

Convex Hamiltonian Energy Surfaces and Their Periodic Trajectories

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Abstract. In this paper we introduce symplectic invariants for convex Hamiltonian energy surfaces and their periodic trajectories and show that these quantities satisfy several nontrivial relations. In particular we show that they can be used to prove multiplicity results for the number of periodic trajectories.

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I. Introduction and Statement of the Main Results

1.1. Dynamical and Geometrical Formulation of the Problem

Denote by $\langle \cdot, \cdot \rangle$ the usual inner product on \mathbb{R}^{2n} and let J be the standard complex structure on \mathbb{R}^{2n} given by the matrix

$$J = \begin{bmatrix} 0_n & 1_n \\ -1_n & 0_n \end{bmatrix}.$$

Associated to $\langle \cdot, \cdot \rangle$ and J is the symplectic form Ω given by

$$\Omega = \langle J \cdot, \cdot \rangle.$$

Assume $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is a smooth map. The so-called associated Hamiltonian vectorfield is defined by the formula

$$dH = X_H \lrcorner \Omega.$$

The corresponding differential equations

$$(HS) \quad \dot{x} = X_H(x)$$

is called a Hamiltonian system. If x solves (HS) then

$$\frac{d}{dt} x = \Omega(X_H(x), X_H(x)) = 0$$

so that H is constant on x . Therefore it is natural to ask for periodic solutions of (HS) having a prescribed energy \bar{H} .

Though the problem of finding a periodic solution with a prescribed energy seems to belong to the theory of dynamical systems, it is possible to formulate it in purely geometrical terms. This can be done in great generality (see [W 2]). Here, however, we shall restrict ourselves to the cases we shall in fact study, namely convex smooth hypersurfaces in \mathbb{R}^{2n} . More precisely we say $S \subset \mathbb{R}^{2n}$ satisfies condition (\mathcal{H}) if the following holds:

$S \subset \mathbb{R}^{2n}$ is a compact C^∞ -manifold bounding a convex region. Moreover S has a nonvanishing Gaussian curvature and S encloses $0 \in \mathbb{R}^{2n}$. The collection of all S satisfying (\mathcal{H}) will be denoted by \mathcal{H} . (H)

The condition that $0 \in \mathbb{R}^{2n}$ is enclosed by S is only some kind of normalisation and has nothing to do with the results obtained.

We defined a 1-form θ on \mathbb{R}^{2n} by

$$\theta(x)h = \frac{1}{2} \langle Jx, h \rangle.$$

Then $d\theta = \Omega$. Denote by λ the restriction of θ to S and put $\omega = d\lambda$. Then $\text{kern}(\omega)$ must be nontrivial since $\dim(S)$ is odd. In fact,

$$\text{kern}(\omega_x) = \mathbb{R}Jn(x),$$

where $n(x)$ is the outward pointing normal vector at $x \in S$ and moreover

$$\lambda(Jn(x)) = \frac{1}{2} \langle Jx, Jn(x) \rangle = \frac{1}{2} \langle x, n(x) \rangle > 0,$$

since (\mathcal{H}) holds. Therefore $\lambda \wedge \omega^{n-1}$ is a volume on S . Hence (S, ω) is a manifold of contact type in the sense of Weinstein, [W 2]. As a consequence of our previous discussion we have the following

Lemma 1. *Let $S \in \mathcal{H}$. Then $\omega = \Omega|_S$ defines a canonical line bundle $\mathcal{L}_S \rightarrow S$, where the fibre over $x \in S$ consists of all those vectors v annihilating ω_x , i.e., $v \lrcorner \omega_x = 0$. Moreover \mathcal{L}_S possesses a canonical orientation induced by the unique vectorfield ζ on S satisfying*

$$\zeta \lrcorner \lambda \equiv 1, \quad \zeta \lrcorner \omega \equiv 0. \tag{1}$$

See [W 2] for the easy proof. Since $\mathcal{L}_S \subset TS$ we have a one dimensional and therefore integrable distribution on S .

Definition 1. Let $S \in \mathcal{H}$. A periodic Hamiltonian trajectory on S is a submanifold Γ of S which is diffeomorphic to S^1 , satisfying

$$T\Gamma = \mathcal{L}_S|_\Gamma.$$

The collection of all Hamiltonian trajectories will be denoted by $\mathcal{T}(S)$.

If $H: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is now a Hamiltonian having $S \in \mathcal{H}$ as a regular energy surface, say $H=1$, then the periodic solutions of the corresponding Hamiltonian system with energy 1 on S are just parametrisations of Hamiltonian trajectories $\Gamma \in \mathcal{T}(S)$. In fact each $x_0 \in \Gamma$ is the initial data for a periodic solution x lying entirely on Γ .

By results of Weinstein [W 1] and Rabinowitz [R 1] it is known that $\mathcal{T}(S) \neq \emptyset$ for $S \in \mathcal{H}$. Knowing that $\mathcal{T}(S) \neq \emptyset$ for $S \in \mathcal{H}$ one can ask for its cardinality. Let $\alpha_i > 0, i = 1, \dots, n$, so that the α_i 's are independent over \mathbb{Z} . Define $S = S(\alpha_1, \dots, \alpha_n)$ by

$$S = \left\{ x \in \mathbb{R}^{2n} \left| \frac{1}{2} \sum_{i=1}^n \alpha_i (x_i^2 + x_{i+n}^2) = 1 \right. \right\}.$$

One easily shows that $\#\mathcal{T}(S) = n$. As far as the cardinality is concerned this is the worst known example. Hence the following conjecture.

Conjecture 1. If $S \in \mathcal{H}$, then $\#\mathcal{T}(S) \geq n$.

A few partial results are known to be true [E-L, E-La, E 1, B-L-M-R], see also [A-M, H 1].

In this paper we shall associate to $S \in \mathcal{H}$ its index interval $\sigma(S)$ which is a compact interval in $(0, \infty)$. We show in particular that $\sigma(S)$ degenerates to a point if $\#\mathcal{T}(S) < \infty$. To the Hamiltonian trajectories $\Gamma \in \mathcal{T}(S)$ we shall associate two positive numbers $\gamma(\Gamma)$ and $\bar{\gamma}(\Gamma)$ which are independent. They are called the *total* and the *mean-torsion* at Γ . In the main result of this paper we shall prove that $\sigma(S)$ and the collections $\{\gamma(\Gamma)\}$ and $\{\bar{\gamma}(\Gamma)\}$ are not independent and that always certain inequalities and equalities have to hold. The inequalities turn out to be optimal. This new approach gives besides new results for Hamiltonian systems a much deeper insight to the problem of periodic Hamiltonian trajectories than previous results. Several open problems are mentioned. For instance, it is shown that if $\mathcal{T}(S)$ is finite, then

$$\sum_{\bar{\gamma}=\sigma} \gamma(\Gamma)^{-1} \geq 1,$$

where $\sigma(S) = \{\sigma\}$. Moreover $\gamma(\Gamma) > 1$ for all $\Gamma \in \mathcal{T}(S)$ if $n \geq 2$. So in particular the above inequality implies that $\# \mathcal{T}(S) \geq 2$ for $n \geq 2$, thus improving the results of [E–La], where this was proven for $n \geq 3$. Further it will be shown that the above inequality is optimal in the sense that there exists an S for which we have equality.

1.2. The Index Interval of an Energy Surface and Torsion Indices for Its Hamiltonian Trajectories

We start with a definition

Definition 2. Denote by $\hat{\mathcal{H}}$ the collection of all maps $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} H &\in C^\infty(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R}), \\ H(\lambda x) &= \lambda^2 H(x) \quad \text{for } \lambda \geq 0 \quad \text{and } x \in \mathbb{R}^{2n}, \\ H''(x) &\geq \alpha_H \text{Id} \quad \forall x \in \mathbb{R}^{2n} \setminus \{0\}, \quad \alpha_H > 0. \end{aligned} \tag{1}$$

Here $H''(x)$ is the linearization of the gradient H' of H at $x \in \mathbb{R}^{2n} \setminus \{0\}$.

The following lemma is obvious:

Lemma 2. *There is a natural bijection $\mathcal{H} \rightarrow \hat{\mathcal{H}}$ associating to $S \in \mathcal{H}$ the unique $H_S \in \hat{\mathcal{H}}$ such that*

$$H_S^{-1}(1) = S. \quad \square$$

Let $H \in \hat{\mathcal{H}}$. Its Fenchel conjugate is the function $H^* \in \hat{\mathcal{H}}$ defined by

$$H^*(y) = \max_{x \in \mathbb{R}^{2n}} (\langle x, y \rangle - H(x)). \tag{2}$$

We equip \mathcal{H} with the metric $d : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}^+$ defined as follows:

$$d(S_1, S_2) = \max_{x \in S_1} \inf_{y \in S_2} |x - y| + \max_{y \in S_2} \inf_{x \in S_1} |x - y|,$$

which is the Hausdorff metric. The map $H \rightarrow H^*$ induces a map $\mathcal{H} \rightarrow \mathcal{H}$ which is continuous for the topology induced by d . Next we introduce a Hilbert space E by

$$E = \left\{ x : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^{2n} \mid x \text{ is absolutely continuous with square integrable derivative and } \int_0^1 x(t) dt = 0 \right\}. \tag{3}$$

The inner product on E is given by

$$(x, y) = \int_0^1 \langle \dot{x}(t), \dot{y}(t) \rangle dt. \tag{4}$$

We associate to $S \in \mathcal{H}$ a $C^{1,1}$ -Hilbert manifold $M_S, M_S \subset E$, by

$$M_S = \left\{ x \in E \mid \int_0^1 H_S^*(-J\dot{x}(t)) dt = 1 \text{ and } \int_0^1 \langle J\dot{x}(t), x(t) \rangle dt < 0 \right\}. \tag{5}$$

Here $C^{1,1}$ -Hilbert manifold means that there exists an atlas so that the overlap maps $\sigma_\alpha \circ \sigma_\beta^{-1}$ are C^1 with a locally Lipschitz derivative. M_S is actually a $C^{1,1}$ -

submanifold of E . The natural S^1 -action on E by phase-shift denoted by

$$S^1 \times E \rightarrow E : (a, x) \rightarrow a * x$$

induces an S^1 -action on M_S . Hence M_S belongs to the category of paracompact S^1 -spaces. We define a smooth map $A \in C^\infty(E, \mathbb{R})$ by

$$A(x) = \frac{1}{2} \int_0^1 \langle J\dot{x}(t), x(t) \rangle dt, \tag{6}$$

and denote by A_S the restriction of A to M_S . For $d \in (-\infty, 0)$ we define

$$M_S^d := A_S^{-1}((-\infty, d]). \tag{7}$$

Note that A is S^1 -invariant. In the following we write (most of the time)

$$G = S^1, \quad E_G = S^\infty, \quad B_G = \mathbb{C}P^\infty, \quad p: S^\infty \rightarrow \mathbb{C}P^\infty \text{ projection.}$$

Then (E_G, p, B_G) is the universal bundle for G -actions. Denote by $M_{S,G}^d$ the “ G -quotient” of M_S^d , that is

$$M_{S,G}^d = (M_S^d \times E_G) / G, \tag{8}$$

where G acts freely in the obvious way on $M_S^d \times E_G$. Hence we have principal bundles

$$M_S^d \times E_G \rightarrow M_{S,G}^d. \tag{9}$$

Denote by $f_S: M_{S,G} \rightarrow B_G$ the up to homotopy uniquely defined classifying map. From the diagram

$$\begin{array}{ccccc} M_S^d \times E_G & \xrightarrow{\text{incl}} & M_S \times E_G & \xrightarrow{\tilde{f}_S} & E_G \\ \downarrow /G & & \downarrow /G & & \downarrow p \\ M_{S,G}^d & \xrightarrow{\text{incl}} & M_{S,G} & \xrightarrow{f_S} & B_G \end{array} \tag{10}$$

and the properties of classifying maps, see [Hu], it follows immediately that the restriction of f_S to $M_{S,G}^d$ denoted by f_S^d can serve as a classifying map for $M_S^d \times E_G \rightarrow M_{S,G}^d$. Denote by \bar{H} the Alexander-Spanier-Cohomology with coefficients \mathbb{Q} . One knows that

$$\bar{H}(B_G) \cong \mathbb{Q}[\eta], \quad \eta \in \bar{H}^2(B_G) \setminus \{0\}. \tag{11}$$

We define for $S \in \mathcal{H}$ a map $\alpha_S: (-\infty, 0) \rightarrow \mathbb{N}$, $\mathbb{N} = \{0, 1, 2, \dots\}$ by

$$\alpha_S(d) = \inf \{k \in \mathbb{N} | (f_S^d)^* (\eta^k) = 0\}, \tag{12}$$

where $(f_S^d)^*: \bar{H}(B_G) \rightarrow \bar{H}(M_{S,G}^d)$. It requires of course some proof that $\alpha_S(d) < \infty$. This will be provided later. For specialists this is clearly the Fadell-Rabinowitz index of M_S^d , see [F-R]. We define a subset $\sigma(S)$ of $\mathbb{R}^+ = [0, +\infty)$ called the *index interval* of S by

$$t \in \sigma(S) \Leftrightarrow \liminf_{d \uparrow 0} \alpha_S(d) |d| \leq t \leq \limsup_{d \uparrow 0} \alpha_S(d) |d|. \tag{13}$$

Denote by \mathcal{C} the collection of all compact intervals in $(0, +\infty)$ which we equip with the Hausdorff topology and Hausdorff metric. As we shall see later the following holds:

Lemma 3. For $S \in \mathcal{H}$ the index interval of S , denoted by $\sigma(S)$ belongs to \mathcal{C} .

A first result which will be proved later is

Theorem 1. (i) The mapping $\mathcal{H} \rightarrow \mathcal{C} : S \rightarrow \sigma(S)$ is continuous for the Hausdorff topologies

(ii) if $\# \mathcal{T}(S) < \infty$, then $\sigma(S) = \{\text{point}\}$.

We mention now an important open problem.

Problem 1. Does there exist $S \in \mathcal{H}$ with $\sigma(S) \neq \{\text{point}\}$?

A positive answer would be extremely interesting because then there exists $\delta > 0$ such that for all $R \in \mathcal{H}$ with $d(S, R) < \delta$ we have $\# \mathcal{T}(R) = \infty$ in view of Theorem 1. If there would exist a dense set Σ in \mathcal{H} with $\sigma(S) \neq \{\text{point}\}$, then in view of Theorem 1 we would have for an open and dense set in \mathcal{H} (for the Hausdorff topology) infinitely many periodic trajectories. Another problem is

Problem 2. Can $\sigma(S)$ be computed without the detour over equivariant cohomology?

Sometimes it is possible to compute $\sigma(S)$. For example for $S = S(\alpha_1, \dots, \alpha_n)$ with $\alpha_i > 0$, we have

$$\sigma(S) = \left\{ \frac{1}{2\pi} \sum_{i=1}^n \alpha_i \right\}, \tag{14}$$

as we shall see later.

Next we introduce the torsion indices for $\Gamma \in \mathcal{T}(S)$, where $S \in \mathcal{H}$. Fix $S \in \mathcal{H}$ and denote by ζ the associated vectorfield defined by

$$\zeta \lrcorner \lambda \equiv 1 \quad \text{and} \quad \zeta \lrcorner \omega \equiv 0 \tag{15}$$

One easily verifies that

$$\zeta(x) = JH'(x), \quad x \in S, \tag{16}$$

where $H'(x)$ is the gradient of $H = H_S$ in \mathbb{R}^{2n} . The right-hand side of (16) defines a Hamiltonian system on \mathbb{R}^{2n} . Let $x : \mathbb{R} \rightarrow \Gamma \in \mathbb{R}^{2n}$ be a solution of $\dot{x} = \zeta(x)$ with minimal period $T > 0$. Then

$$\int \lambda|_{\Gamma} = \int x^* \lambda = \int_0^T 1 dt = T.$$

Definition 3. The volume $V(\Gamma)$ of $\Gamma \in \mathcal{T}(S)$ is defined by

$$V(\Gamma) := \int \lambda|_{\Gamma}. \tag{17}$$

Sometimes $V(\Gamma)$ is also called the action of Γ .

Note that by (15) and the fact that $T_x \Gamma = \mathbb{R} \zeta(x)$, $\lambda|_{\Gamma}$ is a nonvanishing 1-form on Γ and defines therefore a volume-element. Linearizing the Hamiltonian system (HS) around $x : \mathbb{R} \rightarrow \Gamma$ gives

$$\dot{y}(t) = H''(x(t))y(t). \tag{LHS}$$

Denote by $(R(t))_{t \in \mathbb{R}}$, $R(0) = \text{Id}$ the fundamental solution of (LHS). Then $R \in C^\infty(\mathbb{R}, \text{Sp}(n, \mathbb{R}))$. Denote by R^* the adjoint of R defined by

$$\langle R(t) \cdot, \cdot \rangle = \langle \cdot, R^*(t) \cdot \rangle,$$

and define

$$B(t) = (R(t)R^*(t))^{-1/2}R(t). \tag{18}$$

Then B is the “unitary part” of R , see [C–Z 1]. That is, $B(t)$ commutes with J and $|B(t)y| = |y|$ for every $t \in \mathbb{R}$ and $y \in \mathbb{R}^{2n}$. J defines a complex multiplication on \mathbb{R}^{2n} by

$$iy := Jy,$$

turning \mathbb{R}^{2n} into a complex vectorspace of dimension n . Denote by $\det : (\mathbb{R}^{2n})^n \rightarrow \mathbb{C}$ a non-zero complex determinant function. We find a unique continuous map $\Delta_r : \mathbb{R} \rightarrow \mathbb{R}$ characterized by

$$\begin{aligned} \Delta_r(0) &= 0, \\ \det \circ (B(t) \times \dots \times B(t)) &= \exp(2\pi i \Delta_r(t)) \det. \end{aligned} \tag{19}$$

Definition 4. Let $S \in \mathcal{H}$ and $\Gamma \in \mathcal{T}(S)$. The *total torsion* at Γ is the real number

$$\gamma(\Gamma) := \Delta_r(V(\Gamma)). \tag{20}$$

The *mean torsion* at Γ is the number

$$\bar{\gamma}(\Gamma) := \gamma(\Gamma)/V(\Gamma). \tag{21}$$

Now we formulate our main result.

Theorem 2. *Let $S \in \mathcal{H}$. We have:*

- (i) *If $n \geq 2$ then $\gamma(\Gamma) > 1$ for every $\Gamma \in \mathcal{T}(S)$, or equivalently $\bar{\gamma}(\Gamma) > V(\Gamma)^{-1}$.*
- (ii) *Given $t \in \sigma(S)$ there exists a sequence $(\Gamma(k)) \subset \mathcal{T}(S)$ such that $\bar{\gamma}(\Gamma(k)) \rightarrow t$ as $k \rightarrow \infty$.*
- (iii) *Given any $\varepsilon > 0$ denote by $\sigma(S)_\varepsilon$ the open ε -ball around $\sigma(S)$. Then the following inequality is valid.*

$$\sum_{\Gamma \in \mathcal{T}(S), \bar{\gamma}(\Gamma) \in \sigma(S)_\varepsilon} \gamma(\Gamma)^{-1} \geq 1.$$

Theorems 1 and 2 have an obvious

Corollary 1. *If $S \in \mathcal{H}$, $n \geq 2$, then $\#\mathcal{T}(S) \geq 2$. Moreover, if $\#\mathcal{T}(S) < \infty$, then, with $\sigma(S) = \{I\}$ (Theorem 1),*

$$I = \bar{\gamma}(\Gamma_1) = \bar{\gamma}(\Gamma_2)$$

for two suitable $\Gamma_1, \Gamma_2 \in \mathcal{T}(S)$, $\Gamma_1 \neq \Gamma_2$.

