Commun. Math. Phys. 113, 299–339 (1987)

Communications in
Mathematical
Physics

The Decorated Teichmüller Space of Punctured Surfaces

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Abstract. A principal $\mathbb{R}_+^s$-bundle over the usual Teichmüller space of an s times punctured surface is introduced. The bundle is mapping class group equivariant and admits an invariant foliation. Several coordinatizations of the total space of the bundle are developed. There is furthermore a natural cell-decomposition of the bundle. Finally, we compute the coordinate action of the mapping class group on the total space; the total space is found to have a rich (equivariant) geometric structure. We sketch some connections with arithmetic groups, diophantine approximations, and certain problems in plane euclidean geometry. Furthermore, these investigations lead to an explicit scheme of integration over the moduli spaces, and to the construction of a "universal Teichmüller space," which we hope will provide a formalism for understanding some connections between the Teichmüller theory, the KP hierarchy and the Virasoro algebra. These latter applications are pursued elsewhere.

Let F_g^s denote the genus g surface with s points removed, where $2g-2+s>0$, $g \geqq 0$, and $s \geqq 1$. This paper presents a number of results on the Teichmüller space $\mathscr{T}_g^s$ of marked conformal classes of complete finite-area metrics on F_g^s. Actually, we define a principal $\mathbb{R}_+^s$ foliated fibration $\phi: \tilde{\mathscr{T}}_g^s \to \mathscr{T}_g^s$, where the fiber over a point of $\mathscr{T}_g^s$ is the space of all horocycles about the punctures of F_g^s; the total space of the fibration is called the "decorated Teichmüller space." The mapping class group MC_g^s of homotopy classes of orientation-preserving homeomorphisms of F_g^s (which may permute the punctures) acts on $\mathscr{T}_g^s$ and $\tilde{\mathscr{T}}_g^s$, and the map ϕ is equivariant. The theory described below is developed for the decorated Teichmüller space $\tilde{\mathscr{T}}_g^s$, and the analogous results for $\mathscr{T}_g^s$ itself are discussed in an addendum.

Our first result gives a homeomorphism between $\tilde{\mathscr{T}}_g^s$ and $\mathbb{R}_+^q$, $q=6g-6+3s$. Specifically, we assign a positive real number $\lambda(c;\tilde{\Gamma}_m)$ to $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$ and an isotopy class c of arc in F_g^s connecting punctures; fixing a family Δ of such arcs so that each

* Partially supported N.S.F. grant DMS 8601162

component of $F_g^s - \Delta$ is a triangle, the function $\bigtimes_{c \in \Delta} \lambda(c;\cdot)\colon \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+^q$ is shown to be a homeomorphism. Moreover, the functions $\lambda(\cdot;\cdot)$ are natural for the action of MC_g^s.

We next recall a construction from [EP] which gives a canonical assignment of a kind of decomposition $\Delta(\tilde{\Gamma}_m)$ of F_g^s to each point $\tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$. Specifically, $\Delta = \Delta(\tilde{\Gamma}_m)$ consists of a collection of isotopy classes of arcs in F_g^s connecting punctures so that each component of $F_g^s - \Delta$ is a cell (and no two components of Δ are homotopic); such a decomposition Δ is called an "ideal cell decomposition" of F_g^s. There is thus a corresponding decomposition of $\tilde{\mathcal{T}}_g^s$ itself, where the decomposition elements are given by

$$\mathring{\mathscr{C}}(\Delta) = \{\tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s : \Delta(\tilde{\Gamma}_m) = \Delta\}$$

as Δ ranges over all ideal cell decompositions of F_g^s. It is proved that

$$\mathscr{C}_g^s = \{\mathring{\mathscr{C}}(\Delta) : \Delta \text{ is an ideal cell decomposition of } F_g^s\}$$

gives a MC_g^s-invariant cell decomposition of $\tilde{\mathcal{T}}_g^s$. This is analogous to the Harer-Mumford-Thurston cell decomposition (see [Ha]) of $\mathcal{T}_g^s$, but we work in the hyperbolic and [Ha] in the conformal category (see also [BE]). It is furthermore shown that the isotropy group of $\mathring{\mathscr{C}}(\Delta)$ in MC_g^s is exactly the collection of mapping classes $\varphi \in MC_g^s$ so that $\varphi(\Delta)$ is isotopic to Δ. In particular, each $\mathring{\mathscr{C}}(\Delta)$ has a canonical "center" $\tilde{\Gamma}_m(\Delta) \in \tilde{\mathcal{T}}_g^s$ whose conformal symmetry group is exactly the isotropy group of $\mathscr{C}(\Delta)$ in MC_g^s. In case $\mathring{\mathscr{C}}(\Delta)$ is top-dimensional, the corresponding $\phi\tilde{\Gamma}_m(\Delta) \in \mathcal{T}_g^s$ is shown to be an arithmetic group. On the surface F_1^1, we notice a connection between centers of top-dimensional cells and the Markov forms (see [Ca]) of diophantine approximation.

The proof that each $\mathring{\mathscr{C}}(\Delta)$ is actually a cell is rather involved and proceeds as follows. We define an embedding $\tilde{\mathcal{T}}_g^s \subset \mathbb{R}_+^{2q}$ onto an intersection of homogeneous quadrics and a smooth gradient flow on $\mathbb{R}_+^{2q}$ which has this variety as its attracting fixed point set. The trajectories which limit on $\mathring{\mathscr{C}}(\Delta)$ admit a coordinate simplex $\mathring{X}$ as a Poincaré section; in fact, the induced map $\mathring{X} \to \mathring{\mathscr{C}}(\Delta)$ has as inverse the projection from $\mathbb{R}_+^{2q}$ onto a certain linear subspace. Thus, the cell-decomposition $\mathscr{C}_g^s$ of $\tilde{\mathcal{T}}_g^s$ is an essentially linear construction from this point of view.

Finally, we consider the natural action of MC_g^s on our coordinates for $\tilde{\mathcal{T}}_g^s$. MC_g^s is recognized as a subgroup of finite-index in a certain groupoid as in [Mo]. The groupoid is generated by a simple algebraic transformation, which is related to Ptolemy's theorem on Euclidean polygons which inscribe in a circle. As a consequence, we derive a faithful representation of each MC_g^s as a group of rational transformations $\mathbb{R}_+^q \to \mathbb{R}_+^q$. Several examples are pursued in detail. We remark parenthetically that the embedding $\tilde{\mathcal{T}}_g^s \subset \mathbb{R}_+^{2q}$ mentioned above leads to yet another embedding $\tilde{\mathcal{T}}_g^s \subset \mathbb{C}^{2q}$; in case $s=1$, the corresponding action of MC_g^s on coordinates is in fact a faithful representation as a group of analytic motions of $\mathbb{C}^{2q}$.

The work described herein has several applications. First of all, the action of MC_g^s on the complex $\mathscr{C}_g^s$ allows the computation of certain cohomological invariants of MC_g^s; see [HZ, P1]. Furthermore, a problem of current interest in both Mathematics and Physics is the explicit integration of top-dimensional forms over the moduli space $\mathscr{M}_g^s = \mathcal{T}_g^s / MC_g^s$. The combination of the cell-decomposition $\mathscr{C}_g^s$

and the coordinatization of $\tilde{\mathcal{T}}_g^s$ leads to an algorithmic scheme of integration of forms over $\mathcal{M}_g^s$ (see [P2]), provided the form admits a reasonable expression in our coordinates. In particular, the Weil-Petersson Kähler form has been computed with respect to our coordinates in [P2], and it is simply a matter of patience to begin to numerically compute the Weil-Petersson masses of the various moduli spaces. Moreover, the perturbative series techniques of [P1] may well combine with the integration scheme of [P2] to allow such computations by hand. Finally, we mention that there has been some discussion of a "universal Teichmüller space" in the Physics literature (see, for instance [FS]), and several of our constructions suggest a model for such an object.

This paper is organized as follows. Section 1 recalls the basic facts about Minkowski three-space, where most of our constructions take place. Section 2 develops most of our technical machinery on the geometry of the light-cone in Minkowski space. In Sect. 3, the decorated Teichmüller space is defined and several parametrizations of it are developed. For completeness, we work through the "convex hull construction" of [EP] (tailored to our needs) in Sect. 4. Section 5 is devoted to the cell decomposition $\mathscr{C}_g^s$ of $\tilde{\mathcal{T}}_g^s$. Section 6 introduces centers of cells and indicates some connections with plane Euclidean geometry. The rational representation of MC_g^s is discussed in Sect. 7, and the addendum traces through our various constructions and results for the Teichmüller space $\mathcal{T}_g^s$ (as opposed to the decorated Teichmüller space $\tilde{\mathcal{T}}_g^s$). Furthermore, the addendum describes rational representations of planar braid groups mod centers.

1. Minkowski Space and Hyperbolic Geometry

Let V be a real vector space of dimension three with a non-degenerate quadratic form $\langle \cdot , \cdot \rangle$ of type $(2,1)$ so that there is a two-dimensional positive definite subspace and a one-dimensional negative definite subspace. We may choose an orthonormal basis (e_0, e_1, e_2) for V with $\langle e_i \cdot e_j \rangle = 0$ for $i \neq j$ and $-\langle e_0, e_0 \rangle = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1$. The corresponding metric on V admits an expression

$$-dx_0^2 + dx_1^2 + dx_2^2 ,$$

and we define *Minkowski three-space* $\mathbb{M}$ to be V equipped with this metric. The coordinate x_0 on $\mathbb{M}$ will be called the *height*, and a subset of $\mathbb{M}$ will be called *horizontal* if it lies at constant height.

The hyperboloid

$$\{v \in V : \langle v, v \rangle = -1\} = \{x \in \mathbb{M} : -x_0^2 + x_1^2 + x_2^2 = -1\}$$

has two components, and the upper sheet $\mathbb{H}$ (of positive height) is a model for the hyperbolic plane: the form $\langle \cdot , \cdot \rangle$ restricts to a Riemannian metric on tangent spaces to the hyperboloid. An explicit isometry of $\mathbb{H}$ with the Poincaré disk model of the hyperbolic plane is given by radial projection from $(-1,0,0)$ to the unit disk $\mathbb{D}$ about the origin in the plane at height zero. Explicitly, if $x, y \in \mathbb{H}$ and d denotes the Poincaré distance between the projections of x and y to $\mathbb{D}$, then

$$\cosh^2 d = \langle x, y \rangle^2 .$$

The *light-cone* $L \subset \mathbb{M}$ is defined to be

$$L = \{v \in V: \langle v, v\rangle = 0\} = \{x \in \mathbb{M}:\ x_0^2 = x_1^2 + x_2^2\},$$

and the *positive light-cone* is

$$L^+ = \{x \in L:\ x_0 > 0\}.$$

We say a point $x \in \mathbb{M}$ lies "on" L^+ if $x \in L^+$, and we say x lies "inside" L^+ if x lies in the component of $(1,0,0)$ of $\mathbb{M} - L$; otherwise x lies "outside" L^+. Radial projection from $\mathbb{H}$ to $\mathbb{D}$ extends to a map

$$\bar{\cdot}: \mathbb{H} \cup L^+ \to \mathbb{D} \cup S^1_\infty,$$

where $S^1_\infty =$ Frontier $\mathbb{D}$ (in the plane $x_0 = 0$) is the "circle at infinity" of the hyperbolic plane; the fiber of $\bar{\cdot}$ over a point of S^1_∞ is a ray on L^+ from the origin of $\mathbb{M}$. A point $w = (w_0, w_1, w_2) \in L^+$ corresponds to the horocycle

$$h = \{x \in \mathbb{H}:\ \langle w, x\rangle = -1\};$$

the center of $\bar{h}$ is the point $\bar{w} \in S^1_\infty$, and a computation shows that the Euclidean radius of $\bar{h}$ in $\mathbb{D}$ is $(1 + w_0)^{-1}$. Thus, as the height of $w \in L^+$ increases, $\bar{h}$ contracts to its center, and $\bar{\cdot}$ induces a canonical identification of L^+ with the bundle of horocycles over S^1_∞.

The group of linear isomorphisms of $\mathbb{M}$ preserving the quadratic form is the Lie group $O(V) = O(1,n)$. We denote the component of the identity in $O(1,2)$ by $SO^+(1,2)$; this subgroup preserves the orientation of V and the sheet $\mathbb{H}$ of the hyperboloid. $SO^+(1,2)$ (sometimes called the "Möbius group") consists of the orientation-preserving isometries of the hyperbolic plane, and its action on L^+ describes the action of the Möbius group on horocycles. $SO^+(1,2)$ is isomorphic to the group $PSL_2\mathbb{R}$ of invertible two-by-two matrices over $\mathbb{R}$ modulo ± 1. Explicitly, the corresponding action of $PSL_2\mathbb{R}$ on $\mathbb{M}$ is given as follows. Represent $x = (x_0, x_1, x_2) \in \mathbb{M}$ by the symmetric bi-linear form

$$Q = \begin{pmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{pmatrix},$$

and notice that Q is degenerate if and only if $x \in L^+$, Q is indefinite if and only if x and $-x$ lie outside L^+, and Q is positive (negative) definite if and only if x ($-x$, respectively) lies inside L^+. The action of $A \in PSL_2\mathbb{R}$ on $\mathbb{M}$ is given by the usual action on quadratic forms, namely

$$A: Q \mapsto A^t Q A.$$

Hyperbolic elements of $SO^+(1,2)$ are those with an eigenvalue λ so that $|\lambda| \neq 1$. It follows that λ is real and positive with corresponding simple eigenvector (ray) on L^+. There is one other eigenvector on L^+ with eigenvalue λ^{-1} and a third eigenvector outside L^+ with eigenvalue 1. Using the usual correspondence between a point v on the hyperboloid

$$\{v \in \mathbb{M}:\ \langle v, v\rangle = 1\}$$

of one sheet and the (oriented) geodesic $\mathbb{H} \cap v^\perp$, the third eigenvector of a hyperbolic motion corresponds to the invariant geodesic. *Parabolic* transformations have a unique eigenvector on L^+ with eigenvalue 1 and no eigenvector inside L^+. *Elliptic* transformations have all their eigenvalues on the unit circle and one eigenvector inside L^+ (so they have a fixed point in $\mathbb{H}$). A Möbius transformation which is not the identity is hyperbolic (parabolic, elliptic) if and only if the absolute value of the trace of the corresponding element of $PSL_2\mathbb{R}$ is >2 ($=2$, <2, respectively).

