q(F_{\eta'},\ldots,F_{\eta'})\in\Omega^2\left(\frac{\mathscr{A}}{\mathscr{G}}\right)$$

with the real Chern class of the line bundle which admits $\hat{\mathscr{Z}}(A)$ as a local section [37], [47][29]. Hence, (if there are no other obstructions represented by other anomalies) our assumption on the vanishing of the class $[\mathscr{F}_Q]$ implies that the line bundle, which admits $\hat{\mathscr{Z}}(A)$ as a local section, has zero real Chern class. But the torsion information is still missing; namely we are not yet sure that the integral Chern class of our line bundle is zero.

This Chern class is an element of $H^2(\mathscr{A}/\mathscr{G},\mathbf{Z})$, and will be constructed as follows. We will define in a natural (functorial) way a 1-differential character $\mathscr{U}_Q\in\hat{H}^1(\mathscr{A}/\mathscr{G},\mathbf{R}/\mathbf{Z})$ such that $\delta_1\mathscr{U}_Q=\mathscr{F}_Q$. Then the integral 2-class on $\mathscr{A}/\mathscr{G}$ we are looking for, will be identified with the image of $\mathscr{U}_Q$ under the map $\delta_2$.

29 We make the simplifying assumption that there are no other anomalies to worry about

Since we have assumed that $\mathscr{F}_Q \in \Omega^2(\mathscr{A}/\mathscr{G})$ is exact, then in $\mathscr{A}/\mathscr{G}$ we have:

$$\mathscr{F}_Q = d\mathscr{K}, \tag{4.24}$$

and so $\delta_2 \mathscr{U}_Q$ coincides (up to a sign) with $\beta(\mathscr{U}_Q - \tilde{\mathscr{K}})$, where $\beta$ is the Bockstein operator $\beta: H^1(\mathscr{A}/\mathscr{G}, \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{A}/\mathscr{G}, \mathbf{Z})$. Now we pass to the definition of 1-differential characters on $\mathscr{A}/\mathscr{G}$.

Consider the diagram (bundle homomorphism) ([1], Eq. (7.12)):

$$\begin{array}{ccc} \dfrac{P \times \mathscr{A}}{\mathscr{G}} & \xrightarrow{\mathrm{Ev}} & EG \\ \Big\downarrow & & \Big\downarrow \\ \dfrac{M \times \mathscr{A}}{\mathscr{G}} & \xrightarrow{\mathrm{Ev}} & BG, \end{array} \tag{4.25}$$

where $\mathscr{G}$ is the symmetry group.

As has been discussed in [1], Sect. 7, diagram (4.25) holds, strictly speaking, only when we consider the $N$-skeletons of $(M \times \mathscr{A})/\mathscr{G}$; the map Ev itself is then determined by requiring $\mathrm{Ev}^* \xi = \eta'$, where $\xi$ is a universal connection on $EG$ and $\eta'$ is defined as before. Moreover when $\mathscr{G}$ is given by $\mathrm{Diff}^{m,1} M$ then, as usual, $\mathscr{A}$ is to be replaced by $\mathscr{A}^{\mathrm{metric}}$.

Choose the differential character $u_Q(\tau) \in \hat{H}^{n+1}(BG, \mathbf{R}/\mathbf{Z})$ and consider $\underline{\mathrm{Ev}}^* u_Q(\tau) \in \hat{H}^{n+1}((M \times \mathscr{A})/\mathscr{G}, \mathbf{R}/\mathbf{Z})$. For any loop $l$ on $\mathscr{A}/\mathscr{G}$ consider the fiber bundle defined by the diagram

$$\begin{array}{ccc} l^*\left(\dfrac{M \times \mathscr{A}}{\mathscr{G}}\right) & \xrightarrow{\bar{l}} & \dfrac{M \times \mathscr{A}}{\mathscr{G}} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ S^1 & \xrightarrow{l} & \dfrac{\mathscr{A}}{\mathscr{G}}, \end{array} \tag{4.26}$$

where $\bar{l}$ is the canonical covering of $l$. The 1-differential character $\mathscr{U}^\tau_Q \in \hat{H}^1(\mathscr{A}/\mathscr{G}, \mathbf{R}/\mathbf{Z})$ is then define for any loop $l$ as follows:

$$\mathscr{U}^\tau_Q[l(S^1)] \equiv \bar{l}^* \, \underline{\mathrm{Ev}}^* \, u_Q(\tau) \left[ l^*\left(\frac{M \times \mathscr{A}}{\mathscr{G}}\right)\right], \tag{4.27}$$

namely it is given by the evaluation of the differential character $\underline{\mathrm{Ev}}^* u_Q(\tau)$ on the restriction of the bundle $(M \times \mathscr{A})/\mathscr{G}$ to the image of $l$. The definition (4.27) is independent of the choice of the homomorphism (4.25) which induces the connection $\eta'$ on $(P \times \mathscr{A})/\mathscr{G}$ [44]. In particular, if $\mathscr{G}$ is the group of gauge transformations (fixing a point), then we can consider the map $M \times S^1 \xrightarrow{\mathrm{id} \times l} M \times \mathscr{A}/\mathscr{G}$. In this case $\mathscr{U}^\tau_Q[l(S^1)]$ is simply given by $(\mathrm{id} \times l)^* \, \underline{\mathrm{Ev}}^* \, u_Q(\tau)[M \times S^1]$.

If we assume that (4.24) is satisfied, then we call the expression

$$\beta(\mathscr{U}_Q^\tau - \tilde{\mathscr{K}}) = -\delta_2 \mathscr{U}_Q^\tau \in H^2\left(\frac{\mathscr{A}}{\mathscr{G}}, \mathbf{Z}\right) \tag{4.28}$$

a global anomaly (in field theory). Here $\beta$ is the Bockstein operator.

It is evident that, a priori, one cannot consider the left-hand side of (4.28) without assuming that the local anomaly is of non-topological origin, while the right-hand side of (4.28) makes sense in any case. In physics, one is interested in discussing global anomalies only when one has already taken care of local anomalies. Namely it is the left-hand side of (4.28) which is relavant for field theory: it represents the integral Chern class of the line bundle over the orbit space which admits $\hat{\mathscr{Z}}(A)$ as a local section, when we know that the corresponding real Chern class is zero.

Generally speaking, the global anomaly defined above will depend on the class $\tau$, but from the physical point of view we are only interested in finding conditions which guarantee the vanishing of the global anomaly for any choice of $\tau$.

Such a condition is given by requiring that[30]

$$\operatorname{Tor} H^2\left(\frac{\mathscr{A}}{\mathscr{G}}, \mathbf{Z}\right) = 0. \tag{4.29}$$

If we assume, as we always do, that both $H_1(\mathscr{A}/\mathscr{G}, \mathbf{Z})$ and $H_2(\mathscr{A}/\mathscr{G}, \mathbf{Z})$ are finitely generated, then we have $\operatorname{Tor} H^2(\mathscr{A}/\mathscr{G}, \mathbf{Z}) \approx \operatorname{Tor} H_1(\mathscr{A}/\mathscr{G}, \mathbf{Z})$. The last term is in turn equal to the torsion part of $\pi_1(\mathscr{A}/\mathscr{G})$.

We noticed above that the definition (4.27) of the 1-differential character $\mathscr{U}_Q^\tau$ depends on the choice of $\tau \in H^{n+2}(BG, \mathbf{Z})$. From the previous discussion we can argue that we are only interested in the indeterminacy contained in the expression: $\delta_2 \mathscr{U}_Q^\tau \in H^2(\mathscr{A}/\mathscr{G}, \mathbf{Z})$. If either $\operatorname{Tor} H^2(\mathscr{A}/\mathscr{G}, \mathbf{Z}) = 0$ or $\operatorname{Tor} H^{n+2}(BG, \mathbf{Z}) = 0$, then $\delta_2 \mathscr{U}_Q^\tau = \delta_2 \mathscr{U}_Q^{\tau'}$ for any choice of $\tau, \tau' \in H^{n+2}(BG, \mathbf{Z})$ and there is no indeterminacy. On the contrary if both $\operatorname{Tor} H^2(\mathscr{A}/\mathscr{G}, \mathbf{Z})$ and $\operatorname{Tor} H^{n+2}(BG, \mathbf{Z})$ are different from zero, then the expression

$$\delta_2 \mathscr{U}_Q^\tau - \delta_2 \mathscr{U}_Q^{\tau'} \in H^2\left(\frac{\mathscr{A}}{\mathscr{G}}, \mathbf{Z}\right) \tag{4.30}$$

represents a possibly non-zero integral class which is mapped into zero by the canonical map $i: H^2(\mathscr{A}/\mathscr{G}, \mathbf{Z}) \to H^2(\mathscr{A}/\mathscr{G}, \mathbf{R})$.