Proof. Since $\gamma(\Gamma) > 1$ for $n \geq 2$ we infer by (iii) that $\#\mathcal{T}(S) \geq 2$. If now $\#\mathcal{T}(S) < \infty$ then $\sigma(S) = \{I\}$ by theorem 1. Then taking ε sufficiently small in (iii) of Theorem 1 we obtain

$$\sum_{\bar{\gamma}(\Gamma) = I} \gamma(\Gamma)^{-1} \geq 1,$$

which gives the desired conclusion. \square

Let us also note that Theorem 2(ii) implies Theorem 1(ii). Namely if $\sigma(S)$ is different from a point then the set

$$\{\bar{\gamma}(\Gamma) \mid \Gamma \in \mathcal{T}(S)\} \cap \sigma(S)$$

is dense in $\sigma(S)$. Therefore $\#\mathcal{T}(S) = \infty$ in this case. So if $\mathcal{T}(S)$ is finite $\sigma(S)$ consists of a point, which proves Theorem 1(ii).

So Corollary 1 implies conjecture 1 for the case $n=2$. In [E–La] this was claimed too, however due to a faulty argument it was actually unproved (the arguments in [E–La] hold only for $n \geq 3$.) Under the general hypotheses of Theorem 2 the inequality in (iii) is optimal. Namely let $S = S(\alpha_1, \dots, \alpha_n)$ with $\alpha_i > 0$ independent over \mathbb{Z} . Then as we shall see later

$$\begin{aligned} \mathcal{T}(S) &= \{\Gamma_1, \dots, \Gamma_n\}, \quad \text{so } \#\mathcal{T}(S) = n, \\ \sigma(S) &= \{I\}, \quad I = \frac{1}{2\pi} \sum_{i=1}^n \alpha_i, \\ \bar{\gamma}(\Gamma_j) &= I, \\ \gamma(\Gamma_j) &= \sum_{i=1}^n \frac{\alpha_i}{\alpha_j}, \\ V(\Gamma_j) &= \frac{2\pi}{\alpha_j}. \end{aligned} \tag{22}$$

Hence (22) implies

$$\sum_{\bar{\gamma}(\Gamma)=I} \gamma(\Gamma)^{-1} = \sum_{j=1}^n \left(\alpha_j / \sum_{i=1}^n \alpha_i \right) = 1. \tag{23}$$

Problem 3. Is it true if $\mathcal{T}(S) = \{\Gamma_1, \dots, \Gamma_h\}$, i.e., $\#\mathcal{T}(S) < \infty$, that

$$\bar{\gamma}(\Gamma_j) = \bar{\gamma}(\Gamma_i) \quad \text{for all } i, j.$$

We mention another conjecture. Denote by τ_∞ the topology on \mathcal{H} which is induced from the weak Whitney topology on $C^\infty(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R})$ via \mathcal{H} . Then we have

Conjecture 2. For a residual subset \mathcal{H}_1 of \mathcal{H} the following holds: For $S \in \mathcal{H}_1$ the map $\mathcal{T}(S) \rightarrow \mathbb{R} : \Gamma \rightarrow \bar{\gamma}(\Gamma)$ is injective.

A simple corollary of this conjecture is that $\#\mathcal{T}(S) = +\infty$ for $S \in \mathcal{H}_1$, because $\bar{\gamma} : \mathcal{T}(S) \rightarrow \mathbb{R}$ cannot be injective if $\#\mathcal{T}(S) < \infty$ by Corollary 1. Finally we mention the following.

Problem 4. How does $\bar{\gamma}$ behave on periodic Hamiltonian trajectories close to a generic elliptic one? Is it injective?

There is of course some connection between Conjecture 2 and Problem 4.

II. Variational Set Up

II.1. Critical Point Theory

Consider the $C^{1,1}$ -functional $A_S = A|_{M_S}$ on M_S . As Riemannian metric on M_S we take the one induced by (\cdot, \cdot) . Then one verifies easily that for $d \in (-\infty, 0)$

$$\text{If } \|\text{grad } A_S(x_k)\| \rightarrow 0 \text{ and } A_S(x_k) \rightarrow d < 0, \text{ then } (x_k) \subset M_S \text{ is precompact in } M_S. \tag{PS}_d$$

Solving the differential equation

$$x' = -\text{grad } A_S(x)$$

on M_S in forward time we obtain a continuous map,

$$\mathbb{R}^+ \times M_S \rightarrow M_S : (t, x) \rightarrow x * t,$$

which is the restriction of a not necessarily globally defined flow. The map $t \rightarrow A_S(x * t)$ is non-increasing for fixed $x \in M_S$. A well-known consequence of $(PS)_d$ is the following

Lemma 4. *Given an arbitrary neighborhood U of*

$$\text{Cr}(d) = \{x \in M_S | \text{grad } A_S(x) = 0, A_S(x) = d\}$$

there exists $\epsilon > 0$ such that

$$(M_S^{d+\epsilon} \setminus U) * 1 \subset M_S^{d-\epsilon}. \tag{1}$$

Define a semigroup θ by

$$\theta = S^1 \times \mathbb{N}^*, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\} \tag{2}$$

with multiplication

$$(a, k) \times (b, l) = (kb + a, kl), \tag{3}$$

where we take $S^1 = \mathbb{R}/\mathbb{Z}$. θ operates by isometries on E via

$$((a, k) * x)(t) = \frac{1}{k} x(kt + a). \tag{4}$$

One easily verifies that $\theta * M_S = M_S$. Moreover if $\text{Cr}(S)$ denotes the set of critical points of A_S then $\theta * \text{Cr}(S) = \text{Cr}(S)$. But caution, note that A_S is *not* (!) θ -invariant. In fact

$$A_S((a, k) * x) = \frac{1}{k} A_S(x). \tag{5}$$

Moreover θ induces by restriction to $S^1 \sim S^1 \times \{1\}$ the usual S^1 -action. Denote by “ \sim ” the smallest equivalence relation containing the relations

$$x \rightsquigarrow (a, k) * x \text{ for all } (a, k) \in \theta \text{ and } x \in E.$$

Denote by $[x]$ the equivalence class of x

Lemma 5. (i) If $x \in \text{Cr}(S)$, then $[x] \subset \text{Cr}(S)$.

(ii) Given $[x]$ with $x \in M_S$ there exists a unique S^1 -orbit $S^1 * y$ such that for every $z \in S^1 * y$ we have $\theta * z = [x]$. We call such a z a minimal representative for $[x]$.

(iii) There is a canonical bijection $\varphi: \text{Cr}(S)/\sim \rightarrow \mathcal{T}(S)$ which associates to $[x] \in \text{Cr}(S)/\sim$ the Γ defined as follows: Let z be a minimal representative for $[x]$, then there exists a unique constant $c \in \mathbb{R}^{2n}$ such that $|A_S(z)|^{-1}z(t) + c \subset S$ for all $t \in \mathbb{R}$. Put

$$\Gamma = \{A_S(z)^{-1}z(t) + c | t \in \mathbb{R}\}.$$

Clearly Γ does not depend on the choice of the minimal representative.

(iv) If $[x] \in \text{Cr}(S)/\sim$ and z is a minimal representative for $[x]$, then $|A_S(z)| = V(\Gamma)^{-1}$, where $\Gamma = \varphi([x])$. Moreover $|A_S((a, l) * z)| = \frac{1}{lV(\Gamma)}$.

Proof. (ii) Let $u \in [x]$ and denote by G_u the isotropy group of the S^1 -action

$$G_u = \{a \in G | a * u = u\}.$$

Pick $y \in [x]$ with

$$G_y = \min \{ \# G_u | u \in [x] \}.$$

One verifies easily that $S^1 * y$ has the desired properties. In fact $\# G_y = 1$.

(i) If $x \in \text{Cr}(S)$ then we have for some number $\delta \neq 0$,

$$A'x = \delta \Psi'(x),$$

where $\Psi(x) = \int_0^1 H^*(-J\dot{x}(t))dt$ and the prime denotes the gradient in E . Let $k = \# G_x$.

Then y defined by $y(t) = ky\left(\frac{t}{k}\right)$ is a minimal representative for $[x]$. One computes easily

$$A'y = k\delta \Psi'(y).$$

Hence $y \in \text{Cr}(S)$. Moreover with $u = (a, l) * y$,

$$A'u = \frac{k\delta}{l} \Psi'(u),$$

so that again $u \in \text{Cr}(S)$.

(iii) Let $[x] \in \text{Cr}(S)/\sim$ and z a minimal representative. We have $G_z = \{1\}$ and for some $\delta \neq 0$,

$$A'z = \delta \Psi'(z). \tag{6}$$

Taking the inner product with z and using that A' and Ψ' are positively 1-homogeneous we infer since $\Psi(z) = 1$,

$$A_S(z) = \delta. \tag{7}$$

Using (6) we find that for arbitrary $h \in E$,

$$\int_0^1 \langle J\dot{z}(t), h(t) \rangle dt = \delta \int_0^1 \langle H^*(-J\dot{z}(t)), -J\dot{h}(t) \rangle dt. \tag{8}$$

Since h has mean value zero we find a constant $c_1 \in \mathbb{R}^{2n}$ such that

$$z(t) + c_1 = |\delta|H^{*'}(-J\dot{z}(t)). \tag{9}$$

Hence by the Legendre reciprocity formula

$$|\delta|^{-1}H'(z(t) + c_1) = -J\overline{(z\dot{t} + c_1)}. \tag{10}$$

So, defining $z_1(t) = z(|\delta|t) + c_1$ we see that

$$-J\dot{z}_1 = H'(z_1). \tag{11}$$

Therefore the map

$$t \rightarrow H(z_1(t))$$

is constant. Hence the map

$$t \rightarrow H(z(t) + c_1)$$

is constant. Using this and taking the L^2 -inner product of $z(t) + c_1$ with (10) given by (7)

$$|\delta|^{-1}H(z(t) + c_1) = |\delta|.$$

Therefore

$$H(|\delta|^{-1}(z(t) + c_1)) = 1 \quad \forall t \in \mathbb{R},$$

and with $z_2(t) = |\delta|^{-1}(z(|\delta|t) + c_1)$

$$-J\dot{z}_2 = H'(z_2), \quad H(z_2(t)) = 1, \quad t \in \mathbb{R}. \tag{12}$$

By (7) again we conclude from this that

$$t \rightarrow |A(z)|^{-1}z(t) + c$$

with $c = |\delta|^{-1}c_1$ parametrizes an element in $\mathcal{F}(S)$.

Now starting with some Γ and doing the whole procedure backwards we end up with a class $[x] \in \text{Cr}(S)/\sim$.

(iv) Using (12) and the definition of z_2 we see that the minimal period T of z_2 is $|\delta|^{-1} = |A_S(z)|^{-1}$ and that

$$\begin{aligned} V(\Gamma) &= \frac{1}{2} \int_0^T \langle -J\dot{z}_2(t), z_2(t) \rangle dt = \frac{1}{2} \int_0^T 2dt \\ &= T = |\delta|^{-1} = |A_S(z)|^{-1}. \end{aligned}$$

Moreover

$$|A_S((a, l) * z)| = \left| \frac{1}{l} A_S(z) \right| = \frac{1}{lV(\Gamma)}. \quad \square$$

Definition 5. Let $S \in \mathcal{H}$ and $\Gamma \in \mathcal{F}(S)$. The *tower of Γ* , denoted by $\text{tow}(\Gamma)$, is the set

$$\text{tow}(\Gamma) = \varphi^{-1}(\Gamma).$$

Hence in order to show $\#\mathcal{F}(S) \geq n$ we have to show that $\#(\text{Cr}(S)/\sim) \geq n$, or that there are at least n towers!

We use now the Fadell-Rabinowitz index [F–R], denoted by ind . We have already seen that (formula I.12)

$$\text{ind}(M_S^d) = \alpha_S(d). \tag{13}$$

Lemma 6. (i) $d \rightarrow \alpha_S(d)$ is non-decreasing and \mathbb{N} -valued.

$$(ii) \quad \lim_{d \downarrow d_0} \alpha_S(d) = \alpha_S(d_0).$$

$$(iii) \quad \text{ind}(\text{Cr}(d)) \geq \alpha_S(d) - \alpha_S(d^-) \quad \forall d \in (-\infty, 0).$$

In particular if α_S is discontinuous at d , then $\text{Cr}(d) \neq \emptyset$. Moreover if $\alpha_S(d) - \alpha_S(d^-) \geq 2$, then $\text{Cr}(d)$ contains infinitely many S^1 -orbits. Consequently in this case $\# \mathcal{F}(S) = \infty$.

$$(iv) \quad \lim_{d \uparrow 0} \alpha_S(d) = +\infty.$$

Since the proof is essentially contained in [F–R] we can be sketchy.

Proof. $d \rightarrow \alpha_S(d)$ is non-decreasing by the monotonicity property of ind . To see that $\alpha_S(d) < +\infty$, decompose E as follows

$$E = E^- \oplus E^+,$$

where $x \in E^\pm$ is given by

$$x(t) = \sum_{\substack{k \leq 0 \\ k < 0}} \frac{1}{2\pi} \exp(2\pi ktJ)x_k.$$

If $d \in (-\infty, 0)$ one easily finds $N \in \mathbb{N}$ such that $x \in M_S$ and $A_S(x) \leq d$ implies

$$\sum_{k=-N}^{-1} |k| |x_k|^2 \neq 0.$$

Hence the orthogonal projection $P_N: E \rightarrow E_N^-$, where

$$E_N^- = \{x \in E^- \mid x_k = 0 \text{ for } k < -N\}$$

induces an equivariant map

$$M_S^d \rightarrow E_N^- \setminus \{0\}.$$

Hence

$$\alpha_S(d) = \text{ind}(M_S^d) \leq \text{ind}(E_N^- \setminus \{0\}) < \infty$$

by a result in [F–R]. This proves (i).

In order to see (iv) note that there is an equivariant map $S_{E_N^-} \rightarrow M_S^{d(N)}$ ($S_{E_N^-}$ the unit sphere in E_N^-) of the form

$$x \rightarrow f(x)x,$$

where $f: S_{E_N^-} \rightarrow (0, +\infty)$ is a continuous map. Again by a result in [F–R] it follows

$$\text{ind}(M_S^{d(N)}) \geq \text{ind}(S_{E_N^-}).$$

Here $d(N) \rightarrow 0$ as $N \rightarrow \infty$. Since $\text{ind}(S_{E_N^-}) \rightarrow \infty$ as $N \rightarrow \infty$, (iv) follows.

(ii) By the continuity property of the $F - R$ index we find for given $d \in (-\infty, 0)$ an open neighborhood U of M_S^d such that

$$\text{ind}(\bar{U}) = \text{ind}(M_S^d). \tag{14}$$

By a variant of Lemma 4 we find $\varepsilon > 0$ such that

$$M_S^{d+\varepsilon} * 1 \subset M_S^{d-\varepsilon} \cup U \subset U. \tag{15}$$

By the properties of ind we infer from (14) and (15),

$$\begin{aligned} \text{ind}(M_S^d) &= \text{ind}(U) \geq \text{ind}(M_S^{d+\varepsilon} * 1) \\ &\geq \text{ind}(M_S^{d+\varepsilon}) \geq \text{ind}(M_S^d). \end{aligned}$$

This proves (ii).

Assertion (iii) is standard and simple to derive from the properties of ind . \square

Lemma 7. Let $\hat{d} \in (-\infty, 0)$ be a point of discontinuity for α_S . Define k and j by

$$\begin{aligned} k &= \alpha_S(\hat{d}^-) + 1 = \lim_{d \uparrow \hat{d}} \alpha_S(d) + 1, \\ k + j &= \alpha_S(\hat{d}). \end{aligned}$$

Denote by $\varepsilon_0 > 0$ a number which is smaller than the distance of \hat{d} to the closest point of discontinuity d_1 of α_S with $d_1 \neq \hat{d}$. Then we have for $\varepsilon \in (0, \varepsilon_0]$ and $i = k, \dots, k + j$,

$$\bar{H}^{2(i-1)}(M_{S,G}^{\hat{d}+\varepsilon}, M_{S,G}^{\hat{d}-\varepsilon}) \neq 0. \tag{16}$$

Moreover denote by $f: M_{S,G} \rightarrow B_G$ a classifying map and let

$$f^+ : M_{S,G}^{\hat{d}+\varepsilon} \rightarrow B_G \text{ and } f^- : M_{S,G}^{\hat{d}-\varepsilon} \rightarrow B_G$$

be the restrictions. Let

$$a : M_{S,G}^{\hat{d}-\varepsilon} \rightarrow M_{S,G}^{\hat{d}+\varepsilon} \text{ and } b : M_{S,G}^{\hat{d}+\varepsilon} \rightarrow (M_{S,G}^{\hat{d}+\varepsilon}, M_{S,G}^{\hat{d}-\varepsilon})$$

be inclusions. Consider the commutative diagram with exact top row

$$\begin{array}{ccccc} \bar{H}(M_{S,G}^{\hat{d}+\varepsilon}, M_{S,G}^{\hat{d}-\varepsilon}) & \xrightarrow{b^*} & \bar{H}(M_{S,G}^{\hat{d}+\varepsilon}) & \xrightarrow{a^*} & \bar{H}(M_{S,G}^{\hat{d}-\varepsilon}) \\ & & \swarrow (f^+)^* & & \searrow (f^-)^* \\ & & \bar{H}(B_G) & & \end{array} \tag{17}$$

Then there exists a cohomology class

$$\sigma \in \bar{H}^{2(k-1)}(M_{S,G}^{\hat{d}+\varepsilon}, M_{S,G}^{\hat{d}-\varepsilon})$$

with $b^*(\sigma) = (f^+)^*(\eta^{k-1})$. We have moreover

$$(f^-)^*(\eta^m) \cup \sigma \neq 0 \text{ for } m = 0, \dots, j.$$

Proof. Equation (16) follows from our second assertion. Since $k = \alpha_S(\hat{d}^-) + 1$ we see that

$$(f^-)^*(\eta^{k-1}) = 0. \tag{18}$$

By exactness of the row in (17) we find using (18) and

$$a^*(f^+)^*(\eta^{k-1}) = (f^-)^*(\eta^{k-1})$$

that for some $\sigma \in \bar{H}(M_{S,G}^{d+\varepsilon}, M_{S,G}^{d-\varepsilon})$,

$$b^*(\sigma) = (f^+)^*(\eta^{k-1}). \tag{19}$$

Now for $m \in \{0, \dots, j\}$ we compute

$$\begin{aligned} b^*((f^+)^*(\eta^m) \cup \sigma) &= (f^+)^*(\eta^m) \cup b^*(\sigma) \\ &= (f^+)^*(\eta^m) \cup (f^+)^*(\eta^{k-1}) = (f^+)^*(\eta^{k+m-1}). \end{aligned}$$

By our hypothesis $\alpha_S(\hat{d}) = k + j$. Hence

$$(f^+)^*(\eta^{k+j-1}) \neq 0.$$

Since $m \leq j$ we infer therefore that

$$b^*((f^+)^*(\eta^m) \cup \sigma) \neq 0, \tag{20}$$

implying our assertion. \square

II.2. A Finite Dimensional Reduction

Recall the definition of the Hilbert space E . For $N \in \mathbb{N}^*$ we denote by E_N the $4nN$ -dimensional nullspace (G -invariant) defined by

$$E_N = \left\{ x \in E \mid x(t) = \sum_{\substack{k=-N \\ k \neq 0}}^N \frac{1}{2\pi} \exp(2\pi i k t) J x_k \right\}.$$

The orthogonal projection $E \rightarrow E_N$ is denoted by P_N . Moreover we put $Q_N = \text{Id} - P_N$. Define as before $\Psi \in C^{1,1}(E, \mathbb{R})$ by

$$\Psi(x) = \int_0^1 H^*(-J\dot{x}(t)) dt.$$

For $N \in \mathbb{N}^*$ we define an open $C^{1,1}$ -submanifold of $\tilde{M}_S = \{x \in E \mid \Psi(x) = 1\}$ by

$$\tilde{M}_{S,N} = \{x \in \tilde{M}_S \mid P_N x \neq 0\}.$$

For $d_0 \in (-\infty, 0)$ we put moreover

$$\tilde{M}_{S,N}^{d_0} = \tilde{M}_{S,N} \cap M_S^{d_0}.$$

Lemma 8. *There exists a G -invariant $C^{1,1}$ -map*

$$\tau : S_N \times (E_N)^\perp \rightarrow (0, +\infty)$$

such that

$$\sigma : S_N(E_N) \rightarrow \tilde{M}_{S,N} : \sigma(y, z) = \tau(y, z)(y + z)$$

is a $C^{1,1}$ -diffeomorphism onto. Here S_N is the unit sphere in E_N .

