

The Algebra Formed by the Invariant Charges of the Nambu-Goto Theory: Identification of a Maximal Abelian Subalgebra

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Abstract. Continuing the analysis of the algebraic structure of the invariant charges of the Nambu-Goto theory, we identify a complete set of commuting observables for the bosonic strings.

I. Introduction

In a recent publication [1] the present authors have studied the general structure of the algebras $\mathfrak{h}_{\mathcal{P}}^{\pm}$ formed by the conserved observable tensor charges of the bosonic Nambu-Goto string theory under tensor multiplication and Poisson bracket operation. The complete infinite dimensional algebra of infinitesimal classical symmetry transformations

$$\mathfrak{g} = \mathfrak{so}(1, d-1) \oplus (\mathbb{M}^d \oplus \mathfrak{h}_{\mathcal{P}}^+ \oplus \mathfrak{h}_{\mathcal{P}}^-)$$

turned out to be a minimal extension of the Poincaré algebra involving string degrees of freedom. Both $\mathfrak{h}_{\mathcal{P}}^+$ and $\mathfrak{h}_{\mathcal{P}}^-$ are graded algebras

$$\mathfrak{h}_{\mathcal{P}}^{\pm} = \bigoplus_{\ell=0}^{\infty} V^{(\ell)}(\mathfrak{h}_{\mathcal{P}}^{\pm})$$

such that – with the symbol $\{\cdot, \cdot\}$ standing for the Poisson bracket operation –

$$\{V^{(\ell)}(\mathfrak{h}_{\mathcal{P}}^{\pm}), V^{(\ell')}(\mathfrak{h}_{\mathcal{P}}^{\pm})\} \subset V^{(\ell+\ell')}(\mathfrak{h}_{\mathcal{P}}^{\pm}).$$

In this paper we shall identify a *maximal* abelian subalgebra

$$\mathfrak{a}_{\mathcal{P}}^{\pm} \subset \mathfrak{h}_{\mathcal{P}}^{\pm}$$

provided: $\mathcal{P}^2 = m^2 > 0$, whose elements ultimately should serve to distinguish the states of irreducible positive energy representations of \mathfrak{g} .

Since $\mathfrak{h}_{\mathcal{P}}^+$ and $\mathfrak{h}_{\mathcal{P}}^-$ are mutually commuting, isomorphic Poisson bracket algebras, we take the liberty to restrict our attention to $\mathfrak{h}_{\mathcal{P}}^+$, and to suppress the superscript + in the sequel.

In order to state the main result of the present analysis, we introduce for $\mathcal{P}^2 = m^2 > 0$ a set $\mathfrak{a}_{\mathcal{P}}$ of invariant charges. This set is obtained by standard Lorentz boosts from the corresponding set $\mathfrak{a}_{(m, 0, \dots, 0)}$ defined in the momentum rest frame $\mathcal{P}_{\mu} = m\delta_{\mu 0}$ as follows: In addition to a (finite) maximal set of mutually commuting tensor components $\mathcal{R}_{i^{(r)}j^{(r)}}$ – which correspond to the generators of a Cartan subalgebra of the Lie algebra of the little group $\text{SO}(d-1)$ of the Lorentz group $\text{SO}(1, d-1)$ in d dimensional space-time – the set $\mathfrak{a}_{(m, 0, \dots, 0)}$ consists of Lorentz scalar and, in the case of odd space-time dimension d , of Lorentz pseudoscalar invariant charges $\mathcal{A}_{(N)}^{(K)} \in V^{(N-K-1)}(\mathfrak{h}_{(m, 0, \dots, 0)})$ generated by the functionals

$$\text{tr}(\ln \phi_{\lambda T})^K = \sum_{N=1}^{\infty} \lambda^N \mathcal{A}_{(N)}^{(K)}, \quad K = 2, 3, \dots,$$

as well as of all polynomials in those $\mathcal{R}_{i^{(r)}j^{(r)}}$ and $\mathcal{A}_{(N)}^{(K)}$. Here $\phi_{\lambda T}$ denotes the monodromy matrix for the associated Lax pair of linear differential equations [2] with matrix-valued parameters $T^{\mu} = A^{\mu x} T_x = \lambda \Gamma^{\mu}$, $\lambda \in \mathbb{R}$, the constant matrices Γ^{μ} , $\mu = 0, 1, \dots, d-1$, realizing the Dirac algebra

$$[\Gamma^{\mu}, \Gamma^{\nu}]_+ = 2g^{\mu\nu} \cdot \mathbf{1},$$

where in its turn the symbol $[\cdot, \cdot]_+$ denotes the anti-commutator of two matrices.

We show that $\mathfrak{a}_{\mathcal{P}}$ forms a *maximal abelian* subalgebra of $\mathfrak{h}_{\mathcal{P}}$. In particular, the polynomials in \mathcal{R}_{ij} corresponding to the Casimir operators of the little group $\text{SO}(d-1)$ are contained in $\mathfrak{a}_{\mathcal{P}}$ (as polynomials in $\mathcal{A}_{(N)}^{(K)}$).

In Sect. II we shall prove that $\mathfrak{a}_{\mathcal{P}}$ thus defined forms an abelian subalgebra. In Sect. III we shall prove the maximality property of $\mathfrak{a}_{\mathcal{P}}$. We conclude with some remarks concerning the Casimir operators of $\mathfrak{h}_{\mathcal{P}}$ or rather of \mathfrak{g} and their relation to the loop wave equations.

II. The (Abelian) Nature of the Algebra $\mathfrak{a}_{\mathcal{P}}$

Commutativity of the algebra $\mathfrak{a}_{(m, 0, \dots, 0)}$ implies commutativity of the algebra $\mathfrak{a}_{\mathcal{P}}$ and vice versa. Thus it suffices to show that the algebra $\mathfrak{a}_{(m, 0, \dots, 0)}$ is abelian.

The elements $\mathcal{R}_{i^{(r)}j^{(r)}}$ of $\mathfrak{a}_{(m, 0, \dots, 0)}$ commute among each other by their very definition. Moreover, since they act on the invariant charges $\mathcal{L}_{\mu_1 \dots \mu_N}$ like infinitesimal generators of certain rotations in the momentum rest frame

$$\mathcal{P}_{\mu} = m\delta_{\mu 0}, \quad m > 0,$$

they also commute with the remaining elements $\mathcal{A}_{(N)}^{(K)}$ of $\mathfrak{a}_{(m, 0, \dots, 0)}$; the latter ones transforming like scalars or pseudoscalars under Lorentz transformations. Thus it remains to show that the invariant charges $\mathcal{A}_{(N)}^{(K)}$ commute among themselves.

To begin with the proof, we note that the trace $\text{tr} \Gamma^{\mu_1} \dots \Gamma^{\mu_N}$, normalized such that $\text{tr} \mathbf{1} = 1$, can be calculated as a polynomial in metric tensors $g^{\mu_i \mu_j}$ and, if d and $N \geq d$ are odd, in the totally antisymmetric tensor $\varepsilon^{\mu_1 \dots \mu_d}$. These traces are independent of the choice of representation of the Dirac algebra except that, for d odd, $\text{tr} \Gamma^{\mu_1} \dots \Gamma^{\mu_{2n+1}} = 0$ in a non-chiral representation, such that in a non-chiral representation there are no terms involving the ε tensor, whereas in a chiral

representation (e.g. after projection onto the $\gamma^5 = +1$ subspace with $i^{\frac{d-1}{2}} \cdot \gamma^5 = \Gamma^0 \Gamma^1 \dots \Gamma^{d-1}$) terms involving the ε tensor do appear.