Suppose that S is an affine plane in $\mathbb{M}$. We say that S is *elliptic* (*parabolic*, *hyperbolic*) if the conic section $S \cap L$ has the corresponding attribute. The restriction of the form $\langle \cdot, \cdot \rangle$ to S may be definite, degenerate or of type $(1,1)$. If

$$S = \{x \in \mathbb{M} : \langle x, s\rangle = \xi\}$$

for some $0 \neq s \in \mathbb{M}$ and $\xi \in \mathbb{R}$, then these cases correspond to $\langle s, s\rangle < 0$ (elliptic), $\langle s, s\rangle = 0$ (parabolic), and $\langle s, s\rangle > 0$ (hyperbolic), respectively. In the definite case, S has an induced Euclidean structure, and in particular if S is horizontal, then the induced metric is $\sqrt{2}$ times the usual Euclidean metric. It follows that if S is elliptic, then $S \cap L$ is a round circle in the induced structure. An isometry $1 \neq g \in SO^+(1,2)$ preserves an elliptic (parabolic, hyperbolic) affine plane in $\mathbb{M}$ if and only if g is itself elliptic (parabolic, hyperbolic, respectively).

2. The Geometry of the Light-Cone

We begin with a geometric interpretation of the restriction of the pairing $\langle \cdot, \cdot \rangle|_{L^+ \times L^+}$ to $L^+ \times L^+ \subset \mathbb{M} \times \mathbb{M}$.

Lemma 2.1. *Suppose that $u, u' \in L^+$ are non-collinear, and let $h, h' \subset \mathbb{H}$, respectively, denote the corresponding horocycles. If δ denotes the signed Poincaré distance along the geodesic from $\bar{u}$ to $\bar{u}'$ between $\bar{h}$ and $\bar{h}'$, taken with positive sign if $h \cap h' = \emptyset$ and with negative sign if $h \cap h' \neq \emptyset$, then*

$$-\langle u, u'\rangle = 2e^\delta .$$

Proof. To begin, we homogenize two formulas from Sect. 1. Namely, if x and y lie inside L^+, then the rays from the origin through x, y intersect $\mathbb{H}$ in points x', y', respectively, and the Poincaré distance d between $\bar{x}'$ and $\bar{y}'$ satisfies

$$\cosh^2 d = \langle x', y'\rangle^2 / \langle x', x'\rangle \langle y', y'\rangle .$$

Furthermore, if $v \in L^+$, then the cone from the origin over the horocycle in $\mathbb{H}$ corresponding to v is the locus

$$h(v) = \{x \text{ inside } L^+ : \langle v, x\rangle^2 = -\langle x, x\rangle\} .$$

Now, write

$$x = su + (1-s)u' \in h(u) ,$$
$$y = tu + (1-t)u' \in h(u') .$$

Solving for s, t using the conditions $x \in h(u)$, $y \in h(u')$ gives

$$s = \langle u, u' \rangle / (\langle u, u' \rangle - 2),$$
$$t = -2/(\langle u, u' \rangle - 2).$$

It follows that

$$\cosh^2 \delta = \langle x, y \rangle^2 / \langle x, x \rangle \langle y, y \rangle = \left[\frac{\langle u, u' \rangle^2 + 4}{4 \langle u, u' \rangle} \right]^2,$$

whence

$$-\langle u, u' \rangle = 2 e^{\pm \delta}.$$

Finally, $-\langle \cdot, \cdot \rangle$ is continuous on $L^+ \times L^+$ and tends to infinity (zero, respectively) as the heights of its arguments tend to infinity (zero, respectively). The result follows. $\square$

If x, y, z are distinct points in $\mathbb{M}$, then let $\pi(x, y, z)$ denote the affine plane through x, y, and z, and define $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. We next show that if x, y, $z \in L^+$, then ellipticity of $\pi(x, y, z)$ is a linear condition on $\sqrt{-\langle \cdot, \cdot \rangle} \mid_{L^+ \times L^+}$. Indeed, it will evolve that this restriction is a geometrically more natural quantity than the restriction of the pairing itself.

Lemma 2.2. *Let $\{u_i\}_1^3 \subset L^+$ and $\{\lambda_i\}_1^3 \subset \mathbb{R}_+$ be given so that*

$$-\lambda_k^2 = \langle u_i, u_j \rangle, \quad \textit{for} \quad \{i, j, k\} = \{1, 2, 3\},$$

and let $S = \pi(u_1 \cdot u_2, u_3)$. S is elliptic if and only if the three strict triangle inequalities hold amongst λ_1, λ_2, λ_3, S is parabolic if and only if

$$\lambda_i = \lambda_j + \lambda_k, \quad \textit{for some } i, j, k \textit{ with } \{i, j, k\} = \{1, 2, 3\},$$

and S is hyperbolic if and only if some non-strict triangle inequality fails amongst λ_1, λ_2, λ_3.

Proof. The tangent space to S is spanned by $v_1 = u_1 - u_3$ and $v_2 = u_2 - u_3$. Furthermore,

$$\langle v_i, v_i \rangle = 2\lambda_j^2, \quad \text{for} \quad \{i, j\} = \{1, 2\},$$
$$\langle v_1, v_2 \rangle = \lambda_1^2 + \lambda_2^2 - \lambda_3^2.$$

The determinant of this form is

$$-\lambda_1^4 - \lambda_2^4 - \lambda_3^4 + 2\lambda_1^2\lambda_2^2 + 2\lambda_1^2\lambda_3^2 + 2\lambda_2^2\lambda_3^2$$
$$= (\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 + \lambda_3 - \lambda_2)(\lambda_2 + \lambda_3 - \lambda_1)(\lambda_1 + \lambda_2 + \lambda_3).$$

At most one of these factors is not strictly positive, and the lemma follows. $\square$

The next lemma provides the induction step for our basic parametrization theorem.

Lemma 2.3. *If u_1, $u_2 \in L^+$ and λ_1, λ_2, $\lambda_3 \in \mathbb{R}_+$ with $\langle u_1, u_2 \rangle = -\lambda_3^2$, then there exists a unique $u_3 \in L^+$ on each side of $\pi(0, u_1, u_2)$ so that*

$$\langle u_2, u_3 \rangle = -\lambda_1^2, \quad \langle u_3, u_1 \rangle = -\lambda_2^2.$$

As a consequence, there exists a unique ray r on L^+ *on either side of* $\pi(0, u_1, u_2)$ *so that* $\langle u_1, u\rangle = \langle u_2, u\rangle$ *for each* $u \in r$.

Proof. The space W spanned by u_1 and u_2 has type $(1,1)$ (consider the basis u_1+u_2, u_1-u_2), and so $W^\perp$ has type $(1,0)$. Let e be a vector in W with $\langle e,e\rangle = 1$ and solve for

$$u_3 = \alpha_1 u_1 + \alpha_2 u_2 + \beta e,$$

where α_1, α_2, β are unknowns. We find that

$$-\lambda_1^2 = \langle u_2, u_3\rangle = -\alpha_1 \lambda_3^2, \qquad -\lambda_2^2 = \langle u_3, u_1\rangle = -\alpha_2\lambda_3^2.$$

Furthermore, the condition that $\langle u_3, u_3\rangle = 0$ gives

$$\beta = \pm\sqrt{2}\,\lambda_1\lambda_2\lambda_3^{-1}.$$

To see that $u_3 \in L^+$ (instead of $-L^+$), note that $(u_1+u_2)^\perp$ is of type $(2,0)$ separating L^+ from $-L^+$. The condition for a vector x to lie on L^+ is $\langle x, u_1+u_2\rangle < 0$ and $\langle x, x\rangle = 0$, and we have

$$\langle u_3, u_1+u_2\rangle = -\lambda_1^2 - \lambda_2^2 < 0.$$

Finally, the sign of β determines which side of $\pi(0, u_1, u_2)$ the vector u_3 lies on. □

Lemma 2.4. *Given three distinct rays* r_1, r_2, r_3 *from the origin on* L^+, *there are unique* $u_i \in r_i$, $i = 1,2,3$, *so that*

$$\langle u_i, u_j\rangle = -1, \quad \text{for} \quad i \neq j.$$

Proof. Choose $v_i \in r_i$, $i = 1, 2, 3$. We seek α_1', α_2', $\alpha_3' \in \mathbb{R}_+$ so that

$$\langle \alpha_i' v_i, \alpha_j' v_j\rangle = -1, \quad \text{for} \quad i \neq j.$$

These equalities give

$$\alpha_i'\alpha_j' = -\langle v_i, v_j\rangle^{-1}, \quad \text{for} \quad i \neq j,$$

and the unique positive solution is given by

$$\alpha_i' = \sqrt{\frac{-\langle v_j, v_k\rangle}{\langle v_i, v_j\rangle\langle v_i, v_k\rangle}}, \quad \text{where } \{i,j,k\} = \{1,2,3\}. \quad \square$$

Corollary 2.5. $SO^+(1,2)$ *acts transitively on positively oriented triples of distinct rays on* L^+. □

Corollary 2.5 is our analogue of the familiar "three-effectiveness" of the action of the Möbius group on S^1_∞. The next result is of fundamental importance to what follows; it will evolve that part (a) describes the action of the mapping class group on our coordinates, and part (b) describes the faces of cells of a complex on which the mapping class group acts cellularly.

Suppose that S is an affine plane in $\mathbb{M}$ which does not contain the origin, so that $S = \{x \in \mathbb{M}\colon \langle x, s\rangle = -1\}$ for some $0 \neq s \in \mathbb{M}$. We say that $y \in \mathbb{M}$ lies *above* S if S separates y from 0 (i.e., $\langle y, s\rangle < -1$).

Proposition 2.6. *Suppose that $\{u_i\}_1^4 \subset L^+$ are so that any three are linearly independent, u_1 and u_4 lie on different sides of $\pi(0, u_2, u_3)$, and let*

$$-\lambda_{ij}^2 = \langle u_i, u_j \rangle, \quad \textit{for} \quad i<j.$$

(a) *We have the equality*

$$\lambda_{14}\lambda_{23} = \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24}.$$

(b) *u_4 lies above $\pi(u_1, u_2, u_3)$ if and only if*

$$\lambda_{23}(\lambda_{24}\lambda_{34} + \lambda_{12}\lambda_{13}) < \lambda_{14}(\lambda_{13}\lambda_{34} + \lambda_{12}\lambda_{24}).$$

Furthermore, equality holds if and only if $\{u_i\}_1^4$ are coplanar.

Remark. The reader will recognize the similarity between part (a) and the classical Theorem of Ptolemy: a Euclidean quadrilateral of consecutive side lengths A, B, C, D and diagonal lengths E, F inscribes in a circle if and only if

$$EF = AC + BD.$$

Since the restriction of $\langle \cdot, \cdot \rangle$ to an elliptic plane S is Euclidean and $S \cap L$ is a round circle in this structure, Ptolemy's theorem corresponds to the case that $\{u_i\}_1^4$ are coplanar lying in an elliptic plane. Furthermore, Morin has pointed out that the equality in (a) is invariant under scaling each $u_i \in L^+$ independently; this gives a quick proof of (a) from the classical Ptolemy Theorem.

Proof. As before, the space W spanned by u_2, u_3 has type $(1, 1)$, and we let e be a vector in $W^\perp$ with $\langle e, e \rangle = 1$. We write

$$u_1 = \beta e + \alpha_2 u_2 + \alpha_3 u_3,$$

where

$$\begin{aligned} -\lambda_{12}^2 &= \langle u_1, u_2 \rangle = -\alpha_3 \lambda_{23}^2, \\ -\lambda_{13}^2 &= \langle u_1, u_3 \rangle = -\alpha_2 \lambda_{23}^2, \\ \beta &= \pm \sqrt{2}\, \lambda_{12}\lambda_{13}\lambda_{23}^{-1}. \end{aligned}$$

Similarly, we have

$$u_4 = \beta' e + \alpha_2' u_2 + \alpha_3' u_3,$$

where

$$\begin{aligned} \alpha_2' &= \lambda_{34}^2 \lambda_{23}^{-2}, \\ \alpha_3' &= \lambda_{24}^2 \lambda_{23}^{-2}, \\ \beta' &= \mp \sqrt{2}\, \lambda_{24}\lambda_{34}\lambda_{23}^{-1}. \end{aligned}$$

Notice that $\beta\beta' < 0$, since u_1 and u_4 lie on different sides of the plane $\pi(0, u_2, u_3)$.

Now, compute

$$\begin{aligned} \lambda_{14}^2 &= -\langle u_1, u_4 \rangle = \alpha_2 \alpha_3' \lambda_{23}^2 + \alpha_2' \alpha_3 \lambda_{23}^2 + \beta\beta' \\ &= \lambda_{13}^2 \lambda_{24}^2 \lambda_{23}^{-2} + \lambda_{12}^2 \lambda_{34}^2 \lambda_{23}^{-2} + 2\lambda_{24}\lambda_{34}\lambda_{12}\lambda_{13}\lambda_{23}^{-2}, \end{aligned}$$

so

$$\lambda_{14}^2 \lambda_{23}^2 = (\lambda_{24}\lambda_{13} + \lambda_{34}\lambda_{12})^2,$$

proving part (a).