Proof. Define $\tau(y, z)$ by

$$\tau(y, z) = \Psi(y + z)^{-1/2}. \tag{1}$$

Then $\tau(y, z) > 0$ since $y + z \neq 0$, and moreover it is a $C^{1,1}$ -map since this is true for Ψ on $E \setminus \{0\}$. Consequently σ is $C^{1,1}$. It is clear that $P_N \sigma(y, z) \neq 0$, so $\text{im}(\sigma) \subset \tilde{M}_{S,N}$.

Moreover if $u \in \tilde{M}_{S,N}$, define $y = P_N u / \|P_N u\|$ and $z = Q_N u / \|P_N u\|$. Then

$$\sigma(y, z) = u.$$

Clearly the map $u \rightarrow (y, z)$ is $C^{1,1}$ and an inverse to σ . \square

Next we need

Lemma 9. *Given $d_0 \in (-\infty, 0)$, there exists $N_1(d_0) \in \mathbb{N}^*$ such that*

$$M_S^{d_0} \subset \tilde{M}_{S, N_1(d_0)}.$$

Proof. We find $c_1 > 0$ such that

$$\Psi(x) \geq c_1^2 \|x\|^2 \quad \forall x \in E. \tag{2}$$

Hence if $x \in \tilde{M}_S$ we infer

$$\|x\| \leq c_1^{-1}. \tag{3}$$

Now let $x \in M_S^{d_0}$. Then

$$\begin{aligned} A(P_N x) &= A(x) - A(Q_N x) \leq d_0 + \frac{1}{N} \|Q_N x\|^2 \\ &\leq d_0 + \frac{1}{N} \|x\|^2 \leq d_0 + \frac{1}{c_1^2 N} < 0, \end{aligned}$$

if $N > \frac{1}{c_1^2 |d_0|}$. So define

$$N_1(d_0) = \frac{1}{c_1^2 |d_0|} + 1. \quad \square \tag{4}$$

Denote by $c > 0$ a monotonicity constant for Ψ , that is

$$(\Psi'(x) - \Psi'(\bar{x}), x - \bar{x}) \geq c \|x - \bar{x}\|^2 \quad \forall x, \bar{x} \in E. \tag{5}$$

We shall express $\tilde{A} = A|_{\tilde{M}_S}$ by “local coordinates” in $S_N \times (E_N)^\perp$, that is we consider the map of class $C^{1,1}$ given by

$$(y, z) \rightarrow A \circ \sigma(y, z).$$

Define

$$\begin{aligned} \Gamma_y(z) &= A \circ \sigma(y, z), & \sigma_y(z) &= \sigma(y, z), \\ \tau_y(z) &= \tau(y, z). \end{aligned} \tag{6}$$

We equip the vectorbundle $S_N \times (E_N)^\perp \rightarrow S_N$ with the metric $[\cdot, \cdot]$ induced from the inner product on E

$$[(y, z), (y, \bar{z})] := \langle z, \bar{z} \rangle.$$

Lemma 10. *The fibrewise gradient $\Gamma'_y(z)$ with respect to $[\cdot, \cdot]$ is given by*

$$\Gamma'_y(z) = \tau_y(z) Q_N [A'(\sigma_y(z)) - \Gamma'_y(z) \Psi'(\sigma_y(z))]. \tag{7}$$

Proof. We compute

$$\begin{aligned}
 [\Gamma'_y(z), (y, h)] &= D\Gamma_y(z)(h) \\
 &= (A'(\sigma_y(z)), y+z)(\tau'_y(z), h) + (A'(\sigma_y(z)), \tau_y(z)h) \\
 &= \frac{2}{\tau_y(z)} \Gamma_y(z)(\tau'_y(z), h) + \tau_y(z)(A'(\sigma_y(z)), h).
 \end{aligned}
 \tag{8}$$

Moreover

$$\begin{aligned}
 (\tau'_y(z), h) &= -\frac{1}{2}\Psi(y+z)^{-3/2}(\Psi'(y+z), h) \\
 &= -\frac{1}{2}(\tau_y(z)^{-1}\Psi(y+z)^{-1/2})^3\tau_y(z)^3(\Psi'(y+z), h) \\
 &= -\frac{1}{2}\Psi(\sigma_y(z))^3\tau_y(z)^2(\Psi'(\sigma_y(z)), h) \\
 &= -\frac{1}{2}\tau_y(z)^2(\Psi'(\sigma_y(z)), h).
 \end{aligned}
 \tag{9}$$

Hence

$$\tau'_y(z) = -\frac{1}{2}\tau_y(z)^2 Q_N \Psi'(\sigma_y(z)).
 \tag{10}$$

Combining (8) and (10) yields

$$\Gamma'_y(z) = (y, \tau_y(z) Q_N [A'(\sigma_y(z)) - \Gamma_y(z)\Psi'(\sigma_y(z))]).
 \tag{11}$$

Lemma 11. *Given $d_0 \in (-\infty, 0)$ there exists a number $N_2(d_0) \in \mathbb{N}^*$ and a constant $\alpha = \alpha(d_0) > 0$ such that*

$$\alpha \|y - \bar{y}\| \geq \|z_y - z_{\bar{y}}\|
 \tag{12}$$

whenever z_y is a solution of

$$\Gamma'_y(z_y) = 0, \quad \Gamma_y(z_y) \leq d_0 \quad (\text{similarly for } z_{\bar{y}}).
 \tag{13}$$

Proof. Assume $\Gamma'_y(z) = 0$ and $\Gamma_y(z) \leq d_0$. Then

$$A'z = \Gamma_y(z) Q_N \Psi'(y+z),$$

where we used the positively 1-homogeneity of Ψ' . Hence

$$\begin{aligned}
 (A'z, z) &= \Gamma_y(z)(\Psi'(y+z), z) \\
 &= \Gamma_y(z)(\Psi'(y+z) - \Psi'(y), z) + \Gamma_y(z)(\Psi'(y), z) \\
 &\leq \Gamma_y(z)c \|z\|^2 + |\Gamma_y(z)| \|\Psi'(y)\| \|z\| \\
 &\leq d_0 c \|z\|^2 + |\Gamma_y(z)| \|\Psi'(y)\| \|z\|.
 \end{aligned}
 \tag{14}$$

Now, for some constant $c_2 > 0$ independent of x we have

$$\|\Psi'(x)\| \leq c_2 \|x\|.
 \tag{15}$$

Since $\|y\| = 1$, we infer combining (14) and (15),

$$(A'z, z) \leq d_0 c \|z\|^2 + c_2 |\Gamma_y(z)| \|z\|.$$

Now $\sigma_y(z) \in \tilde{M}_S$ and by (3),

$$\|\sigma_y(z)\| \leq c_1^{-1}.$$

Therefore

$$\begin{aligned} (A'z, z) &\leq d_0 c \|z\|^2 + c_2 |A(\sigma_y(z))| \|z\| \\ &\leq d_0 c \|z\|^2 + c_2 \|\sigma_y(z)\|^2 \|z\| \leq d_0 c \|z\|^2 + c_2 c_1^{-2} \|z\|. \end{aligned} \tag{16}$$

Moreover $(A'z, z) \geq -\frac{1}{N} \|z\|^2$. Hence

$$\left(|d_0|c - \frac{1}{N} \right) \|z\|^2 \leq c_2 c_1^{-2} \|z\| =: \frac{1}{2} c_3 \|z\|. \tag{17}$$

So for all $N \geq N^1$, for some suitable N^1 we have the a priori bound (which is independent of N , y or z)

$$\|z\| \leq c_3 \quad \text{if} \quad \Gamma'_y(z) = 0 \quad \text{and} \quad \Gamma_y(z) \leq d_0. \tag{18}$$

Now assume

$$\begin{aligned} \Gamma'_y(z) &= 0, \quad \Gamma'_y(\bar{z}) = 0, \\ \Gamma_y(z) &=: d, \quad \Gamma_y(\bar{z}) =: \bar{d}, \quad \text{with} \quad d, \bar{d} \leq d_0. \end{aligned}$$

Then

$$\begin{aligned} (\bar{d}A'z - dA'\bar{z}, z - \bar{z}) &= (\Psi'(y+z) - \Psi'(\bar{y} + \bar{z}), z - \bar{z}) d\bar{d} \\ &= d\bar{d}(\Psi'(y+z) - \Psi'(y + \bar{z}), z - \bar{z}) \\ &\quad + d\bar{d}(\Psi'(y + \bar{z}) - \Psi'(\bar{y} + \bar{z}), z - \bar{z}). \end{aligned}$$

Now Ψ' is globally Lipschitz continuous. Hence for some constant $c_4 > 0$ independent of y, z and $N \geq N^1$:

$$|(\Psi'(y + \bar{z}) - \Psi'(\bar{y} + \bar{z}), z - \bar{z})| \leq c_4 \|y - \bar{y}\| \|z - \bar{z}\|. \tag{20}$$

Combining (19) and (20) gives

$$d\bar{d}c \|z - \bar{z}\|^2 \leq d\bar{d}c_4 \|y - \bar{y}\| \|z - \bar{z}\| + (\bar{d}A'z - dA'\bar{z}, z - \bar{z}). \tag{21}$$

Moreover

$$\begin{aligned} |(\bar{d}A'z - dA'\bar{z}, z - \bar{z})| &\leq |\bar{d}(A'(z - \bar{z}), z - \bar{z})| + |\bar{d} - d| |(A'\bar{z}, z - \bar{z})| \\ &\leq |\bar{d}| \frac{2}{N} \|z - \bar{z}\|^2 + 2|\bar{d} - d| \frac{1}{N} \|\bar{z}\| \|z - \bar{z}\| \\ &\leq |\bar{d}| \frac{2}{N} \|z - \bar{z}\|^2 + |d - \bar{d}| \frac{2}{N} c_3 \|z - \bar{z}\|. \end{aligned} \tag{22}$$

Combining (21) and (22) yields

$$d\bar{d}c \|z - \bar{z}\|^2 \leq d\bar{d}c_4 \|y - \bar{y}\| \|z - \bar{z}\| + |\bar{d}| \frac{2}{N} \|z - \bar{z}\|^2 + |d - \bar{d}| \frac{2}{N} c_3 \|z - \bar{z}\|. \tag{23}$$

Further

$$\begin{aligned}
 |d - \bar{d}| &= |\tau_y(z)^2 A(y+z) - \tau_{\bar{y}}(\bar{z})^2 A(\bar{y} + \bar{z})| \\
 &\leq |\tau_y(z)^2 - \tau_{\bar{y}}(\bar{z})^2| |A(y+z)| + \tau_{\bar{y}}(\bar{z})^2 |A(y+z) - A(\bar{y} + \bar{z})|, \\
 \text{by (18)} &\leq c_5 |\tau_y(z) + \tau_{\bar{y}}(\bar{z})| |\tau_y(z) - \tau_{\bar{y}}(\bar{z})| \\
 &\quad + \tau_{\bar{y}}(\bar{z})^2 |A(y) - A(\bar{y})| + \tau_{\bar{y}}(\bar{z})^2 |A(z) - A(\bar{z})|.
 \end{aligned}
 \tag{24}$$

We have by (3) and (18)

$$\tau_{\bar{y}}(\bar{z})^2 = \Psi(\bar{y} + \bar{z}) \leq c_1^2 \|\bar{y} + \bar{z}\|^2 \leq c_6,
 \tag{25}$$

and

$$|\tau_y(z) + \tau_{\bar{y}}(\bar{z})| \leq c_7.
 \tag{26}$$

So combining (24), (25), and (26) we obtain

$$|d - \bar{d}| \leq c_5 c_7 |\tau_y(z) - \tau_{\bar{y}}(\bar{z})| + c_6 \|y - \bar{y}\| + c_6 \|z - \bar{z}\|.
 \tag{27}$$

Now for a suitable constant c_8 using that $\|y + z\|$ is bounded and bounded away from zero

$$\begin{aligned}
 |\tau_y(z) - \tau_{\bar{y}}(\bar{z})|^2 &= |\Psi(y+z)^{-1} - \Psi(\bar{y} + \bar{z})^{-1}| \\
 &\leq \Psi(y+z)^{-1} \Psi(\bar{y} + \bar{z})^{-1} |\Psi(y+z) - \Psi(\bar{y} + \bar{z})| \\
 &\leq c_8 \|y - \bar{y}\| + c_8 \|z - \bar{z}\|.
 \end{aligned}
 \tag{28}$$

Using (26) and (27) yields

$$|\tau_y(z) - \tau_{\bar{y}}(\bar{z})| \leq c_9 \|y - \bar{y}\| + c_9 \|z - \bar{z}\|.
 \tag{29}$$

Now combining (27) and (29) yields

$$|d - \bar{d}| \leq c_{10} \|y - \bar{y}\| + c_{10} \|z - \bar{z}\|.
 \tag{30}$$

Now combining (23) and (30) we obtain

$$\left(d\bar{d}c - |\bar{d}| \frac{2}{N} \right) \|z - \bar{z}\| \leq d\bar{d}c_4 \|y - \bar{y}\| + \frac{2}{N} c_3 c_{11} \|z - \bar{z}\| + \frac{2}{N} c_3 c_{11} \|y - \bar{y}\|.$$

Therefore for a suitable constant c_{12} ,

$$\left(d\bar{d}c - |\bar{d}| \frac{2}{N} - \frac{2}{N} c_{12} \right) \|z - \bar{z}\| \leq c_{12} \|y - \bar{y}\|,
 \tag{31}$$

so for a suitable number $N_2(d_0) \geq N^1$ we find $\alpha = \alpha(d_0) > 0$ independent of y, z , and $N \geq N_2(d_0)$ such that

$$\|z - \bar{z}\| \leq \alpha \|y - \bar{y}\|,
 \tag{32}$$

where z is a solution of $\Gamma_y(z) = 0$, $\Gamma_y(z) \leq d_0$ and similarly for \bar{y} and \bar{z} . \square

Define $N(d_0)$ by

$$N(d_0) = \max \left\{ N_1(d_0), N_2(d_0), \frac{2}{\alpha(H)|d_0|} \right\},
 \tag{33}$$

where $\alpha(H) = \alpha(H_S)$ such that $H''(x) \geq \alpha(H_S) \text{Id}$, $\forall x \neq 0$.

Lemma 12. *Let $d_0 \in (-\infty, 0)$ be given and $N \geq N(d_0)$. Assume $y \in S_N$ and $(z_n) \subset (E_N)^\perp$ such that*

$$\Gamma'_y(z_n) \rightarrow 0, \quad \Gamma_y(z_n) \rightarrow d \leq d_0.$$

Then (z_n) is precompact.

Proof. Since $(\Psi(y + z_n))$ is bounded away from zero the sequence $(\tau_y(z_n))$ must be bounded. Let us also show that $(\tau_y(z_n))$ is bounded away from zero. Arguing indirectly and eventually passing to a subsequence we may assume

$$\tau_y(z_n) \rightarrow 0.$$

Hence

$$\|\sigma_y(z_n) - \tau_y(z_n)z_n\| = \|\tau_y(z_n)y\| \rightarrow 0.$$

Consequently

$$|A(\tau_y(z_n)z_n) - A(\sigma_y(z_n))| = |A(\tau_y(z_n)y)| \rightarrow 0.$$

So

$$A(\tau_y(z_n)z_n) \rightarrow d. \tag{34}$$

On the other hand we have the estimate

$$\begin{aligned} |A(\tau_y(z_n)z_n)| &\leq \frac{1}{N} \tau_y(z_n)^2 \|z_n\|^2 \\ &\leq \frac{1}{N} |\tau_y(z_n)^2 \|z_n\|^2 - \|\sigma_y(z_n)\|^2| + \frac{1}{N} \|\sigma_y(z_n)\|^2 \\ &\leq \varepsilon_n + \frac{1}{Nc_1^2}, \end{aligned} \tag{35}$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore taking the limit $n \rightarrow \infty$ we conclude

$$|d| \leq \frac{1}{Nc_1^2}, \tag{36}$$

which gives a contradiction since by (36),

$$\begin{aligned} \frac{1}{|d_0|c_1^2} &\geq \frac{1}{|d|c_1^2} \geq N \text{ [by (33)]} \\ &\geq N_1(d_0) \\ &\geq \frac{1}{|d_0|c_1^2} + 1. \end{aligned}$$

Therefore $(\tau_y(z_n))$ is bounded away from zero and $\Gamma'_y(z_n) \rightarrow 0$ implies

$$Q_N[A'(\sigma_y(z_n)) - d\Psi'(\sigma_y(z_n))] \rightarrow 0. \tag{37}$$

Put $u_n = \sigma_y(z_n)$. Eventually taking a subsequence we may assume that

$$a_n := P_N[A'u_n - d\Psi'(u_n)] \tag{38}$$

converges to some $a \in E_N$ [recall $\dim E_N < \infty$ and clearly $(\|A'u_n - d\Psi'(u_n)\|)$ is bounded]. Consequently for a suitable zero sequence $(\varepsilon_n) \subset E$ we find combining (37) and (38),

$$A'u_n - d\Psi'u_n = a_n + \varepsilon_n \rightarrow a. \tag{39}$$

Since (u_n) is bounded we may assume eventually taking a subsequence that

$$u_n \rightharpoonup u \text{ weakly in } E,$$

$$A'u_n \rightarrow A'u \text{ strongly in } E.$$

So (39) gives using that $\Psi' : E \rightarrow E$ is a homeomorphism,

$$\begin{aligned} \Psi'^{-1} \left(\frac{1}{d} (A'u_n - a_n - \varepsilon_n) \right) &= u_n \\ &\rightarrow \Psi'^{-1} \left(\frac{1}{d} A'u - a \right). \end{aligned}$$

So (u_n) is converging strongly. Hence, with $\sigma_y(z_n) = u_n$ we find $z_n \rightarrow z$ strongly for some z and $\Gamma'_y(z) = 0, \Gamma_y(z) = d$. \square

Therefore we have just proved that Γ_y satisfies (PS)_d for all $d \in (-\infty, d_0]$ if $N \geq N(d_0)$. Hence if $\inf \Gamma_y((E_N)^\perp) \leq d_0$ the infimum is attained. Define

$$\tilde{\Gamma}(y) = \inf \Gamma_y((E_N)^\perp) \tag{40}$$

for every $y \in S_N$ such that the right-hand side in (40) is less than or equals d_0 . So by the previous remark there exists $z_y \in (E_N)^\perp$ with $\tilde{\Gamma}(y) = \Gamma_y(z_y)$. Moreover z_y is uniquely determined by Lemma 11 and the map $y \rightarrow z_y$ is globally Lipschitz continuous.

Define for $N \geq N(d_0)$ a subset $\dot{M}_{S,N}^{d_0}$ of M_S by

$$\dot{\Sigma}_{S,N}^{d_0} = \{ \sigma_y(z_y) \in M_S \mid \tilde{\Gamma}(y) < d_0 \}.$$

Moreover put $\dot{M}_S^{d_0} = \{ x \in M_S \mid A(x) < d_0 \}$. Then $\dot{\Sigma}_{S,N}^{d_0} \subset \dot{M}_S^{d_0}$. The following lemma is crucial.