In either case, $\text{tr}(\ln \phi_{\lambda R})^K$ is a sum of scalar and pseudoscalar charges: $\{\mathcal{A}_{(N)}^{(K)}, \mathcal{B}_{ij}^{(K)}\} = 0$. It remains to show: $\{\text{tr}(\ln \phi_{\lambda R})^K, \text{tr}(\ln \phi_{\mu R})^K\} = 0$, or equivalently: $\{\text{tr} \phi_{\lambda R}^p, \text{tr} \phi_{\mu R}^q\} = 0$ [3]. The canonical Poisson bracket yields

$$\{\text{tr} \phi_{\lambda R}^p, \text{tr} \phi_{\mu R}^q\} = -2pq \cdot \lambda \mu \oint d\sigma \text{tr}(\Gamma_\alpha \phi_{\lambda R}^p(\sigma)) \partial_\sigma \text{tr}(\Gamma^\alpha \phi_{\mu R}^q(\sigma)).$$

The integral $\oint d\sigma \dots$ equals

$$X = \mu \oint d\sigma \text{tr}(\Gamma_\alpha \phi_{\lambda R}^p(\sigma)) u_\beta(\sigma) \text{tr}([\Gamma^\alpha, \Gamma^\beta] \phi_{\mu R}^q(\sigma)),$$

which with the help of $\frac{1}{2}(\Gamma_\alpha u_\beta - \Gamma_\beta u_\alpha) = \frac{1}{4} u^\gamma [\Gamma_\alpha \Gamma_\beta, \Gamma_\gamma]$ can be rewritten as

$$\begin{aligned} X &= \frac{1}{4} \mu \oint d\sigma u^\gamma(\sigma) \text{tr}([\Gamma_\alpha \Gamma_\beta, \Gamma_\gamma] \phi_{\lambda R}^p(\sigma)) \cdot \text{tr}([\Gamma^\alpha, \Gamma^\beta] \phi_{\mu R}^q(\sigma)) \\ &= \frac{1}{8} (\mu/\lambda) \oint d\sigma \partial_\sigma \text{tr}([\Gamma_\alpha, \Gamma_\beta] \phi_{\lambda R}^p(\sigma)) \cdot \text{tr}([\Gamma^\alpha, \Gamma^\beta] \phi_{\mu R}^q(\sigma)). \end{aligned}$$

A quantity X , antisymmetric under $(\lambda, p) \leftrightarrow (\mu, q)$, which equals μ/λ times another quantity with the same antisymmetry, must vanish.

By explicit evaluation of $\text{tr} \Gamma^{\mu_1} \dots \Gamma^{\mu_N}$, and exploiting the cyclic symmetry of $\mathcal{Z}_{\mu_1 \dots \mu_N}^{(K)}$, the following explicit formulae for the nonvanishing $\mathcal{A}_{(N)}^{(K)}$ can be derived:

$$\mathcal{A}_{(2n)}^{(2k)} = (2k)! \sum_G (-1)^{s(G)} \frac{1}{f(G)} \mathcal{Z}_G^{(2k)}, \quad (n \geq k \geq 1),$$

where \sum_G runs over all cyclically distinct contraction schemes of the rank $2n$ tensor $\mathcal{Z}^{(2k)}$ with n metric tensors; G can be visualized by a graph: on a circle equipartitioned by $2n$ vertices corresponding to the Lorentz indices of $\mathcal{Z}^{(2k)}$ in their given cyclic order join any contracted pair of indices by a straight line. Then $s(G)$ equals the number of points of intersection of these lines, and f is the cyclic symmetry factor of G : e.g.

$$\frac{1}{2!} \mathcal{A}_{(4)}^{(2)} = \frac{1}{2} \bigcirc - \frac{1}{4} \otimes \equiv g^{\mu\mu'} g^{\nu\nu'} \left[\frac{1}{2} \mathcal{Z}_{\mu\mu'\nu\nu'}^{(2)} - \frac{1}{4} \mathcal{Z}_{\mu\nu\nu'\mu'}^{(2)} \right].$$

If d is odd, we use the irreducible chiral representation $\Gamma^0 \Gamma^1 \dots \Gamma^{d-1} = i^{\frac{d-1}{2}}$:

$$\begin{aligned} \mathcal{A}_{(d+2n)}^{\left(\frac{d+1}{2} + 2k\right)} &= \left(\frac{d+1}{2} + 2k\right)! i^{\frac{d-1}{2}} \sum_{E,G} (-1)^{s(G)} \frac{g(E)}{d+2n} \frac{1}{f(E,G)} \mathcal{Z}_{(E,G)}^{\left(\frac{d+1}{2} + 2k\right)}, \\ &\left(n, k \geq 0; 2n + \frac{d+1}{2} \geq 2k\right), \end{aligned}$$

where $\sum_{E,G}$ runs over all cyclically distinct choices E of d cyclically ordered indices $\mu_{i_1}, \dots, \mu_{i_d}$ among the $(2n+d)$ indices of \mathcal{Z} to be contracted with $\varepsilon^{\mu_{i_1} \dots \mu_{i_d}}$, and over all distinct contraction schemes G of the remaining $2n$ indices of \mathcal{Z} with n metric tensors. (E, G) can again be visualized by a graph: On a circle equipartitioned by

$(2n + d)$ vertices, first mark the d ε -vertices as indicated by E , then join the other $2n$ vertices by straight lines as indicated by G . $s(G)$ is again the number of points of intersection of the (g contraction) lines, $f(E, G)$ is the cyclic symmetry factor of the whole graph. In order to compute $g(E)$, color the $2n$ arcs on the circle between the $2n$ g -vertices green and red in turn, and call n_r the number of ε -vertices lying on red arcs. Then $g(E) = (-1)^{n_r}(d - 2n_r)$; e.g. $d = 3$:

$$\begin{aligned} \frac{1}{i2!} \mathcal{A}_{(3)}^{(2)} &= \frac{3}{3} \cdot \frac{1}{3} \cdot \text{diagram} \equiv \frac{1}{3} \varepsilon^{\alpha\beta\gamma} \mathcal{L}_{\alpha\beta\gamma}^{(2)}, \\ \frac{1}{i2!} \mathcal{A}_{(5)}^{(2)} &= \frac{3}{5} \cdot \text{diagram} - \frac{1}{5} \cdot \text{diagram} \equiv \varepsilon^{\alpha\beta\gamma} g^{\mu\mu'} \left(\frac{3}{5} \mathcal{L}_{\alpha\beta\gamma\mu\mu'}^{(2)} - \frac{1}{5} \mathcal{L}_{\alpha\beta\mu\gamma\mu'}^{(2)} \right). \end{aligned}$$

$\mathcal{L}_{\mu_1\mu_2\dots\mu_N}^{(K)} = (-1)^{N-K} \mathcal{L}_{\mu_N\dots\mu_2\mu_1}^{(K)}$ (cf. [1]) implies that scalars $\mathcal{A}_{(2n)}^{(K)}$ vanish for odd K , and pseudoscalars $\mathcal{A}_{(d+2n)}^{(K)}$ vanish if $K - \frac{d+1}{2}$ is odd. The following argument shows that the charges $\mathcal{A}_{(d+2n)}^{(K)}$ vanish if $K < \frac{d+1}{2}$: The symmetries of $\mathcal{R}_{\mu_1\dots\mu_N}^t$ [1] imply

$$\ln \phi_T = \sum_{N=1}^{\infty} \mathcal{R}_{\mu_1\dots\mu_N}^t T^{\mu_1} \dots T^{\mu_N} = \sum_{N=1}^{\infty} \frac{1}{N} \mathcal{R}_{\mu_1\dots\mu_N}^t [\dots [T^{\mu_1}, T^{\mu_2}], \dots, T^{\mu_N}].$$

For Dirac matrices, multiple commutators can be worked out to a large extent. Hence

$$\ln \phi_{\lambda T} = \mathbb{P}_{\mu}(\lambda) T^{\mu} + \frac{1}{2} \mathbb{R}_{\mu\nu}(\lambda) [T^{\mu}, T^{\nu}] \equiv \mathbb{P}(\lambda) + \mathbb{R}(\lambda)$$

with power series $\mathbb{P}_{\mu}(\lambda)$, $\mathbb{R}_{\mu\nu}(\lambda) = -\mathbb{R}_{\nu\mu}(\lambda)$, the coefficients of which are certain contracted components of \mathcal{R}^t tensors, the ranks of which are equal to the powers of λ . Hence if $K \leq \frac{d-1}{2}$, $\text{tr}(\ln \phi_{\lambda T})^K$ gets contributions to odd powers of λ only from $\text{tr}(\Gamma^{\mu_1} \dots \Gamma^{\mu_{2n+1}})$, $2n+1 \leq d-2$; such traces vanish.