The expression (4.30) represents, in general, a different type of global anomaly, which may be relevant in a chiral field theory, where the local anomaly corresponding to the polynomial $Q$ has vanishing coefficient, and (4.24) does not necessarily hold.

On the contrary, if (4.24) holds, then (4.30) is given by: $\beta(\mathscr{U}_Q^\tau - \tilde{\mathscr{K}}) - \beta(\mathscr{U}_Q^{\tau'} - \tilde{\mathscr{K}})$. Hence if (4.28) is zero for any $\tau \in H^{n+2}(BG, \mathbf{Z})$ with $i(\tau) = [Q(F_\xi, \ldots, F_\xi)]$, then also (4.30) is zero.

30 We denote as usual, by the symbol Tor the torsion part of an abelian group

From the above discussion, we can conclude that there are no field-theory-global anomalies if the abelianization of $\pi_1(\mathscr{A}/\mathscr{G}) = -\pi_0(\mathscr{G})$ is torsionless.

In order to be able to compute explicitly the 1-differential characters on $\mathscr{A}/\mathscr{G}$ we need a further discussion.

First we define $\forall \psi \in \mathscr{G}$, the space $\mathrm{Map}_\psi(I, \mathscr{A})$ which is given by all $C^\infty$ maps $\varphi$ from $I = [0,1]$ to $\mathscr{A}$, satisfying the following condition:

$$\varphi(0) = A \quad \text{and} \quad \varphi(1) = \psi^* A. \tag{4.31}$$

It is obvious that:

$$\varphi \in \mathrm{Map}_\psi(I, \mathscr{A}) \Rightarrow \pi \circ \varphi \in \Omega\left(\frac{\mathscr{A}}{\mathscr{G}}\right), \tag{4.32}$$

where $\pi: \mathscr{A} \to \mathscr{A}/\mathscr{G}$ is the projection map and $\Omega(\mathscr{A}/\mathscr{G})$ is the loop space of $\mathscr{A}/\mathscr{G}$ (the dependence on the "base point" is always implicitly assumed).

Moreover we have:

**Theorem (4.33).** *Let $\varphi \in \mathrm{Map}_\psi(I, \mathscr{A})$, $\varphi' \in \mathrm{Map}_{\psi'}(I, \mathscr{A})$ and let us assume that $\varphi(0) = \varphi'(0)$. Then $\pi \circ \varphi$ and $\pi \circ \varphi'$ are homotopic (relative to the endpoints) if and only if $\psi$ and $\psi'$ belong to the same connected component of $\mathscr{G}$.*

(The proof is standard.)

It is also obvious that $\forall l \in \Omega(\mathscr{A}/\mathscr{G})$, $\exists \psi \in \mathscr{G}$ and $\varphi \in \mathrm{Map}_\psi(I, \mathscr{A})$ with $l = \pi \circ \varphi$. Namely, one can choose any connection on the bundle $\mathscr{A} \to \mathscr{A}/\mathscr{G}$ and consider the relevant horizontal lift of $l$. Now for any $\psi \in \mathscr{G}$, we denote $\Omega_\psi(\mathscr{A}/\mathscr{G})$ the subset of $\Omega(\mathscr{A}/\mathscr{G})$ given by all the loops which come from elements of $\mathrm{Map}_\psi(I, \mathscr{A})$. The above discussion tells us that $\Omega(\mathscr{A}/\mathscr{G}) = \bigcup_{\psi \in \mathscr{G}} \Omega_\psi(\mathscr{A}/\mathscr{G})$. Moreover if $\psi$ and $\psi'$ belong to the same connected component of $\mathscr{G}$, then $\Omega_\psi(\mathscr{A}/\mathscr{G}) = \Omega_{\psi'}(\mathscr{A}/\mathscr{G})$. Hence if we choose arbitrarily an element $\psi_i$ in each connected component $\mathscr{G}_i$ of $\mathscr{G}$, then the loop space $\Omega(\mathscr{A}/\mathscr{G})$ is decomposed into the disjoint union of $\Omega_{\psi_i}(\mathscr{A}/\mathscr{G})$.

Let us first comment on the gauge case. For any $\psi \in \mathscr{G}$, we consider the manifold $(P \times S^1)_\psi$ constructed from the manifold $P \times I$, by identifying $(p, 0)$ with $(\psi^{-1}(p), 1)$, $\forall p \in P$ [48].

We can then consider the diagram (bundle homomorphism) given by

$$\begin{array}{ccccc} (P \times S^1)_\psi \times \mathrm{Map}_\psi(I, \mathscr{A}) & \xrightarrow{\mathrm{ev}_\psi} & \dfrac{P \times \mathscr{A}}{\mathscr{G}} & \xrightarrow{\mathrm{Ev}} & EG \\ \downarrow & & \downarrow & & \downarrow \\ M \times S^1 \times \mathrm{Map}_\psi(I, \mathscr{A}) & \xrightarrow{\underline{\mathrm{ev}}_\psi} & M \times \dfrac{\mathscr{A}}{\mathscr{G}} & \xrightarrow{\underline{\mathrm{Ev}}} & BG, \end{array} \tag{4.34}$$

where $\underline{\mathrm{ev}}_\psi$ is obtained by combining the identity map on $M$ and the following map:

$$S^1 \times \mathrm{Map}_\psi(I, \mathscr{A}) \xrightarrow{\mathrm{id} \times \pi} S^1 \times \Omega_\psi\left(\frac{\mathscr{A}}{\mathscr{G}}\right) \xrightarrow{\mathrm{ev}} \frac{\mathscr{A}}{\mathscr{G}}.$$

Here $\pi: \mathscr{A} \to \mathscr{A}/\mathscr{G}$ is, as usual, the projection. The map $\mathrm{ev}_\psi$ is in turn defined as follows:

$$\mathrm{ev}_\psi(p, t, \varphi_l) \equiv \tilde{\pi}(p, \varphi_l(t)), \tag{4.35}$$

where $p \in P, t \in I$ and $\varphi_l \in \mathrm{Map}_\psi(I, \mathscr{A})$ with $\pi \circ \varphi_l = l$ and $\tilde{\pi}: P \times \mathscr{A} \to P \times \mathscr{A}/\mathscr{G}$ is the projection. Notice that the above definition of the map $\mathrm{ev}_\psi$ is valid, since we have:

$$\mathrm{ev}_\psi(p, 0, \varphi_l) = \mathrm{ev}_\psi(\psi^{-1}(p), 1, \varphi_l).$$

Obviously $(\mathrm{ev}_\psi, \underline{\mathrm{ev}}_\psi)$ is a $G$-bundle homomorphism. For each $\varphi_l \in \mathrm{Map}_\psi(I, \mathscr{A})$, we will use the same symbol $\varphi_l$ to denote the induced map: $(P \times S^1)_\psi \to (P \times \mathscr{A})/\mathscr{G}$.

We consider now the case when $P = LM^+$ (the bundle of oriented frames), $\mathscr{G} = \mathrm{Diff}_*^{m,1} M$ (the group of orientation preserving diffeomorphisms strongly fixing a point) and $\mathscr{A}$ is replaced by $\mathscr{A}^{\mathrm{metric}}$. In this case we can define $\forall \psi \in \mathscr{G}$, the $(n+1)$-manifold $(M \times S^1)_\psi$ constructed from the manifold $M \times I$, by identifying $(x, 0)$ with $(\psi^{-1}(x), 1)$, $\forall x \in M$. In a similar way we can define the manifold $(LM^+ \times S^1)_\psi$.