For part (b), we may write

$$\pi(u_1, u_2, u_3) = \{x \in \mathbb{M}: \langle x, s \rangle = -1\}$$

for some $0 \neq s \in \mathbb{M}$, since $\{u_i\}_1^3$ are linearly independant. We write

$$s = ae + bu_2 + cu_3,$$

so that

$$\begin{aligned} -1 &= \langle s, u_2 \rangle = -c\lambda_{23}^2, \\ -1 &= \langle s, u_3 \rangle = -b\lambda_{23}^2, \\ -1 &= \langle s, u_1 \rangle = a\beta - b\alpha_3\lambda_{23}^2 - c\alpha_2\lambda_{23}^2, \end{aligned}$$

whence

$$a = \beta^{-1}(\alpha_2 + \alpha_3 - 1) = \pm \sqrt{2}\,\lambda_{12}^{-1}\lambda_{13}^{-1}\lambda_{23}^{-1}(\lambda_{12}^2 + \lambda_{13}^2 - \lambda_{23}^2).$$

The condition

$$-1 > \langle s, u_4 \rangle = a\beta' - \lambda_{23}^2(c\alpha_2' + b\alpha_3')$$

that u_4 lie above $\pi(u_1, u_2, u_3)$ becomes

$$\begin{aligned} \lambda_{23}(\lambda_{12}\lambda_{13} + \lambda_{24}\lambda_{34}) &< \lambda_{23}^{-1}(\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24})(\lambda_{13}\lambda_{34} + \lambda_{12}\lambda_{24}) \\ &= \lambda_{14}(\lambda_{13}\lambda_{34} + \lambda_{12}\lambda_{24}), \end{aligned}$$

as desired. □

The next fact is technical and is used to give coordinates on the putative cells of our complex.

Proposition 2.7. *Suppose that* $\{u_i\}_1^n$ $(n \geqq 4)$ *satisfy the following conditions for* $k = 1, \ldots, n-3$:

(i) *Any three of* $\{u_{k+i}\}_{i=0}^3$ *are linearly independent.*

(ii) u_k, u_{k+3} *lie on different sides of* $\pi(0, u_{k+1}, u_{k+2})$.

(iii) u_{k+3} *lies above* $\pi(u_k, u_{k+1}, u_{k+2})$.

(iv) $\pi(u_k, u_{k+1}, u_{k+2})$ *is either elliptic or parabolic.*

In this case, u_n *lies above* $\pi(u_1, u_2, u_3)$.

Proof. We proceed by induction on n, the case $n = 4$ being trivial. For the inductive step, we simply remove u_{n-1} from the sequence and must show that u_n lies above the plane $\pi(u_{n-4}, u_{n-3}, u_{n-2})$, the other conditions being trivially satisfied. [Notice that the hypotheses do *not* assert the non-hyperbolicity of $\pi(u_{n-2}, u_{n-1}, u_n)$.]

Adopt the notation of Fig. 2.1, where a symbol next to an edge indicates the square root of the negative of the corresponding inner product of points on L^+. By Lemma 2.2 and Proposition 2.6, we have

(1) The triple $\{a, b, e\}$ satisfies all (weak) triangle inequalities, and so does the triple $\{c, d, e\}$.

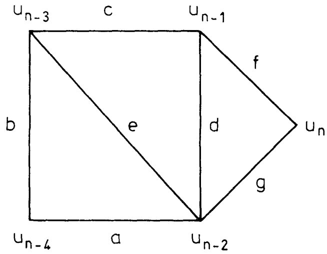

Fig. 2.1

(2) $(ac+bd)(bc+ad) > e^2(ab+cd)$.
(3) $ec(f^2+g^2-d^2) > fg(d^2-c^2-e^2)$.

Now, it follows from repeated application of Proposition 2.6 that it is sufficient to show

$$g^2(ac+bd)(bc+ad) + 2abcefg - cde^2g^2 + e^2ab(f^2-d^2) + defg(a^2+b^2-e^2) > 0 .$$

The inequality (2) gives a lower bound on the first term, so it suffices to show that

$$abe^2(f^2+g^2-d^2) + defg(a^2+b^2-e^2) + 2abcefg \geqq 0 .$$

The inequality (3) then gives a lower bound on the new first term, so it remains to show that

$$cd(a+b-e)(a+b+e) + ab(c+d-e)(c+d+e) \geqq 0 ,$$

which follows from (1). □

We close this section with a geometric interpretation of the quantities α_i', $i=1,2,3$, which were computed in Lemma 2.4. If x and y are distinct points of S^1_∞, let $\gamma\{x,y\}$ denote the (unoriented) Poincaré geodesic $\mathbb{H} \cap [\pi(0,x,y)]$.

Proposition 2.8. *Suppose that $\{u_i\}_1^3 \subset L^+$ are linearly independent, and define*

$$-\lambda_i^2 = \langle u_j, u_k \rangle, \quad \alpha_i = \frac{\lambda_i^2}{\lambda_i \lambda_j \lambda_k}, \quad \textit{for} \;\; \{i,j,k\} = \{1,2,3\} .$$

Then $2\alpha_i$ is the hyperbolic length along the horocycle $\bar{h}(u_i)$ between $\gamma\{\bar{u}_i, \bar{u}_j\}$ and $\gamma\{\bar{u}_i, \bar{u}_k\}$, $\{i,j,k\} = \{1,2,3\}$.

Proof. We first remark that an elementary computation shows that if H is a horocyclic segment of hyperbolic length ε, and δ denotes the hyperbolic distance between the endpoints of H, then

$$\varepsilon/2 = \sinh \delta/2 .$$

We concentrate on computing the hyperbolic distance between

$$\xi_+ = \bar{h}(u_1) \cap \gamma\{\bar{u}_1, \bar{u}_2\} \text{ and } \xi_- = \bar{h}(u_1) \cap \gamma\{\bar{u}_1, \bar{u}_3\}.$$

To this end and in light of Corollary 2.5, we may conjugate in $SO^+(1,2)$ so that

$$u_1 = (\sqrt{2}\,\alpha_1)^{-1}(2,0,2),$$
$$u_2 = (\sqrt{2}\,\alpha_2)^{-1}(1,-1,0),$$
$$u_3 = (\sqrt{2}\,\alpha_3)^{-1}(1,1,0).$$

If we identify a point v on the hyperboloid of one sheet with the geodesic $\mathbb{H} \cap v^\perp$ in the standard way, then $\gamma\{\bar{u}_1, \bar{u}_2\}$ ($\gamma\{\bar{u}_1, \bar{u}_3\}$, respectively) corresponds to $(1,-1,1)$ ($(1,1,1)$, respectively. A routine computation gives

$$\xi_\pm = 2^{-1/2}(\alpha_1 + \alpha_1^{-1}, \pm\alpha_1^{-1}, \alpha_1).$$

Thus, if δ denotes the hyperbolic distance between ξ_+ and ξ_-, and ε the length along $\bar{h}(u_1)$, then

$$\cosh\delta = 1 + \alpha_1^2, \quad \text{so } \varepsilon/2 = \sinh\delta/2 = \alpha_1.$$

The computations involving α_2, α_3 are similar, and the proposition is proved. □

3. Coordinates on the Decorated Teichmüller Space

Consider the compact surface F_g of genus g with a subset $P = \{x_1, x_2, \ldots, x_s\}$ of distinguished points, where $2g - 2 + s > 0$. Let $\mathcal{T}_g^s$ denote the Teichmüller space of $F_g^s = F_g - P$, corresponding to the space of marked complete hyperbolic structures of finite area on F_g^s. We restrict attention to the case where $s \geqq 1$. Each point x_i, $i = 1, \ldots, s$, gives rise to a cusp of F_g^s.

A point of $\mathcal{T}_g^s$ gives rise to an isomorphism $\pi_1 F_g^s \to \Gamma < SO^+(1,2)$, where Γ is a marked discrete group defined up to conjugacy in $SO^+(1,2)$. We will denote a marking on Γ by Γ_m. There is a corresponding covering map $\mathbb{H} \to F_g^s = \mathbb{H}/\Gamma_m$. We will also consider the corresponding group acting on the Poincaré disk $\mathbb{D}$ with covering map $\mathbb{D} \to F_g^s$. The Poincaré metric on $\mathbb{D}$ projects to a metric on F_g^s, which we refer to as the "Γ-Poincaré" metric on F_g^s. We refer to geodesics for the Γ-Poincaré metric as "Γ-geodesics", etc..

Represent a point in $\mathcal{T}_g^s$ by $\Gamma_m < SO^+(1,2)$ and choose a distinguished Γ-horocycle h_i about each cusp x_i. The specification of h_i determines a corresponding Γ-orbit B_i of points on L^+. Explicitly, choose a parabolic $\gamma_i \in \Gamma_m$ corresponding to x_i, let $z_i \in L^+$ be fixed by γ_i and correspond to h_i, and take $B_i = \Gamma z_i$. Each point of B_i has a stabilizer in Γ which is parabolic and infinite-cyclic; different stabilizers for different points of B_i are conjugate in Γ. The stabilizer of $z_i \in B_i$ corresponds to a group $H(i) < \pi_1 F_g^s$ of homotopy classes of loops generated by a loop which circles x_i exactly once.

Thus, a specification of $\Gamma_m \in \mathcal{T}_g^s$ together with a choice h_i of horocycle about each cusp x_i, $i = 1, \ldots, s$, determines an $SO^+(1,2)$-orbit of $(s+1)$-tuple $(\Gamma_m, B_1, B_2, \ldots, B_s)$, where $B_i \subset L^+$ is a Γ-orbit of points corresponding to the Γ-horocycle h_i about x_i. To formalize this notion into an object of our basic interest, we define the *decorated Teichmuller space*,

$$\tilde{\mathcal{T}}_g^s = \{(\Gamma_m, B_1, \ldots, B_s) : \Gamma_m \in \mathcal{T}_g^s\}/SO^+(1,2).$$

The "forgetful" map $\phi: \tilde{\mathcal{T}}_g^s \to \mathcal{T}_g^s$ induced by $(\Gamma_m, B_1, \ldots, B_s) \mapsto \Gamma_m$ is a principal fibration with group $\mathbb{R}_+^s$: $\mathbb{R}_+^s$ acts on the fiber $(B_1, \ldots, B_s)$ by componentwise homothety of points in L^+. $\tilde{\mathcal{T}}_g^s$ is thus a cell of real dimension $6g-6+3s$.

The (full) mapping class group MC_g^s of isotopy classes of (orientation-preserving) homeomorphisms (which may permute the punctures) acts on $\tilde{\mathcal{T}}_g^s$ in the natural way by change of marking; the fibration ϕ is MC_g^s-equivariant. Furthermore, there is a MC_g^s-invariant foliation $\mathcal{F}$ of $\tilde{\mathcal{T}}_g^s$ defined as follows. Let $\varrho_i(\tilde{\Gamma}_m)$ denote the Γ-Poincaré length of the horocycle h_i, where $\tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$ satisfies $\phi(\tilde{\Gamma}_m) = \Gamma_m$. Each $\varrho_i: \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+$ is clearly invariant under the MC_g^s-action, and we consider the foliation $\mathcal{F}$ of $\tilde{\mathcal{T}}_g^s$ by level sets of $F = \times\, \varrho_i: \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+^s$. In particular, the level set $F \equiv \times\, 1$ is MC_g^s-invariant and gives rise to a canonical equivariant section of ϕ.

Fix $\tilde{\Gamma}_m = (\Gamma_m, B_1, \ldots, B_s) \in \tilde{\mathcal{T}}_g^s$. Let c be a homotopy class of path, not necessarily simple, running from x_i to x_j, where we may have $i=j$, and straighten c to a Γ-geodesic C. Such a homotopy class in F_g^s is called an *ideal arc*. If $z \in B_i$, orient C and lift it to a geodesic in $\mathbb{D}$ starting from $\bar{z}$ and ending, say, at a point $\bar{w} \in S_\infty^1$ with $w \in B_j$. (There is only one point of B_j in the fiber of $\bar{\ }$ over $\bar{w}$.) We define the λ-*length* of c (relative to $\tilde{\Gamma}_m$) by

$$\lambda(c; \tilde{\Gamma}_m) = \lambda(c; \Gamma_m, B_1, \ldots, B_s) = \sqrt{-\langle z, w \rangle}\,.$$

Let us now examine how this quantity depends on our choices. The group Γ_m can be changed by conjugating by an isometry $g \in SO^+(1,2)$. The corresponding B_i is transformed to gB_i, z, w are transformed to gz, gw, and the lift to $\mathbb{D}$ of C is transformed by g. Since λ-lengths are a metric quantity in $\mathbb{M}$, the value of λ is unchanged; similarly, choosing z to be another element of B_i does not change the value of λ. *If* $\bar{w}' \in S_\infty^1$ with $w' \in B_j$ is the endpoint of the lift to $\mathbb{D}$ of $-C$ starting from $\bar{z}$, then there exists $\gamma \in \Gamma$ with $\gamma w' = z$, $\gamma z = w$, so the choice of orientation on C does not affect the value of λ.

For each ideal arc c, the λ-length gives a continuous positive real-valued function $\lambda(c)$ defined on $\tilde{\mathcal{T}}_g^s$. We next fix an appropriate finite number of ideal arcs $c_1, \ldots, c_q$ to obtain a map

$$\mathop{\times}_{i=1}^{q} \lambda(c_i): \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+^q\,,$$

which will be shown to be a surjective homeomorphism. An appropriate set of ideal arcs Δ is defined by taking a maximal family of disjointly embedded simple arcs in F_g^s running between distinguished points subject to the condition that no complementary region of Δ in F_g^s is a mono-gon or bi-gon. It follows that each component of F_g^s-Δ is a triangle. Such a family is called and *ideal triangulation*, and Euler characteristic considerations show that there are $q = 6g-6+3s$ ideal arcs in an ideal triangulation of F_g^s.

Theorem 3.1. *If* $\Delta = (c_1, \ldots, c_q)$ *is an ideal triangulation of* F_g^s *then*

$$\mathop{\times}_{i=1}^{q} \lambda(c_i): \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+^q$$

is a homeomorphism.

Proof. We must find the inverse map, so suppose that we are given $(\lambda_1, \ldots, \lambda_q) \in \mathbb{R}_+^q$ and wish to construct a surface. A universal cover of F_g^s is homeomorphic to the Poincaré disk $\mathbb{D}$, which is tesselated by ideal triangles with sides arising from the lifts of the ideal arcs $c_1, \ldots, c_q$. We will map this tesselation into $\mathbb{M}$ so that the vertices map to L^+.