We have computed

$$\begin{aligned} \mathbb{P}_{\mu}(\lambda) &= \frac{1}{2} \sum_{N \text{ odd}} (2\lambda)^N \underbrace{\mathcal{R}_{\mu\alpha\alpha'\dots\beta\beta'}^t}_{N-1} g^{\alpha\alpha'} \dots g^{\beta\beta'} = \lambda \mathcal{P}_{\mu} + \dots, \\ \mathbb{R}_{\mu\nu}(\lambda) &= \frac{1}{4} \sum_{N \text{ even}} (2\lambda)^N \underbrace{\mathcal{R}_{\mu\alpha\alpha'\dots\beta\beta'\nu}^t}_{N-2} g^{\alpha\alpha'} \dots g^{\beta\beta'} = \lambda^2 \mathcal{R}_{\mu\nu}^t + \dots \end{aligned}$$

From $\sum \lambda^N \mathcal{A}_{(N)}^{(2)} = \text{tr}(\ln \phi_{\lambda T})^2 = \mathbb{P}_{\mu} \mathbb{P}^{\mu} - 2 \mathbb{R}_{\mu\nu} \mathbb{R}^{\mu\nu} + (\text{odd rank})$, we conclude

$$\begin{aligned} \mathcal{A}_{(2n)}^{(2)} &= 4^{n-1} \left[g^{\mu\mu'} \sum_{n_1+n_2=n-1} \underbrace{\mathcal{R}_{\mu\alpha\alpha'\dots\beta\beta'}^t}_{2n_1} \underbrace{\mathcal{R}_{\mu'\gamma\gamma'\dots\delta\delta'}^t}_{2n_2} \right. \\ &\quad \left. + \frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} \sum_{n_1+n_2=n-2} \underbrace{\mathcal{R}_{\mu\alpha\alpha'\dots\beta\beta'\nu}^t}_{2n_1} \underbrace{\mathcal{R}_{\nu'\gamma\gamma'\dots\delta\delta'\mu'}^t}_{2n_2} \right] g^{\alpha\alpha'} \dots g^{\beta\beta'} g^{\gamma\gamma'} \dots g^{\delta\delta'}, \end{aligned}$$

i.e.

$$\begin{aligned}
\mathcal{A}_{(2)}^{(2)} &= \mathcal{P}_\mu \mathcal{P}^\mu = m^2, \\
\mathcal{A}_{(2n)}^{(2)} &= 2 \cdot 4^{n-1} \mathcal{P}^\mu \mathcal{R}_\mu^t \underbrace{\mathcal{R}^t \dots \mathcal{R}^t}_{2n-2} g^{\alpha\alpha'} \dots g^{\beta\beta'} + \sum \mathcal{R}^t \mathcal{R}^t \quad (n > 1) \\
&= 4^{n-1} m \mathcal{R}_i^t \underbrace{0 \dots 0}_i \\
&\quad \underbrace{2n-3} \\
&\quad + m \sum \mathcal{R}^t \text{ with more than two space-like indices} \\
&\quad + \sum \mathcal{R}^t \mathcal{R}^t.
\end{aligned}$$

In the last expression we have used the momentum rest frame $\mathcal{P}_\mu = m\delta_{\mu 0}$, as we shall do throughout the sequel.

For the proof of the maximality property of the algebra $\mathfrak{a}_\mathcal{P}$ (Sect. III) it is desirable to gain more insight into the number and the structure of the independent elements of $\mathfrak{a}_{(m, 0, \dots, 0)}$. To this end we shall distinguish in the polynomial ring of the invariant charges $\mathcal{A}_{(N)}^{(K)}$ a countable set of independent polynomials \mathcal{B} . As it turns out finally, the charges $\mathcal{R}_{i^{(r)}j^{(r)}}^t$ and \mathcal{B} form an algebraic basis of $\mathfrak{a}_{(m, 0, \dots, 0)}$, i.e. $\mathfrak{a}_{(m, 0, \dots, 0)}$ is the polynomial ring over $\mathcal{R}_{i^{(r)}j^{(r)}}^t$ and \mathcal{B} .

We turn now to the actual construction of the special elements $\mathcal{B} \in \mathfrak{a}_{(m, 0, \dots, 0)}$: For every invariant charge $\mathcal{L} \in V^{(l)}(\mathfrak{h}_{(m, 0, \dots, 0)})$ we single out ‘‘leading terms’’ as follows: \mathcal{L} is a polynomial of degree $\leq l+1$ in \mathcal{R}^t , considering powers of m as coefficients. Among the monomials of least degree in \mathcal{R}^t (forming the leading part of \mathcal{L}) choose those which contain the maximal p_{\max} -fold product of tensor components \mathcal{R}^t of highest rank N_{\max} and among these choose those which contain the least number s_{\min} of space-like indices. Then

$$4^{n-1} m \mathcal{R}_i^t \underbrace{0 \dots 0}_i \underbrace{2n-3}$$

are the leading terms of $\mathcal{A}_{(2n)}^{(2)}$, and we have already proved the first part of the following statement:

There are polynomials \mathcal{B} in $\mathcal{A}_{(N)}^{(K)}$ the leading terms of which are specified as follows:

$$\begin{aligned}
\mathcal{B}_{(n)}^1(\mathcal{A}) &= m \mathcal{R}_i^t \underbrace{0 \dots 0}_i + \dots \quad (n \geq 0), \\
\mathcal{B}_{(n, k)}^2(\mathcal{A}) &= m^{2x(k)} \mathcal{R}_{i_1 i_2}^t \mathcal{R}_{i_2 i_3}^t \dots \mathcal{R}_{i_{2k-1} i_{2k}}^t \cdot \mathcal{R}_{i_{2k} \underbrace{0 \dots 0}_{2n}}^t i_1 + \dots \\
&\quad \left(n \geq 0, 1 \leq k \leq \left\lfloor \frac{d-2}{2} \right\rfloor \right), \\
\mathcal{B}_{(n)}^3(\mathcal{A}) &= m c^{i_1 \dots i_{d-1}} \mathcal{R}_{i_1 i_2}^t \mathcal{R}_{i_3 i_4}^t \dots \mathcal{R}_{i_{d-4} i_{d-3}}^t \cdot \mathcal{R}_{i_{d-2} \underbrace{0 \dots 0}_{2n}}^t i_{d-1} + \dots \quad (n \geq 0).
\end{aligned}$$

Note that $\mathcal{B}_{(0, k)}^2$ and $\mathcal{B}_{(0)}^3$ can be identified, apart from powers of m , with the $\left\lfloor \frac{d-1}{2} \right\rfloor = \text{rank}(\text{so}(d-1))$ independent Casimir operators of $\text{so}(d-1)$. $\mathcal{B}_{(n)}^3$, of course, only

exist in odd dimensions; they are easily found to be proportional to $\mathcal{A}^{\left(\frac{d+1}{d+2n}\right)}$, since

$$\sum_n \lambda^{d+2n} \mathcal{A}^{\left(\frac{d+1}{d+2n}\right)} = \frac{d+1}{2} \operatorname{tr} \left(\mathbb{P} \mathbb{R}^{\frac{d-1}{2}} \right) = \frac{d+1}{2} i^{\frac{d-1}{2}} e^{\mu_1 \dots \mu_d} \mathbb{P}_{\mu_1} \mathbb{R}_{\mu_2 \mu_3} \dots \mathbb{R}_{\mu_{d-1} \mu_d}.$$

As for the polynomials $\mathcal{B}_{(n,k)}^2$ which for $d > 3$ become independent of the polynomials $\mathcal{B}_{(n)}^1(\mathcal{A})$ and $\mathcal{B}_{(n)}^3(\mathcal{A})$, the argument is more involved.