Analogously to the gauge case we can consider the diagram:

$$\begin{array}{ccccc}
(LM^+ \times S^1)_\psi \times \mathrm{Map}_\psi(I, \mathscr{A}^{\mathrm{metric}}) & \xrightarrow{\mathrm{ev}_\psi} & \dfrac{LM^+ \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}_*^{m,1} M} & \xrightarrow{\mathrm{Ev}} & EGL(n, \mathbf{R})^+ \\
\downarrow & & \downarrow & & \downarrow \\
(M \times S^1)_\psi \times \mathrm{Map}_\psi(I, \mathscr{A}^{\mathrm{metric}}) & \xrightarrow{\underline{\mathrm{ev}}_\psi} & \dfrac{M \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}_*^{m,1} M} & \xrightarrow{\underline{\mathrm{Ev}}} & BGL(n, \mathbf{R})^+,
\end{array} \tag{4.36}$$

where $\mathrm{ev}_\psi$ is defined as in (4.35) and $\underline{\mathrm{ev}}_\psi$ is defined accordingly. Again for each $\varphi_l \in \mathrm{Map}_\psi(I, \mathscr{A})$, we will use the same symbol $\varphi_l$ to denote the induced map: $(M \times S^1)_\psi \to ((M \times \mathscr{A}^{\mathrm{metric}})/\mathrm{Diff}_*^{m,1} M)$. Notice that in the gauge case we are naturally led to considering the bundle:

$$\begin{array}{c}
(P \times S^1)_\psi \\
\big\downarrow \pi \\
M \times S^1,
\end{array} \tag{4.37}$$

while in the gravitational case we are led to considering the bundle:

$$\begin{array}{c}
(LM^+ \times S^1)_\psi \\
\big\downarrow \pi \\
(M \times S^1)_\psi.
\end{array} \tag{4.38}$$

The bundle (4.38) is a reduced bundle of the bundle of linear frames of $(M \times S^1)_\psi$. Both bundles (4.37) and (4.38) are principal $G$-bundles which need not be trivial even if $P$ and $LM^+$ are. More precisely the bundles (4.37) and (4.38) are isomorphic respectively to the bundles $P \times S^1 \to M \times S^1$ and $LM^+ \times S^1 \to M \times S^1$ if and only if $\psi$ is in the connected component of the identity of $\mathscr{G}$.

Notice that when $M = S^n$ and $\psi \in \mathrm{Diff}\, S^n$ is not homotopic to the identity, then $(S^n \times S^1)_\psi$ is the connected sum of $S^n \times S^1$ with an exotic sphere [49].

After the above discussion, we can make the following remarks:

a) In the gauge case, the differential character $\mathscr{U}_Q^\tau$ is completely determined by the differential characters $(\mathrm{id} \times l)^* \underline{\mathrm{Ev}}^* u_Q(\tau) \in \hat{H}^{n+1}(M \times S^1)$ for $l \in \Omega(\mathscr{A}/\mathscr{G})$. In the gravitational case, the differential character $\mathscr{U}_Q^\tau$ is determined by the differential characters $\varphi_l^* \underline{\mathrm{Ev}}^* u_Q(\tau) \in \hat{H}^{n+1}((M \times S^1)_\psi)$ for $\varphi_l \in \mathrm{Map}_\psi(I, \mathscr{A})$. But whenever $\pi \circ \varphi_l = \pi \circ \varphi_l' = l$, we have also $\varphi_l^* \underline{\mathrm{Ev}}^* u_Q(\tau)[(M \times S^1)_\psi] = \varphi_l'^* \underline{\mathrm{Ev}}^* u_Q(\tau)[(M \times S^1)_\psi]$ (see (4.27)). Moreover if $\varphi_1 : (M \times S^1)_{\psi_1} \to (M \times \mathscr{A}^{\mathrm{metric}})/\mathrm{Diff}_*^{m,1} M$ and $\varphi_2 : (M \times S^1)_{\psi_2} \to (M \times \mathscr{A}^{\mathrm{metric}})/\mathrm{Diff}_*^{m,1}$ are such that $\mathrm{Im}(\varphi_1) = \mathrm{Im}(\varphi_2)$, then $\varphi_1^* \underline{\mathrm{Ev}}^* u_Q(\tau)\ [(M \times S^1)_{\psi_1}] = \varphi_2^* \underline{\mathrm{Ev}}^* u_Q(\tau)[(M \times S^1)_{\psi_2}]$. So, in order to determine completely the differential characters on $\mathscr{A}^{\mathrm{metric}}/\mathrm{Diff}_*^{m,1} M$, it is enough to choose one $\psi_i$ in each connected component of $\mathscr{G}$, and consider, on the manifolds $(M \times S^1)_{\psi_i}$, the differential characters $\varphi^* \underline{\mathrm{Ev}}^* u_Q(\tau)$, for $\varphi \in \mathrm{Map}_{\psi_i}(I, \mathscr{A})$.

b) Let us consider the special case when $\mathscr{G}$ is connected. In this case we can choose, for any loop $l$ on $\mathscr{A}/\mathscr{G}$, a map $\varphi_l \in \mathrm{Map}_e(I, \mathscr{A})$, where $e$ is the identity of $\mathscr{G}$. The previous analysis tells us that the differential character $\mathscr{U}_Q^\tau$ is given by

$$\mathscr{U}_Q^\tau[l(S^1)] = \tilde{W}_Q(\varphi_l^* \eta', A_0)[M \times l(S^1)]. \tag{4.39}$$

To prove (4.39), consider a classifying map for the bundle $P(M, G)$ with connection $A_0$, and the projection $p: M \times S^1 \to M$. From (4.17) we learn that:

$$\tilde{W}_Q(\varphi_l^* \eta', A_0)[M \times S^1] = (\varphi_l^* \mathrm{Ev}^* u_Q(\tau) - (f_0 \circ p)^* u_Q(\tau))[M \times S^1].$$

But $(f_0 \circ p)^* u_Q(\tau)$ is zero for dimensional reasons, and so (4.39) is proved. Equation (4.39) tells us that $(\mathscr{U}_Q^\tau - \tilde{\mathscr{K}})$ is the reduction of a real cocycle and so $\beta(\mathscr{U}_Q^\tau - \tilde{\mathscr{K}}) = 0$. Namely we have recovered the fact that, for connected $\mathscr{G}$, we do not have global anomalies.

c) Further comments on the relation between global anomalies in field theory and group cohomology are in order. For $A \in \mathscr{A}$, $\psi \in \mathscr{G}$ we consider a path $p_{A,\psi} : I \to \mathscr{A}$ joining $A$ and $A\psi$. For any $u \in H^1(\mathscr{A}/\mathscr{G}, \mathbf{R}/\mathbf{Z})$ we can define a group theoretical 1-cocycle, with coefficients in the reduction mod $\mathbf{Z}$ of $C^\infty(\mathscr{A})$ (see (4.4)) as follows:

$$\zeta(A; \psi) \equiv u[\pi \circ p_{A,\psi}(I)], \tag{4.40}$$

where $\pi$ is the projection: $\mathscr{A} \to \mathscr{A}/\mathscr{G}$ and $\pi \circ p_{A,\psi}(I)$ is the loop in $\mathscr{A}/\mathscr{G}$ corresponding to the path $p$.

It is immediate to verify that the definition (4.40) is a valid one, since it is independent of the choice of the path $p$ joining $A$ and $A\psi$, due to the contractibility of $\mathscr{A}$.

Let us assume now that $\beta u = 0$, where $\beta$ is the Bockstein operator; then there exists $w \in H^1(\mathscr{A}/\mathscr{G}, \mathbf{R})$, with $\tilde{w} = u$. So we can consider the group theoretical 1-cocycle, with coefficients in $C^\infty(\mathscr{A})$ (not reduced mod $\mathbf{Z}$), given by

$$\eta(A; \psi) \equiv w[\pi \circ p_{A,\psi}(I)], \tag{4.41}$$

where the notation is as in (4.40). Moreover we have $\tilde{\eta}(A; \psi) = \zeta(A; \psi)$. When $u$ or $w$ are zero, then the corresponding group theoretical cocycles are obviously zero.