Fix attention on one triangular region in the universal cover, and suppose that its edges correspond to ideal arcs $c_{I(1)}, c_{I(2)}, c_{I(3)}$ (not necessarily distinct). Fix a tuple r_1, r_2, r_3 of distinct rays on L^+. By Lemma 2.4, there exist points $z_i \in r_i$, $i = 1, 2, 3$, with

$$\langle z_i, z_j \rangle = -\lambda_{I(k)}^2, \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\}.$$

We now inductively map further triangles into $\mathbb{M}$. Each triangle has one side already mapped in, say with vertices $u_1, u_2 \in L^+$. The third vertex u_3 of the triangle is mapped in using Lemma 2.3. The lemma gives two choices of points with the required inner products, and the choice is resolved by the fact that we want the tesselation of the universal cover to map homeomorphically to a tesselation of $\mathbb{D}$. Since one side of $\pi(0, \mathrm{u}_1, u_2)$ already contains points of the lifted tesselation by induction, u_3 must lie on the other side of $\pi(0, u_1, u_2)$. This determines u_3 uniquely.

Each element $\beta \in \pi_1 F_g^s$ acts on the tesselation of the universal cover. Let T be one triangular region in the universal cover, and let τ_1 and τ_2 be the triangles in $\mathbb{M}$ which are the images of T and βT. There is a unique $g(\beta) \in SO^+(1,2)$ taking τ_1 to τ_2 mapping vertices correctly. From the inductive construction of triangles in $\mathbb{M}$, we see that the definition of $g(\beta)$ is independent of the choice of T. The same reasoning shows that

$$g\colon \pi_1 F_g^s \to SO^+(1,2)$$

is a homomorphism.

To see that g is injective with a discrete image, note that the inductive construction above guarantees that the tesselation of the universal cover of F_g^s is mapped injectively to a tesselation of $\mathbb{D}$. Injectivity of g follows immediately. If the image Γ_m of g were not discrete, then there would be a non-trivial element arbitrarily near the identity, and then triangles in $\mathbb{D}$ would overlap.

To complete the discussion of the tesselation and group Γ_m, we claim that the image tesselation $\mathbb{T}$ actually covers all of $\mathbb{D}$. To this end, note first that the inductive definition of $\mathbb{T}$ guarantees that $\mathbb{T}$ is open in $\mathbb{D}$. We show also that $\mathbb{T} \subset \mathbb{D}$ is closed. Each triangle τ in $\mathbb{T}$ is provided with three horocycles centered at the vertices of τ. Furthermore, by Proposition 2.8, there is some $\varepsilon > 0$ so that each horocyclic segment inside τ has length at least ε. It follows easily that $\mathbb{T}$ is closed, so connectivity of $\mathbb{D}$ guarantees that $\mathbb{T} = \mathbb{D}$, as was asserted.

The quotient of the image tesselation by Γ_m is a marked complete hyperbolic surface of finite area. This gives our map from $\mathbb{R}_+^q$ to $\mathscr{T}_g^s$. The map to the fiber of $\tilde{\mathscr{T}}_g^s$ is given by taking the natural Γ-orbits of parabolic fixed points arising as the vertices of the triangles in $\mathbb{M}$. The maps between $\tilde{\mathscr{T}}_g^s$ and $\mathbb{R}_+^q$ are clearly inverse to each other, and the theorem is proved. □

Remark. The maps in Theorem 3.1 between $\tilde{\mathscr{T}}_g^s$ and $\mathbb{R}_+^q$ are linear with respect to scalar multiplication on $\mathbb{R}_+^s$ and $\mathbb{R}_+^q$, giving a homeomorphism between $\mathscr{T}_g^s \times \text{Interior}\,(\sigma^{s-1})$ and $\text{Interior}\,(\sigma^{q-1})$, where σ^n denotes the n-simplex.

One can remove the factor Interior (σ^{s-1}) to get a homeomorphism between $\mathscr{T}_g^s$ itself and Interior (σ^{q-s}) (see Theorem A.1). We also give an interpretation of λ-lengths in terms of the matrix entries of a discrete subgroup of $PSL_2\mathbb{R}$ representing a point of $\mathscr{T}_g^s$ (see the remark after Theorem A.1).

Recall that MC_g^s acts on $\tilde{\mathscr{T}}_g^s$, and if $\varphi \in MC_g^s$, let $\varphi_*\colon \tilde{\mathscr{T}}_g^s \circlearrowleft$ denote the corresponding homeomorphism. Since φ admits a representative which respects Poincaré metrics and λ-lengths are a Poincaré metric quantity by Lemma 2.1, we have

Theorem 3.2. *λ-lengths are natural for the action of MC_g^s in the sense that if $\varphi \in MC_g^s$, $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$, and c is an ideal arc in F_g^s, then*

$$\lambda(c;\tilde{\Gamma}) = \lambda(\varphi c; \varphi_* \tilde{\Gamma}). \quad \square$$

Corollary 3.3. *Suppose that Δ is an ideal triangulation of F_g^s* and *Λ is an assignment of positive real numbers to the ideal arcs of Δ so that (Δ, Λ) determines the point $\tilde{\Gamma} \in \tilde{\mathscr{T}}_g^s$. If $\varphi \in MC_g^s$, then φ induces a one-to-one correspondence between components of Δ and components of $\varphi^{-1}\Delta$. If Λ' denotes the assignment of numbers to components of $\varphi^{-1}\Delta$ induced from Λ by φ, then $(\varphi^{-1}\Delta, \Lambda')$ determines the point $\varphi_*\tilde{\Gamma} \in \tilde{\mathscr{T}}_g^s$.* $\square$

Remark. The action of MC_g^s on $\tilde{\mathscr{T}}_g^s$ with respect to a fixed ideal triangulation is computed in Sect. 7. See also the Addendum.

We close this section with yet another parametrization of $\tilde{\mathscr{T}}_g^s$. Fix an assignment Λ of λ-lengths on the ideal arcs of an ideal triangulation Δ of F_g^s, and let $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$ correspond to (Δ, Λ). Let $\pi\colon \mathbb{D} \to F_g^s$ denote the universal cover with group Γ. Suppose that $T \subset F$ is a triangle in Δ with geodesic sides and choose a lift $\tilde{T}$ of T to $\mathbb{D}$. By an *end* of T in F_g^s, we mean the Γ-orbit of an end (in the usual sense) of the closed convex hull of $\tilde{T}$ in $\mathbb{D}$. The collection of such ends is denoted $\mathscr{E} = \mathscr{E}(\Delta)$. The end $E \in \mathscr{E}$ *abuts* on the puncture x_i of F_g^s if $\pi(E)$ is asymptotic to x_i. Two ends are said to be *equivalent* if they have a common abutment x_i, and the class is denoted $[x_i]$.

Suppose, now, that T has sides $\{c, d, e\} \subset \Delta$. The orientation on $F_g^s \supset T$ induces both a cyclic ordering (c, d, e) on $\{c, d, e\}$ and an orientation $\tilde{c}, \tilde{d}\colon [0,1] \to d, \tilde{e}$ on each side of T (see Fig. 3.1). Consider the end $E \in \mathscr{E}$ of T which meets the *tail* $\tilde{d}[0, \frac{1}{2}]$ of $\tilde{d}$. The end E is said to be *opposite* the ideal arc e, and $\tilde{d}$ is said to *abut* on the end E. We also let $\cdot^{-1}$ denote reversal of paths, so, for instance, $\tilde{d}$ and $\tilde{c}^{-1}$ have a common abutment.

Since $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$ is a decorated group, there are well-defined horocycles h_c, h_d, h_e in $\mathbb{D}$ centered at the vertices of the lift $\tilde{T}$ of T; see Fig. 3.1. The *sector* of the end E of T is the horocyclic segment $\pi(h_e \cap \tilde{T}) \subset F_g^s$. Of course, the sector depends on both $E \in \mathscr{E}$ and $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$.

We define a map

$$I\colon \tilde{\mathscr{T}}_g^s \approx \mathbb{R}_+^q \approx \mathbb{R}_+^\Delta \to \mathbb{R}_+^{2q} \approx \mathbb{R}_+^{\mathscr{E}}$$

and develop the corresponding parametrization of $\tilde{\mathscr{T}}_g^s$. To compute the coordinate entries in the target, suppose that T is a triangle with edges (c, d, e) and the end E of

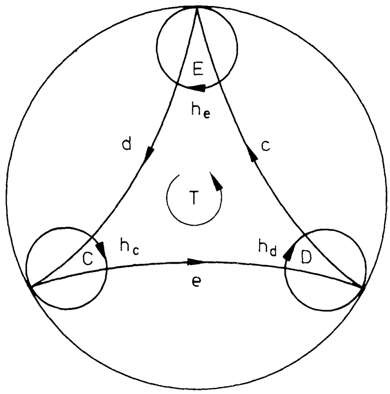

Fig. 3.1

T is opposite e. The *h-length* (for "half horocyclic length") of e for $(\Delta, \Lambda) = \tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$ is defined by

$$h(E, \tilde{\Gamma}_m) = \frac{\Lambda(e)}{\Lambda(c)\,\Lambda(d)}.$$

This defines the map I.

The terminology is motivated by our next result, which follows immediately from the definitions and Proposition 2.8.

Corollary 3.4. *If $(\Delta, \Lambda) = \tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$ and $E \in \mathscr{E}(\Delta)$, then $h(E, \tilde{\Gamma}_m)$ is half the Poincaré length of the sector of E.* □

Remark. It follows that the functions $\varrho_i: \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+$, $i = 1, \ldots, s$, used to define the foliation $\mathscr{F}$ of $\tilde{\mathcal{T}}_g^s$ are easily computed as

$$\varrho_i(\tilde{\Gamma}_m) = 2 \sum_{E \in [x_i]} h(E, \tilde{\Gamma}_m).$$

Returning to the map $I: \tilde{\mathcal{T}}_g^s \to \mathbb{R}_+^{\mathscr{E}}$, we first observe that

$$\Lambda^{-2}(c) = h(C, \tilde{\Gamma}_m)\; h(D, \tilde{\Gamma}_m),$$

so I is an embedding. Moreover, if $e \in \Delta$, then e "abuts" on four ends A, B, C, $D \in \mathscr{E}(\Delta)$ as in Fig. 3.2. The condition

$$h(A, \tilde{\Gamma}_m)\; h(B, \tilde{\Gamma}_m) = h(C, \tilde{\Gamma}_m)\, h(D, \tilde{\Gamma}_m)$$

is called the *coupling equation* of e. We summarize with

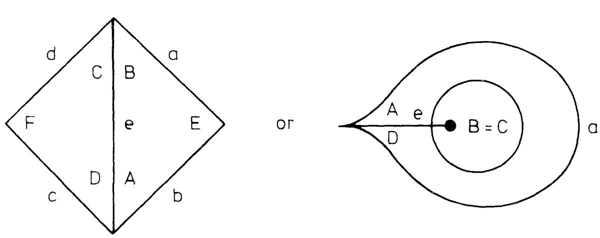

Fig. 3.2

Proposition 3.5. *I: $\tilde{\mathcal{T}}_g^s \to \mathbb{R}_+^{\mathscr{E}}$ is an embedding of $\tilde{\mathcal{T}}_g^s$ onto an intersection of homogeneous quadrics. Explicitly, $I(\tilde{\mathcal{T}}_g^s) \subset \mathbb{R}_+^{\mathscr{E}}$ is characterized by the coupling equations.* □

Several of our subsequent arguments will depend on the "h-length parametrization" of $\tilde{\mathcal{T}}_g^s$ given by Proposition 3.5. As a final parenthetical note, we simultaneously diagonalize the quadratic forms by assigning a complex coordinate $\zeta_{\tilde{e}} \in \mathbb{C}$ to each of the orientations $\tilde{e}$ on $e \in \Delta$ as follows. The orientation of F_g^s induces a canonical orientation on horocycles (as in Fig. 3.1), and if the tail of $\tilde{e}$ separates the sectors of adjacent ends E, E' in the induced order (E, E'), then we define the "strand coordinate",

$$\zeta_{\tilde{e}} = h(E, \tilde{\Gamma}_m) - \sqrt{-1}\, h(E', \tilde{\Gamma}_m).$$

This assignment defines an $\mathbb{R}$-linear embedding J: $\mathbb{R}_+^{\mathscr{E}} \subset \mathbb{C}^{\mathscr{E}}$. The coupling equation on e is equivalent to the condition that

$$\operatorname{Arg} \zeta_{\tilde{e}} = \operatorname{Arg} \zeta_{\tilde{e}^{-1}}.$$

(Arg denotes the principal value of the argument.) Of course, we introduce some further coupling equations: in case T is a triangle in Δ with (canonically) oriented edges (c, d, e), then

$$\begin{aligned} \operatorname{Re} \zeta_{\tilde{c}} &= \operatorname{Im} \zeta_{\tilde{e}^{-1}} \\ \operatorname{Re} \zeta_{\tilde{d}} &= \operatorname{Im} \zeta_{\tilde{c}^{-1}} \\ \operatorname{Re} \zeta_{\tilde{e}} &= \operatorname{Im} \zeta_{\tilde{d}^{-1}}. \end{aligned}$$

For convenience, we may write

$$\begin{aligned} J \circ I \colon \tilde{\mathcal{T}}_g^s &\hookrightarrow (\mathbb{C}^2)^q \\ \colon \tilde{\Gamma}_m &\mapsto \bigtimes_{e \in \Delta} (\zeta_{\tilde{e}} \times \zeta_{\tilde{e}^{-1}}) \end{aligned}$$

and consider the map $\mathbb{C}^{2q} \to \mathbb{C}^{2q}$ induced by the following linear coordinate change on each factor

$$\begin{gathered} \mathbb{C}^2 \to \mathbb{C}^2 \\ (\zeta, \zeta') \mapsto \tfrac{1}{2}(\zeta - \sqrt{-1}\,\zeta', \zeta' - \sqrt{-1}\,\zeta) \end{gathered}$$

In general, if ξ, $\eta \in \mathbb{C}$, then $\operatorname{Arg} \xi = \operatorname{Arg} \eta$ if and only if $|\xi - \sqrt{-1}\,\eta| = |\eta - \sqrt{-1}\,\xi|$; in this way, one imagines the variety determined by the coupling equations as a product $C^q \subset (\mathbb{C}^2)^q$, where $C = \{(z, \omega) \in \mathbb{C}^2 : |z| = |\omega|\}$.