We shall make use of the generating functionals $\mathcal{A}^{(K)}(\lambda) = \sum_{N \text{ even}} \lambda^N \mathcal{A}_{(N)}^{(K)}$, K even, which coincide with $\operatorname{tr}(\ln \phi_{\lambda \Gamma})^K$ in a non-chiral representation of the Dirac algebra. In particular, $\mathcal{A}^{(2)}$ generates $\mathcal{B}_{(n)}^1$. The crucial point to note is this: $\mathcal{A}^{(K)}(\lambda)$ are polynomials in the Lorentz scalars

$$\operatorname{Tr} \mathbb{R}^{2k} \doteq \mathbb{R}_{\mu'_1 \mu'_2}(\lambda) g^{\mu_2 \mu'_3} \mathbb{R}_{\mu_2 \mu_3}(\lambda) g^{\mu_3 \mu'_4} \dots \mathbb{R}_{\mu'_{2k} \mu_1}(\lambda) g^{\mu_1 \mu'_1}$$

and

$$(\mathbb{P} \mathbb{R}^{2k} \mathbb{P}) \doteq \mathbb{P}_{\mu_0}(\lambda) g^{\mu_0 \mu'_0} \mathbb{R}_{\mu_0 \mu_1}(\lambda) g^{\mu_1 \mu'_1} \dots \mathbb{R}_{\mu'_{2k-1} \mu_{2k}}(\lambda) g^{\mu_{2k} \mu'_{2k}} \mathbb{P}_{\mu'_{2k}}(\lambda)$$

only.

It is clear that every homogeneous charge which is a polynomial in $\operatorname{Tr} \mathbb{R}^{2k}$ and $(\mathbb{P} \mathbb{R}^{2k} \mathbb{P})$ has a leading part proportional to $(\lambda m)^{2x}$ times a polynomial in $\operatorname{Tr} \mathbb{R}^{2k}$ and $(\mathbb{R}^{2k})_{00}$. However, the leading part of an invariant charge can be cast into a form which involves only components of \mathcal{R}^l with space-like indices i, j at the extremal positions [1]. Consequently, every such charge is of the form

$$(\lambda m)^{2x} \text{ times a polynomial in } \operatorname{Tr} \mathbb{R}_{\perp}^{2k} + \text{higher degrees in } \mathcal{R}^l,$$

where

$$\operatorname{Tr} \mathbb{R}_{\perp}^{2k} \doteq \mathbb{R}_{i_1 i_2}(\lambda) \mathbb{R}_{i_2 i_3}(\lambda) \dots \mathbb{R}_{i_{2k} i_1}(\lambda).$$

Bearing this in mind, we prove the following claim: For every $k \geq 0$ there is a polynomial \mathcal{B}_k in $\mathcal{A}^{(2k')}$, $k' \leq k+1$, such that

$$\mathcal{B}_k(\mathcal{A}) = (\lambda m)^{2x(k)} \operatorname{Tr} \mathbb{R}_{\perp}^{2k} + \text{higher degrees in } \mathcal{R}^l.$$

Proof by Induction. For $k=0$ the claim is easily verified:

$$\mathcal{B}_0(\mathcal{A}) = \mathcal{A}^{(2)} = \mathbb{P}^2 + 2 \operatorname{Tr} \mathbb{R}^2 = (\lambda m)^2 + O(\mathcal{R}^l).$$

Suppose that the claim is true for all $k' < k$ for some $k > 0$. $\mathcal{A}^{(2k+2)}$ is an invariant charge polynomial in $\operatorname{Tr} \mathbb{R}^{2k'}$ and $(\mathbb{P} \mathbb{R}^{2k'} \mathbb{P})$. Its leading part is proportional to $(\lambda m)^{2y}$ times a polynomial in $\operatorname{Tr} \mathbb{R}_{\perp}^{2k'}$, $1 \leq k' \leq k$, which when viewed as a polynomial in \mathcal{R}^l is homogeneous of total degree $2l \leq 2k$. (Actually, $l=0$). As long as $l < k$, we multiply by a sufficiently high power of $\mathcal{A}^{(2)}$, and replace in the resulting leading part $(\lambda m)^{2x(k')} \operatorname{Tr} \mathbb{R}_{\perp}^{2k'}$ by the appropriate polynomial in $\mathcal{A}^{(2k'')}$, $k'' \leq k'+1 \leq l+1 \leq k$, as provided by the induction hypothesis, plus correction terms of higher degree in \mathcal{R}^l . We subtract from $\mathcal{A}^{(2)x} \mathcal{A}^{(2k+2)}$ the polynomial in $\mathcal{A}^{(2k'')}$ corresponding to the replacements without correction terms, and we are left with another invariant charge, which is a polynomial in $\operatorname{Tr} \mathbb{R}^{2k'}$ and $(\mathbb{P} \mathbb{R}^{2k'} \mathbb{P})$ with a leading part of degree $2l' > 2l$ in \mathcal{R}^l . We continue the procedure until $l' \geq k$.

Now, from the outset, $\mathcal{A}^{(2k+2)}$ got an additive contribution of the form $(\mathbb{P}\mathbb{R}^{2k}\mathbb{P})$ times a nonvanishing coefficient $(-1)^{k2^{4k+1}}(4^{k+1}-1)B_{k+1}$ (B_k = Bernoulli number). None of the subtractions can possibly have cancelled this term. Its presence implies that the leading part of what remains after the above manipulations has degree $2k$ and that it contains among other terms the desired monomial $(\lambda m)^{2y'} \text{Tr} \mathbb{R}_\perp^{2k}$, while all other monomials can, as before, be subtracted after appropriate multiplication with powers of $\mathcal{A}^{(2)}$ by a polynomial in $\mathcal{A}^{(2k')}$, $k'' \leq k$.

Thus, we have constructed

$$\mathcal{B}_k(\mathcal{A}) = (\mathcal{A}^{(2)})^{x(k)} \mathcal{A}^{(2k+2)} + \text{polynomial in } \mathcal{A}^{(2k')}, \quad k' \leq k.$$

Hence the induction hypothesis is also valid for $k+1$. This completes the proof of the claim.

Finally it is clear that, evaluated at order $\lambda^{2x(k)+4k+2n}$, $k \geq 1$, the polynomial $\mathcal{B}_k(\mathcal{A}(\lambda))$ has the leading term proportional to that required for $\mathcal{B}_{(n,k)}^2$.

In the above, the numerical coefficients occurring in all the polynomials were completely independent of the space-time dimension d . However, if we fix d , $\text{Tr} \mathbb{R}_\perp^{2k}$ becomes dependent on $\text{Tr} \mathbb{R}_\perp^{2k'}$, $k' < k$, if $2k > d-1$, due to the Cayley-Hamilton theorem. Since moreover, for odd d ,

$$\mathcal{B}_{(0)}^3 \cdot \mathcal{B}_{(n)}^3 \sim \mathcal{B}_{\left(n, \frac{d-1}{2}\right)}^2 + \text{polynomial in } \mathcal{B}_{(n',k')}^2, \quad n' \leq n, \quad k' < \frac{d-1}{2},$$

we are free to restrict ourselves to the independent $\mathcal{B}_{(n,k)}^2$, $k \leq \left[\frac{d-2}{2}\right]$.

