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Graded Riemann Surfaces

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Abstract. There has been a lot of activity directed at describing super Riemann surfaces and the super Teichmuller spaces that classify them. Most descriptions use a subcategory of G^∞-supermanifolds in which the coordinate charts have a particularly simple form ("de Witt" supermanifolds). This paper considers the more restrictive case of Riemann surfaces in the category of graded manifolds. The gain in doing this is the evident role of the double cover $Sl(2, C)$ of the Lorentz group in the classification of graded Riemann surfaces.

The results are as follows.

1. The group $Sl(2, C)$ plays the role of the Mobius group as the automorphism group of the algebra of "graded rational functions."
2. All graded Riemann surfaces occur as quotients of "simply connected" graded Riemann surfaces by discrete subgroups of $Sl(2, C)$.

Introduction

What super Riemann surfaces should be is a fundamental question in the theory of superstrings. Crane and Rabin [CR, R] have worked on this problem defining super Riemann surfaces using Rogers' category of G^∞-supermanifolds [Ro]. Physical motivation for the problem can be found in these papers and others, e.g. [F]. Their results apply to super Riemann surfaces that are de Witt supermanifolds. Globally each of their super Riemann surfaces is a fibre bundle over its body with a vector space fibre, that is, supermanifold charts are assumed to be open in the de Witt topology [dW, Ba] disallowing nontrivial topology in the "soul" directions. On the other hand the study of super Teichmuller space indicates that more general G^∞-supermanifolds should also be considered relevant since these appear naturally as quotients of simply-connected super Riemann surfaces under appropriate super-group actions.

The only examples that are known are the supermanifolds that are constructed from graded manifolds [Ro, Ba, Br]. Thus it seems sensible to try to work with the analogues of Riemann surfaces in the category of graded manifolds directly.

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Our definition incorporates the notion (see e.g. [CR]) that the body of a super Riemann surface should be an ordinary Riemann surface and that the odd coordinate should transform like a spinor. This is effected by requiring that the "square" of the odd coordinate behave like a form. Our definition requires a super conformal map [CR, F] to preserve this relation between bispinors and differential forms. It is this condition that ensures that the analogue of the Mobius group is $Sl(2, C)$.

The paper is organised as follows: Section 0: Some notation and conventions are introduced. Section 1: Graded Riemann surfaces are defined and some standard examples constructed. The automorphism group of graded rational functions is shown to be $Sl(2, C)$. Section 2: Quotient graded Riemann surfaces are constructed. Section 3: The classification of simply connected graded Riemann surfaces. Section 4: The construction of covering spaces for graded Riemann surfaces is given and the main theorem proved: every graded Riemann surface can be constructed as the quotient of one of the standard simply connected graded Riemann surfaces. Section 5: Comments: A representation of the charge-0 Virasoro algebra is given.

Section 0. Notations, Conventions and Pre-Requisites

All vector spaces are over the field C of complex numbers. By manifold we mean smooth, paracompact hausdorff manifold (without boundary for simplicity). By complex manifold we mean complex analytic manifold. In practice, we will always deal with Riemann surfaces.

Sheaves of functions. Let X be a complex manifold. We will use the sheaves

C_X^∞-smooth complex-valued functions on X,

H_X-analytic functions on X.

We will also need the "graded versions" of these sheaves. In particular, let $\Lambda\theta$ denote the exterior algebra (over C) generated by the single element θ. We will be using the sheaves $C_X^\infty \otimes \Lambda\theta$ and $H_X \otimes \Lambda\theta$.

Sheaves of sections. Let E be a complex analytic line bundle over the complex manifold X. The sheaf of smooth (analytic) sections of E will be denoted $\Gamma_X^\infty(\,, E)$, $(\Gamma_X^\omega(\,, E))$.

Sheaves of forms. For any commutative algebra A we can construct the algebra of forms ΩA to be the universal Z-graded differential algebra generated by A in degree 0 and dA in degree 1. The i^{th} Z-gradation of ΩA is denoted $\Omega^i A$. If $f: A \to B$ is an algebra homomorphism, f induces a map $Df: \Omega A \to \Omega B$. In particular, given a sheaf $\mathscr{A}_X$ of commutative algebras over X, one gets an associated sheaf $\Omega\mathscr{A}_X$.

For X a complex manifold the cases of particular interest are ΩH_X and $\Omega_X = \Omega C_X^\infty$. This last sheaf has a $Z \times Z$ grading with $\Omega_X^i = \bigoplus_{p+q=i} \Omega_X^{p,q}$ according to the splitting $d = \partial + \bar{\partial}$ in the usual manner. We will want to consider complex-valued graded manifolds. These differ from the usual graded manifolds [K] only in that the algebras are algebras over the complex numbers rather than the reals.

Definition 0.1. A graded manifold with complex coefficients is a pair $(X, \mathscr{A})$, where X is a nice (smooth paracompact hausdorff) manifold and $\mathscr{A}$ is a sheaf of Z_2-graded-commutative algebras over C such that

i) There is a surjective map of sheaves of graded C-algebras

$$\varepsilon: \mathscr{A} \to C_X^\infty,$$

where C_X^∞ denotes the sheaf of smooth (not analytic) complex-valued functions.

ii) There is an open cover $\{U_\alpha\}$ of X and sheaf isomorphisms

$$\tau_\alpha: \mathscr{A}_{|U_\alpha} \to C_X^\infty \otimes \Lambda(C^s)_{|U_\alpha},$$

where $\Lambda(C^s)$ is the exterior algebra on s complex generators.