We consider now the elements of $H^1(\mathscr{A}/\mathscr{G}, \mathbf{R}/\mathbf{Z})$ represented by $\mathscr{U}_Q^\tau - \tilde{\mathscr{K}}$ and $\mathscr{U}_Q^\tau - \mathscr{U}_Q^{\tau'}$ (see (4.28) and (4.30)). They define group theoretical 1-cocycles which are reductions mod $\mathbf{Z}$ of 1-cocycles with coefficients in $C^\infty(\mathscr{A})$ if and only if the relevant global anomalies (4.28) and (4.30) vanish. Moreover if $\beta(\mathscr{U}_Q^\tau - \tilde{\mathscr{K}}) = 0$, then, due to the Bockstein exact sequence, there exists a closed 1-form $\mathscr{K}'$ on $\mathscr{A}/\mathscr{G}$ such that $\mathscr{U}_Q^\tau - (\tilde{\mathscr{K}} + \tilde{\mathscr{K}}') = 0$. Obviously also $\mathscr{K} + \mathscr{K}'$ satisfies (4.24); hence the vanishing of the global anomaly (4.28) allows the "redefinition" of the group cocycle determined by $\mathscr{U}_Q^\tau - \tilde{\mathscr{K}}$, so as to have a trivial (in fact zero) cocycle.

d) All the above considerations were motivated by the interpretation of the regularized fermion determinant as a section of a complex line bundle. If we are instead looking for the square root of a real non-vanishing determinant (for Hermitian operators), then we are interested in considering real line bundles over the orbit space. These are classified by $H^1(\mathscr{A}/\mathscr{G}, \mathbf{Z}_2) = \mathrm{Hom}(\pi_1(\mathscr{A}/\mathscr{G}), \mathbf{Z}_2)$. In this case the differential characters, which assign a phase factor to each cycle, do not matter any more. Also there is nothing here which plays the rôle of local anomalies. The sufficient condition which guarantees the absence of global anomalies in this situation, is $\mathrm{Hom}(\pi_1(\mathscr{A}/\mathscr{G}), \mathbf{Z}_2) = 0$, which is the case when $\pi_0(\mathscr{G}) = \mathbf{Z}_p$ with $p$ odd. Witten's $SU(2)$-anomaly [35] arises in this framework. In fact, let us consider a gauge theory with $M = S^4$ and $G = SU(2)$. Then we have $\mathbf{Z}_2 = \pi_4(SU(2)) = \pi_0(\mathscr{G}) = \pi_i(\mathscr{A}/\mathscr{G}) = \mathrm{Hom}(\pi_1(\mathscr{A}/\mathscr{G}), \mathbf{Z}_2)$.

E) *Cobordism and Indeterminacy*

Let us assume that we are given a $G$-principal bundle $P(M^{n+1}, G)$ over an $(n+1)$-dimensional compact manifold $M^{n+1}$, a connection $A$ on $P(M^{n+1}, G)$ and an ad-invariant polynomial $Q$ with $(n/2+1)$-entries. Let $f$ be any map which induces the bundle $P(M^{n+1}, G)$ with connection $A$ from a $G$-universal bundle with connection $\xi$. The problem we want to discuss is the following: knowing only the connection $A$, are we able to detect the differential character $f^* u_Q(\tau)$? Here, $f$ is an unspecified map which is supposed to induce the $G$-bundle $P(M^{n+1}, G)$ with connection $A$.

From (4.16) we learn that

$$\pi^* f^* u_Q(\tau) = \widetilde{TQ}(A)|_{Z_{n+1}(P)}, \tag{4.42}$$

namely we are able to determine $f^* u_Q(\tau)$ up to $\ker \pi^*$, where $\pi^*$ is considered as a map from $H^{n+1}(M^{n+1}, \mathbf{R}/\mathbf{Z})$ to $H^{n+1}(P, \mathbf{R}/\mathbf{Z})$.

If $\ker \pi^*$ is zero, then $f^* u_Q(\tau)$ is completely determined by (4.42). If $\ker \pi^* = H^{n+1}(M^{n+1}, \mathbf{R}/\mathbf{Z}) \approx \mathbf{R}/\mathbf{Z}$, then Eq. (4.42) tells us only that $TQ(A)$ is the image of an integral class of $P$ and $f^* u_Q(\tau)$ is not completely determined by (4.42).

Finally if $\ker \pi^*$ is a proper subgroup of $H^{n+1}(M^{n+1}, \mathbf{R}/\mathbf{Z}) \approx \mathbf{R}/\mathbf{Z}$ different from zero, then it must be finite [11]. Notice that the finite subgroups of $\mathbf{R}/\mathbf{Z} \approx U(1)$ are represented by the $p$-th roots of the identity in the unit circle. In this last case Eq (4.42) leaves a rational indeterminacy.

Another way of seeing the uncertainty in the determination of the differential characters involves cobordism [44]. Assume that there exists a compact manifold $N$ such that $\partial N = M^{n+1}$ and assume moreover that there exists a $G$-bundle $\hat{P}$ over

$N$ with connection $\hat{A}$ such that $\hat{P}$ restricted to $\partial N$ gives the bundle $P$ with connection $A$. Under this hypothesis we can write

$$f^* u_Q(\tau)[M^{n+1}] = \left( \int_N Q(F_{\hat{A}}, \dots, F_{\hat{A}}) \right)^{\sim}, \tag{4.43}$$

where $(\ )^{\sim}$ denotes the reduction mod $\mathbf{Z}$. Equation (4.43) allows us to determine uniquely the value of the differential character $f^* u_Q(\tau)$ over the fundamental cycle $M^{n+1}$.

We can have also that $\partial N$ is given by $k$-copies of $M^{n+1}$ [31] and in this case again a rational indeterminacy arises. In fact we have, due to the definition of differential characters

$$k f^* u_Q(\tau)[M^{n+1}] = \left( \int_N Q(F_{\hat{A}}, \dots, F_{\hat{A}}) \right)^{\sim}. \tag{4.44}$$

Let us assume again that Eq. (4.43) holds. We know moreover that, if $N$ is connected, then $Q(F_{\hat{A}}, \dots, F_{\hat{A}})$ is exact on $N$, namely we have $Q(F_{\hat{A}}, \dots, F_{\hat{A}}) = d\hat{H}$, and due to Stokes theorem:

$$f^* u_Q(\tau)[M^{n+1}] = \left( \int_{M^{n+1}} \hat{H} \right)^{\sim}, \tag{4.45}$$

that is

$$f^* u_Q(\tau) = \tilde{\hat{H}}. \tag{4.46}$$

More generally given a principal $G$-bundle $P(M^{n+1}, G)$ with connection $A$, we can consider a situation in which:

i) we are given a Lie group $\bar{G}$ such that $G$ is a subgroup of $\bar{G}$;
ii) there exists a compact connected manifold $N$, with $\partial N = M^{n+1}$ and a $\bar{G}$-bundle $\bar{P}$ over $N$ with connection $\bar{A}$;
iii) $P(M^{n+1}, G)$ is a reduced bundle of the restriction of $\bar{P}$ to $\partial N$; moreover the connection $\bar{A}$, restricted to $\bar{P}|_{\partial N}$ is reducible to $A$;
iv) there exists a polynomial $\bar{Q}$ on Lie $\bar{G}$, which, when evaluated on the elements of Lie $G$, is equal to $Q$;

Under the above hypothesis, taking into account the functoriality of $f^* u_Q(\tau)$, we can conclude that (4.45) and (4.46) still hold, when $\hat{H}$ is replaced by a form $\bar{H}$ such that $d\bar{H} = \bar{Q}(F_{\bar{A}}, \dots, F_{\bar{A}})$.

Now we come back to global anomalies in field theory. Let us choose an element $\psi_i$ in each connected component of $\mathscr{G}$ and consider the relevant bundles (4.37) or (4.38) over $M \times S^1$ and $(M \times S^1)_{\psi_i}$ respectively. If the conditions i)–iv) are satisfied for $M^{n+1} \equiv M \times S^1$ and for each bundle (4.37) determined by the chosen $\psi_i$'s, and for each manifold $M^{n+1} \equiv (M \times S^1)_{\psi_i}$ with the relevant bundle (4.38), then the differential character $\mathscr{U}_Q^{\tau}$ (4.27) is completely determined by (4.46).

Notice that if an equation like (4.43) is satisfied or $M^{n+1} \equiv M \times S^1$ or for

31 If these $k$-copies are identified, we speak of $\mathbf{Z}/k$-manifolds [50]

$M^{n+1} \equiv (M \times S^1)_{\psi_i}$ then we can draw the conclusion that $\beta(\mathscr{U}_Q^{\tau} - \mathscr{U}_Q^{\tau'}) = 0$ for any choice of $\tau, \tau' \in H^{n+2}(BG, \mathbf{Z})$, with $i(\tau) = i(\tau') = [Q(F_\xi, \ldots, F_\xi)]$, and so there can be no global anomalies of the form (4.30).

A few final remarks:

a) In all the discussion above we have admittedly kept aside any consideration concerning universality. What we had in mind to do was simply to exploit the fact that a given anomaly was of "non-topological origin." But field theory requires locality (that is universality) and anomaly cancellation in field theory is permitted only if we do not introduce new fields. The expression (4.30) fits the universality criterion, while the expression (4.28) represents a universal object only when the given polynomial $Q$ is such that $Q(F'_\eta, \ldots, F'_\eta) = 0$ as a form and not simply as a cohomology class.