Lemma 13. *Let $d_0 \in (-\infty, 0)$ and $N \geq N(d_0)$. Then $\dot{\Sigma}_{S,N}^{d_0}$ is a strong G -deformation retract of $\dot{M}_S^{d_0}$ by a G -homotopy $r : [0, 1] \times \dot{M}_S^{d_0} \rightarrow \dot{M}_S^{d_0}$ such that*

- $s \rightarrow A(r(s, x))$ is nonincreasing,
- $r(0, x) = x \quad \forall x,$
- $r(s, x) = x \quad \forall t \in [0, 1] \quad \forall x \in \dot{\Sigma}_{S,N}^{d_0},$
- $r(s, \cdot)$ is G -equivariant.

Proof. Consider the $C^{1,1}$ -map

$$\sigma : S_N \times (E_N)^\perp \rightarrow \tilde{M}_{S,N} \supset M_{S,N}^{d_0}.$$

The preimage of $\dot{\Sigma}_{S,N}^{d_0}$ consists of all $(y, z) \in S_N \times (E_N)^\perp$ such that

$$\Gamma_y(z) < d_0.$$

We solve the parameter dependent differential equation

$$z' = -\Gamma'_y(z) \quad (\Gamma'_y(\cdot) \text{ is locally Lipschitz continuous}),$$

$$z(0) = z_0$$

where $\Gamma_y(z) < d_0$. By the (PS)_d-condition, $d \leq d_0$, since Γ_y is bounded from below and has only one critical point z_y with

$$\Gamma'_y(z_y) = 0, \quad \Gamma_y(z_y) < d_0,$$

we infer that $\lim_{t \rightarrow \infty} z(t) = z_y$. Define

$$\tilde{r}: [0, 1] \times \dot{\theta}^{d_0} \rightarrow \dot{\theta}^{d_0}$$

with $\dot{\theta}^{d_0} = \{(y, z) | \Gamma_y(z) < d_0\}$ by

$$\tilde{r}(s, (y, z_0)) = \begin{cases} \left(y, z \left(\frac{s}{1-s} \right) \right) & \text{if } s \in [0, 1) \\ (y, z_y) & \text{if } s = 1. \end{cases}$$

Then \tilde{r} is a continuous G -equivariant homotopy. Define \tilde{r}_1 by

$$\tilde{r}(s, (y, z)) = (y, \tilde{r}_1(s, (y, z))).$$

Then

$$s \rightarrow \Gamma_y(\tilde{r}_1(s_1(y, z))) \text{ is non-increasing.}$$

Defining $r: [0, 1] \times \dot{M}_S^{d_0} \rightarrow \dot{M}_S^{d_0}$ by

$$r(s, x) = \sigma \circ r(s, \sigma^{-1}(x))$$

gives the desired map. \square

By our construction $\dot{\Sigma}_{S,N}^{d_0}$ is G -homeomorphic to an open subset of S_N , say U , by the map

$$U \rightarrow \dot{\Sigma}_{S,N}^{d_0}: y \rightarrow \tau_y(z_y)(y + z_y). \tag{41}$$

Since U carries as an open subset of S_N the induced C^∞ -differentiable structure coming from the standard differentiable structure, we can equip $\dot{\Sigma}_{S,N}^{d_0}$ with a smooth differentiable structure uniquely characterized by the requirement that the map in (41) is a C^∞ -diffeomorphism. From now on we think of $\dot{\Sigma}_{S,N}^{d_0}$ as being equipped with this differentiable structure.

Lemma 14. *$A|_{\dot{\Sigma}_{S,N}^{d_0}}$ is of class $C^{1,1}$. Moreover the critical points of $A|_{\dot{\Sigma}_{S,N}^{d_0}}$ are exactly the critical points of $A|_{\dot{M}_S^{d_0}}$. Moreover the G -action on $\dot{\Sigma}_{S,N}^{d_0}$ is smooth near to critical orbits. Also $A|_{\dot{\Sigma}_{S,N}^{d_0}}$ is smooth near a critical orbit.*

Proof. By the definition of the differentiable structure on $\dot{\Sigma}_{S,N}^{d_0}$ we have to show that the map $y \rightarrow \tilde{\Gamma}(y) = \Gamma_y(z_y)$ is of class $C^{1,1}$ in order to establish that $A|_{\dot{\Sigma}_{S,N}^{d_0}}$ is $C^{1,1}$. For this we equip S_N with the Riemannian metric induced by our inner product (\cdot, \cdot) on E . We shall show that

$$\tilde{\Gamma}'(y) = \tau_y(z_y)P_N[A'(\sigma_y(z_y)) - A(\sigma_y(z_y))\Psi'(\sigma_y(z_y))]. \tag{42}$$

From this we shall infer since τ and σ are C^1 and $y \rightarrow z_y$ is Lipschitz continuous, that \tilde{F}' is of class $C^{0,1}$, i.e., \tilde{F} is of class $C^{1,1}$. We compute with $\tilde{F}'(x)$ defined by (42),

$$\begin{aligned} \tilde{F}(y_1) - \tilde{F}(y_0) - (\tilde{F}'(y_0), y_1 - y_0) &= \Gamma_{y_1}(z_1) - \Gamma_{y_0}(z_0) - (\tilde{F}'(y_0), y_1 - y_0) \\ &\leq \Gamma_{y_1}(z_0) - \Gamma_{y_0}(z_0) - (\tilde{F}'(y_0), y_1 - y_0) \\ &= (\tau_{y_1}(z_0)^2 - \tau_{y_0}(z_0)^2)A(y_1 + z_0) \\ &\quad + \tau_{y_0}(z_0)^2(A(y_1) - A(y_0)) - (\tilde{F}'(y_0), y_1 - y_0) \\ &= (\Psi^{-1}(y_1 + z_0) - \Psi^{-1}(y_0 + z_0))A(y_1 + z_0) \\ &\quad + \tau_{y_0}(z_0)^2(A(y_1) - A(y_0)) - (\tilde{F}'(y_0), y_1 - y_0). \end{aligned}$$

Now dividing the above inequality by $\|y_1 - y_0\|$ and taking the lim sup for $y_1 \rightarrow y_0$ we infer

$$\limsup_{y_1 \rightarrow y_0} (\tilde{F}(y_1) - \tilde{F}(y_0) - (\tilde{F}'(y_0), y_1 - y_0)) / \|y_1 - y_0\| \leq 0, \tag{43}$$

where we use that $(\Psi^{-1}(y_1 + z_0) - \Psi^{-1}(y_0 + z_0)) / \|y_1 - y_0\|$ can be replaced in the limit by

$$-\Psi(y_0 + z_0)^{-2} \left(\Psi'(y_0 + z_0), \frac{y_1 - y_0}{\|y_1 - y_0\|} \right).$$

Similarly one proves that

$$\liminf_{y_1 \rightarrow y_0} (\tilde{F}(y_1) - \tilde{F}(y_0) - (\tilde{F}'(y_0), y_1 - y_0)) / \|y_1 - y_0\| \geq 0. \tag{44}$$

Note that we had in principle to work in local coordinates to establish that \tilde{F} is differentiable at y_0 and has $\tilde{F}'(y_0)$ given in (42) as gradient. However, taking an exponential chart

$$\exp_{y_0}^{-1} : \exp_{y_0}(W) \rightarrow W \subset T_{y_0}U$$

for a suitable small zero neighborhood W , we see that

$$T \exp_{y_0}^{-1}(y_0) : T_{y_0}M \rightarrow T_0 T_{y_0}M \cong T_{y_0}M$$

is the identity so that actually (43) and (44) imply the assertion in the approach using local coordinates. So we have till now proved that (42) gives indeed the gradient. Since by construction of \tilde{F} we have

$$Q_N[A'(\sigma_y(z_y)) - A(\sigma_y(z_y))\Psi'(\sigma_y(z_y))] = 0, \tag{45}$$

we infer that

$$\text{grad } A_S(\sigma_y(z_y)) = 0, \tag{46}$$

if $\Gamma'(y) = 0$. On the other hand if $\text{grad } A_S(x) = 0$ with $A_S(x) < d_0$, then writing $x = \sigma_y(z)$ we see that z is a critical point of $\Gamma_y(z)$, so that by our previous discussion $z = z_y$. Hence y is a critical point of \tilde{F} .

Next we have to prove the assertion concerning the smoothness of the G -action and of $A|_{\dot{\Sigma}_{S,N}^{d_0}}$ near a critical orbit.

By construction

$$A'z_y - A(\sigma_y(z_y))Q_N\Psi'(\sigma_y(z_y)) = 0. \tag{47}$$

Define for $k \in \mathbb{N}^*$,

$$\Delta_k = \{x \in C^k(S^1, \mathbb{R}^{2n}) \cap E \mid \dot{x}(t) \neq 0 \ \forall t \in S^1\}.$$

One verifies easily that we have the following commutative diagram:

$$\begin{array}{ccc} \Delta_k & \xrightarrow{\Psi'} & \left\{ \dot{x} \in C^k(S^1, \mathbb{R}^{2n}) \mid \int_0^1 x(t) dt = 0 \right\} = C^k(S^1, \mathbb{R}^{2n}) \cap E \\ \text{incl} \downarrow & & \downarrow \text{incl} \\ E & \xrightarrow{\Psi'} & E \end{array},$$

where the top arrow is a smooth map. We have to exclude x with $\dot{x}(t) = 0$ because H'' and $H^{*''}$ do not exist at zero. Here the space Δ_k and $C^k(S^1, \mathbb{R}^{2n}) \cap E$ are of course equipped with the C^k -topology. Define a map by

$$(y, z) \rightarrow A'z - A(\sigma_y(z))Q_N\Psi''(y+z)$$

for $y \in S_N$ and $z \in \Delta_k \cap (E_N)^\perp$ such that $(y+z)(t) \neq 0 \ \forall t \in S^1$. So the map is in particular smooth around pairs (y, z) such that $\sigma_y(z)$ is a critical point of $A|_{M_S^{d_0}}$. The partial differential with respect to z at $\sigma_y(z_y)$ is given by

$$(y, (z, h)) \rightarrow A'h - A(\sigma_y(z_y))Q_N\Psi''(y+z_y)h, \tag{48}$$

where $\Psi''(y+z_y)$ is given by

$$\begin{aligned} h \rightarrow & \int_0^t JH^{*''}(-J(\dot{y}(\tau) + \dot{z}_y(\tau)))(-J\dot{h}(\tau))dt \\ & - \int_0^1 \left(\int_0^t JH^{*''}(-J(\dot{y}(\tau) + \dot{z}_y(\tau)))(-J\dot{h}(\tau))d\tau \right) dt. \end{aligned}$$

By the definition of N it follows that the E -extension of the map (48)

$$(E_N)^\perp \rightarrow (E_N)^\perp : h \rightarrow A'h - A(\sigma_y(z_y))Q_N\Psi''(y+z_y)h$$

is an isomorphism. Now let $\tilde{z} \in (E_N)^\perp \cap C^k(S^1, \mathbb{R}^{2n})$, and pick $h \in (E_N)^\perp$ with

$$A'h - A(\sigma_y(z_y))Q_N\Psi''(y+z_y)h = \tilde{z}.$$

By a simple regularity argument it follows that $h \in C^k(S^1, \mathbb{R}^{2n}) \cap (E_N)^\perp$. So by the open mapping theorem the map given in (48) as a map of the h -variable is a topological isomorphism. By the implicit function theorem there exists a smooth map $C^k \rightarrow C^k : y \rightarrow \hat{z}_y$ defined for y close to a critical orbit of \tilde{F} such that

$$A'\hat{z}_y = A(\sigma_y(\hat{z}_y))Q_N\Psi''(y+\hat{z}_y).$$

By uniqueness $z_y = \hat{z}_y$. Since $k \in \mathbb{N}^*$ was arbitrary we see that the points in $\Sigma_{S, N}^{d_0}$ close (“close” is independent of k) to a critical orbit belong to $C^\infty(S^1, \mathbb{R}^{2n}) \cap E$. Moreover the map $y \rightarrow \Gamma_y(\sigma_y(z_y))$ is smooth for y close to a critical orbit. So $A|_{\Sigma_{S, N}^{d_0}}$ is smooth near critical orbits. S^1 acts smoothly on S_N , so it acts smoothly on $\{\sigma_y(z_y)\}$, provided the y are close to a critical orbit. In fact, close to a critical orbit the map $y \rightarrow \sigma_y(z_y)$ is smooth and

$$a * \sigma_y(z_y) = \sigma_{a*y}(z_{a*y}),$$

implying our assertion. \square

Recall that a critical point x of A_S satisfies $\dot{x}(t) \neq 0, t \in S^1$. Therefore the following definition makes sense.

Definition 6. Let x be a critical point of A_S . The (formal) Hessian at x is the quadratic form

$$Q_x(h) = \frac{1}{2} \int_0^1 \langle J\dot{h}(t), h(t) \rangle dt - \frac{1}{2} A(x) \int_0^1 \langle H^{**}(-J\dot{x}(t))J\dot{h}(t), J\dot{h}(t) \rangle dt, \tag{49}$$

where $h \in T_x M_S$.

Clearly Q_x has a finite index $m^-(x)$ which is the maximal dimension of a linear space in $T_x M_S$ on which Q_x is negative definite, and a finite nullity $m^0(x)$, which must be of course bounded by $2n$. We call $m^-(x)$ and $m^0(x)$ the *formal index* and $m^0(x)$ the *formal nullity* of the critical point x .

We shall show that there is a close relation between $m^-(x), m^0(x)$ and the index and nullity of x as a critical point of $A|\dot{\Sigma}_{S,N}^{d_0}$ for d_0 sufficiently close to 0, $d_0 < 0$. More precisely,

Lemma 15. *Let $d_0 < 0$ and $N \geq N(d_0)$. Let $x \in \dot{\Sigma}_{S,N}^{d_0}$ be a critical point of $A|\dot{\Sigma}_{S,N}^{d_0}$ and denote its index and nullity by $i^-(x)$ and $i^0(x)$ respectively. Then*

$$i^-(x) = m^-(x) \quad \text{and} \quad i^0(x) = m^0(x). \tag{50}$$

This is quite standard and we will be somewhat sketchy. See also [E 1] for a related result for a different reduction method.

Proof. By definition we have

$$\tilde{F}'(y) = \tau_{y,(z_y)} P_N[A'_y - \tilde{F}(y)\Psi'(y + z_y)]. \tag{51}$$

Let $y_0 + z_{y_0} = x$. Then \tilde{F} is smooth near y_0 by our previous discussion. Differentiating (51) at y_0 gives for $h \in T_{y_0} S_N$,

$$\tilde{F}''(y_0)h = \tau_{y_0,(z_{y_0})} P_N[A'(h) - \tilde{F}(y_0)\Psi''(y_0 + z_{y_0})(h + z'_{y_0}h)]. \tag{52}$$

On the other hand by the construction of \tilde{F} we have

$$0 = \tau_{y,(z_y)} Q_N[A'z_y - \tilde{F}(y)\Psi'(y + z_y)]. \tag{53}$$

By the proof in Lemma 14 the map $y \rightarrow z_y$ is smooth in the C^k -setting if y is close to y_0 . So we infer differentiating (53)

$$0 = \tau_{y_0,(z_{y_0})} Q_N[A'z'_{y_0}h - \tilde{F}(y_0)\Psi''(y_0 + z_{y_0})(h + z'_{y_0}h)]. \tag{54}$$

Combining (52) and (54) gives therefore

$$\frac{1}{2}(\tilde{F}''(y_0)h, h) = \tau_{y_0,(z_{y_0})} Q_x(h + z'_{y_0}h). \tag{55}$$

This implies in particular that

$$\begin{aligned} \text{index}(\tilde{F}''(y_0)) &= i^-(x) \text{ since } \tilde{F} \text{ is a local coordinate description of } A|\dot{\Sigma}_{S,N}^{d_0} \\ &\leq m^-(x) \text{ by (55),} \end{aligned}$$

$$\text{nullity}(\tilde{F}''(y_0)) \leq m^0(x). \tag{56}$$

On the other hand assume X is a linear subspace of $T_x M_S$ with Q_x being negative definite on X . Then by the definition of N we infer that $P_N u \neq 0$ for $u \in X \setminus \{0\}$. Note that $z'_{y_0} h$ is defined by the minimum problem [as a unique solution which follows from the definition of $N(d_0)$ in (33)]

$$\min_{v \in (E_N)^\perp} (A(v) - \tilde{\Gamma}(y_0)^{\frac{1}{2}}(\Psi''(y_0 + z_{y_0})(h + v), h + v)). \tag{57}$$

Let $v_h = z'_{y_0} h$. Then defining a subspace \tilde{X} of $T_x M_S$ by

$$\tilde{X} = \{P_N u + v_{P_N u} | u \in X\},$$

we see that $Q_x|_{\tilde{X}}$ is negative definite and by construction $\dim \tilde{X} = \dim X$. So

$$\text{index}(\tilde{\Gamma}''(y_0)) \geq m^-(x), \tag{58}$$

and similarly

$$\text{nullity}(\tilde{\Gamma}''(y_0)) \geq m^0(x). \quad \square \tag{59}$$

II.3. Critical Points with Prescribed Formal Index

Definition 7. The discontinuity sequence $(\hat{d}_k)_{k \in \mathbb{N}^*}$ for $\alpha_S, S \in \mathcal{H}$, denoted by $\text{dis}(S)$, is the non-decreasing sequence consisting of all points $d < 0$ at which α_S is not continuous. Moreover each point d is repeated according to its multiplicity $\alpha_S(d) - \alpha_S(d^-)$.

The aim of this section is to prove the following:

Proposition 1. Let $k \in \mathbb{N}^*, j \in \mathbb{N}$ and define $\hat{d}_0 = -\infty$. Assume

$$\hat{d}_{k-1} < \hat{d}_k = \dots = \hat{d}_{k+j} < \hat{d}_{k+j+1}. \tag{1}$$

Then there exist $\Gamma_k, \dots, \Gamma_{k+j} \in \mathcal{T}(S)$ mutually different and numbers l_k, \dots, l_{k+j} in \mathbb{N}^* such that

$$|\hat{d}_i| = \frac{1}{V(\Gamma_i)l_i}, \quad |m^-(x_i^{l_i}) - 2i| \leq 2n + 1 \tag{2}$$

for every $i \in \{k, \dots, k+j\}$. Here x_i denotes a minimal representative for Γ_i , and $x_i^{l_i} := (1, l_i) * x_i$ denotes the l_i^{th} iterate of x_i .

The rather involved proof is based on a sequence of Lemmata.

We fix $d_0 > \hat{d}_{k+j+1}$, $N \geq N(d_0)$ and denote by $\varepsilon_0 > 0$ a number satisfying

$$0 < \varepsilon_0 < \min\{\hat{d}_{k+j+1} - \hat{d}_{k+j}, \hat{d}_k - \hat{d}_{k-1}\}. \tag{3}$$

Proposition 1 will be a consequence of the following:

Proposition 2. Under the assumptions of Proposition 1 there exist for $i \in \{k, \dots, k+j\}$ critical points \bar{x}_i of A_S with $A(\bar{x}_i) = \hat{d}$, $\hat{d} := \hat{d}_k = \dots = \hat{d}_{k+j}$ such that

$$m^-(\bar{x}_i) \leq 2(i-1) \leq m^-(\bar{x}_i) + m^0(\bar{x}_i) - 1. \tag{4}$$

If $j \geq 1$, we have in addition

Given any integer b and positive number $\delta_0 > 0$, the \bar{x}_i can be chosen in such a way that a δ_0 -ball around \bar{x}_i contains at least b critical points on different orbits on level \hat{d} . (5)

We don't claim that the \bar{x}_i are mutually different.