This finishes the introduction of the special elements \mathcal{B} of $\mathfrak{a}_{(m,0,\dots,0)}$. From the preceding discussion we have gained some insight into the substance of the algebra $\mathfrak{a}_{(m,0,\dots,0)}$ or rather of $\mathfrak{a}_\mathcal{A}$.

III. The Maximality Property of the Algebra $\mathfrak{a}_\mathcal{A}$

Maximality of the algebra $\mathfrak{a}_{(m,0,\dots,0)}$ implies maximality of the algebra $\mathfrak{a}_\mathcal{A}$ and vice versa. Thus it suffices to show that the algebra $\mathfrak{a}_{(m,0,\dots,0)}$ is maximal.

As a preparation for the proof for general space-time dimension d , we elucidate the argument first for the case of three space-time dimensions where the situation is especially transparent. The algebra $\mathfrak{a}_{(m,0,0)}$ contains at least the polynomial ring over the special elements

$$\mathcal{B}_{(n)}^1(\mathcal{A}) = m \mathcal{R}_{i_0 \dots i_{2n+1}}^1 + \dots, \quad (n \geq 0),$$

$$\mathcal{B}_{(n)}^3(\mathcal{A}) = 2m \mathcal{R}_{1 \ 0 \dots 0 \ 2}^t + \dots, \quad (n \geq 0).$$

The Cartan subalgebra of $\mathfrak{so}(3-1)$ consists of one generator only. The tensor component corresponding to this generator: \mathcal{R}_{12}^t (or rather $-\frac{1}{2}\mathcal{R}_{12}^t$) occurs already as the $n=0$ term in the series $\mathcal{B}_{(n)}^3(\mathcal{A})$. We shall show, now, that any

invariant charge \mathcal{L} commuting with the special elements $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$ is contained in the polynomial ring over $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$.

Proof by Contradiction. Suppose that there exists an invariant charge \mathcal{L} commuting (i.e. in involution) with all special elements $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$, without itself being a polynomial in these elements.

Without loss of generality, we may assume that the degree q of the leading part of \mathcal{L} (viewed as a polynomial in \mathcal{R}^t) cannot be decreased by adding to \mathcal{L} a suitable polynomial in $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$; $q = q_{\min} > 0$. Otherwise we would replace the invariant charge \mathcal{L} of the hypothesis by a sum of \mathcal{L} and such a polynomial in $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$.

We focus our attention on the leading part of \mathcal{L} , a homogeneous polynomial of degree q_{\min} in the linearly independent tensor components \mathcal{R}^t with space-like indices at the extremal positions [1]. If \mathcal{L} is in involution with $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$, then, at the very least, the modified Poisson brackets $\{\cdot, \cdot\}^*$ (cf. [1]) between the leading part of \mathcal{L} and the leading parts of $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$, respectively, must vanish.

In the leading part of \mathcal{L} we express the factors \mathcal{R}^t with just two space-like indices as linear combinations of

$$\text{i) } (\mathcal{R}_{2n+1}^t \underbrace{0 \dots 0}_1 + \mathcal{R}_{2n+1}^t \underbrace{0 \dots 0}_2), \quad (\mathcal{R}_{2n+1}^t \underbrace{0 \dots 0}_1 - \mathcal{R}_{2n+1}^t \underbrace{0 \dots 0}_2), \quad \text{and} \quad \mathcal{R}_{2n+1}^t \underbrace{0 \dots 0}_2 \quad \text{in case of}$$

odd rank,

$$\text{ii) } \mathcal{R}_{2n}^t \underbrace{0 \dots 0}_2 \quad \text{in case of even rank.}$$

For the modified Poisson brackets between the leading part of \mathcal{L} and the leading parts of $\mathcal{B}_{(n)}^1$ and $\mathcal{B}_{(n)}^3$, respectively, we consider \mathcal{O} , the leading part of \mathcal{L} , as a polynomial in the variables:

$$(\mathcal{R}_{2v+1}^t \underbrace{0 \dots 0}_1 - \mathcal{R}_{2v+1}^t \underbrace{0 \dots 0}_2), \quad \mathcal{R}_{2v+1}^t \underbrace{0 \dots 0}_2$$

and the linearly independent ‘‘standard components’’ $\mathcal{R}_{N,i}^t$ with three and more space-like indices (two of which occupy the extremal positions), after having replaced the factors $(\mathcal{R}_{2v+1}^t \underbrace{0 \dots 0}_1 + \mathcal{R}_{2v+1}^t \underbrace{0 \dots 0}_2)$ and $\mathcal{R}_{2v}^t \underbrace{0 \dots 0}_2$ by the leading parts

of $\mathcal{B}_{(v)}^1$ and $\mathcal{B}_{(v)}^3$, respectively, and by the standard components $\mathcal{R}_{2v+3,j}^t$ and $\mathcal{R}_{2v+2,j}^t$, respectively, with three and more space-like indices, and, subsequently, having counted the leading parts of $\mathcal{B}_{(v)}^1$ and $\mathcal{B}_{(v)}^3$, respectively, as coefficients.

By assumption, the degree \hat{q}_{\min} of this polynomial $\hat{\mathcal{O}}$ is larger than zero. We look for the contributions from those monomials which contain variables $\mathcal{R}_{N,i}^t$ (of the specified type) of highest rank N_{\max} . Among these monomials we select those in which the variables $\mathcal{R}_{N_{\max},i}^t$ of rank N_{\max} carry a maximal number p_{\max} of times the least number s_{\min} of space-like indices. If $s_{\min} = 2$, then the variables $\mathcal{R}_{N_{\max},i}^t$ in question must have odd rank $N_{\max} = 2v_0 + 3$ and they occur in the monomials under consideration in the form: p_1 times $(\mathcal{R}_{2v_0+1}^t \underbrace{0 \dots 0}_1 - \mathcal{R}_{2v_0+1}^t \underbrace{0 \dots 0}_2)$ and p_2 times $\mathcal{R}_{2v_0+1}^t \underbrace{0 \dots 0}_2$: $p_1 + p_2 = p_{\max}$.

We further concentrate on the contribution from those monomials for which one of the variables $(\mathcal{R}_1^t \underbrace{0 \dots 0}_1 - \mathcal{R}_2^t \underbrace{0 \dots 0}_2)$ and $\mathcal{R}_1^t \underbrace{0 \dots 0}_2$, say $(\mathcal{R}_1^t \underbrace{0 \dots 0}_1 - \mathcal{R}_2^t \underbrace{0 \dots 0}_2)$, occurs a maximal number of times, here: $p_{1 \max}$ times. If more than one monomial of this kind gives a net contribution, then the residual factors in the remaining variables \mathcal{R}^t may be assumed to be linearly independent.

When we compute the modified Poisson bracket between the leading part of \mathcal{L} and $\mathcal{R}_1^t \underbrace{0 \dots 0}_2$ for sufficiently large values of n , we find that the right-hand side received a non-vanishing contribution from monomials, each one containing the variables $(\mathcal{R}_1^t \underbrace{0 \dots 0}_1 - \mathcal{R}_2^t \underbrace{0 \dots 0}_2)(p_{1 \max} - 1)$ times, the variables $\mathcal{R}_1^t \underbrace{0 \dots 0}_2$ $(p_{\max} - p_{1 \max})$ times and the factor $\mathcal{R}_1^t \underbrace{0 \dots 0}_1 \underbrace{0 \dots 0}_2$ once, the residual factors in the remaining variables \mathcal{R}^t being linearly independent. This contribution cannot be compensated by any other term occurring on the right-hand side of the modified Poisson bracket in question. Hence the least number of space-like indices s_{\min} must satisfy the bound $s_{\min} \geq 3$.