Remarks. 1) The functions ϱ_i in the definition of the foliation $\mathscr{F}$ of $\tilde{\mathcal{T}}_g^s$ are easily computed in strand coordinates as

$$\varrho_i(\tilde{\Gamma}_m) = \frac{2}{1 + \sqrt{-1}} \sum \zeta_{\tilde{e}},$$

where the sum is over all oriented edges $\tilde{e}$ whose tails abut on x_i, $i = 1, \ldots, s$.

2) If h_i is the i^{th} distinguished horocycle determined by $\tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$, then the

"combinatorial length" of h_i is defined as $\#\{\text{components of } h_i - \Delta\}$. Provided this quantity is *not* equivalent to zero mod four, then one can easily $\mathbb{C}$-linearly solve for the h-lengths of ends of Δ that abut on x_i from the strand coordinates $\{\zeta_{\tilde{e}}: \tilde{e} \text{ abuts on } x_i, e \in \Delta\}$. In particular, when $s=1$, $q=6g-3$, so the combinatorial length of h_1 is $2q \equiv 2(4)$. Thus, h-lengths are $\mathbb{R}$-valued $\mathbb{C}$-linear functions of strand coordinates in case $s=1$. See Remark 4 after Corollary 7.4.

4. The Convex Hull Construction

Lemma 4.1. *Suppose $\Gamma_m \in \mathcal{T}_g^s$, and let $u \in L^+$. The orbit Γu is discrete in $L^+ \cup \{0\}$ if and only if u is a fixed point of some parabolic transformation in Γ. In particular, 0 is an accumulation point of Γu if and only if u is not a parabolic fixed point.*

Proof. First suppose that u is a parabolic fixed point and choose an embedded horocycle about the corresponding cusp of F_g^s. Since the horocycle is embedded, the Euclidean radius of a lift to $\mathbb{D}$ is bounded away from one. Since the height h of a point h in L^+ is related to the Euclidean radius r of the corresponding horocycle by $r=(1+h)^{-1}$, it follows that 0 is not an accumulation point of Γu.

Conversely, suppose that 0 is not an accumulation point of Γu, and let K be the complement in F_g^s of a union of disjointly embedded horoballs, one about each cusp. Choose $\beta > 0$, so that if the height of a point on L^+ exceeds β, then the corresponding horocycle in $\mathbb{D}$ is disjoint from a fixed lift of K. Now, choose $\alpha \in \mathbb{R}_+$, so the height of any point of $\Gamma\alpha u$ exceeds β. It follows that the horocycle on F_g^s corresponding to αu lies inside $F_g^s - K$, which implies that u is a parabolic fixed point.

It remains to show that if Γu accumulates at a point, say v, on L^+, then Γu accumulates at 0. To see this, choose a sequence $\{\gamma_i\}$ of hyperbolic elements of Γ so that the contracting eigenvector (ray) of γ_i on L^+ tends to the ray from 0 through v. If $\{u_i\} \subset \Gamma u$ is a sequence which accumulates at v, then $\{\gamma_i u_i\}$ accumulates at 0. □

Remarks. 1) In fact, the action of Γ on L^+ is ergodic; see [EP].

2) Another proof of discreteness of Γu for u a parabolic fixed point comes from the fact that the hyperbolic length spectrum of F_g^s is discrete [Ab] with an application of Lemma 2.1.

If $\tilde{\Gamma}_m = (\Gamma_m, B_1, \ldots, B_s) \in \tilde{\mathcal{T}}_g^s$ (see Sect. 2), then we define $\mathcal{B} = B_1 \cup \ldots \cup B_s$ and let C be the closed convex (Euclidean) hull of $\mathcal{B}$ in $\mathbb{M}$.

Lemma 4.2. *$L^+ \cap C$ is the set of points of the form αz, where $\alpha \geqq 1$ and $z \in \mathcal{B}$.*

Proof. If $u \in L^+$ is not of the stated form, then we may choose $\alpha > 1$ so that αu is also not of the stated form. Let T be the tangent plane to L^+ at αu, and let $A \subset T$ be the horizontal line through αu. We may rotate T slightly about A so that the rotated plane separates u from $\mathcal{B}$, since there are only finitely many point of $\mathcal{B}$ below the height of αu by Lemma 4.1. Therefore, $u \notin C$.

Conversely, if $u \in \mathcal{B}$ and $\alpha > 1$, choose a sequence $\{\gamma_i\}$ of hyperbolic elements of Γ whose expanding eigenvectors on L^+ tend to the ray from the origin through u. Since $\mathcal{B}$ is discrete, the height of $\gamma_i u$ tends to infinity, so αu is in the closed convex hull of $\{\gamma_i u\}$. □

Lemma 4.3. *Each ray r from the origin inside L^+ meets ∂C exactly once.*

Proof. Since the projection $\bar{\mathscr{B}} \subset S^1_\infty$ of $\mathscr{B}$ is dense, the projection to $\mathbb{D}$ of the point $r \cap \mathbb{H}$ lies in the hyperbolic convex hull of a finite set $\{\bar{z}_1, \ldots, \bar{z}_k\} \subset \bar{\mathscr{B}}$. It follows that r meets the convex hull of $\{z_1, \ldots, z_k\} \subset \mathscr{B}$ and so meets C. By discreteness of $\mathscr{B}$, there are points $\{w_1, \ldots, w_k\} \subset \mathscr{B}$ of arbitrarily large height with $\bar{w}_i$ arbitrarily near $\bar{z}_i$, $i = 1, \ldots, k$. It follows that every point of r beyond the first intersection with C lies in C. □

Proposition 4.4. *The boundary of C inside L^+ consists of a countable set $\Phi_1, \Phi_2, \ldots$ of codimension-one "faces," each of which is the convex hull of a finite number of points in $\mathscr{B}$. Each face lies in an elliptic plane, and the set of faces is locally-finite inside L^+.*

Proof. Let $z_0 \in \partial C - L^+$ and let S be a support plane for C at z_0. If S were hyperbolic, then since $\bar{\mathscr{B}}$ is dense, we could find points of $\mathscr{B}$ on either side of S, which is absurd. If $S = \{x \in \mathbb{M}\colon \langle x, s\rangle = -1\}$ were parabolic, then $s \in L^+$ (since $\langle z_0, s\rangle = -1$ and z_0 lies inside L^+), and s could not be a multiple of any point of $\mathscr{B}$ (since $\langle s, s\rangle = 0$ and C lies above S); by Lemma 4.1, there are $\gamma_j \in \Gamma$ so that $\gamma_j s$ tends to 0, but

$$-1 \geqq \langle \gamma_j^{-1} z, s\rangle = \langle z, \gamma_j s\rangle \to 0$$

for $z \in \mathscr{B}$, which is absurd. It follows that S is elliptic.

We claim that there is a support plane at z_0 which contains three affinely independent points of $\mathscr{B}$. Indeed, suppose some line A in the support plane S contains $S \cap \mathscr{B}$, and rotate S about A until a point of $\mathscr{B}$ is encountered. (As before, only a finite number of points of $\mathscr{B}$ lie below the elliptic plane S.) Performing at most two such rotations, we arrive at such a support plane.

It remains to show that the set of faces is locally-finite inside L^+. To this end, suppose that K is a compactum lying inside L^+ meeting the faces $\Phi_1, \Phi_2, \ldots$. Choose $x_i \in \Phi_i \cap K$ converging to x, so that the plane of Φ_i converges to a limit plane W containing x. As a limit of support planes, W is itself a support plane of C, whence W is elliptic. By discreteness of $\mathscr{B}$, the faces $\Phi_1, \Phi_2, \ldots$ cannot all be distinct, as desired. □

If $\tilde{\Gamma}_m = (\Gamma_m, B_1 \ldots, B_s) \in \tilde{\mathscr{T}}^s_g$, let $\Delta(\tilde{\Gamma}_m)$ denote the collection of geodesics on F^s_g arising from the edges of ∂C inside L^+. Explicitly, if $z, w \in \mathscr{B}$, then the geodesic in $\mathbb{D}$ connecting $\bar{z}, \bar{w} \in S^1_\infty$ projects to a geodesic arc connecting cusps of F^s_g; $\Delta(\tilde{\Gamma}_m)$ consists of the geodesic arcs that arise in this way from the endpoints of edges of ∂C inside L^+.

Theorem 4.5. *$\Delta(\tilde{\Gamma}_m)$ consists of a finite collection of simple geodesic arcs disjointly embedded in F^s_g connecting punctures. Furthermore, components of $F^s_g - \Delta(\tilde{\Gamma}_m)$ are simply connected.*

The isotopy class of such a decomposition is called an *ideal cell decomposition* of F^s_g.

Proof. Suppose that $c_1, c_2 \in \Delta(\tilde{\Gamma}_m)$ (perhaps with $c_1 = c_2$) and $c_1 \cap c_2 \neq \emptyset$. There are lifts e_1, e_2 of c_1, c_2 to $\mathbb{M}$ (with $\partial e_1, \partial e_2 \subset \mathscr{B}$) so that the endpoints $\partial \bar{e}_1$ separate $\partial \bar{e}_2$ on S^1_∞. Since the construction of C was Γ-equivariant, e_1 and e_2 are edges of ∂C

inside L^+. By discreteness of $\mathscr{B}$ in L^+, there is a point $z \in \mathscr{B}$ whose height exceeds that of ∂e_1 and ∂e_2. It follows that e_1 and e_2 cannot both be edges of ∂C, and this contradiction shows that $\Delta(\tilde{\Gamma}_m)$ consists of disjointly embedded arcs.

If $\gamma \in \Gamma$ leaves invariant a face of ∂C, then γ preserves the plane of the face since it acts linearly. Since the plane of a face is elliptic by Proposition 4.4, γ must be elliptic (see Sect. 1), which is absurd for $\Gamma_m \in \mathscr{T}_g^s$. It follows that complementary components of $\Delta(\tilde{\Gamma}_m)$ are simply connected, and the theorem is proved. □

Remarks. 1) The ideal cell decomposition $\Delta(\tilde{\Gamma}_m)$ depends on the choice of orbits $B_1, \ldots, B_s$. There is thus an $(s-1)$-parameter family of ideal cell decompositions. In particular, if $s=1$, then the ideal cell decomposition is unique.

2) At the expense of choosing a distinguished cusp of F_g^s, we give a convex hull construction for a kind of decomposition of F_g^s associated to a point of $\mathscr{T}_g^s$ (see Theorem A.2).

3) Many of the arguments in this section generalize readily to the setting of finite-volume hyperbolic n-manifolds with cusps, $n \geqq 3$ (see [EP]).

4) The inner product on $\mathbb{M}$ induces a Euclidean structure on each face since the plane of each face is elliptic (see Sect. 1). These combine to give a canonical (non-complete) Euclidean structure on F_g^s associated to $\Gamma_m \in \tilde{\mathscr{T}}_g^s$.

5. The Cell Decomposition of the Decorated Teichmüller Space

Suppose that $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$. The convex hull construction of Sect. 4 determines a canonical ideal cell decomposition (i.c.d.) $\Delta(\tilde{\Gamma}_m) \subset F_g^s$. Conversely, if Δ is a fixed i.c.d. of F_g^s, then we define

$$\mathring{\mathscr{C}}(\Delta) = \{\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s : \Delta(\tilde{\Gamma}_m) = \Delta\},$$
$$\mathscr{C}(\Delta) = \{\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s : \Delta(\tilde{\Gamma}_m) \subseteq \Delta\}.$$

By definition, $\mathscr{C}(\Delta_1) \cap \mathscr{C}(\Delta_2) \neq \emptyset$, if and only if $\Delta_1 \cap \Delta_2$ is an i.c.d. of F_g^s, and in this case, $\mathscr{C}(\Delta_1) \cap \mathscr{C}(\Delta_2) = \mathscr{C}(\Delta_1 \cap \Delta_2)$. Our immediate goal is to characterize $\mathscr{C}(\Delta)$, $\mathring{\mathscr{C}}(\Delta)$ in terms of λ-lengths on Δ in the special case that Δ is an ideal triangulation (i.t.) of F_g^s.