As with ordinary graded manifolds, a map between graded manifolds $f:(X, \mathscr{A}) \to (X', \mathscr{A}')$ is an algebra homomorphism $f: \mathscr{A}'(X') \to \mathscr{A}(X)$.

In this paper X will always be a Riemann surface, and the odd dimension of $(X, \mathscr{A})$ (the number s) will be 1.

Ringed Spaces. These are just pairs $(X, \mathscr{A})$ where X is a space and $\mathscr{A}$ is a sheaf of rings over X. Maps of ringed spaces are defined to be pairs $(f, \phi):(X, \mathscr{A}) \to (Y, \mathscr{B})$ such that $f: X \to Y$ is continuous and ϕ is a family of ring homomorphisms,

$$\phi_U: \mathscr{B}(U) \to \mathscr{A}(f^{-1}U),$$

for U open in Y, which commute with restrictions.

Graded manifolds are ringed spaces and it has been shown (see [Br]) that a graded manifold map $f:(X, \mathscr{A}) \to (Y, \mathscr{B})$ is a map of ringed spaces and conversely.

Riemann Surfaces. A Riemann surface is a connected complex manifold X of complex dimension one. It is equivalent to demand that X be an orientable 2-real-dimensional manifold with a given conformal class of Riemannian metrices (see e.g. [N]).

Mobius Group. This group, denoted M, is the group of all fractional linear transformations of C:

$$z \to \frac{az + b}{cz + d}, \quad ad - bc = 1.$$

Since this gives a surjective map $\lambda: Sl(2; C) \to M$ with kernel $\{\pm I\}$, we see that M is isomorphic to the Lorentz group.

The Characterisation of Riemann Surfaces. The facts that we wish to generalise are the following: First, every simply connected Riemann surface is conformally equivalent to one of C (the complex plane), $C \cup \infty = S$ (the Riemann sphere) or H (the upper half plane in C). Second, every Riemann surface with non-trivial fundamental group is conformally equivalent to a quotient of C or H by the action of a subgroup of M acting freely and discontinuously. See [FK] chapter IV for details.

Section 1

Ordinary Riemann surfaces are just connected one-dimensional complex manifolds.

That is, they are manifolds X for which the sheaf of smooth complex-valued functions contains a distinguished subsheaf H_X of holomorphic functions. One logical generalisation to the graded case would be that a graded manifold $(X, \mathscr{A})$ with complex coefficients is a graded Riemann surface if X is a Riemann surface and $\mathscr{A}$ contains a distinguished subsheaf $\mathscr{B}$ of graded holomorphic functions.

This definition of graded Riemann surface would place no restrictions on the behaviour of the odd variable θ. To ensure that θ retains its spinorial nature, a more restrictive definition is required.

Definition 1.1. A graded Riemann surface is a graded manifold $(X, \mathscr{A})$ with complex coefficients such that X is an ordinary Riemann surface and $\mathscr{A}$ contains a distinguished subsheaf $\mathscr{B}$ (of graded holomorphic functions) such that

i) The trivialisations τ_α restrict to isomorphisms of sheaves of algebras

$$\tau_\alpha : \mathscr{B}_{|U_\alpha} \to H_{|U_\alpha} \otimes \Lambda\theta_\alpha .$$

ii) There is an isomorphism of sheaves

$$k : \mathscr{B}_1 \otimes \mathscr{B}_1 \to \Omega^1 H_X .$$

Remark. Because the odd dimension is one ε restricts to an isomorphism $\varepsilon : \mathscr{A}_0 \to C_X^\infty$. Condition i) implies that ε also restricts to an isomorphism $\varepsilon : \mathscr{B}_0 \to H_X$. This is because for each pair (α, β), the automorphism $\tau_\beta \cdot \tau_\alpha^{-1}$ necessarily restricts to the identity on even elements. It is clear, moreover, that we can find invertible *holomorphic* functions $h_{\alpha\beta}$ so that $\tau_\beta \cdot \tau_\alpha^{-1}(f_0 + f_1\theta_\alpha) = f_0 + h_{\alpha\beta} f_1 \theta_\beta$. [This is the data for a holomorphic line bundle]. C_X^∞ is a right H_X-module and in fact one may show that $\mathscr{A}_1 = C_X^\infty \otimes_{H_X} \mathscr{B}_1$. In particular, the graded manifold sheaf $\mathscr{A}$ is recovered from the sheaf of graded holomorphic functions $\mathscr{B}$. Finally k extends to an isomorphism $k : \mathscr{A}_1 \otimes \mathscr{A}_1 \to \Omega_X^{1,0}$ [the holomorphic 1-forms on X].

Maps of graded Riemann surfaces must preserve this structure.

Definition 1.2. A graded manifold map $f : (X, \mathscr{A}) \to (X', \mathscr{A}')$ between two graded Riemann surfaces is graded analytic if the associated map of Z_2-ringed spaces [K, Br] restricts to a map of Z_2-ringed spaces

$$(f, \phi) : (X, \mathscr{B}) \to (X', \mathscr{B}')$$

and if the following diagrams commute:

$$\begin{array}{ccc} \mathscr{B}'_1(U') \otimes \mathscr{B}'_1(U') & \to & \Omega H_{X'}(U') \\ \phi_1 \otimes \phi_1 \downarrow & & \downarrow Df \\ \mathscr{B}_1(f^{-1}(U')) \otimes \mathscr{B}_1(f^{-1}(U')) & \to & \Omega H_X(f^{-1}(U')) \end{array}$$

One can show, in that case, that $f : X \to X'$ is analytic $[(\phi^*)_0 : C^\infty(X') \to C^\infty(X)$ is just the algebra map f^* and this restricts to a map $H(X') \to H(X)]$.