The monomials on which we focus our attention now have the form of a maximal p_{\max} -fold product of standard components $\mathcal{R}_{N_{\max}, i}^t$ of maximal rank N_{\max} and minimal number $s_{\min} \geq 3$ of space-like indices with residual factors in the variables \mathcal{R}^t of lower rank. Should several monomials with an identical structure in the standard components $\mathcal{R}_{N_{\max}, i}^t$ of maximal rank N_{\max} and minimal number of space-like indices give a net contribution, then the residual factors in the variables \mathcal{R}^t of lower rank may be assumed to be linearly independent. The effect of the modified Poisson bracket operation with either $(\mathcal{R}_1^t \underbrace{0 \dots 0}_1 + \mathcal{R}_2^t \underbrace{0 \dots 0}_2)$ or $\mathcal{R}_1^t \underbrace{0 \dots 0}_2$ for a definite value of n on the leading part of \mathcal{L} , as far as the generation

of terms with the least number of space-like indices is concerned, consists of inserting $(2n + 1)$ or $(2n)$ zero indices, respectively, in a definite way into $\mathcal{R}_{N, j}^t$ such that the extremal positions are still occupied by space-like indices and such that under this insertion for sufficiently large fixed values of n , different standard components $\mathcal{R}_{N, j}^t$ go over into independent linear combinations of standard components of correspondingly higher rank. Thus, when we compute the modified Poisson bracket of the leading part of \mathcal{L} with $(\mathcal{R}_1^t \underbrace{0 \dots 0}_1 + \mathcal{R}_2^t \underbrace{0 \dots 0}_2)$ and $\mathcal{R}_1^t \underbrace{0 \dots 0}_2$, respectively, we find – looking at the terms on the right-hand side

which contain $(p_{\max} - 1)$ times a standard component $\mathcal{R}_{N_{\max}, i}^t$ of rank N_{\max} and once a standard component $\mathcal{R}_{N_{\max} + 2n + 1, j}^t$ or $\mathcal{R}_{N_{\max} + 2n, j}^t$ of rank $N_{\max} + 2n + 1$ or $N_{\max} + 2n$, respectively – a non-vanishing contribution from each one of the selected monomials $\left(\prod_{r=1}^{p_{\max}} \mathcal{R}_{N_{\max}, i_r}^t \right)$ times a product of variables \mathcal{R}^t of rank $< N_{\max}$

of the leading part of \mathcal{L} . These contributions do not mutually cancel nor can they be compensated by other terms occurring on the right-hand side of the modified Poisson bracket.

Thus the hypothesis, in the form $\hat{q}_{\min} > 0$, is seen to lead to a contradiction with the required commutativity of the debatable invariant charge \mathcal{L} and the special elements $\mathcal{B}_{(m)}^1$ and $\mathcal{B}_{(m)}^3$. Consequently, in three space-time dimensions, all invariant charges commuting with the special elements $\mathcal{B}_{(m)}^1(\mathcal{A})$ and $\mathcal{B}_{(m)}^3(\mathcal{A})$ are contained in the polynomial ring over $\mathcal{B}_{(m)}^1(\mathcal{A})$ and $\mathcal{B}_{(m)}^3(\mathcal{A})$. In other words, the polynomial ring under consideration forms a maximal abelian subalgebra $\mathfrak{a}_{(m,0,0)} = \mathfrak{a}_{\mathcal{P}}$ of the algebra $\mathfrak{h}_{\mathcal{P}}$ of all invariant charges in three space-time dimensions.

For more than three space-time dimensions, the principal idea of the proof of the maximality of the algebra $\mathfrak{a}_{(m,0,\dots,0)}$ is the same as for three dimensions, although the details of the argument are much more involved.

As before, we shall deduce a contradiction from the hypothesis that there exists an invariant charge \mathcal{L} which commutes with all the tensor components $\mathcal{R}_{i(r)j(r)}^t$ as well as with all the special elements $\mathcal{B}_{(n)}^{1,3}(\mathcal{A})$ and $\mathcal{B}_{(n,k)}^2(\mathcal{A})$, but which does not belong to the polynomial ring in $\mathcal{R}_{i(r)j(r)}^t$, $\mathcal{B}_{(n)}^{1,3}(\mathcal{A})$, and $\mathcal{B}_{(n,k)}^2(\mathcal{A})$.

After having subtracted from \mathcal{L} a suitable element of this polynomial ring, we may assume that \mathcal{L} , regarded as a polynomial in \mathcal{R}^t , has a leading part \mathcal{O} of minimal degree $q = q_{\min} > 0$. As before, in order that the Poisson brackets of \mathcal{L} and all the elements of the polynomial ring vanish, at the very least, the modified Poisson brackets between the leading part \mathcal{O} of \mathcal{L} and the leading parts of $\mathcal{B}_{(n)}^{1,3}(\mathcal{A})$, $\mathcal{B}_{(n,k)}^2(\mathcal{A})$ and the tensor components $\mathcal{R}_{i(r)j(r)}^t$ must vanish. Like in the three dimensional case, in the leading part of \mathcal{L} we replace the combinations

$$\mathcal{R}_{i \underbrace{0 \dots 0}_{2v+1}}^t, \quad \mathcal{R}_{i_1 i_2}^t \mathcal{R}_{i_2 i_3}^t \cdots \mathcal{R}_{i_{2k-1} i_{2k}}^t \cdot \mathcal{R}_{i_{2k} \underbrace{0 \dots 0}_{2v}}^t$$

and

$$\varepsilon^{i_1 \dots i_{d-1}} \mathcal{R}_{i_1 i_2}^t \mathcal{R}_{i_3 i_4}^t \cdots \mathcal{R}_{i_{d-4} i_{d-3}}^t \mathcal{R}_{i_{d-2} \underbrace{0 \dots 0}_{2v}}^t i_{d-1}$$

by the leading parts of $\mathcal{B}_{(v)}^1$, $\mathcal{B}_{(v,k)}^2$, and $\mathcal{B}_{(v)}^3$, respectively, and by those standard components $\mathcal{R}_{N,j}^t$ which are linearly independent of the above combinations, and we count the leading parts of $\mathcal{B}_{(v)}^{1,3}$, $\mathcal{B}_{(v,k)}^2$ as well as the tensor components $\mathcal{R}_{i(r)j(r)}^t$ as coefficients.

Thus we are led to consider the leading part of \mathcal{L} as a polynomial in the remaining standard components as variables [1]. By assumption, the degree \hat{q}_{\min} of this polynomial \mathcal{O} is larger than zero. This way of looking at things is motivated by the task of computing the modified Poisson brackets in question. For the time being, we concentrate on the leading terms of the polynomial \mathcal{O} : If the least number s_{\min} of space-like indices occurring in the (specified) variables \mathcal{R}^t of maximal rank $N_{\max} > 2$ present in the leading terms is equal to two, then the latter terms have the form

$$\text{i) } r_{(i_1 j_1) \dots (i_p j_p)} \cdot \mathcal{R}_{i_1 \underbrace{0 \dots 0}_{2v_0+1}}^t \cdots \mathcal{R}_{i_p \underbrace{0 \dots 0}_{2v_0+1}}^t \text{ for odd values of } N_{\max} = 2v_0 + 3 \text{ with}$$

maximal $p = p_{\max}$,

$$\text{ii) } r_{[i_1 j_1] \dots [i_p j_p]} \cdot \mathcal{R}_{i_1 \underbrace{0 \dots 0}_{2v_0}}^t \cdots \mathcal{R}_{i_p \underbrace{0 \dots 0}_{2v_0}}^t \text{ for even values of } N_{\max} = 2v_0 + 2 \text{ with}$$

maximal $p = p_{\max}$, where $r_{(i_1 j_1) \dots (i_p j_p)}$ and $r_{[i_1 j_1] \dots [i_p j_p]}$ are polynomials in the previously specified variables \mathcal{R}^t which either involve more than two space-like indices or are of rank $N < N_{\max}$, the polynomials being symmetric and antisymmetric, respectively, in each one of the bracketed pairs of indices and being symmetric under the interchange of brackets.