To establish notation, fix an arc e in the i.t. Δ, and consider a lift $\tilde{e}$ of e to $\mathbb{D}$. $\tilde{e}$ separates two triangles $\tilde{S}$, $\tilde{T}$ of the lift $\tilde{\Delta}$ of Δ to $\mathbb{D}$, and we adopt the notation of Fig. 5.1 (0) for the arcs in $\partial\tilde{S}$, $\partial\tilde{T}$. It may be that $\pi\tilde{S} = \pi\tilde{T}$, where $\pi: \mathbb{D} \to F_g^s$ is the canonical projection, and $a' = \pi(\tilde{a}), \ldots, d' = \pi(\tilde{d})$ need not be distinct; see Fig. 5.1, where we enumerate the various cases. In any case, if $\Lambda \in \mathbb{R}_+^\Delta = \{\Lambda : \Delta \to \mathbb{R}_+\}$, then we say Λ satisfies the *(strict) face condition* on $e \in \Delta$ if the following inequality holds:

$$\begin{aligned}&\Lambda(a')\Lambda(b')[\Lambda^2(c') + \Lambda^2(d') - \Lambda^2(e')]\\ &\quad + \Lambda(c')\Lambda(d')[\Lambda^2(a') + \Lambda^2(b') - \Lambda^2(e')] > 0.\end{aligned}$$

The strict face condition on e is indicated in Fig. 5.1 in the various cases (where we identify an arc with its Λ-value for convenience). We will also refer to "face equality" (corresponding to equality above) and the "weak" face condition (corresponding to the weak inequality $\geqq$ above) on e for Λ. Furthermore, if $\Delta' \subset \Delta$,

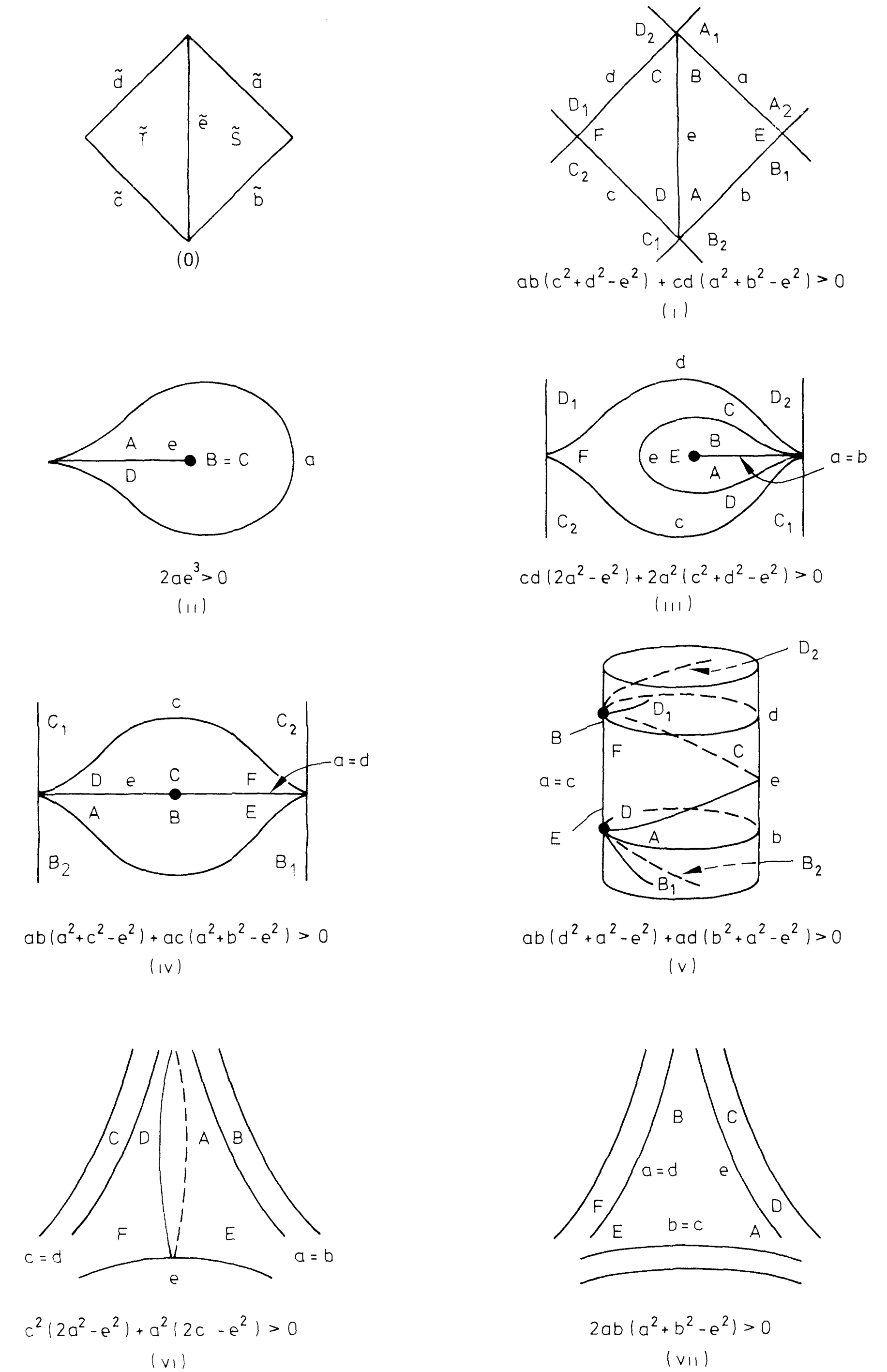
(0)
ab(c²+d²-e²) + cd(a²+b²-e²) > 0
(i)
2ae³ > 0
(ii)
cd(2a²-e²) + 2a²(c²+d²-e²) > 0
(iii)
a = b
ab(a²+c²-e²) + ac(a²+b²-e²) > 0
(iv)
a = d
ab(d²+a²-e²) + ad(b²+a²-e²) > 0
(v)
a = c
c = d
a = b
c²(2a²-e²) + a²(2c -e²) > 0
(vi)
a = d
b = c
2ab(a²+b²-e²) > 0
(vii)

Fig. 5.1

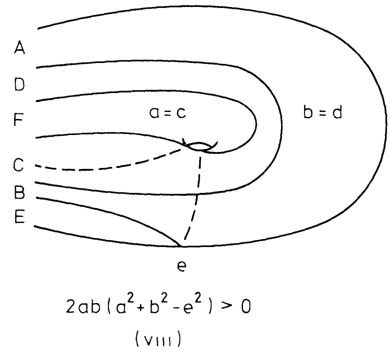

Fig. 5.1 (continued)

then we will say that $\Lambda \in \mathbb{R}_+^4$ satisfies the "face relations on Δ rel Δ'" if the strict face conditions hold for Λ on each $e \in \Delta' \subset \Delta$, and the face equalities hold on each $e \in \Delta - \Delta'$. In particular, we say simply that the "face relations" hold for Δ if the strict face conditions hold on each $e \in \Delta$.

Theorem 5.1. *Suppose that Δ is an* i.t. *of F_g^s*, and let $\mathbb{R}_+^4 \ni \Lambda = \tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$. *Then a necessary and sufficient condition for $\tilde{\Gamma}_m \in \mathring{\mathscr{C}}(\Delta)$ is that Λ satisfy the face relations for Δ. Furthermore, if $\Delta' \subset \Delta$ is an* i.c.d., *then a necessary and sufficient condition for $\tilde{\Gamma}_m \in \mathring{\mathscr{C}}(\Delta') \subset \mathscr{C}(\Delta)$ is that Λ satisfy the face relation for Δ* rel *Δ'.*

Proof of Necessity. Fix $\tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$, recall the construction of $\Delta(\tilde{\Gamma}_m)$ from $\tilde{\Gamma}_m$ (in Sect. 4), and let $\mathscr{B} \subset L^+$ denote the discrete subset corresponding to the decoration of horocycles. Δ lifts to a collection of Euclidean geodesics in $\mathbb{M}$ connecting points of $\mathscr{B}$. If $\tilde{e}$ is such a lift of $e \in \Delta$ separating triangles $\tilde{S}$, $\tilde{T}$ in the lift, then $\tilde{e}$ is extremal in the hull of $\mathscr{B}$ and so in particular in the hull of $\tilde{S} \cup \tilde{T}$. Comparison of the face condition with Proposition 2.6b thus guarantees necessity. The proof of necessity in the second assertion is analogous. □

Before we undertake a proof of sufficiency, we develop some generalities. Fix an i.t. Δ of F_g^s. Suppose that $(T_j)_1^n$ is a *cycle* of triangles in the sense that $T_j \cap T_{j+1} = e_j$, for all j, where we henceforth regard the index j as cyclic, so for instance, $T_{n+1} = T_1$. If the edges of T_j are $\{e_{j-1}, e_j, b_j\}$, $j = 1, \ldots, n$, then the collection $\{b_j\}_1^n \subset \Delta$ is called the *boundary* of the cycle $(T_j)_1^n$.

Lemma 5.2. *Suppose the (weak) face conditions hold for $\mathbb{R}_+^4 \ni \Lambda = \tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$ on each $e \in \Delta$. Then all three strict triangle inequalities on $\{\Lambda(c), \Lambda(d), \Lambda(e)\}$ hold whenever there is a triangle in Δ with sides c, d, e.*

Proof. To get a contradiction, we suppose for instance that $\Lambda(e) \geqq \Lambda(c) + \Lambda(d)$, and adopt the usual notation for the edges adjacent to e (see Fig. 5.1). Thus,

$$\Lambda^2(c) + \Lambda^2(d) - \Lambda^2(e) \leqq -2\Lambda(c)\Lambda(d),$$

so the face condition on e gives

$$0 \leqq \Lambda(a)\Lambda(b)\left[(\Lambda(c) - \Lambda(d))^2 - \Lambda^2(e)\right],$$

and we find a second edge-triangle pair so that the triangle inequality fails. It follows that there is a cycle $(T_j)_1^n$ of triangles in Δ so that a strict triangle inequality fails at the edge-triangle pair (T_j, e_j), for all j. As before, let $\{b_j\}_1^n$ denote the ideal arcs of Δ in the boundary of the cycle. We have

$$\Lambda(e_{j+1}) \geqq \Lambda(b_j) + \Lambda(e_j), \qquad j = 1, \ldots, n.$$

Upon summing these inequalities and cancelling like terms, we obtain

$$0 \geqq \sum_{j=1}^{n} \Lambda(b_j),$$

which is absurd for $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$. □

Proof of Sufficiency in Theorem 6.1. To prove sufficiency in the first claim, we suppose that $\Lambda \in \mathbb{R}_+^\Delta$ satisfies the face relation on Δ (and hence the "triangle inequality" condition of Lemma 6.2) and prove that $\tilde{\Gamma}_m = (\Delta, \Lambda) \in \mathring{\mathscr{C}}(\Delta)$. To this end, adopt the notation in the proof of necessity, so that $\mathscr{B} \subset L^+$ arises from $\Lambda \in \mathbb{R}_+^\Delta$. By Proposition 2.2, the triangle inequality condition is equivalent to ellipticity of the affine planes spanned by triples in $\mathscr{B}$ arising as the vertices of a lift of a triangle in Δ. Furthermore, we saw above that the face condition is equivalent to "local extremality." Finally, from the inductive definition of $\mathscr{B} \subset L^+$ in Theorem 3.1, it follows by induction and an appeal to Proposition 2.7 that $\tilde{\Gamma}_m \in \mathring{\mathscr{C}}(\Delta)$. The proof of sufficiency in the second assertion is analogous. □

Recall the h-length parametrization of $\tilde{\mathscr{T}}_g^s$ given in Proposition 3.5. A pleasant algebraic fact relating h-lengths and the face condition is the observation that the face condition is linear in h-length coordinates. Indeed, suppose first that $e \in \Delta$ separates two triangles $S \neq T$ in Δ with edges (a, b, e), (c, d, e), respectively, where $\#\{a, b, c, d\} = 4$, and let $(\alpha, \beta, \varepsilon)$ $((\gamma, \delta, \varphi)$, respectively) denote the h-lengths of the ends of S opposite (a, b, e) (of T opposite (c, d, e), respectively); see Fig. 5.2a. We see that the (strict) face condition on e is equivalent to

$$\alpha + \beta + \gamma + \delta > \varepsilon + \varphi,$$

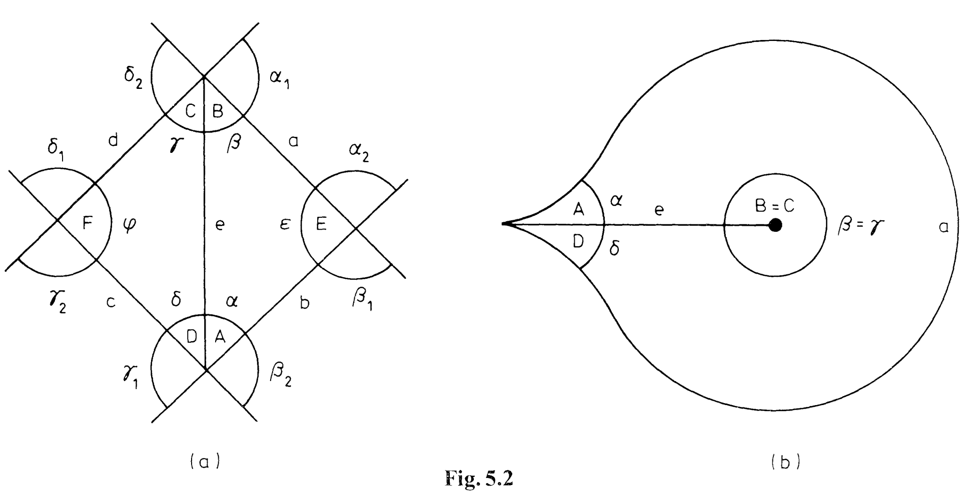

Fig. 5.2

by dividing the former by $\Lambda(a)\Lambda(b)\Lambda(c)\Lambda(d)\Lambda(e)$. Since the various cases (indicated in Fig. 5.1) give rise to linear quotients, the claim follows.

For each $e \in \Delta$, we next define a pair of vectors $B_e, C_e \in \mathbb{R}^{\mathscr{E}}$, where $\mathscr{E} = \mathscr{E}(\Delta)$ is the set of ends of Δ (see Sect. 3). Adopt the notation of Fig. 5.2 for the ends $A, B, C, D \in \mathscr{E}$ on which e abuts. B_e and C_e each lie in the coordinate subspace of $\mathbb{R}^{\mathscr{E}}$ corresponding to A, B, C, D (in this order), and B_e (C_e, respectively) has entries $(1,1,1,1)$ ($(1,-1,1,-1)$, respectively); the ends A, B, C, D need not be distinct. See Fig. 5.3a.

Lemma 5.3. *$\{B_e, C_e : e \in \Delta\}$ is a basis for $\mathbb{R}^{\mathscr{E}}$. Furthermore, suppose*

$$z = x + y = \sum_{e \in \Delta} x_e B_e + \sum_{e \in \Delta} y_e C_e,$$

$x_e, y_e \in \mathbb{R}$. Then z satisfies the face relation on Δ rel Δ' if and only if $x_e > 0$ for $e \in \Delta'$ and $x_e = 0$ for $e \in \Delta - \Delta'$.

Proof. The span of $\{B_e, C_e : e \in \Delta\}$ is clearly identical with the span of the vectors $\left\{B'_e = \frac{B_e + C_e}{2},\ C'_e = \frac{B_e - C_e}{2} : e \in \Delta\right\}$. Let us fix a triangle T in Δ, say with ends (A, B, E). There are exactly three vectors among $\{B'_e, C'_e : e \in \Delta\}$ with a non-zero projection into the subspace of $\mathbb{R}^{\mathscr{E}}$ corresponding to (A, B, E); namely, $(1,1,0)$, $(0,1,1)$, and $(1,0,1)$. See Fig. 5.3b. Insofar as these projections are linearly independent, $\{B'_e, C'_e : e \in \Delta\}$, and hence $\{B_e, C_e : e \in \Delta\}$ forms a linearly independent set, proving the first part.