Proof of Proposition 1 assuming Proposition 2. Assume first $j=0$. Then by Lemma 5 we find that the first part of (2) holds for $\Gamma_k = \varphi([\bar{x}_k])$. Moreover from (4) we infer

$$|m^-(\bar{x}_k) - 2k| \leq 2n + 1.$$

If x_k is a minimal representative of $[\bar{x}_k]$ we may assume for some

$$l_k \in \mathbb{N}^* : \bar{x}_k = x_k^{l_k},$$

and the second part of (2) is proved. If now $j \geq 1$ we can argue as follows. Define

$$x_k := \bar{x}_k, \quad \Gamma_k = \varphi([x_k]).$$

Then (2) holds for $i=k$. Assume x_k, \dots, x_i are constructed so that $\Gamma_k, \dots, \Gamma_i$, where $\Gamma_a = \varphi([x_a])$, satisfy (2) and are mutually different. We have to find x_{i+1} if $i < k+j$ so that $\Gamma_k, \dots, \Gamma_{i+1}$ are mutually different and verify (2). Pick the \bar{x}_{i+1} from Proposition 2. If the G -orbit of \bar{x}_{i+1} is different from the orbits belonging to x_k, \dots, x_i we define $x_{i+1} := \bar{x}_{i+1}$ and are done. So assume \bar{x}_{i+1} belongs to $G * x_{i_0}$ for some $i_0 \in \{k, \dots, i\}$.

Pick $b \geq j + 1$ and $\delta_0 > 0$ such that all critical points on level \hat{d} being δ_0 -close to x_k, \dots, x_i have a Morse index m^- satisfying

$$m^-(x_l) \leq m^- \leq m^-(x_l) + m^0(x_l) - 1 \quad \text{for } l = k, \dots, i. \tag{6}$$

(The -1 comes from the fact that we have a nontrivial S^1 -action.) Now according to (5) we can take a new \bar{x}_{i+1} corresponding to $b \geq j + 1$ and δ_0 as above. If \bar{x}_{i+1} coincides again with some of the x_k, \dots, x_i we find a critical point x_{i+1} different from the orbits $G * x_k, \dots, G * x_i$ on level \hat{d} which is δ_0 close to one of the critical points in $\{x_k, \dots, x_i\}$. It satisfies by (6)

$$m^-(\bar{x}_{i+1}) \leq m^-(x_{i+1}) \leq m^-(\bar{x}_{i+1}) + m^0(\bar{x}_{i+1}) - 1.$$

Now combining (4) and (6) gives

$$2i + 1 - m^0(\bar{x}_{i+1}) \leq m^-(x_{i+1}) \leq 2i + m^0(\bar{x}_{i+1}) - 1. \tag{7}$$

Since $m^0(x_{i+1}) \leq 2n$, this yields

$$|m^-(x_{i+1}) - 2(i + 1)| \leq 2n + 1.$$

We take x_{i+1} for our new \bar{x}_{i+1} and the second part of (2) is proved.

Define

$$\Sigma := \Sigma_{S,N}^{d_0}, \tag{8}$$

and let for $d \in (-\infty, d_0)$

$$\Sigma^d := \{x \in \Sigma \mid \tilde{A}(x) \leq d\}, \tag{9}$$

where

$$\tilde{A} = A|_{\Sigma}. \tag{10}$$

For $c < d < d_0$ the inclusion

$$(\Sigma^d, \Sigma^c) \rightarrow (M_S^d, M_S^c) \tag{11}$$

is a G -homotopy equivalence by Lemma 13. Denote by $\text{Cr}(\hat{d})$, where $\hat{d} := \hat{d}_k = \dots = \hat{d}_{k+j}$, the set of critical points of \tilde{A} on level \hat{d} .

Given $\delta > 0$ we define an equivalence relation on $\text{Cr}(\hat{d})$ by

$$\begin{aligned} \hat{x} \sim_{\delta} \bar{x} \text{ iff there exists a finite sequence } (\tilde{x}_i)_{i=0 \dots m+1} \subset \text{Cr}(\hat{d}) \text{ with} \\ \tilde{x}_0 = \hat{x}, \tilde{x}_{m+1} = \bar{x} \text{ and } \|\tilde{x}_i - \tilde{x}_{i+1}\| < \delta. \end{aligned} \tag{12}$$

By the compactness of $\text{Cr}(\hat{d})$ there are only a finite number of equivalence classes. The Riemannian metric on S_N induced by the inner product on E induces a Riemannian metric for Σ . We denote by

$$\mathbb{R}^+ \times \Sigma \rightarrow \Sigma : (t, x) \rightarrow x * t \tag{13}$$

the restriction of the minus-gradient flow associated to \tilde{A} , that is

$$x' = -\text{grad} \tilde{A}(x). \tag{14}$$

We shall also denote by $x * t$ for $t < 0$ the image of x in backward time as long as the flow is defined on $[t, 0]$. Note that Σ^d is compact for every $d < d_0$.

Now fix $\delta > 0$ and denote by $[u_1]_{\delta}, \dots, [u_{m(\delta)}]_{\delta}$ the mutually disjoint equivalence classes of $\text{Cr}(\hat{d})$. Note that every $[u_i]_{\delta}$ is G -invariant, open and closed in $\text{Cr}(\hat{d})$.

We find $\varepsilon(\delta) \in (0, \varepsilon_0)$ and compact G -neighborhoods K_i in Σ of $[u_i]_{\delta}$ such that

$$\text{The } G\text{-action and } \tilde{A} \text{ are smooth on an invariant neighborhood of } K_i, \tag{15}$$

$$K_i \cap K_j = \emptyset \text{ for } i \neq j, \tag{16}$$

$$\text{dist}(\partial K_i, [u_i]_{\delta}) \leq \delta \text{ and } \text{ind}(K_i) = \text{ind}([u_i]_{\delta}), \tag{17}$$

$$\begin{aligned} \partial K_i \cap \{x \in \Sigma | \tilde{A}(x) \in [\hat{d} - \varepsilon(\delta), \hat{d} + \varepsilon(\delta)], D\tilde{A}(x) = 0\} \\ \subset \partial K_i \cap \{x \in \Sigma | \tilde{A}(x) = \hat{d} - \varepsilon(\delta) \text{ or } \tilde{A}(x) = \hat{d} + \varepsilon(\delta)\}, \end{aligned} \tag{18}$$

$$\begin{aligned} \text{If } a, b \geq 0, x \in K_i \text{ and for } t \in [-a, b] \tilde{A}(x * t) \subset [\hat{d} - \varepsilon(\delta), \hat{d} + \varepsilon(\delta)], \\ \text{then } x * [-a, b] \subset K_i. \end{aligned} \tag{19}$$

We define $K_i^- = K_i \cap \Sigma^{\hat{d} - \varepsilon(\delta)}$.

Lemma 16. *The inclusion*

$$\coprod (K_i, K_i^-) \rightarrow (\Sigma^{\hat{d} + \varepsilon(\delta)}, \Sigma^{\hat{d} - \varepsilon(\delta)}) \tag{20}$$

induces an isomorphism in equivariant cohomology. Here \coprod denotes disjoint union.

Results of this type are well known if the critical orbits are isolated. That they are isolated is however not assumed here.

Proof. We use the strong excision property of Alexander-Spanier cohomology. Since we shall work in the finite-dimensional manifold Σ we can define equivariant cohomology by taking the G -product with $E_G^k = S^{2k-1}$ instead of E_G for some sufficiently large k . The inclusion (20) induces a bijection

$$(\coprod K_i) \setminus (\coprod K_i^-) \rightarrow (\Sigma^{\dot{d}-\varepsilon(\delta)} \cup \coprod K_i) \setminus \Sigma^{\dot{d}-\varepsilon(\delta)}. \tag{21}$$

Moreover if we take the G -product of the data involved in (21) with E_G^k, k large, we obtain a similar assertion to (21):

$$(\coprod K_{i,G}) \setminus (\coprod K_{i,G}^-) \rightarrow (\Sigma_G^{\dot{d}-\varepsilon(\delta)} \cup \coprod K_{i,G}) \setminus \Sigma_G^{\dot{d}-\varepsilon(\delta)}. \tag{22}$$

Moreover the inclusion

$$\coprod K_{i,G} \rightarrow \Sigma_G^{\dot{d}-\varepsilon(\delta)} \cup \coprod K_{i,G}$$

is a closed map since a closed set in the left-hand space is compact. Recall that the suffix G means product with E_G^k for k large enough. By the strong excision the inclusion in (20), say $j = \{j_i\}$, induces an isomorphism

$$\bar{H}_G(\Sigma^{\dot{d}-\varepsilon(\delta)} \cup \coprod K_i, \Sigma^{\dot{d}-\varepsilon(\delta)}) \rightarrow \bigoplus_i \bar{H}_G(K_i, K_i^-). \tag{23}$$

Here $\bar{H}_G(X) := \bar{H}(X_G)$ by definition. \bar{H}_G is called an equivariant cohomology theory. This construction is due to Borel, [B]. By condition (18), using the map $*: \mathbb{R}^+ \times \Sigma \rightarrow \Sigma$, we can easily construct a continuous map

$$r: [0, 1] \times \Sigma^{\dot{d}+\varepsilon(\delta)} \rightarrow \Sigma^{\dot{d}+\varepsilon(\delta)}$$

such that

$$\begin{aligned} r(0, \cdot) &= \text{Id}, \\ r(t, x) &= x \quad \forall t \in [0, 1] \quad \forall x \in \Sigma^{\dot{d}-\varepsilon(\delta)}, \\ r(1, x) &\in \Sigma^{\dot{d}-\varepsilon(\delta)} \cup \coprod K_i \quad \forall x \in \Sigma^{\dot{d}+\varepsilon(\delta)}, \\ r(t, \cdot) &\text{ is } G\text{-equivariant}, \\ r([0, 1] \times (\Sigma^{\dot{d}-\varepsilon(\delta)} \cup (\coprod K_i))) &\subset \Sigma^{\dot{d}-\varepsilon(\delta)} \cup (\coprod K_i). \end{aligned} \tag{24}$$

Using (24) we obtain the following G -homotopy commutative diagrams:

$$\begin{array}{ccc} (\Sigma^{\dot{d}-\varepsilon(\delta)} \cup (\coprod K_i), \Sigma^{\dot{d}-\varepsilon(\delta)}) & \xrightarrow{\text{incl}} & (\Sigma^{\dot{d}+\varepsilon(\delta)}, \Sigma^{\dot{d}-\varepsilon(\delta)}) \\ r(1, \cdot) \circ \text{incl} \sim \text{id} \swarrow & & \searrow r(1, \cdot) \\ (\Sigma^{\dot{d}-\varepsilon(\delta)} \cup (\coprod K_i), \Sigma^{\dot{d}-\varepsilon(\delta)}) & & \\ \\ (\Sigma^{\dot{d}+\varepsilon(\delta)}, \Sigma^{\dot{d}-\varepsilon(\delta)}) & \xrightarrow{r(1, \cdot)} & (\Sigma^{\dot{d}-\varepsilon(\delta)} \cup (\coprod K_i), \Sigma^{\dot{d}-\varepsilon(\delta)}) \\ \text{incl} \circ r(1, \cdot) \sim \text{id} \swarrow & & \searrow \text{incl} \\ (\Sigma^{\dot{d}+\varepsilon(\delta)}, \Sigma^{\dot{d}-\varepsilon(\delta)}) & & \end{array} .$$

So incl is a G -homotopy equivalence. Combining this fact with (23) we see that the inclusion

$$\coprod (K_i, K_i^-) \rightarrow (\Sigma^{\dot{d}+\varepsilon(\delta)}, \Sigma^{\dot{d}-\varepsilon(\delta)})$$

induces an isomorphism in equivariant cohomology. Denote the inclusion

$$(K_i, K_i^-) \rightarrow (\Sigma^{\hat{d} + \varepsilon(\delta)}, \Sigma^{\hat{d} - \varepsilon(\delta)})$$

by j_i . Hence

$$\bar{H}_G(\Sigma^{\hat{d} + \varepsilon(\delta)}, \Sigma^{\hat{d} - \varepsilon(\delta)}) \xrightarrow{\langle j_i^*, \dots, j_{m(\delta)}^* \rangle} \bigoplus_i \bar{H}_G(K_i, K_i^-). \quad \square \tag{24}$$

By Lemma 13 the inclusion

$$(\Sigma^{\hat{d} + \varepsilon(\delta)}, \Sigma^{\hat{d} - \varepsilon(\delta)}) \rightarrow (M_S^{\hat{d} + \varepsilon(\delta)}, M_S^{\hat{d} - \varepsilon(\delta)}) \tag{25}$$

induces an isomorphism in equivariant cohomology. We can combine this with (24). Consider the commutative diagram [recall Lemma 7, (24), (25)]

$$\begin{array}{ccccc} \bar{H}_G(M_S^{\hat{d} + \varepsilon(\delta)}, M_S^{\hat{d} - \varepsilon(\delta)}) & \xrightarrow{b^*} & \bar{H}_G(M_S^{\hat{d} + \varepsilon(\delta)}) & \xrightarrow{a^*} & \bar{H}_G(M_S^{\hat{d} - \varepsilon(\delta)}) \\ \langle j_i^*, \dots, j_{m(\delta)}^* \rangle \Downarrow & & \downarrow & \swarrow (f^+)^* & \uparrow (f^-)^* \\ \bigoplus \bar{H}_G(K_i, K_i^-) & \longrightarrow & \bigoplus \bar{H}_G(K_i) & \xleftarrow{\langle (f_i^+)^*, \dots, (f_{m(\delta)}^+)^* \rangle} & \bar{H}(B_G) \end{array} \tag{26}$$

where f_i, f^\pm are induced by a classifying map (see Lemma 7), everything else is induced by an inclusion. Recall the cohomology class σ exhibited in Lemma 7. We easily infer from (26) that for some $i_0 \in \{1, \dots, m(\delta)\}$,

$$j_{i_0}^*((f^+)^*(\eta^m) \cup \sigma) \neq 0 \quad \text{for } m=0, \dots, j. \tag{27}$$

Hence, using that $j_{i_0}^*(f^+)^* = (f_0^+ j_{i_0})^* = f_{i_0}^*$, and defining $\sigma_{i_0} \in \bar{H}_G^{2(k-1)}(K_{i_0}, K_{i_0}^-)$ by $\sigma_{i_0} = j_{i_0}^*(\sigma)$ we infer

$$f_{i_0}^*(\eta^m) \cup \sigma_{i_0} \neq 0 \quad \text{for } m=0, \dots, j. \tag{28}$$

Moreover the nontrivial cohomology given in (28) ‘‘lives’’ above or on level \hat{d} , namely we have the commutative diagram ($d \in [\hat{d} - \varepsilon(\delta), \hat{d} + \varepsilon(\delta)]$),

$$\begin{array}{ccc} \bar{H}_G(\Sigma^{\hat{d} + \varepsilon(\delta)}, \Sigma^{\hat{d} - \varepsilon(\delta)}) & \longrightarrow & \bar{H}_G(\Sigma^d, \Sigma^{\hat{d} - \varepsilon(\delta)}) \\ \Downarrow & & \Downarrow \\ \bigoplus \bar{H}_G(K_i, K_i^-) & \longrightarrow & \bigoplus \bar{H}_G(K_i^d, K_i^-) \end{array} .$$

That the vertical arrow on the right is an isomorphism follows as in the proof of Lemma 16. Now if $d < \hat{d}$ the cohomology classes $(f^+)^*(\eta^m) \cup \sigma$ are mapped to zero by the top-horizontal arrow. Consequently, the restrictions of the $f_{i_0}^*(\eta^m) \cup \sigma_{i_0}$ to $\bar{H}_G(K_{i_0}^d, K_{i_0}^-)$ for $d < \hat{d}$ are zero. Moreover if $d > \hat{d}$, the cohomology classes $(f^+)^*(\eta^m) \cup \sigma$ are mapped to a nonzero class, since everything remains true if we replace $\hat{d} + \varepsilon(\delta)$ by d . Hence we have proved the first part of

Lemma 19. *For $m \in \{0, \dots, j\}$ the cohomology classes*

$$f_{i_0}^*(\eta^m) \cup \sigma_{i_0} \in \bar{H}_G^{2(k-1+m)}(K_{i_0}, K_{i_0}^-)$$

are nonzero. However the restriction for $d < \hat{d}$ to $\bar{H}_G^{2(k-1+m)}(K_{i_0}^d, K_{i_0}^-)$ is zero. Moreover if $j \geq 1$ then K_{i_0} contains infinitely many critical orbits on level d , in fact $\text{ind}([u_{i_0}]) \geq j + 1$.

Proof. We have $f_{i_0}^*(\eta^m) \neq 0$ for $m=0, \dots, j$ in $\bar{H}_G^{2m}(K_{i_0})$. Hence

$$\text{ind}(K_{i_0}) \geq j + 1.$$

By construction, see (17), we have

$$\text{ind}(K_{i_0}) = \text{ind}([u_{i_0}]_\delta).$$

Therefore

$$\text{ind}([u_{i_0}]_\delta) \geq j + 1 \geq 2,$$

which implies our assertion. \square

Now using (15), (17), (18), (19), a result by Wasserman, [Wa], and an equivariant partition of unity argument, there is a G -invariant smooth map \hat{A} defined on a neighborhood of K_{i_0} such that

$$\tilde{A} \text{ is } C^\infty\text{-close to } \hat{A}, \tag{29}$$

$$\hat{A} \text{ coincides with } \tilde{A} \text{ on a neighborhood of } K_{i_0}, \tag{30}$$

$$\text{The critical } S^1\text{-orbits on levels between } \hat{d} - (\varepsilon(\delta)/2) \text{ and } \hat{d} + (\varepsilon(\delta)/2) \text{ are nondegenerate,} \tag{31}$$

$$\begin{aligned} &\text{The inclusion } (\{x \in K_{i_0} \mid \hat{A}(x) \leq d\}, K_{i_0}^-) \rightarrow (K_{i_0}, K_{i_0}^-) \text{ induces a map} \\ &\text{in equivariant cohomology mapping } f_{i_0}^*(\eta^m) \cup \sigma_{i_0}, m=0, \dots, j, \\ &\text{to non-zero classes of } d \geq \hat{d} + \frac{\varepsilon(\delta)}{4}, \text{ and to zero classes for} \\ &d \leq \hat{d} - \frac{\varepsilon(\delta)}{4}. \end{aligned} \tag{32}$$

Note that (32) is true if (29), and (30) hold.

Define a map $\beta: [\hat{d} - \varepsilon(\delta), \hat{d} + \varepsilon(\delta)] \rightarrow \mathbb{Z}$ by

$$\begin{aligned} \beta(d) = \max(\{m \in \{0, \dots, j\} \mid f_{i_0}^*(\eta^m) \cup \sigma_{i_0} \text{ induces a non-zero class} \\ \text{in } \bar{H}_G^{2(k-1+m)}(\{x \in K_{i_0} \mid \hat{A}(x) \leq d\}, K_{i_0}^-)\}) \cup \{-1\}. \end{aligned}$$

By the construction of \hat{A} we have

$$\beta(d) = -1 \quad \text{for } d \leq \hat{d} - \frac{\varepsilon(\delta)}{4}, \quad \beta(d) = j \quad \text{for } d \geq \hat{d} + \frac{\varepsilon(\delta)}{4}. \tag{33}$$

Lemma 20. *There exists a sequence $d_i, 0 \leq i \leq j$, such that*

$$\hat{d} - \frac{\varepsilon(\delta)}{4} < d_0 \dots < d_j \leq \hat{d} + \frac{\varepsilon(\delta)}{4} \tag{34}$$

and β is discontinuous at d_i . Moreover

$$\beta(d_i^+) - \beta(d_i^-) = 1, \quad \beta(d_0^-) = -1. \tag{35}$$

Proof. This is of course a replica of the proof of the corresponding properties of α_g . Note that by (30) K_{i_0} has property (19) with respect to the minus-gradient flow associated to \hat{A} . Equations (34) and (35) follow from the fact that (31) holds, so that

there can be only a finite number of critical orbits between levels $\hat{d} - \varepsilon(\delta)/2$ and $\hat{d} + \varepsilon(\delta)/2$. (If $d_i = d_{i+1}$ for some i , then there would be infinitely many orbits on level d_i). \square

Our aim is to show that there exists a critical point of \hat{A} in K_{i_0} on level d_i having index $2(k-1+i)$. From this Proposition 2 will follow easily. For this we have to recall some facts from equivariant Morse theory [Bo, Hi], as well as some local results concerning the Poincaré polynomial of a nondegenerate orbit. The reader can also use the note by Viterbo [V]. Combining a local version (in K_{i_0}) of Lemma 7 with Lemma 20 and using a localization technique in K_{i_0} similar to the procedure within this chapter (however somewhat simpler) together with the nondegeneracy we obtain

Lemma 21. *For d_i as in Lemma 20 there exists a critical point u_i of \hat{A} in K_{i_0} on level d_i such that*

$$\bar{H}_G^{2(k-1+i)}(N_i, \dot{N}_i) \neq 0, \quad i=0, \dots, j, \tag{36}$$

where $N_i \rightarrow G * u_i$ denotes the negative bundle and \dot{N}_i is the negative bundle with the zero-section deleted.