   In the next subsection we will discuss global anomalies in sigma-models where, as we know from Sect. 1, local anomaly cancellation is possible if the relevant Weil homomorphism of the target space gives zero when applied to the given ad-invariant polynomial $Q$. Our analysis of global anomalies in sigma-models, will benefit greatly from our analysis of global anomalies in field theory.

b) Our discussion has been simplified by the fact that we were dealing with one polynomial $Q$ at a time. In reality one has a combination of ad-invariant polynomials with rational coefficients given by characteristic classes related to the index of some elliptic operator, like the Dirac operator (in which case the $\hat{A}$-class is to be considered) or the Hirzebruch signature (in which case the class constructed with Hirzebruch polynomials is to be considered). But the arguments above concerning the rôle of torsion need not to be essentially modified.

c) While talking about cobordism, we meant cobordism of manifolds with principal bundles over them. In the gravitational case, the problem can be reduced to a problem of cobordism defined in terms of manifolds endowed with structures on the tangent bundle. We start by noticing that the bundle $(TM \times S^1)_\varphi$, associated to $(LM^+ \times S^1)_\varphi$, is stably equivalent to $T(M \times S^1)_\varphi$, which is associated to $L(M \times S^1)_\varphi^+$. So, as far as characteristic classes (or numbers) are concerned, we can work as well with $L(M \times S^1)_\varphi^+$. If $M$ is a compact Riemannian spin-manifold (as in our case), so is $(M \times S^1)_\varphi$. If $N$ is such that $\partial N = (M \times S^1)_\varphi$, then we know that the differential characters in which we are interested are associated to polynomials which give rise to Pontrjagin classes of $N$. In this way we are led to study oriented cobordism. Some technical details are needed here (for instance we have to ask that the metric on the collar of $\partial N$ is the product metric), but we will not discuss these problems now.

   Spin-cobordism is eventually needed since we are really working with spin-bundles and spin-connections (see [1], Sect. 7). Moreover spin-cobordism allows us to establish a connection between differential characters and the $\eta$-invariant of the Dirac operator on the $(n+2)$-dimensional manifold $N$ [51].

F) *Global Anomalies in Sigma-Models and Generalized Wess–Zumino Term*

Let $T$ be a compact target space, without boundary, endowed with a principal bundle $P$ with connection $\xi$. We consider the usual evaluation map:

$$\mathrm{ev}: M \times \mathrm{Map}(M, T) \to T, \tag{4.47}$$

choose an ad-invariant polynomial $Q$ with $(n/2+1)$-entries and assume that there exists an $(n+1)$-form $H$ on $T$, such that

$$Q(F_\xi, \dots, F_\xi) = dH. \tag{4.48}$$

All other notations will be the same as in Sect. 1. By analogy with the previous analysis of global anomalies in field theory, we could immediately argue that the presence of global anomalies is connected with the possibility that the real class $[Q(F_\xi, \dots, F_\xi)]$ is the image of a non-zero integral class. And we know that in order to avoid this, we have to require that $\mathrm{Tor}\, H^{n+2}(T, \mathbf{Z}) = 0$. Recall, by the way, that $\mathrm{Tor}\, H^{n+2}(T, \mathbf{Z}) \approx \mathrm{Tor}\, H_{n+1}(T, \mathbf{Z})$.

We rather prefer to arrive at the same conclusion starting from our generalized Wess–Zumino terms. Consider again diagram (1.15) and consider the relevant generalized Wess–Zumino term $B$ ((1.20) and Theorem (1.19)). We want the final effective action (with the inclusion of the (generalized) Wess–Zumino term) to be a functional on the true degrees of freedom of the theory (namely the space of maps $\mathrm{Map}(M, T)$). The anomaly which is supposed to be cancelled by the Wess–Zumino term is a form on the group of the (induced) guage transformations. So one of the basic requirements we have to ask, in order not to introduce extra degrees of freedom, is that the functional $\exp(2\pi i \int_M B)$, once restricted to the loop space $\Omega_{f_0}(\mathrm{Map}(M, T))$, be in fact a functional over the group of (induced) gauge transformations.

This is completely equivalent to requiring that for any loop $l \in \Omega_{f_0}(\mathrm{Map}(M, T))$ which induces the identity over $\mathrm{Aut}_v f_0^* P$, namely such that $\tau_2(l) = \text{identity}$ (see (1.12)), we have that $\left(\int_M J^* B\right)(l)$ is an integer. The isomorphism induced by $\xi$ between $f_0^* P \times \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ and $\pi_1^* \mathrm{ev}^* P$ tells us that, if $\tau_2(l) = \text{identity}$, then $l^* P$ is isomorphic to $f_0^* P \times S^1$. We denote by $\bar{l}$ the canonical covering of the map $l$ in the following diagram

$$\begin{array}{ccc} f_0^* P \times S^1 & \xrightarrow{\bar{l}} & P \\ \downarrow \pi & & \downarrow \pi \\ M \times S^1 & \xrightarrow{l} & T. \end{array} \tag{4.49}$$

Requiring that $\left(\int_M J^* B\right)(l)$ is an integer is the same as requiring that

$$\int_{M \times S^1} W_Q(\bar{l}^* \xi, A_0) - l^* H$$

is an integer. Equivalently we want to ask that, for any loop $l$ such that $\tau_2(l) = \text{identity}$,

$$\tilde{W}_Q(\bar{l}^* \xi, A_0) - l^* \tilde{H} \tag{4.50}$$

is exact as a cocycle in $H^{n+1}(M \times S^1, \mathbf{R}/\mathbf{Z})$. But, due to (4.17), the cohomology class of (4.50) is in turn the same as the cohomology class represented by

$$l^*(u_Q(\xi) - \tilde{H}), \tag{4.51}$$

where $u_Q(\xi)$ is one of the differential characters in $\hat{H}^{n+1}(T, \mathbf{R}/\mathbf{Z})$ such that $\delta_1 u_Q(\xi) = Q(F_\xi, \ldots, F_\xi)$.

A sufficient condition which guarantees that (4.51) is exact for any loop $l$ with $\tau_2(l) =$ identity is the exactness of $u_Q(\xi) - \tilde{H}$. Assume now that $\operatorname{Tor} H^{n+2}(T, \mathbf{Z}) = 0$. Then $\beta(u_Q(\xi) - \tilde{H}) = 0$ and due to the Bockstein exact sequence (4.7), $u_Q(\xi) - \tilde{H}$ must be the reduction of a real class $\lambda \in H^{n+1}(T, \mathbf{R})$ represented by a closed differential form $H_\lambda$. Since $dH_\lambda = 0$, we have also that $d(H + H_\lambda) = Q(F_\xi, \ldots, F_\xi)$, and so by redefining [37]

$$H' = H + H_\lambda, \tag{4.52}$$

we obtain that

$$\int_{M \times S^1} W_Q(\bar{l}^* \xi, A_0) - l^* H'$$

is an integer for any loop $l \in \Omega_{f_0}(\mathrm{Map}(M, T))$ with

$$\tau_2(l) = \text{identity}.$$

Hence, if we require that the torsion part of $H^{n+2}(T, \mathbf{Z})$ is zero, then the exponential of $2\pi i$ times the integral of the generalized Wess–Zumino term $B$, restricted to each gauge orbit, descends to a functional on the group of induced gauge transformations.

More generally we can prove, under the above assumption on $\operatorname{Tor} H^{n+2}(T, \mathbf{Z})$, that the functional $\exp\left(2\pi i \int_M B\right)$ descends to a functional over the space of induced bundle homomorphisms, namely on the image of the map $\tau_0$ (1.10) (see also (1.19)). In order to verify this, it is enough to notice that, if two paths $p_1, p_2 \in \mathscr{P}_{f_0}(\mathrm{Map}(M, T))$ are such that $\tau_0(p_1) = \tau_0(p_2)$, then their endpoints must be equal and so, by combining them, one obtains a loop which induces the identity in $\mathrm{Aut}_v\, h_0^* P$.

Hence, if the restriction of $\exp\left(2\pi i \int_M B\right)$ to the loop space $\Omega_{f_0}(\mathrm{Map}(M, T))$ descends to a functional on the group of induced gauge transformations, then the term $\exp\left(2\pi i \int_M B\right)$ descends to a function on the image of the map $\tau_0$. Notice that the Dirac operator (coupled to the induced connection) is parametrized by the space $\mathrm{Im}(\tau_0) \subset \mathrm{Hom}(f_0^* P, P)$ and transforms covariantly under the group $\mathrm{Im}(\tau_2)$; hence the relevant determinant will be represented by a section of a line bundle over $\mathrm{Map}(M, T)_{f_0}$.