Example 1.3.

i) Let D be a connected open subset of the complex plane. Then $(D, C_D^\infty \otimes \Lambda\theta)$ is a graded Riemann surface with the sheaf of graded holomorphic functions given by $\mathscr{B} = H_D \otimes \Lambda\theta$ and k defined by setting $k(\theta \otimes \theta) = dz$.

ii) Consider the Riemann sphere S as complex projective space, and let E denote the tautological line bundle over S. This is the space

$$E=\left\{\left((a,b),\frac{a}{b}\right):a,b\in C \text{ not both zero}\right\}$$

with projection $p:E\to S$ defined by $p((a,b),a/b)=a/b$. This is a complex line bundle as is the exterior bundle ΛE. Define $\mathscr{A}$ [respectively $\mathscr{B}$] to be the sheaf of smooth [respectively holomorphic] sections of ΛE. The claim is that $(S,\mathscr{A})$ is a graded Riemann surface.

$\mathscr{B}$ is certainly a graded holomorphic subsheaf of $\mathscr{A}$. We have only to define $k:\mathscr{B}_1\otimes\mathscr{B}_1\to\Omega^1 H$. This can be done by choosing explicit coordinates.

Cover S by the open sets U_0, U_∞ excluding the points 0, ∞ respectively. Choose coordinates $z_0=1/z$ on U_0 and $z_\infty=z$ on U_∞. Choose sections $\theta_0(z)=((i,i/z),z)$, $\theta_\infty(z)=((z,1),z)$ over U_0, U_∞ respectively. Notice that on $U_0\cap U_\infty$, $i/z\theta_\infty(z)=\theta_0(z)$.

For an arbitrary open set U, an element of $\mathscr{A}_1$ [respectively $\mathscr{B}_1$] is given by a pair $(f_0\theta_0,f_\infty\theta_\infty)$, where f_i is a smooth [holomorphic] function on $U\cap U_i$, such that $i/zf_0=f_\infty$ on $U\cap U_0\cap U_\infty$. An element of $\Omega^1 H(U)$ is a pair (fdz_0,gdz_∞) such that $fdz_0=gdz_\infty$ on $U\cap U_0\cap U_\infty$. Since $z_0=1/z$, $z_\infty=z$ on $U\cap U_0\cap U_\infty$ it must be that $fd(1/z)=-f(z)z^{-2}dz=g(z)dz$, thus $g(z)=-f(z)z^{-2}$.

Define $k:\mathscr{B}_1\otimes\mathscr{B}_1\to\Omega^1 H$ by setting

$$k_U((f_0\theta_0,f_\infty\theta_\infty)\otimes(g_0\theta_0,g_\infty\theta_\infty))=(f_0g_0dz_0,f_\infty g_\infty dz_\infty).$$

This is well-defined since $f_\infty g_\infty=((i/z)f_0)((i/z)g_0)=-f_0g_0z^{-2}$. That k is an isomorphism is easy to verify.

Observe that $(C,C^\infty\otimes\Lambda\theta)$ includes in $(S,\mathscr{A})$ in the obvious way.

Example 1.4: *Automorphisms of* $(S,\mathscr{A})$. We would like to state that the automorphism group of $(S,\mathscr{A}=\Gamma_S^\infty(\,,\Lambda E))$ is $Sl(2;C)$. Because automorphisms of $(S,\mathscr{A})$ are defined in terms of automorphisms of the smooth sections (rather than the meromorphic sections—see Sect. 5) this is not quite accurate. We do have the following proposition.

Proposition 1.5. *Let Δ be an open set in C.*

i) *Let Γ be a subgroup of $Sl(2,C)$ such that the image $\bar{\Gamma}$ in M acts on Δ. Then there is an inclusion*

$$K:\Gamma\to\mathrm{Aut}(\Delta,\mathscr{A}_{|\Delta})$$

determined by

$$K\begin{bmatrix} a & b\\ c & d\end{bmatrix}(\sigma_0(z)+\sigma_1(z))=\sigma_0\left(\frac{az+b}{cz+d}\right)+\frac{1}{cz+d}\sigma_1\left(\frac{az+b}{cz+d}\right)$$

for $\sigma=\sigma_0+\sigma_1$ *in* $\mathscr{A}(\Delta)=\mathscr{A}(\Delta)_0+\mathscr{A}(\Delta)_1$, $\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ *in* Γ.

ii) *Let Γ be a group of graded analytic automorphisms of $(\Delta,\mathscr{A}_{|\Delta})$ and let $\bar{\Gamma}$ be its image in the group of analytic automorphisms of Δ. Assume that $\bar{\Gamma}$ is a subgroup of M (Mobius group). Then there is an inclusion $L:\Gamma\to Sl(2;C)$ making the following* *diagram commute*

$$\begin{array}{ccc} \Gamma & \xrightarrow[L]{} & Sl(2.C) \\ \downarrow & & \downarrow \\ \bar{\Gamma} & \longrightarrow & M \end{array}$$

Proof.

i) Since $(az+b)/(cz+d) \neq \infty$ is in Δ for z in Δ, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Γ, $1/(cz+d) \neq \infty$.

Thus $1/(cz+d)$ is analytic on Δ and $K\begin{bmatrix} a & b \\ c & d \end{bmatrix}\sigma$ is in $\mathscr{A}(\Delta)$.

That $K\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an algebra homomorphism can be verified directly. It is also easy to check that the map satisfies condition (ii) for graded analytic maps using the fact that $d((az+b)/(cz+d)) = (1/(cz+d)^2)dz$.