The required vanishing of the modified Poisson brackets in question implies the vanishing of the result obtained by contracting only the space indices of the factors $\mathcal{R}_{i_1 0 \dots 0 j_1}^t \dots \mathcal{R}_{i_p 0 \dots 0 j_p}^t$ in the leading terms of $\hat{\mathcal{C}}$ and the factors $\mathcal{R}_{i_1 0 \dots 0 j_1}^t$ in the leading terms of $\mathcal{B}_{(n,1)}^2$. From this we infer the restrictions

$$i) \quad r_{(i,j)(i_2 j_2) \dots (i_p j_p)} \mathcal{R}_{jk}^t = \mathcal{R}_{ij}^t r_{(j,k)(i_2 j_2) \dots (i_p j_p)}$$

and

$$ii) \quad r_{[i][i_2 j_2] \dots [i_p j_p]} \mathcal{R}_{jk}^t = \mathcal{R}_{ij}^t r_{[jk][i_2 j_2] \dots [i_p j_p]},$$

which, in turn, imply that

i) $r_{(i_1 j_1)(i_2 j_2) \dots (i_p j_p)}$ is a linear combination of symmetrized products of the tensors $\delta_{i_r j_r}$, $(\mathcal{R}^{t2})_{i_r j_r}$, $(\mathcal{R}^{t4})_{i_r j_r}$, ..., symmetric under the replacement $(i_r, j_r) \leftrightarrow (i_s, j_s)$, and

ii) $r_{[i_1 j_1][i_2 j_2] \dots [i_p j_p]}$ is a linear combination of symmetrized products of the tensors $(\mathcal{R}^t)_{i_r j_r}$, $(\mathcal{R}^{t3})_{i_r j_r}$, ..., and, in case of odd space-time dimensions, in $e^{k_1 \dots k_{d-3} i_r j_r} \cdot \mathcal{R}_{k_1 k_2}^t \dots \mathcal{R}_{k_{d-4} k_{d-3}}^t$, symmetric under the replacement $[i_r, j_r] \leftrightarrow [i_s, j_s]$.

By construction, leading terms of type ii) with such polynomials $r_{[i_1 j_1] \dots [i_p j_p]}$ do not occur in the polynomial $\hat{\mathcal{C}}$. This statement also applies for the case $N_{\max} = 2$, where the leading terms turn out to be polynomials in $\mathcal{R}_{i(r)j(r)}$ and the Casimir operators of the group of spatial rotations in the momentum rest frame.

In order to rule out the appearance of leading terms of type i) with polynomials $r_{(i_1 j_1) \dots (i_p j_p)}$ specified above and different from the special case $p=1$, $r_{(ij)}$ proportional to δ_{ij} , we exploit the required vanishing of the modified Poisson bracket between the reinterpreted leading part of \mathcal{L} and the element $\mathcal{B}_{(1)}$. Here besides the leading terms we have to consider also those monomials of $\hat{\mathcal{C}}$ which involve the maximal $p = p_{\max}$ -fold product of variables \mathcal{R}^t of maximal rank $N_{\max} = 2\nu_0 + 1$, $(p_{\max} - 1)$ factors of which have two space-like indices and one factor of which has four space-like indices. Thus the relevant terms are:

$$\begin{aligned} & r_{(i_1 j_1) \dots (i_p j_p)} \mathcal{R}_{i_1 \underbrace{0 \dots 0}_{2\nu_0+1} j_1}^t \cdot \dots \cdot \mathcal{R}_{i_p \underbrace{0 \dots 0}_{2\nu_0+1} j_p}^t + \sum_{a_1 + b_1 + c_1 = 2\nu_0 - 1} S_{i_1(a_1)j_1(b_1)k_1(c_1)l_1; (i_2 j_2) \dots (i_p j_p)} \\ & \times \mathcal{R}_{i_1 \underbrace{0 \dots 0}_{a_1} j_1 \underbrace{0 \dots 0}_{b_1} k_1 \underbrace{0 \dots 0}_{c_1} l_1}^t \cdot \mathcal{R}_{i_2 \underbrace{0 \dots 0}_{2\nu_0+1} j_2}^t \cdot \dots \cdot \mathcal{R}_{i_p \underbrace{0 \dots 0}_{2\nu_0+1} j_p}^t, \end{aligned}$$

where S_{\dots} are polynomials in the previously specified variables which either involve three, five or more space-like indices or are of rank $N < N_{\max}$, the polynomials being symmetric under the inversion of each set of indices separately

$$(i_r, j_r) \rightarrow (j_r, i_r),$$

$$i_1(a_1)j_1(b_1)k_1(c_1)l_1 \rightarrow l_1(c_1)k_1(b_1)j_1(a_1)i_1,$$

and symmetric under the interchange of the sets of indices

$$(i_r, j_r) \leftrightarrow (i_s, j_s).$$

The vanishing of the modified Poisson bracket implies the vanishing of the result obtained by contracting all indices of the factors $\mathcal{R}_{i_r, \underbrace{0, \dots, 0}_{2v_0+1} j_r}$, but only the *space-like* indices of the factor $\mathcal{R}_{i_1, \underbrace{0, \dots, 0}_{a_1} j_1, \underbrace{0, \dots, 0}_{b_1} k_1, \underbrace{0, \dots, 0}_{c_1} \ell_1}$ with the indices of $\mathcal{R}_{i_0 i}$. We have selected the element $\mathcal{B}_{(1)}^1$ because it is “in a sufficiently general position” and because, moreover, it has the advantage of having a leading part which consists of the leading terms $m \cdot \mathcal{R}_{i_0 i}$ only.

It suffices to compute the contribution of those terms arising from the indicated contractions which contain $(p_{\max} - 1)$ factors of the type $\mathcal{R}_{i, \underbrace{0, \dots, 0}_{2v_0+1} j}$ and one factor of the type $\mathcal{R}_{i', \underbrace{0, \dots, 0}_{a'} j', \underbrace{0, \dots, 0}_{b'} k', \underbrace{0, \dots, 0}_{c'} \ell'}$, $a' + b' + c' = 2v_0$ with three of the four indices i', j', k', ℓ' assuming the same value, say 1, and the remaining index assuming a different value, say 2. This contribution cannot be cancelled by any other term on the right-hand side of the modified Poisson bracket. Hence, it has to vanish. The upshot of this is a set of linear equations relating the polynomials $r_{(12)(i_2 j_2) \dots (i_p j_p)}$, $s_{2(a_1)1(b_1)1(c_1)1; (i_2 j_2) \dots (i_p j_p)}$ and $s_{1(a_1)2(b_1)1(c_1)1; (i_2 j_2) \dots (i_p j_p)}$. This set of equations turns out to be consistent only if $r_{(12)(i_2 j_2) \dots (i_p j_p)}$ vanishes. However, this is definitely not the case for the previously specified polynomials. Thus, also leading terms of type i) do not occur in $\hat{\mathcal{O}}$.

Summarizing the above discussion, we have ruled out the possibility $s_{\min} = 2$ for the leading terms of $\hat{\mathcal{O}}$, the reinterpreted leading part of \mathcal{L} . The argument for the case $s_{\min} \geq 3$ follows the same line as the one for three space-time dimensions. It suffices to invoke the vanishing of the modified Poisson brackets between the leading terms of $\hat{\mathcal{O}}$ and the leading terms of the elements $\mathcal{B}_{(n)}^1$, to infer the vanishing of the former terms.