In analogy to (4.27), we can define differential characters on $\mathrm{Map}(M, T)$. In fact, any loop $l \in \Omega_{f_0}(\mathrm{Map}(M, T))$ can be seen both as a map $l: S^1 \to \mathrm{Map}(M, T)$

and as a map $l: M \times S^1 \to T$. So a 1-differential character $\mathscr{U}_Q$ on Map$(M, T)$ can be defined as follows:

$$\mathscr{U}_Q[l(S^1)] \equiv l^* u_Q(\xi)[M \times S^1]. \tag{4.53}$$

The global anomaly in the given sigma-model is then defined as the class $\delta_2 \mathscr{U}_Q$. It is determined by

$$\beta(u_Q(\xi) - \tilde{H}) = -\delta_2 u_Q(\xi) \in H^{n+2}(T, \mathbf{Z}), \tag{4.54}$$

where $\beta$ is the Bockstein operator and $H$ satisfies (4.48). The class (4.54) depends obviously on the choice of $u_Q(\xi)$, but, differently from the global anomaly in field theory, it does not depend on the space-time manifold $M$. In other words it is a "universal object" in the sense of Sect. 1. Again, if we require Tor $H^{n+2}(T, \mathbf{Z}) = 0$, then (4.54) is zero.

Summarizing, in calculating sigma-model anomalies we consider a class $\omega$ in $H^{n+2}(T, \mathbf{Z})$ whose image $i(\omega)$ in $H^{n+2}(T, \mathbf{R})$ is given by $[Q(F_\xi, \ldots, F_\xi)]$. From our discussion of Sect. 1 we can conclude that in order to have the cancellation of the corresponding local anomaly, we have to require that $i(\omega) = 0 \in H^{n+2}(T, \mathbf{R})$. The cancellation of both the global and the local anomaly follows when $\omega$ itself is zero. Obviously if Tor $H^{n+2}(T, \mathbf{Z}) = 0$, then the local anomaly cancellation implies that there is also no global anomaly. Notice that this condition is automatically satisfied when the world-sheet of a string is imbedded in a four dimensional ambient manifold $T$.

In this connection we notice that we can pullback any class $\omega \in H^{n+2}(T, \mathbf{Z})$ via the evaluation map, yielding a class in $H^{n+2}(M \times \mathrm{Map}(M, T), \mathbf{Z})$. If we denote by $Z_2^{M,T}$ a "singular 2-cycle" in Map$(M, T)$, we can obtain, by evaluating $\mathrm{ev}^* \omega$ on cycles of the form $M \times Z_2^{M,T}$ an element of $H^2(\mathrm{Map}(M, T), \mathbf{Z})$. If we identify the latter class with the Chern class of a complex line bundle over Map$(M, T)$, then, in analogy with our previous discussion of global anomalies in field theory, we can conclude that the cancellation of all local and global anomalies in sigma models allows us to define a complex non-vanishing section of this line bundle, i.e. a complex non-vanishing functional on Map$(M, T)$, representing an invariant fermion functional integral obtained by correcting the original fermion determinant by means of the Wess–Zumino term. By requiring Tor $H^{n+2}(T, \mathbf{Z}) = 0$ and $[Q(F_\xi, \ldots, F_\xi)] = 0 \in H^{n+2}(T, \mathbf{R})$, we guarantee that the ad-invariant polynomial $Q$ does not provide any obstruction to the definition of such a functional on Map$(M, T)$, for any $n$-dimensional compact manifold $M$.

It is also worthwhile noticing that in the case of the imbedded string (see Sect. 3), the above topological constraints on the target space (i.e. on the ambient manifold) guarantee also that the polynomial $Q$ does not provide any obstruction to the trivialization of the determinant line bundle, defined on the space of the imbeddings, modulo the diffeomorphisms of the world-sheet of the string. So we can conclude that the vanishing of the first Pontrjagin class of the ambient manifold plus the requirement that its third homology group is torsionless, allows the cancellation of the diffeomorphism-sigma-model anomalies of the string and the absence of the relevant global anomalies.

G) *Sigma-Models with Target T such that* $\mathrm{Tor}\, H_{n+1}(T, \mathbf{Z}) \neq 0$

Let us assume, for the moment that the free part of $H_{n+1}(T, \mathbf{Z})$ is zero. In this case $H_{n+1}(T, \mathbf{Z}) = \mathbf{Z}_{p_1} \oplus \mathbf{Z}_{p_2} \oplus \cdots \oplus \mathbf{Z}_{p_r}$. If $p$ is the l.c.m. of the $p_i$, then we have:

$$l^*(u_Q(\xi) - \tilde{H}))[M \times S^1] \in \frac{\mathbf{Z}}{p}, \tag{4.55}$$

where as usual $Q(F_\xi, \ldots, F_\xi) = dH$ on $T$[32]. Obviously homotopic loops in $\Omega_{f_0}(\mathrm{Map}(M, T))$ yield the same number in $\mathbf{Z}/p$. In this case there is no global anomaly if $Q(F_\xi, \ldots, F_\xi)$ has periods which are multiples of $p$. Roughly speaking, this means that if the "normalized" polynomial $Q$ is further multiplied by $p$, then (4.55) is zero. Now the coefficient in front of the polynomial $Q$ depends on the matter fields we are considering (i.e. on their respective representations of the group $G$).

If $\mathrm{Free}\, H_{n+1}(T) \neq 0$, then the situation does not change substantially. We are in fact allowed to add to $H$ any closed differential form. Hence, if $Q(F_\xi, \ldots, F_\xi)$ has periods which are multiples of $p$, then we have $\beta(u_Q(\xi) - \tilde{H}) = 0$, where $\beta$ is the Bockstein operator. So there exists a closed differential form $H'$, such that $u_Q(\xi) - (\tilde{H} + \tilde{H}') = 0$.

Hence, as happens for local anomalies also global anomalies can be cancelled accordingly to the matter fields which are present in the theory (see e.g. [40, 41]).

When $\mathrm{Tor}\, H_{n+1}(T)$ is not necessarily zero, then we may be interested in looking for sufficient conditions which could guarantee the absence of global sigma-model anomalies, for a specific space-time manifold $M$. For instance, let $f_0$ be a given map: $M \to T$ and let us assume that for any loop $l \in \Omega_{f_0}(\mathrm{Map}(M, T))$ there exist:

a) a compact connected manifold $N_l$ with $\partial N_l = M \times S^1$,
b) a map $h_l: N_l \to T$ which, when restricted to $\partial N_l = M \times S^1$, is equal to $l$.

The above maps $h_l$ induce principal $G$-bundles over $N_l$ which satisfy the condition $w_{N_l}(Q) = 0$, where $w_{N_l}$ is the relevant Weil homomorphism. Hence, taking into account definition (4.53) we have

$$\mathcal{U}_Q(l(S^1)) = l^* u_Q(\xi)[\partial N_l] = u_Q(\xi)[\partial h_l(N_l)] = \left( \int_{M \times l(S^1)} \mathrm{ev}^* H \right)^{\sim};$$

where $\mathrm{ev}: M \times \mathrm{Map}(M, T) \to T$ is the usual evaluation map. So $\mathcal{U}_Q$ is the reduction of a real cochain and there is no global sigma-model anomaly. In conclusion,

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32 More specifically we can denote by $\Omega_{f_0}(\mathrm{Map}(M, T))_*$ the space

$$\{l \mid l \in \Omega_{f_0}(\mathrm{Map}(M, T)) \text{ and } l(m, t) = f_0(m) \text{ for a given } m \in M \text{ and for each } t \in S^1\}.$$

Each loop $l \in \Omega_{f_0}(\mathrm{Map}(M, T))_*$ can be seen as a map $M \wedge S^1 \to T$, where $M \wedge S^1$ is the suspension of $M$. Then there exists a group homomorphism [13]

$$\pi_0(\Omega_{f_0}(\mathrm{Map}(M, T))_*) \to H_{n+1}(T, \mathbf{Z})$$

given by $[l] \mapsto [l_* \pi_*(c)]$, where $c$ is the fundamental cycle in $M \times S^1$, $\pi: M \times S^1 \to M \wedge S^1$ is the projection and $l_*, \pi_*$ denote the induced maps in homology bordism [52] can be relevant in determining the absence of sigma-model global anomalies, for a given space time manifold $M$.

## 5. Comments

We have seen that anomalies in field theory are naturally connected with the cohomology induced by suitable evaluation maps. Anomalies can have a "topological" significance only when the evaluation map has a topological significance, i.e. only when the evaluation map allows us to compute some of the true (De Rham) cohomology of the spaces of maps we are considering.