Now observe that

$$K\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot K\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = K\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

For

$$\begin{aligned}
&K\begin{bmatrix} a & b \\ c & d \end{bmatrix} K\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}(\sigma_0(z) + \sigma_1(z)) \\
&= K\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\sigma_0\left(\frac{a'z+b'}{c'z+d'}\right) + \frac{1}{c'z+d'}\sigma_1\left(\frac{a'z+b'}{c'z+d'}\right)\right) \\
&= \sigma_0\left(\frac{a'\left(\dfrac{az+b}{cz+d}\right)+b'}{c'\left(\dfrac{az+b}{cz+d}\right)+d'}\right) + \left(\frac{1}{c'\left(\dfrac{az+b}{cz+d}\right)+d'}\right)\left(\frac{1}{cz+d}\right)\sigma_1\left(\frac{a'\left(\dfrac{az+b}{cz+d}\right)+b'}{c'\left(\dfrac{az+b}{cz+d}\right)+d'}\right) \\
&= \sigma_0\left(\frac{(a'a+b'c)z+(a'b+b'd)}{(c'a+d'c)z+(c'b+d'd)}\right) + \frac{1}{(c'a+d'c)z+(c'b+d'd)} \\
&\quad \cdot \sigma_1\left(\frac{(aa'+b'c)z+(a'b+b'd)}{(c'a+d'c)z+(c'b+dd')}\right) \\
&= K\left(\begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)(\sigma_0(z) + \sigma_1(z)).
\end{aligned}$$

This guarantees that K is an (anti) homomorphism of $Sl(2;C)$ into $\mathrm{Aut}(\mathscr{A}(\Delta))$ and hence a homomorphism into $\mathrm{Aut}(\Delta, \mathscr{A})$.

To see that K is injective suppose $K\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\sigma) = \sigma$ for all σ in $\mathscr{A}(\Delta)$. Then

$K\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\sigma_0) = \sigma_0$ implies that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm I$. Since $K(-I)(\sigma_1) = -\sigma_1$ the kernel of K is the identity.

ii) Assume Γ is a group of automorphisms with the given property and let γ be in Γ. By the assumption $\bar{\Gamma} \subset M$, $(\gamma\sigma_0)(z) = \sigma_0((az+b)/(cz+d))$, and we have, for f in $H(\Delta)$,

$$D\gamma(f(z)dz) = f\left(\frac{az+b}{cz+d}\right)d\left(\frac{az+b}{cz+d}\right) = f\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^2}dz.$$

Write $\gamma(\theta_\infty) = f_\infty \theta_\infty$. Then for $f\theta_\infty \otimes \theta_\infty \in \mathscr{B}(\Delta)_1 \otimes \mathscr{B}(\Delta)_1$,

$$D\gamma k(f\theta_\infty \otimes \theta_\infty) = D\gamma(f(z)dz) = f\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^2}dz,$$

$$k(\gamma(f\theta_\infty \otimes \theta_\infty)) = f\left(\frac{az+b}{cz+d}\right)f_\infty^2 dz.$$

By condition ii) for graded analytic maps $f_\infty = \pm 1/(cz+d)$. Define $L(\gamma) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $f_\infty = 1/(cz+d)$. It is not difficult to verify that $\gamma = K.L(\gamma)$.

Section 2. Quotient Graded Riemann Surfaces

Our last examples of graded Riemann surfaces are quotients of the Riemann surfaces described in the last section by the action of subgroups of $Sl(2; C)$.

More explicitly, let $\bar{\Gamma}$ be a (discrete) subgroup of the Mobius group M, and suppose Δ is a connected, and simply connected region of C on which $\bar{\Gamma}$ acts discontinuously without fixed points. Suppose also that Γ is a discrete subgroup of $Sl(2, C)$ mapping isomorphically onto $\bar{\Gamma}$. Let X denote the Riemann surface $\Delta/\bar{\Gamma}$ with covering map $\pi: \Delta \to X$. The intention is to build a sheaf $\mathscr{A}$ over X making $(X, \mathscr{A})$ a graded Riemann surface. The sheaf $\mathscr{A}$ will be constructed as invariant subalgebras under the action of Γ.

Lemma 2.1. *For any open set U in X there is an action of Γ on the algebras $C^\infty(\pi^{-1}(U)) \otimes \Lambda\theta[H(\pi^{-1}(U)) \otimes \Lambda\theta)]$*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} f(u) = f\left(\frac{au+b}{cu+d}\right),$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \theta(u) = \frac{\theta}{cu+d}.$$

Proof. This is essentially a corollary of Proposition 1.5.

Now we can define our sheaves $\mathscr{A}, \mathscr{B}$ to be the subalgebras of invariant elements of $C^\infty \otimes \Lambda\theta$, $H \otimes \Lambda\theta$ respectively. Explicitly define

$$\mathscr{A}(U) = \left\{\phi \text{ in } C^\infty(\pi^{-1}(U)) \otimes \Lambda\theta : \begin{bmatrix} a & b \\ c & d \end{bmatrix}\phi = \phi \text{ for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } \Gamma\right\},$$

$$\mathscr{B}(U)=\left\{\phi \text{ in } H(\pi^{-1}(U))\otimes\Lambda\theta:\begin{bmatrix} a & b \\ c & d\end{bmatrix}\phi=\phi \text{ for } \begin{bmatrix} a & b \\ c & d\end{bmatrix} \text{ in } \Gamma\right\}.$$

Proposition 2.2.
1) *$(X,\mathscr{A})$ is a graded Riemann surface.*
2) *The inclusion $\mathscr{A}(X)\to C^\infty(\Delta)\otimes\Lambda\theta$ defines a map of graded Riemann surfaces*

$$(\Delta, C^\infty_\Delta\otimes\Lambda\theta)\to(X,\mathscr{A}).$$

Proof.
1) To check that $(X,\mathscr{A})$ is a graded manifold it is necessary to define $\varepsilon:\mathscr{A}\to C^\infty_X$ and to find an open cover over which $\mathscr{A}$ is trivial.