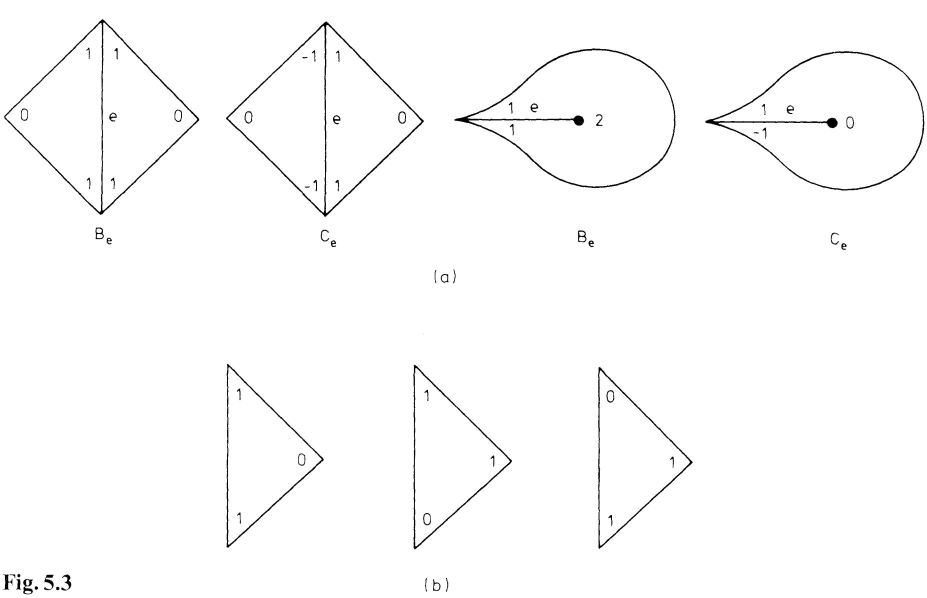

Fig. 5.3

Since the face condition is linear, the second part follows at once from the fact that the face equality $\alpha+\beta+\gamma+\delta=\varepsilon+\varphi$ holds on every edge for any C_e, $e\in\Delta$. $\square$

Define the following subspaces of $\mathbb{R}^{\mathscr{E}}$:

$$\begin{aligned} \mathrm{X} &= \{\Sigma x_e B_e \colon x_e \in \mathbb{R}\}\,, \\ \mathrm{Y} &= \{\Sigma y_e C_e \colon y_e \in \mathbb{R}\}\,, \\ \bar{\mathrm{X}} &= \{\Sigma x_e B_e \colon x_e \geqq 0\}\,, \\ \mathring{\mathrm{X}} &= \{\Sigma x_e B_e \colon x_e > 0\}\,, \end{aligned}$$

and remark that $\bar{\mathrm{X}}$ has a natural structure as a cone on a simplex. Indeed, the (open) faces of $\bar{\mathrm{X}}$ correspond to subsets $\Delta'\subset\Delta$, where the face relation holds on Δ rel Δ'. A face F of $\bar{\mathrm{X}}$ is said to be *finite* if the corresponding subset $\Delta'=\{e\colon x_e\neq 0\}$ of Δ is an i.c.d., and we define

$$\mathrm{X}^+=\mathring{\mathrm{X}}\cup\{\text{faces } F \text{ of } \bar{\mathrm{X}}\colon F \text{ is finite}\}\subset\bar{\mathrm{X}}\,.$$

We regard $\tilde{\mathscr{T}}_g^s$ as the subset of $\mathbb{R}_+^{\mathscr{E}}\subset\mathbb{R}^{\mathscr{E}}$ determined by the coupling equations (as in Proposition 3.5), and consider the projection Π of $\mathbb{R}^{\mathscr{E}}$ along Y onto X.

Theorem 5.4. *For each* i.t. Δ *of* F_g^s, *the projection* Π *induces a homeomorphism*

$$\Pi\colon \mathscr{C}(\Delta)\to\mathrm{X}^+$$

which maps $\mathring{\mathscr{C}}(\Delta)$ *to* $\mathring{\mathrm{X}}$. *If* $\Delta'\subset\Delta$ *is an* i.c.d., *then* Π *maps* $\mathring{\mathscr{C}}(\Delta')$ *to the corresponding (open) finite face of* X^+.

The argument involves an "energy functional"

$$\begin{aligned} K\colon\ & \mathbb{R}_+^{\mathscr{E}}\to\mathbb{R} \\ &\colon z\mapsto\Sigma(\ln\alpha\beta/\gamma\delta)^2\,, \end{aligned}$$

where the sum is over all $e\in\Delta$ and α, β, γ, δ denote the h-lengths of the ends of Δ on which e abuts (see Fig. 5.2). Clearly, K is non-negative, homogeneous, and smooth; furthermore, $K(z)=0$ if and only if z satisfies the coupling equations.

Suppose $z\in\mathbb{R}_+^{\mathscr{E}}$ and consider the affine subspace $\mathrm{Y}_z=\{y+z\colon y\in\mathrm{Y}$ and $y+z\in\mathbb{R}_+^{\mathscr{E}}\}$. The gradient ∇K restricts to a vector-field, denoted $\nabla K|_{\mathrm{Y}}$, on Y_z; consider the negative-time flow $z_t=(\nabla K|_{\mathrm{Y}})_{-t}(z)$, so energy is decreasing along trajectories (the system is "dissipative").

Claim 1. Fix $z\in\mathbb{R}_+^{\mathscr{E}}$. If $\Pi z\in\mathrm{X}^+$, then $\lim_{t\to\infty} z_t=z_\infty$ exists, where $z_\infty\in\mathbb{R}_+^{\mathscr{E}}$.

Proof. Since K is homogeneous, the limit $[z_\infty]\in P\mathbb{R}^{\mathscr{E}}$ exists projectively, and there are two cases: either $[z_\infty]\in P\mathbb{R}_+^{\mathscr{E}}$, or perhaps $[z_\infty]\in P[(\mathbb{R}_+\cup\{0\})^{\mathscr{E}}-\mathbb{R}_+^{\mathscr{E}}]$. To prove the claim, we show that the second case is absurd for $\Pi z\in\mathrm{X}^+$. To this end, suppose $[z_t]\in P\mathbb{R}_+^{\mathscr{E}}$ are normalized by the condition $\sup_{E\in\mathscr{E}}\{z_t(E)\}=1$. Notice that $\Pi z_t=\Pi z\in\mathrm{X}^+$ by definition.

Suppose $z_t(A)\to 0$ for some arc $e\in\Delta$ with adjacent ends A, B, C, D (as in Fig. 5.2). Since z_t is dissipative, in case (a), we must have $z_t(C)\to 0$ or $z_t(D)\to 0$; in case (b), $z_t(D)\to 0$. There is thus a cycle of triangles $(T_j)_1^n$ with boundary $(\mathrm{b}_j)_1^n$ so that $z_t(B_j)\to 0$, $j=1,\ldots,n$, where B_j is the end of Δ which is opposite b_j in T_j.

Now, for any $z \in \mathbb{R}_+^{\mathscr{E}}$ with $\Pi z = \Sigma x_e B_e$, we easily compute that

$$x_e = \tfrac{1}{2}(\alpha + \beta + \gamma + \delta - \varepsilon - \varphi)$$

in case (a), and $x_e = \frac{1}{2}\beta$ in case (b). In particular, if z vanishes on the ends opposite a cycle of triangles $(T_j)_1^n$ as above, then the sum

$$\sum_{j=1}^{n} x_{e_j} = \sum_{j=1}^{n} z(B_j)$$

telescopes.

Finally, since $\Sigma z_t(B_j) \to 0$, it follows that Σx_{e_j} vanishes, which contradicts $\Pi z \in \mathrm{X}^+$. □

Claim 2. If $x \in X^+$, then $\mathrm{Y}_x \neq \emptyset$.

Proof. The proof is by induction on the number N of vanishing coordinates, and the basis step $N = 0$ is trivial. For the induction, suppose first that e separates two triangles of Δ and adopt the notation of Fig. 5.2a for the nearby ends and h-lengths. If, for instance, $\alpha = 0$ and $\gamma\delta \neq 0$, then it is easy to deform z along Y and decrease N by at least one. Similarly, if e does not separate and we adopt the notation of Fig. 5.2b, then if $\alpha \neq 0$, $\delta = 0$ or $\alpha = 0$, $\delta \neq 0$, we can again easily decrease N. We are led to a cycle of triangles so that x vanishes on the boundary of the cycle, and this contradiction as before establishes the claim. □

Let us enumerate the arcs $e_1, e_2, \ldots, e_q \in \Delta$ once and for all, and define

$$(\ln \psi_i)^2 = (\ln \psi_{e_i})^2 = \left(\ln \frac{\alpha\beta}{\gamma\delta}\right)^2$$

to be the corresponding term of K; suppose $z = x + \Sigma y_e C_e$, $x \in \mathrm{X}^+$, and let $y_i = y_{e_i}$, $i = 1, \ldots, q$.

Claim 3. $z \in \mathbb{R}_+^{\mathscr{E}}$ is a zero of K if and only if z is a fixed point of $(\nabla K|_{\mathrm{Y}})_t$.

Proof. We compute

$$\frac{\partial}{\partial y_j} K = \sum_{i=1}^{q} \frac{2}{\psi_i} (\ln \psi_i) \frac{\partial \psi_i}{\partial y_j},$$

so a zero of K is automatically a zero of $\nabla K|_{\mathrm{Y}}$.

Conversely, suppose $K(z) \neq 0$ for $z \in \mathbb{R}_+^{\mathscr{E}}$. Thus, $\psi_e(z) \neq 1$ for some $e \in \Delta$, and we may choose an arc e so that $(\ln \psi_e)^2$ is greatest. Suppose first that e does not separate two triangles of Δ, and adopt the notation of Fig. 5.2b for the nearby ends and h-lengths. Let us make the convention that C_e has projection $(1, -1)$ into the (α, δ)-subspace. Compute

$$\frac{1}{2}\frac{\partial K}{\partial y_e}(z) = \frac{1}{2}\frac{\partial}{\partial t}\bigg|_{t=0} \left(\ln \frac{\alpha + t}{\alpha - t}\right)^2 = (\alpha^{-1} - \delta^{-1}) \ln \frac{\alpha}{\delta},$$

so if e does not separate triangles of Δ, then $\nabla K|_{\mathrm{Y}}(z) \neq 0$.

In case e does separate triangles of Δ, adopt the notation of Fig. 5.2a for nearby ends and h-lengths. Let us suppose first that all the ends pictured in the figure are distinct [case (i) of Fig. 5.1] and make the convention that C_e has projection $(1,1,-1,-1)$ into the $(\alpha,\beta,\gamma,\delta)$-subspace. We compute

$$\begin{aligned}\frac{1}{2}\frac{\partial K}{\partial y_e}(z)&=\frac{1}{2}\frac{\partial}{\partial t}\bigg|_{t=0}\left[\left(\ln\frac{(\alpha+t)(\beta+t)}{(\gamma-t)(\delta-t)}\right)^2+\left(\ln\frac{\varepsilon(\alpha+t)}{\beta_1\beta_2}\right)^2\right.\\&\quad\left.+\left(\ln\frac{\varepsilon(\beta+t)}{\alpha_1\alpha_2}\right)^2+\left(\ln\frac{\varphi(\gamma-t)}{\delta_1\delta_2}\right)^2+\left(\ln\frac{\varphi(\delta-t)}{\gamma_1\gamma_2}\right)^2\right]\\&=(\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1})\ln\frac{\alpha\beta}{\gamma\delta}+\frac{1}{\alpha}\ln\frac{\varepsilon\alpha}{\beta_1\beta_2}\\&\quad+\frac{1}{\beta}\ln\frac{\varepsilon\beta}{\alpha_1\alpha_2}-\frac{1}{\gamma}\ln\frac{\varphi\gamma}{\delta_1\delta_2}-\frac{1}{\delta}\ln\frac{\varphi\delta}{\gamma_1\gamma_2}.\end{aligned}$$

Since $(\ln\varphi_e)^2=\left(\ln\frac{\alpha\beta}{\gamma\delta}\right)^2$ is greatest, we must have $\partial K/\partial y_e(z)\neq 0$ unless

$$\psi_a^2=\psi_b^2=\psi_c^2=\psi_d^2=\psi_e^2\,.$$

Furthermore, we may suppose $\alpha\beta>\gamma\delta$, and it follows that

$$\begin{aligned}\alpha_1\alpha_2&>\beta\varepsilon, &\gamma_1\gamma_2&>\delta\varphi\,,\\ \beta_1\beta_2&>\alpha\varepsilon, &\delta_1\delta_2&>\gamma\varphi\,.\end{aligned}$$

Finally, if $f\in\{a,b,c,d\}$, then these inequalities are asymmetric in e and f; it follows easily that if f is of type (i), then $\partial K/\partial y_f(z)\neq 0$.

Armed with this computation, we can handle the various cases of Fig. 5.1 (iii)–(viii) in turn. For instance, if e is of type (iii), then $\alpha_2=\varepsilon=\beta_1$, $\alpha_1=\alpha$, $\beta_2=\beta$, and

$$\begin{aligned}\frac{1}{2}\frac{\partial K}{\partial y_e}(z)&=(\alpha^{-1}-\beta^{-1})\ln\frac{\alpha}{\beta}+(\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1})\ln\frac{\alpha\beta}{\gamma\delta}\\&\quad-\gamma^{-1}\ln\frac{\gamma\varphi}{\delta_1\delta_2}-\delta^{-1}\ln\frac{\delta\varphi}{\gamma_1\gamma_2},\end{aligned}$$

so $\frac{\partial K}{\partial y_e}(z)\neq 0$ in this case. We leave the analogous routine computations in cases (iv)–(viii) to the untiring reader. □

Claim 4. Each zero $z\in\mathbb{R}_+^{\mathscr{E}}$ of $\nabla K|_Y$ with $\Pi z\in X^+$ is non-degenerate with index one.