But also if this is the case, as in gauge theories, there are true anomalies which do not correspond to any non-trivial cohomology class of our space of maps. Nevertheless we have shown that, independently of the topological significance of anomalies, the evaluation map methods developed in this and the previous paper can give us the correct framework for computing the coefficients according to the family's index theorem, even when the objects we are considering represent trivial cohomology classes. This is specially relevant for gravitational anomalies.

Another bonus we gained, by considering the evaluation map, is the possibility of showing similarities between gauge theories, sigma-models and strings. The cohomology induced by the evaluation map is the relevant object for anomaly calculations in all these cases.

Conformal anomalies in string theory can be thought of as special cases of holomorphic anomalies, but this gauge interpretation of conformal anomalies cannot be extended to the case of higher dimensional manifolds. Moreover, requiring the cancellation of the holomorphic anomalies for strings is equivalent to requiring the cancellation over the moduli (Teichmüller) space.

From the point of view of anomalies, sigma-models are very similar to gauge theories; in fact, gauge theories can be envisaged as limiting cases of sigma-models when the target space approaches the classifying space. This similarity however leaves room for important differences, which render sigma-models much more flexible from the point of view of chiral anomaly cancellation. Indeed we have shown that, under suitable geometrical constraints on the target space, we can add local counterterms to the quantum action which cancel the corresponding anomalies. These terms have been called "generalized Wess–Zumino terms." They are functionals in the path space of the maps into the target space. They are particularly important in sigma-models which represent a (super) string propagating in the background provided by the zero modes of the string itself.

The mechanism proposed in the literature in order to cancel chiral anomalies in these models is essentially the same mechanism à la Green–Schwarz as in the effective field theories, a mechanism which is based on the properties of the 2-form field $B$. We have seen that it is possible to cancel the same anomalies by means of generalized Wess–Zumino terms, which come up very naturally in the context of sigma-models and represents complicated interactions of the bosonic part of the superstring with the background geometry. They allow us to avoid postulating problematic properties for the background field $B$. Of course this means also that the effective action of the relevant sigma-model is incomplete without the addition of such Wess–Zumino terms.

The price we have to pay in order to be allowed to add such Wess–Zumino terms consists in constraints on the background geometry. This feature is far from negative and has to be interpreted as a selective criterion for the background geometry. As is suggested in the literature, conformal invariance should provide the equations of motion for the background fields. However explicit information concerning the geometry and topology of the space-time do not seem to be within reach of this method. The constraints necessary for the existence of a generalized Wess–Zumino term are a way to obtain such information. Additional information comes from the requirement that the quantum action obtained after adding the generalized Wess–Zumino term be really a functional defined on the true degrees of freedom of the theory. In this way, we come across the problem of global anomalies. We have analyzed this problem in terms of differential characters and found sufficient conditions for the absence of global anomalies in field theories and sigma-models of the string. In the latter case, they translate into additional constraints on the topology of the space-time.

We have extended the previous analysis to the diffeomorphism type sigma-model anomalies of the string, which we have described in Sect. 3.

Finally we should try to compare the results we have found with still another method of calculating chiral anomalies in string theory in terms of the background fields: the direct string loop calculations of amplitudes with external legs representing the zero modes of the string itself [53–56]. These calculations, which are limited to one-loop and to a flat background, do not reveal the presence of any anomaly. Although a close comparison with our results is far from straightforward, we can interpret this as a support to our attitude of adding generalized Wess–Zumino terms, which represent properties of the string rather than properties of the background field $B$. On the other hand, since our results are independent of the genus of the surface, they add support to the conjecture that anomalies are absent to any string loop order in the sense of references [53–56].

## Appendix I. On the Evaluation Map for Diff $M$

Let $M$ be an $n$-dimensional, compact, Riemannian oriented manifold, let $\Omega^p(M)$ be the space of $p$-forms on $M$ and let $Z_k(M)$ be the space of smooth $k$-cycles in $M$.

We consider the evaluation map:

$$\mathrm{ev}: M \times \mathrm{Diff}\, M \to M. \tag{AI.1}$$

We know already tha

$$\mathrm{ev}^*: \Omega^p(M) \to \Omega^p(M \times \mathrm{Diff}\, M) \tag{AI.2}$$

is a monomorphism in cohomology. We can establish a more stringent result by considering for all $\chi^p \in \Omega^p(M)$, the following maps:

$$\begin{aligned} \chi_l^p : Z_{p-l}(M) &\to \Omega^l(\mathrm{Diff}\, M) \\ c \quad &\mapsto \int_c \underbrace{i_{(\cdot)} \cdots i_{(\cdot)}}_{l \text{ terms}} \psi^* \chi^p . \end{aligned} \tag{AI.3}$$

Here the notation is as in (2.2)′ of [1], that is $\psi \in \mathrm{Diff}\, M$, $i_{(\cdot)}$ is the map $X \to i_X$ for $X \in \mathrm{diff}\, M$, and vectors $Y \in T_\psi\, \mathrm{Diff}\, M$ are pulled back to the origin (i.e. we consider $\psi_*^{-1} Y \in T_{\mathrm{id}}\, \mathrm{Diff}\, M \approx \mathrm{diff}\, M$).

Let us define now, $\forall \chi_l^p$, the map:

$$\begin{aligned} \delta\chi_l^p : Z_{p-l}(M) &\to \Omega^{l+1}(\mathrm{Diff}\, M), \\ c &\mapsto \delta(\chi_l^p(c)). \end{aligned} \tag{AI.4}$$

Here $\delta(\chi_l^p(c))$ means the exterior derivative, in $\mathrm{Diff}\, M$, of the $l$-form $\chi_l^p(c)$.

We can now state:

**Theorem AI.5.** *The following conditions are satisfied (for* $1 \leqq l \leqq p$*):*

(a) $\delta\chi_l^p = (-1)^l (d\chi)_{l+1}^{p+1}\ \forall l$,
(b) $\chi_l^p = \eta_l^p$ *for some* $l \Rightarrow \chi^p = \eta^p$.

*Here $d$ is the exterior derivative in $\Omega^*(M)$, and the index $p+1$ in* (a) *only reminds us that $d\chi$ is a $(p+1)$-form.*

*Proof.* A direct calculation proves (a)[33]. In order to prove (b), we have to prove first the following:

**Lemma AI.6.** *Let $\chi \in \Omega^p(M)$ and let $1 \leqq l \leqq p$, then:*

$$d(\underbrace{i_X i_Y \cdots i_Z}_{l \text{ terms}} \chi) = 0 \quad \forall X, Y, \ldots, Z \in \mathrm{Diff}\, M \Rightarrow \chi = 0.$$

*Proof* [57]. It is enough to consider $l = 1$ and to work in local coordinates $\{x_1, \ldots, x_n\}$. Let $X \equiv x_k(\partial/\partial x_i)$, and let the following equality hold:

$$0 = d i_X \chi = dx_k \wedge i_{\partial/\partial z_i}\chi + x_k d(i_{\partial/\partial x_i}\chi).$$

But we can choose at $u \in M, x_k(u) = 0$, so we have

$$0 = dx_k \wedge i_{\partial/\partial x_i}\chi \quad \forall x_k,$$

and hence

$$i_{\partial/\partial x_i}\chi = \lambda \cdot dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Since $i_{\partial/\partial x_i}\chi$ is a $(p-1)$-form, the parameter $\lambda$ must be zero and so we have:

$$d i_Z \chi = 0 \quad \forall Z \in \mathrm{diff}\, M \Rightarrow \chi = 0. \quad \square$$

Coming back to the proof of Theorem AI.5 we have:

$$\chi_l^p = \eta_l^p \quad \text{for some} \quad l \Rightarrow d\left(\underbrace{i_{(\,)} \cdots i_{(\cdot)}}_{(l+1) \text{ terms}} \psi^*(\chi^p - \eta^p)\right) = 0 \Rightarrow \chi^p - \eta^p = 0,$$

and so the proof is completed. $\square$

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33 That is (a) follows from the following equality:

$$\delta \underbrace{i_{(\cdot)} \cdots i_{(\cdot)}}_{l \text{ terms}} \psi^* \chi^p = d \underbrace{i_{(\cdot)} \cdots i_{(\cdot)}}_{(l+1) \text{ terms}} \psi^* \chi^p + (-1)^l \underbrace{i_{(\cdot)} \cdots i_{(\cdot)}}_{(l+1) \text{ terms}} d\psi^* \chi^p$$

As a corollary to Theorem AI.5 we have:

$$\delta\chi_l^p = 0 \quad \text{for some} \quad l \Rightarrow d\chi^p = 0, \tag{AI.7}$$

$$\chi_l^p = \delta\varphi_{l-1}^{p-1} \quad \text{for some} \quad l \Rightarrow \chi^p = d\varphi^{p-1}. \tag{AI.8}$$

## Appendix II. Covariant Anomalies

Let $P(M, G)$ be a principal $G$-bundle, let $\mathscr{G} = \mathrm{Aut}_v P$ and let $\mathrm{ev}: P \times \mathscr{G} \to P$ be the evaluation map.