For the first problem, observe that

$$C^\infty_X(U)=\left\{f\in C^\infty_\Delta(\pi^{-1}(U)):\begin{bmatrix} a & b \\ c & d\end{bmatrix}f=f \text{ for all} \begin{bmatrix} a & b \\ c & d\end{bmatrix} \text{ in } \Gamma\right\}.$$

Thus the map $C^\infty_\Delta(\pi^{-1}(U))\otimes\Lambda\theta\to C^\infty_\Delta(\pi^{-1}(U))$ restricts to a suitable augmentation ε on $\mathscr{A}$.
2) For the second choose an open cover U_α of X such that $\pi^{-1}(U_\alpha)=U_{\alpha0}\times\Gamma$, where $U_{\alpha0}$ is an open subset of Δ mapping diffeomorphically onto U_α via π. Thus

$$C^\infty_\Delta(\pi^{-1}(U_\alpha))\otimes\Lambda\theta=\prod_\Gamma C^\infty_\Delta(\bar{\gamma}U_{\alpha0})\otimes\Lambda\theta.$$

An element (ϕ_γ) in $\prod_\Gamma C^\infty_\Delta(\bar{\gamma}U_{\alpha0})\otimes\Lambda\theta$ is in $\mathscr{A}(U_\alpha)$ if $\gamma'\phi_\gamma=\phi_{\gamma\gamma'}$ for γ,γ' in Γ. From this it can be shown that the inclusion $\mathscr{A}(U_\alpha)\to C^\infty_X(U_\alpha)\otimes\Lambda\theta$ is an isomorphism.

Clearly $\mathscr{B}$ is a suitable distinguished subsheaf. We define

$$k_U:\mathscr{B}(U)_1\otimes\mathscr{B}(U)_1\to\Omega^1H_X(U)$$

to be the restriction of the map of sheaves over Δ,

$$\tilde{k}_V:(H_\Delta(V)\otimes\theta)\otimes(H_\Delta(V)\otimes\theta)\to\Omega^1H_\Delta(V),$$

determined by $\tilde{k}(\theta\otimes\theta)=dz$. It remains to check that $\tilde{k}$ sends invariant elements in $(H_\Delta(\pi^{-1}U)\otimes\theta)\otimes(H_\Delta(\pi^{-1}U)\otimes\theta)$ to invariant elements in $\Omega^1H_\Delta(\pi^{-1}U)$. The result follows from the observation that $\Omega^1H_X(U)$ is the set of invariant elements in $\Omega^1H_\Delta(\pi^{-1}U)$.

Check this directly. Elements ρ, τ are in $\mathscr{B}(U)$ if and only if, over $\{U_\alpha\}$, $\rho=(\rho_\alpha)$, $\tau=(\tau_\alpha)$ and $\gamma'\rho_\gamma=\rho_{\gamma\gamma'}$, $\gamma'\tau_\gamma=\tau_{\gamma\gamma'}$ as above. Explicitly writing $\rho_\gamma=f_\gamma\theta$, $\tau_\gamma=g_\gamma\theta$,

$(f_\gamma,g_\gamma\in H)$, $\gamma'=\begin{bmatrix} a' & b' \\ c' & d'\end{bmatrix}$,

$$\gamma'\rho_\gamma=\gamma'(f_\gamma(z)\theta)=f_\gamma\left(\frac{a'z+b'}{c'z+d'}\right)\frac{\theta}{c'z+d'},$$

$$\gamma'\tau_\gamma=\gamma'(g_\gamma(z)\theta)=g_\gamma\left(\frac{a'z+b'}{c'z+d'}\right)\frac{\theta}{c'z+d'}.$$

Thus $f_{\gamma\gamma'}(z) = f_\gamma((a'z+b')/(c'z+d'))1/(c'z+d')$ and $g_{\gamma\gamma'}(z) = g_\gamma((a'z+b')/(c'z+d'))1/(c'z+d')$; also $\tilde{k}(\rho \otimes \tau) = (f_\gamma g_\gamma dz)$.

An element $(h_\gamma dz)$ in $\Omega^1 H_\Delta(\pi^{-1} U \cap U_\alpha)$ is invariant if and only if

$$\gamma'(h_\gamma dz) = h_\gamma\left(\frac{a'z+b'}{c'z+d'}\right) d\left(\frac{a'z+b'}{c'z+d'}\right) = h_\gamma\left(\frac{a'z+b'}{c'z+d'}\right)\frac{1}{(c'z+d')}dz;$$

that is $h_{\gamma\gamma'}(z) = h_\gamma((a'z+b')/(c'z+d'))1/(c'z+d')^2$.

Now observe that $(f_\gamma g_\gamma dz)$ has precisely this property; thus $\tilde{k}$ restricts to a sheaf map k as desired.