We need now some information about the Morse index of the u_i .

Lemma 22. *The Morse index of u_i as given in Lemma 21 is*

$$m^-(u_i) = 2(k-1+i). \tag{37}$$

By the nondegeneracy of u_i the nullity is exactly one:

$$m^0(u_i) = 1. \tag{38}$$

Proof. Denote by $N_{i,x}$ the fibre over $x \in G * u_i$ and consider the trivial vectorbundle

$$N_x \times S^\infty \xrightarrow{p} S^\infty. \tag{39}$$

The isotropy group G_x of x is a \mathbb{Z}_l , $l = \text{ord } G_x$. Let g be a generator for G_x . Then $gN_{i,x} = N_{i,x}$ and G_x acts on the vectorbundle (39) in the obvious way. Of course we take the standard action on S^∞ . p commutes with the action and taking quotients we obtain

$$\zeta := (N_{i,x} \times S^\infty)/G_x \rightarrow S^\infty/G_x =: L^\infty, \tag{40}$$

where L^∞ is an infinite dimensional lens-space. Clearly we have the commutative diagram

$$\begin{array}{ccc} (N_i \times E_G)/G & \xrightarrow{\quad} & \zeta \\ \downarrow & & \downarrow \\ (G * u_i \times E_G)/G & \xrightarrow{\quad} & L^\infty \end{array}$$

where the horizontal maps are isomorphisms. (So we have a vectorbundle isomorphism.) Now $\zeta \rightarrow L^\infty$ is \mathbb{Q} -orientable iff $N_i \rightarrow G * u_i$ is \mathbb{Q} -orientable.

We start with computing $\bar{H}_G(G * u_i)$. We have

$$(G * u_i \times E_G)/G \simeq L^\infty = S^\infty/G_x = S^\infty/\mathbb{Z}_e.$$

By a result in [B] we infer

$$\bar{H}(S^\infty/\mathbb{Z}_l, \mathbb{Q}) \xrightarrow{\pi^*} [\bar{H}(S^\infty; \mathbb{Q})]^{\mathbb{Z}_l} = (\mathbb{Q}, 0), \tag{42}$$

where $\pi: S^\infty \rightarrow S^\infty/\mathbb{Z}_l$ is the projection and

$$[\bar{H}(S^\infty, \mathbb{Q})]^{\mathbb{Z}_l} = \{a \in \bar{H}(S^\infty, \mathbb{Q}) \mid g^* a = a \ \forall g \in \mathbb{Z}_l \cong G_x\}.$$

The exact equivariant cohomology triangle for the pair (N_b, \dot{N}_i) is

$$\begin{array}{ccc} \bar{H}_G(N_b, \dot{N}_i) & \longrightarrow & \bar{H}_G(N_b) \\ & \delta^* \swarrow & \searrow \\ & \bar{H}_G(\dot{N}_i) & \end{array} \tag{43}$$

Since N_i G -retracts fibrewise to $G * u_i$ we have

$$\bar{H}_G(\dot{N}_i) = (\mathbb{Q}, 0).$$

So (43) gives

$$\begin{array}{ccc} \bar{H}_G(N_b, \dot{N}_i) & \longrightarrow & (\mathbb{Q}, 0) \\ & \delta^* \swarrow & \searrow \\ & \bar{H}_G(\dot{N}_i) & \end{array} . \tag{44}$$

Since $(\dot{N}_i \times E_G)/G \simeq (\dot{N}_{i,x} \times E_G)/G_x$, we obtain again by a result in [B]

$$\begin{aligned} H_G(\dot{N}_i) &= \bar{H}((\dot{N}_{i,x} \times E_G)/G_x) \stackrel{\pi^*}{\simeq} [\bar{H}(\dot{N}_{i,x} \times E_G)]^{\mathbb{Z}_l} \\ &= [\bar{H}(\dot{N}_{i,x})]^{\mathbb{Z}_l} \quad (S^\infty \text{ contractible}). \end{aligned} \tag{45}$$

If all $g \in \mathbb{Z}_l$ induce an orientation preserving (op) map, we have with $\dim N_{i,x} = a$,

$$[\bar{H}(\dot{N}_{i,x})]^{\mathbb{Z}_l} = (\mathbb{Q}, 0) \oplus (\mathbb{Q}, a-1) \quad (\text{if op}), \tag{46}$$

if one is orientation reversing (or),

$$[\bar{H}(\dot{N}_{i,x})]^{\mathbb{Z}_l} = (\mathbb{Q}, 0) \quad (\text{if or}). \tag{47}$$

So if $\mathbb{Z}_l \cong G_x$ acts orientation preserving on N_i which is equivalent to $N_i \rightarrow G * u_i$ is orientable, we infer combining (44), (45), (46),

$$\begin{array}{ccc} \bar{H}_G(N_b, \dot{N}_i) & \longrightarrow & (\mathbb{Q}, 0) \\ & \delta^* \swarrow & \nwarrow \text{injective} \\ & (\mathbb{Q}, 0) \oplus (\mathbb{Q}, a-1) & \end{array}$$

which implies

$$\bar{H}_G(N_b, N_i) = (\mathbb{Q}, a) \quad \text{if } N_i \rightarrow G * u_i \text{ is orientable.} \tag{48}$$

If $N_i \rightarrow G * u_i$ is not orientable then a similar argument based on (47) gives

$$\bar{H}_G(N_b, N_i) = 0 \quad \text{if } N_i \rightarrow G * u_i \text{ is non-orientable.} \tag{49}$$

Now by assumption $\bar{H}_G^{2(k-1+i)}(N_b, \dot{N}_i) \neq 0$. So we must be in case (48) with $2(k-1+i) = a$. This proves (37). Equation (38) is clear. \square

Proof of Proposition 2. Since \hat{A} is arbitrarily C^∞ -close to \tilde{A} , we find in view of Lemma 22 for $i \in \{k, \dots, k+j\}$ a critical point \bar{x}_i of \tilde{A} on level \hat{d} such that

$$m^-(\bar{x}_i) \leq 2(i-1) \leq m^-(\bar{x}_i) + m^0(\bar{x}_i) - 1, \\ \text{ind}[\bar{x}_i] \geq j + 1.$$

Since $\delta > 0$ is arbitrarily given, (5) follows immediately. \square

II.4. The Index Interval

We shall show in this section that $\sigma(S)$ is a compact interval in $(0, +\infty)$ and that the map $S \rightarrow \sigma(S)$ is continuous.

Lemma 23. *Let $\hat{S} = \left\{ x \in \mathbb{R}^{2n} \mid \frac{1}{2} \sum_{i=1}^{2n} x_i^2 = 1 \right\}$. Then*

$$\sigma(\hat{S}) = \left\{ \frac{n}{2\pi} \right\}.$$

Proof. Given $x_0 \in \hat{S}^1$ the map $t \rightarrow \exp(2\pi Jt)x_0$ parametrises an element in $\mathcal{T}(\hat{S})$, and every $\Gamma \in \mathcal{T}(\hat{S})$ can be obtained this way. Dividing out the S^1 -action in S we obtain a bijection

$$\hat{S}/S^1 \xrightarrow{\cong} \mathcal{T}(\hat{S}), \quad [x_0] \rightarrow \{\exp(2\pi tJ)x_0 \mid t \in \mathbb{R}\}.$$

We compute $V(\Gamma) = \frac{1}{2} \int_0^1 2\pi |x_0|^2 dt = \frac{1}{2} 4\pi = 2\pi$ for $\Gamma \in \mathcal{T}(\hat{S})$. Therefore the critical levels for $A_{\hat{S}}$ must be of the form $-\frac{1}{2\pi l}$, $l \in \mathbb{N}^*$. Since our critical point problem is a linear eigenvalue problem, one easily computes (a variant of the Courant-Hilbert min-max principle)

$$\hat{d}_1 = \hat{d}_2 = \dots = \hat{d}_n < \hat{d}_{n+1} = \dots = \hat{d}_{2n} < \hat{d}_{2n+1} = \dots = \hat{d}_{3n} \text{ etc.},$$

where

$$d_m = -\frac{1}{2\pi l}.$$

Hence

$$\lim_{l \rightarrow \infty} |d_m| ln = \frac{n}{2\pi}.$$

This implies as one easily sees,

$$\lim_{\uparrow 0} \alpha_S(d) |d| = \frac{n}{2\pi}. \quad \square$$

Proof of Theorem 1 (i). $\sigma(S) \in \mathcal{C}$ and $S \rightarrow \sigma(S)$ is continuous.

Let \hat{S} as in Lemma 23. For $b > 0$ denoting by bS the image of $S \in \mathcal{H}$ under the map $z \rightarrow bz$, we see that

$$H_{bS} = b^{-2} H_S.$$

This implies

$$\alpha_{bS}(d) = \alpha_S(b^2 d), \quad d \in (-\infty, 0).$$

Consequently

$$\inf\sigma(bS) = b^{-2} \inf\sigma(S), \quad \sup\sigma(bS) = b^{-2} \sup\sigma(S).$$

Write $R > S$ iff R encloses S . Assume $R > S$, then $H_S \geq H_R$, and consequently $H_S^* \leq H_R^*$. So we find a G -map of the form $x \rightarrow f(x)x$ from $M_R^d \rightarrow M_S^d$. This implies $\alpha_S \geq \alpha_R$. Hence we have

$$R > S \Rightarrow \alpha_S \geq \alpha_R. \tag{1}$$

This implies in particular

$$R > S \Rightarrow \inf\sigma(S) \geq \inf\sigma(R) \quad \text{and} \quad \sup\sigma(S) \geq \sup\sigma(R). \tag{2}$$

Now given $S \in \mathcal{H}$ we find $\delta > 0$ such that

$$\delta^{-1}\hat{S} > S > \delta\hat{S}. \tag{3}$$

Hence by Lemma 23 and the previous discussion

$$\sigma(S) \subset \left[\delta^2 \frac{n}{2\pi}, \delta^{-2} \frac{n}{2\pi} \right].$$

So we know that $\sigma(S) \in \mathcal{C}$. Next we show the continuity $S \rightarrow \sigma(S)$. Assume $\varepsilon > 0$ is given. For $\delta \in (0, 1)$ define $U_{S, \delta}$ by

$$R \in U_{S, \delta} \quad \text{iff} \quad (1 - \delta)S < R < (1 + \delta)S.$$

Then $(U_{S, \delta})_{S \in \mathcal{H}, \delta \in (0, 1)}$ is a basis for the topology on \mathcal{H} . By our previous discussion we have for $R \in U_{S, \delta}$,

$$\begin{aligned} (1 + \delta)^{-2} \inf\sigma(S) &\leq \inf\sigma(R) \leq (1 + \delta)^2 \inf\sigma(S), \\ (1 + \delta)^{-2} \sup\sigma(S) &\leq \sup\sigma(R) \leq (1 + \delta)^2 \sup\sigma(S). \end{aligned}$$

Therefore, we have for sufficiently small δ ,

$$d(R, S) < \varepsilon \quad \forall R \in U_{S, \delta}.$$

This proves the continuity. \square

Lemma 24. For $S \in \mathcal{H}$ we have

$$\alpha_S(d) - \alpha_S(d^-) \leq n \quad \forall d \in (-\infty, 0).$$

Proof. Arguing indirectly assume for some $d \in (-\infty, 0)$ we have

$$\alpha_S(d) - \alpha_S(d^-) \geq n + 1.$$

Then, denoting by $\text{Cr}(d)$ the critical set of A_S on level d , we have

$$\text{ind}(\text{Cr}(d)) \geq n + 1$$

by Lemma 6 (iii). By a result in [F-R, Proposition 6.12] (use that $\text{ind} = \text{ind}_\eta^* + 1$, $\dim_\eta = 2$)

$$2(\text{ind}(\text{Cr}(d)) - 1) \leq \dim(\text{Cr}(d)/G).$$

Therefore

$$2n \leq \dim(\text{Cr}(d)/G).$$

Since $\dim(\text{Cr}(d)/G) \leq 2n - 1$, we obtain a contradiction. \square

A consequence is the following useful

Lemma 25.

$$\begin{aligned} \inf \sigma(S) &= \liminf_{k \rightarrow \infty} |\hat{d}_k|k, \\ \sup \sigma(S) &= \limsup_{k \rightarrow \infty} |\hat{d}_k|k. \end{aligned}$$

Proof. We have

$$\alpha_S(d) |d| \geq \alpha_S(d) |\bar{d}| \geq (\alpha_S(\bar{d}) - n) |\bar{d}|, \tag{4}$$

where $\bar{d} \geq d$ is the closest point of discontinuity for α_S on the right of d . Defining $\underline{d} \leq d$ similarly we obtain

$$\alpha_S(d) |d| \leq \alpha_S(\underline{d}) |\underline{d}|. \tag{5}$$

Hence (4) and (5) imply our assertion. \square

III. Index Sequence and Torsion at a Hamiltonian Trajectory

III.1. Index Sequence and Winding Number

Let $S \in \mathcal{H}$ and pick $\Gamma \in \mathcal{T}(S)$. Denote by $x: \mathbb{R} \rightarrow S$ a solution of $\dot{x} = JH'(x)$ with $x(0) \in \Gamma$, where $H = H_S$. Consequently $x(t) \in \Gamma$ for all $t \in \mathbb{R}$. As we have already seen the minimal period T of x satisfies $T = V(\Gamma)$. We study now the linearisation of $\dot{x} = JH'(x)$ along x , which is

$$\dot{y}(t) = JH''(x(t))y(t). \tag{LHS}$$

Definition 8. Two times $t_1 < t_2$ are called *conjugate* along x if the linearised problem (LHS) possesses a solution $y: [t_1, t_2] \rightarrow \mathbb{R}^{2n}$ satisfying $y(t_1) = y(t_2)$. The *multiplicity* of t_2 with respect to t_1 is the number of linearly independent solutions of (LHS) satisfying $y(t_1) = y(t_2)$. If $t_1 = 0$, we define $m(t)$ for $t > 0$ as follows:

$$m(t) = \begin{cases} 0 & \text{if } t \text{ is not conjugate to } 0. \\ \text{multiplicity of } t & \text{if } t \text{ is conjugate to } 0. \end{cases} \tag{1}$$

Now we are in the position to associate to $\Gamma \in \mathcal{T}(S)$ an index sequence as follows

$$(\mathbb{N}^* = \mathbb{N} \setminus \{0\}, \mathbb{N} = \{0, 1, \dots\}).$$

Definition 9. Let $\Gamma \in \mathcal{T}(S)$. The *index sequence* of Γ denoted by $i_\Gamma = (i_\Gamma^k)_{k \in \mathbb{N}^*}$ is defined as follows:

$$i_\Gamma^k = \sum_{0 < s < kV(\Gamma)} m(s). \tag{2}$$

In [E 1–E 3] the reader will find the basic properties of the index sequence. An alternative but equivalent definition of the index sequence can be given as follows.

For $k \in \mathbb{N}^*$ denote by F_k the Hilbert space,

$$F_k = \left\{ y: [0, kV(\Gamma)] \rightarrow \mathbb{R}^{2n} \mid y \in H^1([0, kV(\Gamma)]; \mathbb{R}^{2n}) \text{ and } \int_0^{kV(\Gamma)} y(t) dt = 0 \right\}.$$

Define a quadratic form Q_k on F_k by

$$Q_k(y) = \int_0^{kV(\Gamma)} [\langle J\dot{y}(t), y(t) \rangle + \langle H^{*''}(-J\dot{x}(t))J\dot{y}(t), J\dot{y}(t) \rangle] dt.$$

Then it has been shown in [E 3] or [E-H 1] that i_R^k is just the number of negative squares of Q_k , or with other words the maximal dimension of a linear subspace of F_k , so that the restriction of Q_k to that subspace is negative definite. Moreover there is a formula relating i_R to i_R^k and the Floquet multiplier of the time- T -map of the fundamental solution of (LHS)

$$i_R^k = \sum_{w=1}^k j(w), \tag{3}$$

where j is a map from the unit circle $\{z \in \mathbb{C} \mid |z|=1\}$ in \mathbb{C} into the non-negative integers, which is described in detail in [E 2]. Equation (3) implies that

$$\lim_{k \uparrow \infty} \frac{1}{k} i_R^k = \frac{1}{2\pi} \int j(w) dw =: \hat{i}_R. \tag{4}$$

We call \hat{i}_R the mean index of Γ . Now using results in [C-Z 1, C-Z 2] we can relate \hat{i}_R to a winding number. In [C-Z 1] Conley and Zehnder introduced an index based on a winding number and related to previous work by Duistermaat [Du] and Cushman-Duistermaat [Cu-Du]. From facts which can be found in [C-Z 1, p. 651] and formula (1.17) in [C-Z 2] we have for a constant $C > 0$ independent of Γ (note that our Δ_Γ is $\frac{1}{2}$ times Conley-Zehnder's Δ)

$$|i_R^k - 2\Delta_\Gamma(kV(\Gamma))| \leq C \quad \forall k \in \mathbb{N}. \tag{5}$$

Since, as shown in [C-Z 2, p. 652] $\Delta_\Gamma(kV(\Gamma)) = k\Delta_\Gamma(V(\Gamma))$, we infer combining (5) with

$$\Delta_\Gamma(kV(\Gamma)) = k\gamma(\Gamma), \tag{6}$$

the following:

Lemma 26. *For $\Gamma \in \mathcal{T}(S)$ we have*

$$\hat{i}_R = 2\gamma(\Gamma). \tag{7}$$

Proof. Using (5) and (6) we have

$$\left| \frac{1}{k} i_R^k - 2\gamma(\Gamma) \right| \leq C \frac{1}{k}.$$

Taking the limit gives (7). \square

In the following we study in more detail the quantity \hat{i}_R to obtain information concerning $\gamma(\Gamma)$ and $\bar{\gamma}(\Gamma)$

Lemma 27. *Let $\dim \ker(R(T) - \text{Id}) = d$. Then*

$$j(-1) \geq d + i_T^1.$$

Here $R(t)$ is the fundamental solution of (LHS) with $R(0) = \text{Id}$.