Here the notation will be the same as in Sect. 2 and 5 of [1]. In particular we set $n = \dim M$, with $n$ even.

Consider on $P$ a universal $k$-form $\chi$ given by an ad-invariant polynomial $Q$ on $\mathrm{Lie}\, G$, whose entries are filled with $A, F, [A, A], [F, A], [A, [F, A]]$, etc.. According to Sect. 2 of [1], we have:

$$\mathrm{ev}^* \chi = \psi^* \chi + i_{(\cdot)} \psi^* \chi - i_{(\cdot)} i_{(\cdot)} \psi^* \chi + \cdots + (-1)^{(n(n+1)/2)} \underbrace{i_{(\cdot)} \cdots i_{(\cdot)}}_{k \text{ terms}} \psi^* \chi. \tag{AII.1}$$

Now $A$ is a pseudotensorial form (in the sense of [4]) while $F$ is a tensorial form. But $\forall X \in \mathrm{Lie}\, G, i_X A$ is a tensorial form while $i_X F = 0$. Hence if $\chi$ is not basic, there exists a unique $h$, with $1 \leqq h \leqq k$, such that:

$$\underbrace{i_{(\cdot)} \cdots i_{(\cdot)}}_{h \text{ terms}} \psi^* \chi \tag{AII.2}$$

is a basic non-zero $(k-h)$-form on $P$. Such a number $h$ is simply given by the "number of $A$'s" which appear in $\chi$ (i.e. in the entries of $Q$).

We are specially interested in finding all different forms (AII.2) in the case $k = n + 1$, $h = 1$. In this case $Q$ can have either $(n/2 + 1)$-entries, i.e:

$$\chi = Q(A, F, \ldots, F), \tag{AII.3}$$

or $Q$ can have $n/2$-entries, i.e.:

$$\chi = Q([F, A], F, \ldots, F). \tag{AII.4}$$

But $[F, A] = dF$, due to the Bianchi identity, and so (AII.4) is identically zero. Hence the unique solution to our problem is

$$i_{(\cdot)} \psi^* \chi = i_{(\cdot)} \psi^* Q(A, F, \ldots, F), \tag{AII.5}$$

which is called "the covariant anomaly" [58–59].

The covariant anomaly (AII.5) is a basic $n$-form on $P$, and so it is closed, e.g. $di_X \chi = 0$ $\forall X \in \mathrm{Lie}\, G$.

A direct calculation shows that, if $\chi'$ is any universal form (of the same kind as $\chi$) then the form $i_{(\cdot)} \psi^* \chi$ given by (AII.5) cannot be written as $di_{(\cdot)} \chi'$.

In order to understand how the covariant anomaly enters the calculations concerning chiral anomalies, consider the expression of the anomaly with a background connection (see Sect. 3 of [1]):

$$j_{(\cdot)} W_Q(\psi^* A, A_0). \tag{AII.6}$$

If in (AII.6) we set $A = A_0$ and $\psi$ = identity, then we obtain the covariant anomaly. It is easy to see that the covariant anomaly, despite Eq. (3.5) in [1], does not satisfy the consistency condition.

This is not a contradiction. In fact when we put in (AII.6) $A_0 = A$, then we obtain a form on $M \times \mathcal{G}$ which contains both $\psi^* A$ and $A$. That is, we obtain a form on $M \times \mathcal{G}$ which is not the pullback, via the evaluation map of a (universal) form on $P$.

## Appendix III. A Remark on the Absence of Global Gauge Anomalies for Spin(32) and $E_8$ Gauge Theories and on the Classification of $E_8$-Bundles

Let $P$ be a $G$-bundle over $M$ such that $\mathrm{Aut}_v^m P$ is weakly homotopic to $\mathrm{Map}^m(M, G)$. In order to understand better the conditions for the absence of global anomalies, we recall that for a topological space $X$ with $\pi_1(X) = \pi_2(X) = 0$ one can find a family of approximating spaces $X_n, n \geqq 3$ (Postnikov approximation, see e.g. [13]) such that:

$$\begin{array}{ccccc} & & \downarrow & & \\ K(\pi_5, 5) & \longrightarrow & X_5 & \longrightarrow & K(\pi_6, 7) \\ & & \downarrow p_4 & & \\ K(\pi_4, 4) & \xrightarrow{i_4} & X_4 & \xrightarrow{k_4} & K(\pi_5, 6) \\ & & \downarrow p_3 & & \\ K(\pi_3, 3) & = & X_3 & \longrightarrow & K(\pi_4, 5), \end{array} \tag{AIII.1}$$

where $K(\pi_n, k)$ are Eilenberg-Mac Lane spaces and $\pi_n \equiv \pi_n(X)$;

$$\begin{array}{ccc} K(\pi_n, n) & \xrightarrow{i_n} & X_n \\ & & \downarrow p_{n-1} \\ & & X_{n-1} \xrightarrow{k_{n-1}} K(\pi_n, n+1) \end{array} \tag{AIII.2}$$

is a fibration, with projection $p_{n-1}$ and fiber $K(\pi_n, n)$, where $i_n$ is the inclusion map and $k_{n-1}$ the inducing map. If $M$ is any $CW$ complex of dimension $n$, then

$$[M, X]_* = [M, X_n]_*,$$

where $[X, Y]_*$ represents the homotopy classes of maps from the pointed space $X$ to the pointed space $Y$. In this sense $X_n$ represents an approximation to $X$.

For a ten-dimensional space $M$ satisfying the above requirements, we have therefore the following set of exact sequences (obtained from the above fibrations):

$$\begin{array}{c} [M, X]_* \\ \| \\ H^{10}(M, \pi_{10}) \to [M, X_{10}]_* \to [M, X_9]_* \to H^{11}(M, \pi_{10}), \\ H^9(M, \pi_9) \to [M, X_9]_* \to [M, X_8]_* \to H^{10}(M, \pi_9), \end{array}$$

$$\begin{array}{ccccccc} \cdots & \to & \cdots & \to & \cdots & \to & \cdots, \\ \cdots & \to & \cdots & \to & \cdots & \to & \cdots, \\ H^4(M,\pi_4) & \to & [M,X_4]_* & \to & H^3(M,\pi_3) & \to & H^5(M,\pi_4), \end{array}$$

since $[M,K(\pi,n)]_* = H^n(M,\pi)$. In the case of $X = E_8$, we have $\pi_1 = \pi_2 = 0, \pi_3 = \mathbf{Z}$, $\pi_4 = \cdots = \pi_{14} = 0$. Therefore

$$[M,X]_* = [M,X_{10}]_* = H^3(M,\mathbf{Z}).$$

In the case of $X = \mathrm{Spin}(32)$, we have $\pi_1 = \pi_2 = 0, \pi_3 = \mathbf{Z}, \pi_4 = \pi_5 = \pi_6 = 0$ and $\pi_7 = \mathbf{Z}, \pi_8 = \pi_9 = \mathbf{Z}_2$ and $\pi_{10} = 0$. We can conclude that there are no global anomalies for $E_8$-gauge theories if $H^3(M,\mathbf{Z})$ is torsionless. If we are concerned with $E_8 \times E_8$-bundle over $M$, then sufficient conditions for the absence of global anomalies can be found, by noticing that

$$[M,E_8 \times E_8]_* = H^3(M,\mathbf{Z} \oplus \mathbf{Z}) = H^3(M,\mathbf{Z}) \oplus H^3(M,\mathbf{Z}).$$

In the case of Spin(32) the requirements are more stringent since sufficient conditions which imply the absence of global anomalies are met if $H^3(M,\mathbf{Z}) = H^7(M,\mathbf{Z}) = H^8(M,\mathbf{Z}) = H^8(M,\mathbf{Z}_2] = H^9(M,\mathbf{Z}_2] = H^{10}(M,\mathbf{Z}_2) = 0$.

By setting $X = BE_8$ in the above diagram and using $\pi_i(BE_8) = \pi_{i-1}(E_8)$, one easily finds that the principal bundles over $M$ ($\dim M \leqq 14$) with fiber $E_8$ are classified by $H^4(M,\mathbf{Z})$.

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