Section 3. The Classification of Simply Connected Graded Riemann Surfaces

Say a graded Riemann surface $(X, \mathscr{A})$ is simply connected if X is simply connected. We will show that a simply connected graded Riemann surface is graded analytically equivalent to $(C, C^\infty \otimes \Lambda\theta)$, $(H, C^\infty \otimes \Lambda\theta)$, where H is the upper half plane, or $(S, \Gamma(\,,\Lambda E))$-Example 1.3.

We will need the following lemma.

Lemma 3.1. *Let $(X, \mathscr{A})$ be a graded Riemann surface with $\mathscr{B}$ the designated sheaf of graded holomorphic functions. Suppose X is simply connected, and $\{U_\alpha\}$ is an open cover of X. Suppose B_α is a subset of $\mathscr{B}(U_\alpha)_1$ such that given b_α in B_α and u in $U_\alpha \cap U_\beta$ there is b_β in B_β with $b_\alpha = b_\beta$ in a neighbourhood of u, then there is b in $\mathscr{B}(X)_1$ with $b = b_\alpha$ on restriction to each U_α.*

Proof. The critical fact about $\mathscr{B}(X)_1$ is that it is the set of holomorphic sections of a holomorphic line bundle. The proof is identical to the proof of the special case of holomorphic functions proved in [Be]. The idea is to look at the covering space by germs $\mathscr{H}$ of functions in $\{B_\alpha\}$. The condition on the B_α ensures that the covering is regular. Any component $\mathscr{H}_0$ of $\mathscr{H}$ is then diffeomorphic to X (since X is simply-connected) and provides a suitable section b.

If $(X, \mathscr{A})$ is a simply connected graded Riemann surface the uniformisation theorem provides an analytic diffeomorphism,

$$\tau : \Delta \to X.$$

From τ one may easily construct a graded Riemann surface over Δ which is graded analytically equivalent to $(X, \mathscr{A})$. It follows that we may take $X = \Delta$ without loss of generality.

We next use the lemma to extend the uniformisation theorem to a characterisation of simply connected graded Riemann surfaces.

Proposition 3.2. *Let Δ be C, H or S and let $(\Delta, \mathscr{A})$ be a graded Riemann surface. Then*
1. *$\Delta = C$ or H and $(\Delta, \mathscr{A})$ is graded analytically equivalent to $(\Delta, C^\infty \otimes \Lambda\theta)$.*
2. *$\Delta = S$ and $(\Delta, \mathscr{A})$ is graded analytically equivalent to $(\Delta, \Gamma(\,,\Lambda E))$ of Example 1.3 ii).*

Proof.
1. Let $\{U_\alpha\}$ be an open cover of Δ by discs over which $\mathscr{B}$ is trivial. In this case global

triviality of $\mathscr{B}$ will imply global triviality of $\mathscr{A}$ by the remark after Definition 1.1. For any disc U define B_U by

$$B_U = \{b \in \mathscr{B}(U)_1 : k(b \otimes b) = dz\}$$

and write B_α for B_{U_α}. We check that the sets B_α satisfy the hypotheses of the lemma.

Using the trivialisation

$$\tau_\alpha : \mathscr{B}(U_\alpha) \to H(U_\alpha) \otimes \Lambda\theta_\alpha$$

and the fact that k is an isomorphism, we can write $k^{-1}(dz) = g_\alpha \theta_\alpha \otimes \theta_\alpha$, where g_α is an invertible element of $H(U_\alpha)$. g_α has square roots and it is apparent that for any disc $U \subset U_\alpha$, $B_U = \{\pm \sqrt{g_\alpha}\theta_\alpha\}$. This guarantees that the conditions are satisfied.

Choose b in $\mathscr{B}(\Delta)_1$ such that $k(b \otimes b) = dz$. Then define a graded manifold map $\Psi : (\Delta, \mathscr{A}) \to (\Delta, C^\infty \otimes \Lambda\theta)$ by setting

$$\Psi(f_0 + f_1\theta) = f_0 + f_1 b.$$

From $\mathscr{A}_1 = C^\infty \otimes \mathscr{B}_1$ it is not difficult to check that this gives a graded Riemann equivalence.

2. Use the technique of part 1 to choose b_∞, b_0 in $\mathscr{B}(U_\infty)_1$, $\mathscr{B}(U_0)_1$ such that $k(b_\infty \otimes b_\infty) = dz_\infty$, $k(b_0 \otimes b_0) = dz_0$. As before these provide trivialisations,

$$\mathscr{B}_{|U_\infty} \to H_{|U_\infty} \otimes \Lambda b_\infty, \quad \mathscr{B}_{|U_0} \to H_{|U_0} \otimes \Lambda b_0.$$

On $U_0 \cap U_\infty$ write $b_0 = h(z) b_\infty$ and compute $dz_0 = k(b_0 \otimes b_0) = k(hb_\infty \otimes h_\infty) = h^2 dz_\infty$. Thus $h^2 = -1/z$ and $h(z) = \pm i/z$. By replacing b_0 by $-b_0$ if necessary we may assume that $h(z) = i/z$.

This allows us to write any element in $\mathscr{A}(U)_1$ (U open in Δ) as a pair (fb_0, gb_∞) with $f \in C^\infty(U \cap U_0)$ and $g \in C^\infty(U \cap U_\infty)$ so that $f.i/z = g$ on $U \cap U_0 \cap U_\infty$. It is not hard to check that the map

$$\Psi : \mathscr{A}(S) \to \Gamma(S, \Lambda E)$$
$$\Psi(fb_0, gb_\infty) \to (f\theta_0, g\theta_\infty)$$

defines an equivalence of graded Riemann surfaces.