Proof. F_1 possesses a $(d + i_T^1)$ -dimensional subspace X such that $Q_1|_X \leq 0$. X admits the Q_1 -orthogonal decomposition

$$X = X_1 \oplus X_2,$$

where X_1 is spanned by the functions in the kernel of Q_1 and X_2 is spanned by the eigenfunctions belonging to negative eigenvalues. Let $\{y_1, \dots, y_d\}$ be a Q_1 -orthogonal basis for X_1 and $\{y_{d+1}, \dots, y_{d+i_T^1}\}$ a Q_1 -orthogonal basis for X_2 . We define $Y_j \subset F_2$ for $j = 1, 2, 3$ by

$$Y_1 = \{y \in F_2 \mid \dot{y} = \dot{z} \text{ on } [0, V(\Gamma)] \text{ for some } z \in X_1 \text{ and } \dot{y} = 0 \text{ otherwise}\},$$

$$Y_2 = \{y \in F_2 \mid \dot{y} = \dot{z} \text{ on } [0, V(\Gamma)] \text{ for some } z \in X_2 \text{ and } \dot{y} = 0 \text{ otherwise}\},$$

$$Y_3 = \{y \in F_2 \mid \dot{y} = 0 \text{ on } [0, V(\Gamma)] \text{ and } \dot{y} = \dot{z}(\cdot - V(\Gamma)) \text{ for some } z \in X_2\}.$$

Then the Y_j are mutually Q_2 -orthogonal in F_2 and a simple calculation shows

$$Q_2|_{Y_1 \oplus Y_2 \oplus Y_3} \leq 0.$$

Moreover $Q_2(y) = 0$ implies $y \in Y_1$ if $y \in Y_1 \oplus Y_2 \oplus Y_3$. Since Y_1 does not contain an eigenfunction since y is constant on $(V(\Gamma), 2V(\Gamma))$, we infer the existence of a linear subspace Y of F_2 such that

$$Q_2(y) < 0 \quad \text{if } y \in Y \setminus \{0\},$$

$$\dim Y = \dim(Y_1 \oplus Y_2 \oplus Y_3) = d + 2i_T^1.$$

Therefore

$$j(-1) = i_T^2 - i_T^1 \geq d + 2i_T^1 - i_T^1 = d + i_T^1,$$

as required. \square

Lemma 28. *There is an integer $\delta \in [0, d]$ such that*

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \neq 0}} j(e^{i\varepsilon}) = i_1 + n + \delta.$$

Proof. See [E 1] or [E-La]. \square

Corollary 2. $j(w) \geq 2$ except for a finite number of points.

Proof. By Lemma 27 we have $j(-1) \geq d \geq 2$. That $d \geq 2$ follows from the 2-homogeneity of H , since T is conjugate to 0. It has been shown in [E 1] that any point of discontinuity of j must be a Floquet multiplier of x , and that if $w \neq \pm 1$ is a Floquet multiplier with $|w| = 1$, p times Krein-positive and q Krein-negative then

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \neq 0}} (j(we^{i\varepsilon}) - j(we^{-i\varepsilon})) = q - p.$$

Assume w is a point in the upper half-sphere which is not a Floquet multiplier such that $j(w) \leq 1$. Since $j(-1) \geq 2$ by Lemma 27 we have $w \neq -1$ and the arc in the upper half-sphere $[-1, w]$ must contain a Floquet multiplier. Hence the arc $[w, 1]$ can contain at most $n-2$ Floquet multipliers, hence

$$j(w) \geq i_{\Gamma}^1 + n - (n-2) \geq 2. \quad \square$$

Corollary 3. *If $n \geq 3$ then $\hat{i}_{\Gamma} > 2$.*

Proof. We have $j(w) \geq 2$ except at a finite number of points. By Lemma 28 $j(w) > 3$ for $w \neq 1$ close to 1. Hence

$$i_{\Gamma} = \frac{1}{2\pi} \int j(w) dw > 2. \quad \square$$

We develop now a special argument to extend Corollary 3 to the case $n = 2$.

Lemma 29. *Assume $R(t_0)$ has a simple eigenvalue $e^{i\theta_0}$ with $0 < \theta_0 < \pi$. Then there are neighborhoods U of t_0 and V of θ_0 and a C^1 -map $t \rightarrow \theta(t)$ from U to V such that for any $t \in U$ $e^{i\theta(t)}$ is the only eigenvalue of $R(t)$ with $\theta(t) \in V$. We have*

$$\frac{d\theta}{dt} > 0 \quad \text{if } e^{i\theta(t)} \text{ is Krein-positive,}$$

$$\frac{d\theta}{dt} < 0 \quad \text{if } e^{i\theta(t)} \text{ is Krein-negative.}$$

Proof. Krein has proved similar results when $R(t_0)$ is perturbed by increasing the Hamiltonian (that is, changing $H''(x(t))$ to $H''(x(t)) + \varepsilon Q(t)$, with $Q(t)$ positive definite (see [S-Y, Chap. III]). Here we perturb $R(t_0)$ by changing t_0 to some neighboring t , but the argument is quite similar.

By standard perturbation theory, there is a C^1 -map $t \rightarrow w(t)$, defined on a neighborhood of U , such that $w(t)$ is the only eigenvalue of $R(t)$ close to $e^{i\theta_0}$. Since $R(t)$ is symplectic and $w(t)$ is a simple eigenvalue, it cannot leave the unit circle, so $w(t) = e^{i\theta(t)}$. We can also choose for each t an eigenvector $y(t)$ in such a way that the map $t \rightarrow y(t)$ is C^1 .

Now write

$$R(t)y(t) = e^{i\theta(t)}y(t)$$

and differentiate:

$$\dot{R}(t)y(t) + R(t)\dot{y}(t) = ie^{i\theta(t)}\dot{\theta}(t)y(t) + e^{i\theta(t)}\dot{y}(t).$$

Hence

$$\begin{aligned} (R(t) - e^{i\theta(t)})\dot{y}(t) &= ie^{i\theta(t)}\dot{\theta}(t)y(t) - \dot{R}(t)y(t) \\ &= ie^{i\theta(t)}\dot{\theta}(t)y(t) - JH''(x(t))R(t)y(t) \\ &= e^{i\theta(t)}(i\dot{\theta}(t) - JH''(x(t)))y(t). \end{aligned}$$

We take the Hermitian product with $Jy(t)$. The left-hand side vanishes since

$$\begin{aligned} (R(t)\dot{y}(t), Jy(t)) &= (\dot{y}(t), R(t)^*Jy(t)) = (\dot{y}(t), JR^{-1}(t)y(t)) \\ &= (\dot{y}(t), Je^{-i\theta(t)}y(t)) = e^{i\theta(t)}(\dot{y}(t), Jy(t)). \end{aligned}$$

Therefore we are left with

$$i(y(t), Jy(t))\dot{\theta}(t) = (H''(x(t))y(t), y(t)).$$

The right-hand side is positive. It is known that the Hermitian form $-iJ$ does not vanish on the eigenvector $y(t)$, and by definition its sign defines the Krein-sign of the eigenvalue $e^{i\theta(t)}$:

- if $i(y(t), Jy(t)) > 0$, then $e^{i\theta(t)}$ is Krein-positive,
- if $i(y(t), Jy(t)) < 0$, then $e^{i\theta(t)}$ is Krein-negative.

Hence the result. \square

Before proceeding, we must make an excursion into index-theory. Take w on the unit circle and $t > 0$. Consider the Hermitian form

$$(Qy, y) = \int_0^t \langle J\dot{y}(\tau), y(\tau) \rangle d\tau + \int_0^t \langle H^{*''}(-J\dot{x}(\tau))J\dot{y}(\tau), J\dot{y}(\tau) \rangle d\tau$$

on the complex Hilbert space

$$H_w^1(0, t) = \{y \in H^1(0, t; \mathbb{C}^{2n}) \mid y(t) = wy(0)\}.$$

This form is the sum of a positive definite term (for $w \neq 1$) and a compact term. Hence it has a finite index. We call it $j(w, t)$. Note that $j(w) = j(w, T)$ in our previous notation. Clearly $j(w, t)$ cannot change without Q degenerating, which happens only if w is an eigenvalue of $R(t)$.

Definition 10. Let w be on the complex unit circle. We call $t > 0$ w -conjugate to 0 along x if w is an eigenvalue of $R(t)$. Note that Definition 8 is concerned with 1-conjugate times t . Denote by $m(w; t_1, t_2)$ for $t_1 < t_2$ the number of $s \in (t_1, t_2)$ which are w -conjugate to 0, each counted with multiplicity. (The multiplicity is of course defined similar to that in Definition 8.)

Assume t is not w -conjugate to 0 and $w \neq 1$, then j is constant in a neighborhood of (w, t) . If $w = 1$ and t is not 1-conjugate to 0, we have

$$\lim_{\substack{\theta \rightarrow 0 \\ \theta \neq 0}} j(e^{i\theta}, t) = j(1, t) + n. \tag{8}$$

To see this we determine y from \dot{y} by the formula

$$y(0) = (w - 1)^{-1} \int_0^t \dot{y}(s) ds$$

and \dot{y} spans the whole of L^2 . We can therefore rewrite Q as a Hermitian form over L^2 ,

$$(Qy, y) = \int_0^t \left[\left\langle Jy(s), \int_0^s y(\tau) d\tau \right\rangle + \left\langle H^{*''}(-J\dot{x}(s))Jy(s), Jy(s) \right\rangle \right] ds + (\bar{w} - 1)^{-1} \left\langle J \int_0^t y(s) ds, \int_0^t y(s) ds \right\rangle.$$

We can split L^2 into $L_0^2 \oplus \mathbb{C}^{2n}$, where L_0^2 is the space of \mathbb{C}^{2n} -valued L^2 -functions with mean value zero and \mathbb{C}^{2n} denotes the space of constant functions. The restriction of

Q to \mathbb{C}^{2n} has index n and the restriction of Q to L_0^2 has index $j(1, t)$. If w is close enough to 1 the index of Q will be $j(1, t) + n$. Thus we have proved

Lemma 29. *If $n=2$ then if t is not 1-conjugate to zero:*

$$\lim_{\substack{\theta \rightarrow 0 \\ \theta \neq 0}} j(e^{i\theta}, t) = j(1, t) + n. \quad \square$$

Lemma 30. *Assume $w=1$ is a double eigenvalue of $R(T)$. Then there are neighborhoods V of 1 and U of T and a continuous map $t \rightarrow \theta(t)$ from U to \mathbb{R} such that*

- (i) $\theta(t) \neq 0$ for $t \neq T$ and $e^{i\theta(t)} \in V \forall t \in U$.
- (ii) The restriction of $\theta(t)$ to $U \setminus \{T\}$ is C^1 .
- (iii) For $t \in U$ $e^{i\theta(t)}$ and $e^{-i\theta(t)}$ are the only eigenvalues of $R(t)$ belonging to V .

Proof. T is clearly conjugate to 0 with multiplicity 2 as we have previously seen. Conjugate points are known to be isolated [E 2, E 3] so that there is a neighborhood U' of T with

$$\text{Ker}(R(t) - \text{Id}) = \{0\} \quad t \in U' \setminus \{T\}.$$

We consider the equation

$$\det(R(t) - w \text{Id}) = 0. \tag{9}$$

The left-hand side is a polynomial in w with smooth coefficients in t . For $t = T$ there is a double root $w = 1$. Choose a disk V around $w = 1$ containing no other root. Then there exists an open neighborhood $U \subset U'$ of T such that whenever $t \in U$ and $t \neq T$. Eq. (9) has two simple roots in V . Since $R(t)$ is symplectic these roots must either be both real

$$\varrho(t) \text{ and } \varrho(t)^{-1} \text{ with } 0 < \varrho(t) \leq 1 \tag{10}$$

or both on the unit circle

$$e^{i\theta(t)} \text{ and } e^{-i\theta(t)} \text{ with } 0 \leq \theta(t) < \pi. \tag{11}$$

The functions $\varrho(t)$ and $\theta(t)$ must be C^1 on $U \setminus \{T\}$. This leaves us with four possibilities

- (a) real roots for all $t \in U$.
- (b) real roots for $t < T$, complex roots for $t > T$.
- (c) complex roots for $t < T$, real roots for $t > T$.
- (d) complex roots for all $t \neq T$.

We may choose U to be an interval containing T . By the preceding lemma $\dot{\theta}(t)$ will have a constant sign on each of the half-intervals $U \cap \{t < T\}$ and $U \cap \{t > T\}$. It follows that a complex eigenvalue $w = e^{i\theta}$ can occur at most once on each side of T . In other words, for each $w \in V$ with $|w| = 1$ and, $w \neq 1$, Eq. (9), now considered as an equation in t has at most two solutions t_1 and t_2 in U , one with $t_1 < T$ and one with $t_2 > T$. If there are exactly two we have case (d).

We now use index theory. Choose an interval $[t_1, t_2] \subset U$ with $t_1 < T < t_2$. Since T is 1-conjugate to 0 with multiplicity two, we have

$$j(1, t_2) = j(1, t_1) + 2. \tag{12}$$

Since neither t_1 nor t_2 is 1-conjugate to zero, it follows that there is a neighborhood W of 1 contained in V with

$$j(w, t_2) = j(w, t_1) + 2, \quad w \in W.$$

So, whenever $w \in W$ and $w \neq 1$, Eq. (9) must have two solutions in $(S_1, S_2) \subset U$. We are therefore in case (d) and Lemma 30 is proved. \square

Still in the case $n=2$ we have

Lemma 31. *Assume $\ker(R(T) - \text{Id})$ is two-dimensional. Then*

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} j(e^{i\varepsilon}) = i_T^1 + 3.$$

Proof. Take $t < T$ in U and consider $\theta(t)$ which was defined in Lemma 30. We have $\theta(t) > 0$ and $\theta(T) = 0$, so $e^{i\theta(t)}$ is Krein-negative by Lemma 29. Fix $\theta_1 \in (0, \pi)$ so that for all $t \in U$ with $t < T$ the only eigenvalue of $R(t)$ of the form $e^{i\theta}$, $0 < \theta \leq \theta_1$ is $\theta(t)$. Set $\theta_2(t) = \frac{1}{2}\theta(t)$ and $w_1 = e^{i\theta_1}$ and $w_2(t) = e^{i\theta_2(t)}$. We have

$$j(w_1) = \lim_{\varepsilon \rightarrow 0} j(e^{i\varepsilon}).$$

Between w_1 and $w_2(t)$ there is a single Floquet-multiplier $e^{i\theta(t)}$, which is Krein-negative. The change in $j(\cdot, t)$ is then $+1$, see [E 1]:

$$j(w_1, t) - j(w_2(t), t) = +1.$$

Now let $t \rightarrow T$. Since $R(t)$ never has eigenvalue w_1 , we have

$$j(w_1, t) = j(w_1, T) = j(w_1).$$

On the other hand we have

$$j(w_2(t), t) = j(1, t) + 2.$$

Since there are no 1-conjugate points to 0 in (t, T) we infer

$$j(1, t) = j(1, T) = i_T^1.$$

Comparing the four equalities we get

$$j(w_1) = j(w_2(t), t) + 1 = j(1, t) + 2 + 1 = i_T^1 + 3. \quad \square$$

Corollary 4. *If $n=2$ we have*

$$\hat{i}_T > 2.$$

Proof. Since $j(w) \geq 3$ if w close to 1 and the value of $j(w)$ can drop by at most 1 for $w \neq 1$ (since there can be at most one simple multiplier $w \neq 1$ on the upper half circle) we infer $j(w) \geq 2$ for $w \neq 1$. Hence

$$\hat{i}_T = \frac{1}{2\pi} \int j(w) dw > 2.$$

Proof of Theorem 2(i). Corollary 3 and 4 give $\hat{i}_T > 2$. Since by Lemma 26 we have $\hat{i}_T = 2\gamma(\Gamma)$, we find

$$\gamma(\Gamma) > 1 \quad \text{if } n \geq 2. \quad \square$$

Finally we need a result connecting the indices i_T^k and the formal Morse index defined in Definition 6 (II.2).

Proposition 3. *Let x_0 be a critical point of A_S such that $\varphi([x_0]) = \Gamma$ (see II.1). Let z be a minimal representative for $[x_0]$ and $|A(x_0)| = \frac{1}{kV(\Gamma)}$ (see II.1, Lemma 5). Then*

$$m^-(x_0) = i_T^k.$$

Proof. By Definition 6 in II.2 $m^-(x_0)$ is the index of the quadratic form on $T_{x_0}M_S$,

$$\begin{aligned} Q_{x_0}(h) &= \frac{1}{2} \int_0^1 \langle J\dot{h}(t), h(t) \rangle dt \\ &\quad - \frac{1}{2} A(x_0) \int_0^1 \langle H^{*''}(-J\dot{x}_0(t))J\dot{h}(t), J\dot{h}(t) \rangle dt \\ &= \frac{1}{2} \int_0^1 \langle J\dot{h}(t), h(t) \rangle dt \\ &\quad + \frac{1}{2} \frac{1}{kV(\Gamma)} \int_0^1 \langle H^{*''}(-J\dot{x}_0(t))J\dot{h}(t), J\dot{h}(t) \rangle dt. \end{aligned} \tag{13}$$

Now the right-hand side of (13) defines a quadratic form on E (see I.2 for the definition of E). One easily verifies that this new quadratic form which we denote again by Q_{x_0} has the same index $m^-(x_0)$. Carrying out a change of variable and a rescaling of x_0 [similar to II.1, Lemma 5 (iii)] we obtain for a suitable constant $c \in \mathbb{R}^{2n}$,

$$x(t) = V(\Gamma)z \left(\frac{t}{V(\Gamma)} \right) + c,$$

which solves $-J\dot{x} = H'(x)$ and $x(t) \in \Gamma \ \forall t \in \mathbb{R}$. Moreover x has minimal period $V(\Gamma) = T$. It is now straightforward to verify that the index of Q_k associated to x (see III.1) is the same as $m^-(x_0)$. By the definition of i_T^k this implies the desired result. \square

III.2. Computation of Total and Mean Torsion

Let S be the surface given by $H = 1$, where

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \frac{1}{2} \sum_{i=1}^n \alpha_i (q_i^2 + p_i^2). \tag{1}$$

Here the α_i are positive and independent over \mathbb{Z} . Denote by $e_i, i = 1, \dots, 2n$, the standard orthonormal basis for \mathbb{R}^{2n} . We obtain that the only Hamiltonian trajectories on S are those given by the following parametrisation:

$$\Gamma_j: x_j(t) = \sqrt{\frac{2}{\alpha_j}} \exp(2\pi t J) e_j. \tag{2}$$

Then with $V_j = V(\Gamma_j)$,

$$V_j = - \int_0^1 \left\langle \frac{1}{2} J \dot{x}_j(t), x_j(t) \right\rangle dt = \int_0^1 \frac{2\pi}{\alpha_j} dt = \frac{2\pi}{\alpha_j}. \tag{3}$$

Now (LHS) is given by a linear time independent differential equation

$$\dot{y} = J \begin{bmatrix} \alpha_1 & & & & 0 \\ & \ddots & & & \\ & & \alpha_n & & \\ & & & \alpha_1 & \\ 0 & & & & \alpha_n \end{bmatrix} y =: JAy. \tag{4}$$

Hence $R(t) = \exp(tJA)$. Note that $R(t)$ commutes with J and moreover $|R(t)y| = |y|$. Now

$$\det_J(R(t) \times \dots \times R(t)) = e^{2\pi i \Delta(t)} \det_J \tag{5}$$

is equivalent to

$$e^{i \binom{n}{j} \alpha_j t} = e^{i 2\pi \Delta(t)}, \quad \Delta(0) = 0. \tag{6}$$

Therefore

$$\Delta(t) = \frac{1}{2\pi} \left(\sum_{j=1}^n \alpha_j \right) t. \tag{7}$$

Consequently with $\gamma_j = \gamma(\Gamma_j)$ and $\bar{\gamma}_j = \bar{\gamma}(\Gamma_j)$ we infer

$$\gamma_j = \sum_{i=1}^n \frac{\alpha_i}{\alpha_j}, \quad \bar{\gamma}_j = \frac{1}{2\pi} \sum_{i=1}^n \alpha_i. \tag{8}$$

Note that $\sum_{i=1}^n V_i^{-1} = \frac{1}{2\pi} \sum_{i=1}^n \alpha_i = \bar{\gamma}_j, j = 1, \dots, n$.