Thus the hypothesis of the existence of an invariant charge \mathcal{L} with the required commutativity properties and the degree $q_{\min} > 0$ is seen to lead to a contradiction. Hence, the polynomial ring in $\mathcal{R}_{i^{(r)} j^{(r)}}$ and $\mathcal{B}_{(n)}^{1,3}(\mathcal{A})$ and $\mathcal{B}_{(n,k)}^2(\mathcal{A})$ forms a maximal abelian subalgebra $\mathfrak{a}_{(m,0,\dots,0)}$ of the algebra $\mathfrak{h}_{(m,0,\dots,0)}$. Thus, finally, we conclude that the algebra $\mathfrak{a}_{\mathcal{L}}$ is a maximal abelian subalgebra of the algebra $\mathfrak{h}_{\mathcal{L}}$ for all integer space-time dimensions $d \geq 3$.

IV. Conclusions

The preceding identification of a maximal abelian subalgebra $\mathfrak{a}_{\mathcal{L}}$ of the algebra $\mathfrak{h}_{\mathcal{L}}$ of the invariant (conserved observable) charges of the Nambu-Goto string theory meets a prerequisite for a systematic investigation of the unitary positive energy representations of \mathfrak{g} . Besides, this identification may also help to construct non-groundstate wave-functionals $\psi(\mathcal{C})$ of the Nambu-Goto theory satisfying the renormalized loop wave equation in WKB-approximation [4]. These wave-functionals $\psi(\mathcal{C})$ could be specified as simultaneous eigenwave-functionals of all

elements of $\mathfrak{a}_{\mathcal{P}}$ corresponding to non-trivial eigenvalues within the allowed spectrum. Presumably, this requirement effectively suppresses the undesirable formation of spikes or branches [5].

As a by-product of the analysis of this paper, we realize that there are no Casimir operators in the algebra $\mathfrak{h}_{\mathcal{P}}$. If such Casimir operators existed, without loss of generality, we could arrange each element of a basis of them to be contained in the intersection of $\mathfrak{a}_{\mathcal{P}}$ with a definite stratum $V^{(\ell)}(\mathfrak{h}_{\mathcal{P}})$. However, in this intersection, there are no central charges! Actually, as exemplified by the non-abelian two-dimensional Lie algebra, it is not all that unusual that the enveloping algebra of a given algebra contains no Casimir operators.

In [2] we suggested to interpret the loop wave equations of the Nambu-Goto theory as an infinite collection of representation conditions imposed on the Casimir operators of $\mathfrak{h}_{\mathcal{P}}^{\pm}$. We were led to this suggestion by the fact that the ‘‘operators’’ on the left-hand side of the loop wave equations have vanishing Poisson brackets with the elements \mathcal{Z} of $\mathfrak{h}_{\mathcal{P}}^{\pm}$ (i.e. by the invariance property of the charges \mathcal{Z}) and by the analogy with the point particle case. In order to implement this suggestion, it is thus necessary to relate the operators in question to the Casimir operators of the algebras $\mathfrak{h}_{\mathcal{P}}^{\pm}$. Since, for the relevant sector of the classical theory, we were able to express the invariant content of the left-hand side of the loop wave equations in terms of the invariant charges \mathcal{Z} using limiting processes and division [6], we expect to hit upon the Casimir operators of $\mathfrak{h}_{\mathcal{P}}^{\pm}$ when we pass to the closure of $\mathfrak{h}_{\mathcal{P}}^{\pm}$ and its (corresponding) algebra of fractions [7]. It is the construction of the Casimir operators and their relation to the constraints that constitute one of the crucial tests for the feasibility of our algebraic program. We shall continue our studies in this direction.

V. Appendix

We want to point out a deeper algebraic structure underlying the abelian charges $\mathcal{A}_{(N)}^{(K)}$ generated by $\text{tr}(\ln \phi_{\lambda I})^K$: From the modified Poisson bracket (cf. [1]) among $\ln \phi_{\lambda I}$:

$$\begin{aligned} \{\ln \phi_{\lambda I} \otimes \ln \phi_{\mu I}\}^* &= \lambda \mu [\ln \phi_{\lambda I} \otimes \mathbf{1} - \mathbf{1} \otimes \ln \phi_{\mu I}, \Gamma^\alpha \otimes \Gamma_\alpha] \\ &\quad - \frac{\lambda \mu}{\lambda^2 - \mu^2} [\ln \phi_{\lambda I} \otimes \mathbf{1} + \mathbf{1} \otimes \ln \phi_{\mu I}, (\lambda^2 + \mu^2) \Gamma^\alpha \otimes \Gamma_\alpha - \frac{1}{4} \lambda \mu [\Gamma^\alpha, \Gamma^\beta] \otimes [\Gamma_\alpha, \Gamma_\beta]] \end{aligned}$$

we deduce the graded Lie algebra relations

$$\{\mathbb{R}_{AB}^{\ell_1}, \mathbb{R}_{CD}^{\ell_2}\}^* = -2f_{AB,CD}^{EF} \mathbb{R}_{EF}^{\ell_1 + \ell_2}, \quad \ell_1, \ell_2 \geq 0$$

with the structure constants $f_{AB,CD}^{EF}$ of the de Sitter algebra $\mathfrak{so}(1, d)$. Here, capital indices run from 0 to d , and \mathbb{R}_{AB}^ℓ are defined by

$$\begin{aligned} \mathbb{R}_{\mu\nu}(\lambda) &\doteq \sum_{\ell=0,2,\dots} \lambda^{\ell+2} \mathbb{R}_{\mu\nu}^\ell \quad \text{for } \mu, \nu = 0, 1, \dots, d-1; \\ 2\mathbb{R}_{d\mu}(\lambda) &= -2\mathbb{R}_{\mu d}(\lambda) \doteq \mathbb{P}_\mu(\lambda) = \lambda \mathcal{P}_\mu + \sum_{\ell=1,3,\dots} \lambda^{\ell+2} \cdot 2\mathbb{R}_{d\mu}^\ell, \quad \mathbb{R}_{d\mu}^\ell = -\mathbb{R}_{\mu d}^\ell, \end{aligned}$$

for the components referring to an additional, space-like direction d . Accordingly, the range of the gradation index ℓ of \mathbb{R}_{AB}^ℓ is

$$\begin{aligned} \ell &= 0, 2, 4, \dots & \text{if } A \neq d \neq B, \\ \ell &= 1, 3, 5, \dots & \text{if } A = d \text{ or } B = d, \end{aligned}$$

which is compatible with the above Lie algebra relations since $\text{SO}(1, d)/\text{SO}(1, d-1)$ is a symmetric space.

Now, since $I^{\alpha\beta} = \frac{1}{4}[\Gamma^\alpha, \Gamma^\beta]$, $I^{d\alpha} = -I^{\alpha d} = \frac{1}{2}\Gamma^\alpha$ are generators of a representation of $\text{so}(1, d)$, it follows that the charges

$$\text{tr}(\ln \phi_{\lambda T})^K = 2^K \text{tr}(\mathbb{R}_{AB}(\lambda) I^{AB})^K$$

are polynomials in the $\text{so}(1, d)$ -invariant “generalized $\text{so}(1, d)$ Casimir operator” charges

$$\begin{aligned} &\mathbb{R}_{A_1 A_2}(\lambda) g^{A_2 A_2} \mathbb{R}_{A_2 A_3}(\lambda) \dots \mathbb{R}_{A_{2k} A_1}(\lambda) g^{A_1 A_1}, \\ &(e^{A_1 \dots A_{d+1}} \mathbb{R}_{A_1 A_2}(\lambda) \dots \mathbb{R}_{A_d A_{d+1}}(\lambda), \text{ if } d+1 \text{ is even}). \end{aligned}$$

This structure might prove helpful for the construction of the relevant representations of \mathfrak{h}_φ .

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