Section 4. Covering Surfaces for Graded Riemann Surfaces

In Sect. 2 we defined the quotient of a simply connected graded Riemann surface by a suitably acting subgroup of $Sl(2; C)$ not containing $-I$. This section reverses that process: given a graded Riemann surface $(X, \mathscr{A})$ we construct a simply connected graded Riemann surface $(\tilde{X}, \tilde{\mathscr{A}})$ with a graded Riemann map $(\pi, \tilde{\pi}) : (\tilde{X}, \tilde{\mathscr{A}}) \to (X, \mathscr{A})$.

Construction. Let $(X, \mathscr{A})$ be a graded Riemann surface and let $\tilde{X}$ be the universal covering space of X with covering map $\pi : \tilde{X} \to X$. Take $\Gamma = \pi_1(X)$, recalling that Γ acts on $\tilde{X}$ by analytic diffeomorphisms so that $X = \tilde{X}/\Gamma$.

Fix a trivialising cover $\{U_\alpha\}$ of X for which $\pi^{-1}(U_\alpha) = \Gamma \times U_{\alpha 0}$ with $\pi : U_{\alpha 0} \to U_\alpha$ a diffeomorphism. Then the sets $\{\gamma U_{\alpha 0}\}$ with $\gamma \in \Gamma$ form an open cover for $\tilde{X}$. Define

$$\tilde{\mathscr{A}}(\gamma U_{\alpha 0}) = \mathscr{A}(U_\alpha), \quad \tilde{\mathscr{B}}(\gamma U_{\alpha 0}) = \mathscr{B}(U_\alpha),$$

where $\mathscr{B}$ is the sheaf of graded holomorphic functions on X.

This defines sheaves $\tilde{\mathscr{A}}, \tilde{\mathscr{B}}$ on $\tilde{X}$. Explicitly elements of $\tilde{\mathscr{A}}(U)$ [respectively $\tilde{\mathscr{B}}(U)$] are elements $(a_{\gamma,\alpha})$ with $a_{\gamma,\alpha}$ in $\mathscr{A}(U_\alpha \cap \pi U)$ [respectively $\mathscr{B}(U_\alpha \cap \pi U)$] such that $a_{\gamma,\alpha} = a_{\gamma',\alpha'}$ on $\pi(U \cap \gamma U_{\alpha_0} \cap \gamma' U_{\alpha_0})$. It is not hard to check that $(\tilde{X}, \tilde{\mathscr{A}})$ is a graded manifold (one can write elements of the sheaf $C^\infty_{\tilde{X}}$ in just the same way). Notice that for any open set $U \subset \gamma U_{\alpha_0}$, π determines an isomorphism

$$\pi^*: H_X(\pi U) \to H_{\tilde{X}}(U).$$

Then there is an induced isomorphism

$$\tilde{\mathscr{B}}(U) = \mathscr{B}(\pi U) \to H_X(\pi U) \otimes \Lambda\theta \to H_{\tilde{X}}(U) \otimes \Lambda\theta$$

as required for local triviality.

To define k, first observe that the map π^* induces isomorphisms

$$\Omega\pi^*: \Omega H_X(\pi U) \to \Omega H_{\tilde{X}}(U).$$

Then define $\tilde{k}_{\gamma,\alpha}$ over $U \subset \gamma U_{\alpha_0}$ to be the composition

$$\tilde{\mathscr{B}}(U)_1 \otimes \tilde{\mathscr{B}}(U)_1 \to \mathscr{B}(\pi U)_1 \otimes \mathscr{B}(\pi U)_1 \xrightarrow[k]{} \Omega^1 H_X(\pi U) \xrightarrow[D\pi^*]{} \Omega^1 H_{\tilde{X}}(U).$$

Since the maps $\tilde{k}_{\gamma,\alpha}$, $\tilde{k}_{\gamma',\alpha'}$ agree over $U \subset \gamma U_{\alpha_0} \cap \gamma' U_{\alpha_0'}$, they define a sheaf map $\tilde{k}: \tilde{\mathscr{B}}_1 \otimes \tilde{\mathscr{B}}_1 \to \Omega^1 H_{\tilde{X}}$. Since the maps locally are isomorphisms, $\tilde{k}$ is an isomorphism. Thus $(\tilde{X}, \tilde{\mathscr{A}})$ is a simply connected graded Riemann surface.

Notice that for any open set V in X, Γ acts on $\tilde{\mathscr{A}}(\pi^{-1}V)$ by the formula

$$\gamma'(a_{\gamma,\alpha}) = (b_{\gamma'',\alpha''}), \quad \text{where} \quad b_{\gamma'\gamma,\alpha} = a_{\gamma,\alpha}, \quad \gamma \quad \text{in} \quad \Gamma.$$

(To check that this action is well defined, observe that over $U_{\alpha_0} \cap U_{\alpha_0''}$, $b_{\gamma'\gamma,\alpha} = a_{\gamma,\alpha} = a_{\gamma'',\alpha''} = b_{\gamma'\gamma'',\alpha''}$ as required.)

This allows us to identify $\mathscr{A}(V)$ with the set of invariant elements of $\mathscr{A}(\pi^{-1}V)$. Let us write the inclusion

$$\tilde{\pi}: \mathscr{A}(V) \to \tilde{\mathscr{A}}(\pi^{-1}V), \quad \pi(a_{\gamma,\alpha}) = a \text{ restricted to } V \cap U_\alpha.$$

In particular we have

$$\tilde{\pi}: \mathscr{A}(X) \to \tilde{\mathscr{A}}(\tilde{X}).$$

This gives us a map of graded manifolds and it is straightforward to verify the following proposition:

Proposition 4.1. *The graded manifold map $(\pi, \tilde{\pi}): (\tilde{X}, \tilde{\mathscr{A}}) \to (X, \mathscr{A})$ defined above is a map of graded Riemann surfaces.*

Remark. $(\tilde{X}, \tilde{\mathscr{A}})$ has a universal property amongst all graded Riemann surfaces that cover $(X, \mathscr{A})$ that extends the universal property of $\tilde{X}$ over all regular covers of X.