Definition 11. Let S be defined by $H \in \hat{\mathcal{H}}$. We call S (r, R) -pinched with $0 \leq r \leq R$ if

$$x \in S \Rightarrow r \leq |x| \leq R, \quad \frac{1}{R^2} \text{Id} \leq \frac{1}{2} H''(x) \leq \frac{1}{r^2} \text{Id} \quad \forall x \in S. \tag{9}$$

Proposition 4. *If $S \in \mathcal{H}$ is (r, R) -pinched then for every $\Gamma \in \mathcal{F}(S)$, we have*

$$\frac{n}{\pi R^2} \leq \bar{\gamma}(\Gamma) \leq \frac{n}{\pi r^2}. \tag{10}$$

Proof. If $A(t)$ is a symmetric positive definite matrix depending continuously on $t \in \mathbb{R}$ we can solve

$$\dot{R} = JA(t)R, \quad R(0) = \text{Id}$$

and can associate to A and $T > 0$ a winding number $\Delta_A(T)$ just as in the definition of $\gamma(\Gamma)$. From the variational characterisation of the index sequence it is immediate that

$$A \geq B \Rightarrow B^{-1} \geq A^{-1} \Rightarrow \Delta_A(t) \geq \Delta_B(t). \tag{11}$$

Now $\bar{\gamma}(\Gamma) = \Delta_\Gamma(V(\Gamma))/V(\Gamma) = \lim_{t \rightarrow \infty} \Delta_\Gamma(t)/t$. Hence

$$\lim_{t \rightarrow \infty} \frac{\Delta_{\frac{2}{R^2} \text{Id}}(t)/t \leq \bar{\gamma}(\Gamma) \leq \lim_{t \rightarrow \infty} \frac{\Delta_{\frac{2}{r^2} \text{Id}}(t)/t, \tag{12}$$

$\frac{2}{R^2}$ Id corresponds to $\bar{H}(q_1, \dots, q_n, p_1, \dots, p_n) = \frac{1}{R^2} \sum_{j=1}^n (q_j^2 + p_j^2)$ and similarly for $\frac{2}{r^2}$ Id. For a sphere of radius Γ we can compute as before.

$$V(\Gamma) = \pi r^2, \quad \bar{\gamma}(\Gamma) = \frac{n}{\pi r^2}, \quad \gamma(\Gamma) = n. \tag{13}$$

So combining (12) and (13) gives

$$\frac{n}{\pi R^2} \leq \bar{\gamma}(\Gamma) \leq \frac{n}{\pi r^2}, \tag{14}$$

as required. \square

IV. Proof of the Main Results

IV.1. Two Basic Theorems

The following together with Proposition 1 is a key step in the proof of the main results.

Definition 12. Let $S \in \mathcal{H}$. We call a Hamiltonian trajectory $\Gamma \in \mathcal{T}(S)$ *k-essential*, where $k \in \mathbb{N}^*$, if there exists $l \in \mathbb{N}^*$ such that

$$|\hat{d}_k| = \frac{1}{lV(\Gamma)}, \quad |i_R^l - 2k| \leq 2n + 1. \tag{1}$$

Here $\text{dis}(S) = (\hat{d}_k)_{k \in \mathbb{N}^*}$ is the discontinuity sequence (see def. 7).

We have

Theorem 3. *Let $S \in \mathcal{H}$. There exists a sequence $\Gamma(k)_{k \in \mathbb{N}^*} \subset \mathcal{T}(S)$ such that*

$$\Gamma(k) \text{ is } k\text{-essential}. \tag{2}$$

Moreover if $\hat{d}_k = \dots = \hat{d}_{k+j}$ for some $j \geq 1$, then the $\Gamma(k) \dots \Gamma(k+j)$ are mutually different.

Proof. We construct the $\Gamma(k)$ inductively as follows. Denote by $(k_i)_{i \in \mathbb{N}^*}$ the sequence of “jump points” for the sequence $(\hat{d}_k)_{k \in \mathbb{N}^*}$,

$$\hat{d}_{k_i} < \hat{d}_{k_{i+1}}. \tag{3}$$

Assume $\Gamma(k)$ for $k = 1, \dots, k_i$ is constructed. We have to find $\Gamma(k_{i+1}) \dots \Gamma(k_{i+1})$ mutually different so that $\Gamma(k)$ is *k-essential* for $k \in \{k_i + 1, \dots, k_{i+1}\}$. By Proposition 1 there exist mutually different $\Gamma(k_i + 1), \dots, \Gamma(k_{i+1})$ such that

$$\hat{d}_{k_{i+1}} = \frac{1}{V(\Gamma_i)l_i}, \quad |m^-(x_i^{l_i}) - 2i| \leq 2n + 1, \quad i \in \{k_i + 1, \dots, k_{i+1}\} \tag{4}$$

for suitable $l_i \in \mathbb{N}^*$, where x_i denotes a minimal representative for $\Gamma(i)$ and $x_i^{l_i}$ is the l_i -th iterate. By Proposition 3 we have

$$m^-(x_i^{l_i}) = i_{\Gamma(i)}^{l_i}. \tag{5}$$

So combining (4) and (5) gives the desired result.

Moreover we have

Theorem 4. *Given $S \in \mathcal{H}$ there exists a constant $c = c(S) > 0$ such that the following holds:*

If $\Gamma \in \mathcal{T}(S)$ is k -essential then

$$|\widehat{\gamma}(\Gamma) - |\widehat{d}_k|k| \leq c|\widehat{d}_k|^{1/2}. \tag{6}$$

Proof. We have for some $l \in \mathbb{N}^*$,

$$|\widehat{d}_k| = \frac{1}{lV(\Gamma)}, \quad i_\Gamma^l \in [2k - 2n - 1, 2k + 2n + 1]. \tag{7}$$

In the formula

$$i_\Gamma^l = \sum_{w^l=1} j(w), \tag{8}$$

we must have

$$j(w) \in [i_\Gamma^l, i_\Gamma^l + 2n]. \tag{9}$$

Hence

$$\left| \frac{l\widehat{i}_\Gamma - i_\Gamma^l}{2lV(\Gamma)} \right| \leq 2nl \frac{1}{2lV(\Gamma)} = \frac{n}{V(\Gamma)}. \tag{10}$$

The set $\Omega := G \setminus \{\text{Floquet multipliers}\}$ can be written as

$$\Omega = \bigcup_{\lambda} U_{\lambda}, \tag{11}$$

where the U_{λ} are open intervals on $S^1 = G$ which are mutually disjoint. Moreover $\#\{\lambda\} \leq 2n - 1$. On U_{λ} j takes the value j_{λ} . Denote by $\#_{\lambda}$ the number of $w \in U_{\lambda}$ such that $w^l = 1$. We have the estimate

$$[la_{\lambda} - 1] \leq \#_{\lambda} \leq [la_{\lambda} + 1], \tag{12}$$

where $[\]$ denotes the integer part and a_{λ} is the length of U_{λ} , where we put the uniform measure of total measure one on G . Since

$$\widehat{i}_\Gamma = \sum_{\lambda} j_{\lambda} a_{\lambda},$$

we find for a fixed λ

$$lj_{\lambda}a_{\lambda} - [la_{\lambda} + 1]j_{\lambda} \leq lj_{\lambda}a_{\lambda} - \sum_{\substack{w \in U_{\lambda} \\ w^l=1}} j(w) \leq lj_{\lambda}a_{\lambda} - [la_{\lambda} - 1]j_{\lambda}.$$

This implies

$$\left| lj_{\lambda}a_{\lambda} - \sum_{\substack{w \in U_{\lambda} \\ w^l=1}} j(w) \right| \leq 2j_{\lambda}. \tag{13}$$

Using (13) we obtain

$$\begin{aligned} |l\widehat{i}_\Gamma - i_\Gamma^l| &\leq \sum_{\lambda} \left| lj_{\lambda}a_{\lambda} - \sum_{\substack{w \in U_{\lambda} \\ w^l=1}} j(w) \right| + \sum_{\substack{w \notin \Psi \\ w^l=1}} j(w) \\ &\leq (2n) \cdot 2 \cdot (i_\Gamma^l + 2n) + 2n(i_\Gamma^l + 2n) \\ &\leq 12n^2(i_\Gamma^l + 1) \leq c_1(i_\Gamma^l + 1), \end{aligned} \tag{14}$$

where c_1 is independent of Γ and only depending on $S \in \mathcal{H}$ (actually only on $\dim S$). So we conclude from (14) and (7),

$$\left| \frac{\widehat{l}_\Gamma - i_\Gamma^l}{2lV(\Gamma)} \right| \leq \frac{c_1}{2} (i_\Gamma^l + 1) |\widehat{d}_k|. \tag{15}$$

We have by (9)

$$i_\Gamma^l = \sum_{w=1}^l j(w) \geq li_\Gamma. \tag{16}$$

Combining (15) and (16) gives

$$\left| \frac{\widehat{l}_\Gamma - i_\Gamma^l}{2lV(\Gamma)} \right| \leq \frac{c_1}{2} \left(\frac{i_\Gamma^l}{l} + 1 \right) |\widehat{d}_k|. \tag{17}$$

Equations (7) and (17) together yield for a suitable constant, $c_2 = c_2(S) > 0$,

$$\left| \frac{\widehat{l}_\Gamma - i_\Gamma^l}{2lV(\Gamma)} \right| \leq \frac{c_1}{2} \left(\frac{2k + 2n + 1}{l} + 1 \right) |\widehat{d}_k| \leq c_2 \left(k|\widehat{d}_k| \frac{1}{l} + |\widehat{d}_k| \right). \tag{18}$$

Now the sequence $\{k|\widehat{d}_k| \mid k \in \mathbb{N}^*\}$ is bounded by some constant $c_3 = c_3(S) > 0$ by Theorem 1 (i). So (18) and (10) imply for some constant $c_4 = c_4(S) > 0$,

$$\left| \bar{\gamma}(\Gamma) - \frac{i_\Gamma^l}{2lV(\Gamma)} \right| \leq c_4 \left(\frac{1}{l} + |\widehat{d}_k| \right), \quad \left| \bar{\gamma}(\Gamma) - \frac{i_\Gamma^l}{2lV(\Gamma)} \right| \leq c_4 \frac{1}{V(\Gamma)}. \tag{19}$$

Moreover by (7)

$$\bar{\gamma}(\Gamma) - \frac{1}{2}(i_\Gamma^l + 2n + 1) \frac{1}{lV(\Gamma)} \leq \bar{\gamma}(\Gamma) - |\widehat{d}_k|k \leq \bar{\gamma}(\Gamma) + \frac{1}{2}(i_\Gamma^l + 2n + 1) \frac{1}{lV(\Gamma)}.$$

Therefore we have for some constant $c_5 = c_5(S) > 0$,

$$\begin{aligned} |\bar{\gamma}(\Gamma) - |\widehat{d}_k|k| &\leq \left| \bar{\gamma}(\Gamma) - \frac{i_\Gamma^l}{2lV(\Gamma)} \right| + \left| \frac{i_\Gamma^l}{2lV(\Gamma)} - |\widehat{d}_k|k \right| \\ &\leq \left| \bar{\gamma}(\Gamma) - \frac{i_\Gamma^l}{2lV(\Gamma)} \right| + c_5 \frac{1}{lV(\Gamma)} \\ &\leq \left| \bar{\gamma}(\Gamma) - \frac{i_\Gamma^l}{2lV(\Gamma)} \right| + c_5 |\widehat{d}_k|. \end{aligned} \tag{20}$$

Equations (19) and (20) combined give the two estimates:

$$|\bar{\gamma}(\Gamma) - |\widehat{d}_k|k| \leq c_6 \left(\frac{1}{l} + |\widehat{d}_k| \right), \quad |\bar{\gamma}(\Gamma) - |\widehat{d}_k|k| \leq c_6 \left(\frac{1}{V(\Gamma)} + |\widehat{d}_k| \right), \tag{21}$$

for some constant $c_6 = c_6(S) > 0$. From (21) we deduce using (7) again

$$|\bar{\gamma}(\Gamma) - |\widehat{d}_k|k|^2 \leq c_6^2 \left(|\widehat{d}_k| + \left(\frac{1}{l} + \frac{1}{v(\Gamma)} \right) |\widehat{d}_k| + |\widehat{d}_k|^2 \right). \tag{22}$$

Since $|\widehat{d}_k| \leq \inf A_S(M_S)$, $V(\Gamma)^{-1} \leq \inf A_S(M_S)$ we find for a suitable constant $c = c(S) > 0$ finally

$$|\bar{\gamma}(\Gamma) - |\widehat{d}_k|k|^2 \leq c^2 |\widehat{d}_k|, \tag{23}$$

which implies the desired result. \square

IV.2. Proof of the Main Theorem

We have already proved Theorem 1 (i). Moreover we know by Lemma 25 that

$$\inf \sigma(S) = \liminf k|\hat{d}_k|, \quad \sup \sigma(S) = \limsup k|\hat{d}_k|. \tag{1}$$

Since we have already seen after the statement of Theorem 2 that Theorem 1 (ii) is a corollary of Theorem 2, we have only to prove Theorem 2. Moreover Theorem 2 (i) has been already proved in III.1.

Proof of Theorem 2 (ii). Let $t \in \sigma(S)$. We shall construct a monotonic sequence $(k_l) \subset \mathbb{N}^*$ such that

$$\lim_{l \rightarrow \infty} |\hat{d}_{k_l}|k_l = t.$$

Then picking by Theorem 3 a k_l -essential $\Gamma(k_l) \in \mathcal{F}(S)$, we have by Theorem 4

$$|\tilde{\gamma}(\Gamma(k_l)) - |\hat{d}_{k_l}|k_l| \leq c|\hat{d}_{k_l}|^{1/2}.$$

Since $\hat{d}_{k_l} \rightarrow 0$ as $l \rightarrow \infty$ and $|\hat{d}_{k_l}|k_l \rightarrow t$, we infer

$$\tilde{\gamma}(\Gamma(k_l)) \rightarrow t \quad \text{as } l \rightarrow \infty.$$

If $\sigma(S) = \{I\}$ we have by definition

$$|\hat{d}_k|k \rightarrow I \quad \text{as } k \rightarrow \infty,$$

and are done. So assume

$$\inf \sigma(S) < \sup \sigma(S).$$

Constructing (k_l) inductively assume k_l has been constructed such that

$$k_l > k_{l-1}, \quad \|\hat{d}_{k_l}|k_l - t| < \frac{1}{l}.$$

We shall now construct k_{l+1} such that

$$k_{l+1} > k_l, \quad \|\hat{d}_{k_{l+1}}|k_{l+1} - t| < \frac{1}{l+1}.$$

We find $k^* > k_l$ such that

$$\|\hat{d}_{k^*}|k^* < \inf \sigma(S) + \frac{1}{l+1}, \quad |\hat{d}_{k^*}| < \frac{1}{2(l+1)}.$$

Using the monotonicity of (\hat{d}_k) we find for $a \in \mathbb{N}$,

$$\begin{aligned} (k^* + a + 1)|d_{k^*+a+1}| - (k^* + a)|d_{k^*+a}| &\leq (k^* + a + 1)|d_{k^*+a}| - (k^* + a)|d_{k^*+a}| \\ &\leq |d_{k^*+a}| \leq |d_{k^*}| \leq \frac{1}{2(l+1)}. \end{aligned}$$

Since there exists $a_0 \geq 0$ such that

$$(k^* + a_0)|d_{k^*+a_0}| > \sup \sigma(S) - \frac{1}{l+1},$$

we see that the balls

$$B_{\frac{1}{l+1}}((k^* + a) |d_{k^*+a}|) \quad \text{for } a=0, \dots, a_0$$

cover $\sigma(S)$. Hence we find $k_{l+1} \in \{k^*, \dots, k^* + a_0\}$ with the desired property. \square

Proof of Theorem 2 (iii). We have to show that for $S \in \mathcal{H}$ the following inequality holds:

$$\sum_{\Gamma \in \mathcal{F}(S), \bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \gamma(\Gamma)^{-1} \geq 1 \quad \forall \varepsilon > 0.$$

Fix $\varepsilon > 0$. By Theorem 3 there exists a sequence $(\Gamma(k)) \subset \mathcal{F}(S)$ such that

$$\Gamma(k) \text{ is } k\text{-essential}, \tag{2}$$

and

$$\begin{aligned} &\text{If } \hat{d}_k = \dots = \hat{d}_{k+j} \text{ for some } j \geq 1 \\ &\text{then } \Gamma(k), \dots, \Gamma(k+j) \text{ are mutually different.} \end{aligned} \tag{3}$$

By Theorem 4 we find $k_0 \in \mathbb{N}^*$ such that for every $k \geq k_0$,

$$|\bar{\gamma}(\Gamma(k)) - |\hat{d}_k|k| < \varepsilon. \tag{4}$$

Denote by $K_r(d)$ the number

$$K_r(d) = \# \left\{ l \in \mathbb{N}^* \mid \frac{1}{lV(\Gamma)} \geq |d| \right\}. \tag{5}$$

We have

$$K_r(d) = \left\lceil \frac{1}{V(\Gamma)|d|} \right\rceil. \tag{6}$$

By construction, for $k \geq k_0$,

$$\begin{aligned} k - k_0 &\leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} K_r(\hat{d}_k) = \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \left\lceil \frac{1}{V(\Gamma)\hat{d}_k} \right\rceil \\ &\leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{V(\Gamma)|\hat{d}_k|}. \end{aligned} \tag{7}$$

Dividing (7) by k , we obtain

$$1 - \frac{k_0}{k} \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{V(\Gamma)} \frac{1}{k|\hat{d}_k|}. \tag{8}$$

Take a monotonic sequence $(k_l) \subset \mathbb{N}^*$ such that

$$|d_{k_l}|k_l \geq \sup \sigma(S) - \delta$$

for a given $\delta \in (0, \sup \sigma(S))$. Then

$$1 - \frac{k_0}{k_l} \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{V(\Gamma)} \frac{1}{\sup \sigma(S) - \delta}.$$

Taking the limit $l \rightarrow \infty$ gives

$$1 \leq \frac{1}{\sup \sigma(S) - \delta} \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} V(\Gamma)^{-1}.$$

Since δ has been arbitrary

$$1 \leq \frac{1}{\sup \sigma(S)} \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} V(\Gamma)^{-1}.$$

If $\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)$, we have $\bar{\gamma}(\Gamma) \leq \sup \sigma(S) + \varepsilon$. Hence

$$1 \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{V(\Gamma)(\bar{\gamma}(\Gamma) - \varepsilon)}. \tag{9}$$

Since $\bar{\gamma}(\Gamma) \geq \inf \sigma(S) - \varepsilon$, we can write for some $\delta_\varepsilon > 0$ with $|\delta_\varepsilon| \leq c_1 \varepsilon$ for some constant $c_1 > 0$ independent of ε ,

$$\bar{\gamma}(\Gamma) - \varepsilon \geq (1 - \delta_\varepsilon) \bar{\gamma}(\Gamma).$$

Using this in (9) gives

$$1 \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{V(\Gamma)(1 - \delta_\varepsilon) \bar{\gamma}(\Gamma)}.$$

Therefore

$$1 - \delta_\varepsilon \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{V(\Gamma) \bar{\gamma}(\Gamma)} = \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \frac{1}{\gamma(\Gamma)}. \tag{10}$$

Now let $0 < \varepsilon_1 < \varepsilon$. Then

$$1 - \delta_{\varepsilon_1} \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_{\varepsilon_1}(S)} \gamma(\Gamma)^{-1} \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \gamma(\Gamma)^{-1}.$$

Since $\varepsilon_1 > 0$ was arbitrary and $\delta_{\varepsilon_1} \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$, we find

$$1 \leq \sum_{\bar{\gamma}(\Gamma) \in \sigma_\varepsilon(S)} \gamma(\Gamma)^{-1},$$

completing the proof of Theorem 2 (iii). \square

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