We can now state and prove our main result

Theorem 4.2. *Let $(X, \mathscr{A})$ be a graded Riemann surface, and suppose that X is not simply connected.*

1. *There is a covering map*

$$\pi: (\Delta, C^\infty_\Delta \otimes \Lambda\theta) \to (X, \mathscr{A}),$$

where $\Delta = C$ or H.
2. *There is an isomorphism τ^Γ of graded Riemann surfaces which makes the following diagram commute:*

$$\begin{tikzcd}
(\Delta, C^\infty \otimes \Lambda\theta) \arrow[r, "\tau"'] \arrow[d] \arrow[dr, "\pi"] & (\tilde{X}, \tilde{\mathscr{A}}) \arrow[d, "\tilde{\pi}"] \\
(\Delta/\bar{\Gamma}, C^\infty \otimes \Lambda\theta)^\Gamma \arrow[r, "\tau^\Gamma"'] & (X, \mathscr{A}),
\end{tikzcd}$$

where Γ is isomorphic to $\pi_1(X)$ and Γ is a subgroup of $Sl(2; C)$ (acting as in Lemma 2.1) which maps isomorphically onto $\bar{\Gamma} \subset M$.

Proof.
1. By 3.2 $(\tilde{X}, \tilde{\mathscr{A}})$ is isomorphic to $(\Delta, C^\infty \otimes \Lambda\theta)$ by some map τ. We may establish part 1 using the composition $\pi = \tilde{\pi} \cdot \tau$.
2. We wish to use τ to identify the action of $\pi_1(X)$ on $(\tilde{X}, \tilde{\mathscr{A}})$ described above with the action of a subgroup of $Sl(2, C)$ acting on $(\Delta, C_\Delta^\infty \otimes \Lambda\theta)$.

Identify $\pi_1(X)$ with a subgroup Γ of automorphisms of $(\tilde{X}, \tilde{\mathscr{A}}) = (\Delta, C_\Delta^\infty \otimes \Lambda\theta)$. For $\gamma \in \Gamma$, $\bar{\gamma}$ is an analytic automorphism of Δ, and hence $\bar{\gamma}$ is in M. Thus Γ satisfies the conditions of Proposition 1.5 and can be identified with its image under L in $Sl(2, C)$.

Section 5. Comments

One may ask what virtue there is in repeating, in a restrictive framework, what has been ably done by others in a more general fashion. The motivation for doing this work lies in the following observations.
1. The "symmetry group" of $(S, \mathscr{A})$ is precisely $Sl(2, C)$ rather than some larger supergroup with $Sl(2; C)$ as its body. More precisely if we consider the sheaf of meromorphic sections of ΛE it can be shown that this sheaf is $R \otimes \Lambda\theta$, where R is the sheaf of rational functions. The map $K: Sl(2; C) \to \mathrm{Aut}(R \otimes \Lambda\theta)$ given by

$$K\begin{bmatrix} a & b \\ c & d \end{bmatrix}(r(z) + s(z)\theta) = r\left(\frac{az+b}{cz+d}\right) + s\left(\frac{az+b}{cz+d}\right)\frac{\theta}{cz+d}$$

defines an (anti)-isomorphism onto the subalgebra of automorphisms which preserve the extended map

$$k: \mathscr{M} \otimes \mathscr{B}_1 \otimes \mathscr{B}_1 \to \mathscr{M} \otimes \Omega^1 H_s,$$

where $\mathscr{M}$ is the algebra of meromorphic functions.
2. It is worth calculating the Lie algebra of derivations of $R \otimes \Lambda\theta$ which "preserve" k; that is, (even) derivations $\partial = \partial_0 + \partial_1$ for which the following diagram commutes:

$$\begin{tikzcd}
R\theta \otimes R\theta \arrow[r, "k"'] \arrow[d, "\partial_1 \otimes 1 + 1 \otimes \partial_1"'] & R \otimes dz \arrow[d, "D\partial_0"] \\
R\theta \otimes R\theta \arrow[r] & R \otimes dz
\end{tikzcd}$$

Since ∂_0 is in Der R, $\partial_0 = r_0(z)\partial_z$ for $r \in R$. Commutativity of the diagram implies that if we write $\partial_1(\theta) = r_1\theta$ $(r_1 \in R)$, then

$$D\partial_0 k(\theta \otimes \theta) = D\partial_0 dz = d(r_0(z)) = r_0'(z)dz,$$
$$k(1 \otimes \partial_1 + \partial_1 \otimes 1)(\theta \otimes \theta) = k(2r_1\theta \otimes \theta) = 2r_1 dz.$$

Thus $r_1(z) = 1/2r_0'(z)$.

Consider the subspace of $\mathrm{Der}(R \otimes \Lambda\theta)$ generated by the elements

$$L_k = -x^{k+1}\frac{\partial}{\partial x} - \frac{(k+1)}{2}x^k\theta\frac{\partial}{\partial\theta}.$$

One can verify directly that

$$[L_k, L_h] = (k-h)L_{k+h}.$$

This provides a representation of the charge-0 Virasoro algebra by derivations of $R \otimes \Lambda\theta$.

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