Proof. Suppose that $z\in\mathbb{R}_+^{\mathscr{E}}$ satisfies $\nabla K|_Y(z)=0$, so $K(z)=0$ by Claim 3. Compute

$$\frac{1}{2}\frac{\partial^2 K}{\partial y_k\partial y_j}(z)=\frac{\partial}{\partial y_k}\sum_{i=1}^{q}\frac{\ln\psi_i}{\psi_i}\frac{\partial\psi_i}{\partial y_j}=\sum_{i=1}^{q}\frac{\partial\psi_i}{\partial y_j}\bigg|_z\frac{\partial\psi_i}{\partial y_k}\bigg|_z,$$

and define the matrix

$$A_{ij}=\frac{\partial\psi_i}{\partial y_j}\bigg|_z,\qquad i,j=1,2,\ldots,q\,.$$

The Hessian of K at z is therefore $2A^tA$, and the claim follows provided A is non-singular.

Adopting the notation of Fig. 5.2, define

$$\eta_{ii} = \begin{cases} \alpha\beta, & \text{if } e = e_i \text{ is type (a)} \\ a \ , & \text{if } e = e_i \text{ is type (b)}; \end{cases}$$

for $i = 1, \ldots, q$, set

$$A' = (\eta_{ii})(A_{ij}),$$

and remark that invertibility of A' implies the desired invertibility of A. To see that A' is non-singular, we suppose that

$$0 = \sum_{i=1}^{q} \xi_i \left(\eta_{ii} \frac{\partial \psi_i}{\partial y_j} \right)_{j=1}^{q}$$

is a relation amongst the rows, where $\xi_i \in \mathbb{R}$. Fix an arc $e \in \Delta$; we show below that $\xi_e = 0$, and there are several cases (those of Fig. 5.1) depending on the topology of Δ near e. These cases are considered in turn.

Cases (ii) *and* (iii). Adopt the notation of case (iii) and define

$$\begin{aligned} C_a &= C_b = (-1, 1) \text{ in } (\alpha, \beta)\text{-coordinates} \\ C_c &= (1, 1, -1, -1) \text{ in } (\gamma, \varphi, \delta_1, \delta_2)\text{-coordinates}, \\ C_d &= (1, 1, -1, -1) \text{ in } (\delta, \varphi, \gamma_1, \gamma_2)\text{-coordinates}, \end{aligned}$$

with C_e as before. Compute the four-by-four minor of A' corresponding to (a, c, d, e) to be

$$B = \begin{bmatrix} \alpha+\beta & 0 & 0 & \alpha+\beta \\ 0 & \gamma_1+\gamma_2+\varphi+\gamma & \delta & -\varphi \\ 0 & \gamma & \delta_1+\delta_2+\varphi+\gamma & -\varphi \\ 0 & \gamma & \delta & \alpha+\beta+\gamma+\delta \end{bmatrix}.$$

In particular, the two-by-two minor corresponding to (a, e) is non-singular. Since the only non-zero entries of $(\partial\psi_a/\partial y_j|_z)_{j=1}^{q}$ lie in this subspace, we conclude that the coefficient $\xi_a = 0$ if a is of type (ii).

Furthermore, the three-by-three minor corresponding to the $(1,1)$ entry of B row-reduces to

$$\begin{bmatrix} \gamma_1+\gamma_2-\gamma+\varphi+\delta & 0 & -(\alpha+\beta+\gamma+\delta+\varphi) \\ 0 & \delta_1+\delta_2-\delta+\varphi+\gamma & -(\alpha+\beta+\gamma+\delta+\varphi) \\ \gamma & \delta & (\alpha+\beta+\gamma+\delta+\varphi) \end{bmatrix}.$$

Finally, each diagonal entry is positive since $\Pi z \in X^+$ and $z \in \mathbb{R}_+^{\mathscr{E}}$; expanding by minors along the first row, each term is positive; and one concludes that the matrix is non-singular. It follows as before that $\xi_e = 0$ if e is of type (iii).

Case (iv). Adopt the notation of case (iv), and define

$$C_a = C_d = (1,1,-1,-1) \text{ in } (\gamma, \varphi, \beta, \varepsilon)\text{-coordinates},$$
$$C_b = (1,1,-1,-1) \text{ in } (\beta_1, \beta_2, \alpha, \varepsilon)\text{-coordinates},$$
$$C_c = (1,1,-1,-1) \text{ in } (\delta, \varphi, \gamma_1, \gamma_2)\text{-coordinates}.$$

The four-by-four minor of A' corresponding to (a, b, c, e) is computed to be

$$\begin{bmatrix} \beta+\varepsilon+\gamma+\varphi & -\beta & \gamma & \varepsilon-\varphi \\ -\alpha & \alpha+\varepsilon+\beta_1+\beta_2 & 0 & \varepsilon \\ \delta & 0 & \delta+\varphi+\gamma_1+\gamma_2 & -\varphi \\ \delta-\alpha & -\beta & \gamma & \alpha+\beta+\gamma+\delta \end{bmatrix},$$

which row-reduces to

$$\begin{bmatrix} \alpha+\beta+\varepsilon+\gamma+\varphi-\delta & 0 & 0 & \varepsilon-(\alpha+\beta+\gamma+\delta+\varphi) \\ -\alpha & \alpha+\varepsilon+\beta_1+\beta_2-\beta & 0 & \varepsilon \\ \delta & 0 & \delta+\varphi+\gamma_1+\gamma_2-\gamma & -\varphi \\ \delta-\alpha & -\beta & \gamma & (\alpha+\beta+\gamma+\delta) \end{bmatrix}$$

Again the diagonal is positive, and one concludes (after expanding by minors along the first row) that $\xi_e = 0$ if e is of type (iv).

Case (v). This case is computationally identical with the previous one.

Case (i). Adopt the notation of case (i), and define

$$C_a = (1,1,-1,-1) \text{ in } (\alpha_1, \alpha_2, \beta, \varepsilon)\text{-coordinates},$$
$$C_b = (1,1,-1,-1) \text{ in } (\beta_1, \beta_2, \alpha, \varepsilon)\text{-coordinates},$$
$$C_c = (1,1,-1,-1) \text{ in } (\delta, \varphi, \gamma_1, \gamma_2)\text{-coordinates},$$
$$C_d = (1,1,-1,-1) \text{ in } (\gamma, \varphi, \delta_1, \delta_2)\text{-coordinates}.$$

The five-by-five minor of A' corresponding to (a, b, c, d, e) is

$$\begin{bmatrix} \alpha_1+\alpha_2+\beta+\varepsilon & -\beta & 0 & 0 & \varepsilon \\ -\alpha & \beta_1+\beta_2+\alpha+\varepsilon & 0 & 0 & \varepsilon \\ 0 & 0 & \gamma_1+\gamma_2+\delta+\varphi & \delta & -\varphi \\ 0 & 0 & \gamma & \delta_1+\delta_2+\gamma+\varphi & -\varphi \\ -\alpha & -\beta & \gamma & \delta & \alpha+\beta+\gamma+\delta \end{bmatrix},$$

which row-reduces to

$$\begin{bmatrix} \alpha_1+\alpha_2+\beta+\varepsilon+\alpha & -(\beta_1+\beta_2+\varepsilon+\alpha+\beta) & 0 & 0 & 0 \\ -\alpha & \varepsilon+\alpha+\beta_1+\beta_2 & 0 & 0 & \varepsilon \\ 0 & 0 & \gamma_1+\gamma_2-\gamma+\varphi+\delta & \delta-\varphi-\gamma-\delta_1-\delta_2 & 0 \\ 0 & 0 & \gamma & \varphi+\gamma+\delta_1+\delta_2 & -\varphi \\ 0 & -(\alpha+\beta+\varepsilon+\beta_1+\beta_2) & \gamma & \delta & \alpha+\beta+\gamma+\delta-\varepsilon \end{bmatrix}.$$

Expanding by minors on the first row, one easily concludes that each term is positive; as before, we find that $\xi_e = 0$ if e is of type (i).

Cases (vi)–(viii). These are left as easy exercises. □

Proof of Theorem 5.4. Fix $x \in \mathrm{X}^+$ and consider the flow $\nabla K|_{\mathrm{Y}}$ on Y_x; $\mathrm{Y}_x \neq \emptyset$ by Claim 2. The cycle of triangles argument in Claim 1 shows that the direction field of $\nabla K|_{\mathrm{Y}}$ extends continuously to

$$\overline{\mathrm{Y}}_x = \{y \in \mathrm{Y} : x + y \in (\mathbb{R}_+ \cup \{0\})^{\mathscr{E}}\}$$

in such a way that the extension points into Y_x at the ideal points $\overline{\mathrm{Y}}_x - \mathrm{Y}_x$. Insofar as $\overline{\mathrm{Y}}_x$ is convex, Claim 3 together with the Poincaré-Hopf index theorem imply that $\nabla K|_{\mathrm{Y}}$ has a unique attracting zero in Y_x.

We define a map

$$\sigma : \mathrm{X}^+ \to \mathscr{C}(\Delta)$$

$$: x \mapsto \lim_{t \to \infty} (\nabla K|_{\mathrm{Y}})_{-t} x',$$

where $x - x' \in \mathrm{Y}$ and $x' > 0$. By Claims 1–4, σ is well-defined; moreover, σ is clearly continuous, and $\Pi \circ \sigma$ is the identity on X^+ by construction. To finish the proof, we show that σ is onto. To this end, suppose $z \in \mathbb{R}_+^{\mathscr{E}}$, $K(z) = 0$, $x = \Pi z \in \mathrm{X}^+$, and define $z' = \sigma x$. It follows that $K(z') = 0$ from Claim 3 and $z' \in \mathbb{R}_+^{\mathscr{E}}$ from Claim 1. Claim 4 finally gives $z = z'$, so $\sigma x = z' = z$, as desired. □

Remark. Because of the formal similarity with classical mechanics, it would be interesting if the dissipative flow on $\mathbb{R}_+^{\mathscr{E}}$ were Hamiltonian for some reasonable extension to $\mathbb{R}_+^{\mathscr{E}}$ of the (known; see [P2]) Weil-Petersson Kähler form on the variety $\mathscr{V} \subset \mathbb{R}_+^{\mathscr{E}}$.

As an immediate consequence of Theorem 5.4 and Corollary 3.3, we find

Theorem 5.5. *If Δ is an* i.c.d. *of F_g^s, then $\mathring{\mathscr{C}}(\Delta)$ is an open cell of dimension $\#\Delta \cdot \{\mathscr{C}(\Delta)$: Δ is an* i.c.d. *of $F_g^s\}$ is a MC_g^s-invariant cell decomposition of $\tilde{\mathscr{T}}_g^s$ itself. Furthermore, the isotropy group of $\mathscr{C}(\Delta)$ in MC_g^s is isomorphic to the (finite) group of mapping classes of F_g^s leaving Δ invariant.* □

By definition, the complex

$$\mathscr{C}_g^s = \{\mathscr{C}(\Delta) \colon \Delta \text{ is an i.c.d. of } F_g^s\}$$

is isomorphic to the poset of i.c.d.'s of F_g^s with the relation of inclusion. Following Harer [Ha], we define the *arc complex* $\mathscr{A}_g^s$ of F_g^s to be the simplicial complex whose p-simplices correspond to collections Δ of disjointly embedded families of $(p+1)$ ideal arcs in F_g^s so that no arc, nor any pair of arcs, in Δ bounds a disc in F_g^s. Of course, MC_g^s acts on $\mathscr{A}_g^s$ in the natural way.

Now, the cell-decomposition $\mathscr{C}_g^s$ of $\tilde{\mathscr{T}}_g^s$ induces a cell-decomposition $\mathscr{C}_g^s/\mathbb{R}_+$ of $\tilde{\mathscr{T}}_g^s/\mathbb{R}_+$, and we identify cells of $\mathscr{C}_g^s/\mathbb{R}_+$ with corresponding cells of $\mathscr{A}_g^s$ in the natural way. Clearly, $\mathscr{A}_g^s - \mathscr{C}_g^s/\mathbb{R}_+$ is a subcomplex of $\mathscr{A}_g^s$, and the identification of cells in $\mathscr{C}_g^s/\mathbb{R}_+$ with cells of $\mathscr{A}_g^s$ induces a MC_g^s-equivariant inclusion $\mathscr{C}_g^s/\mathbb{R}_+ \to \mathscr{A}_g^s$.

6. Centers of Cells, Cyclic Euclidean Polygons, and the Construction of Matrix Groups

Suppose that Δ is an i.t. of F_g^s. The assignment $\Lambda \equiv 1$ of λ-lengths to each $e \in \Delta$ determines a point $\tilde{\Gamma}_m(\Delta)$, called the *center* of $\mathring{\mathscr{C}}(\Delta)$. It follows immediately from naturality (Corollary 3.3) that if $\varphi \in MC_g^s$, then $\varphi_* \tilde{\Gamma}_m(\Delta) = \tilde{\Gamma}_m(\varphi^{-1}\Delta)$. Thus, the conformal symmetry group of $\tilde{\Gamma}_m(\Delta)$ (preserving the decoration) is naturally isomorphic to the topological symmetry group of mapping classes leaving Δ invariant, which is itself naturally isomorphic to the isotropy group of $\mathring{\mathscr{C}}(\Delta)$ in MC_g^s.

Proposition 6.1. *Suppose that Δ is an* i.t. *of F_g^s. Then the point $\Gamma_m(\Delta) = \phi\tilde{\Gamma}_m(\Delta) \in \mathscr{T}_g^s$ is arithmetic, where $\phi: \tilde{\mathscr{T}}_g^s \to \mathscr{T}_g^s$ is the canonical fibration* (see Sect. 2). *That is, $\Gamma_m(\Delta)$ is conjugate in $PSL_2\mathbb{R}$ to a subgroup of finite index in $PSL_2\mathbb{Z}$.*

Proof. Since $\Gamma_m = \Gamma_m(\Delta)$ is finite co-area, it suffices to show that Γ_m is conjugate to a subgroup of $PSL_2\mathbb{Z}$. To this end, let T be the triangle in $\mathbb{M}$ with vertices $2^{-1/2}(1,1,0)$, $2^{-1/2}(-1,-1,0)$, $2^{-1/2}(2,0,2)$. If u, v, $w \in L^+$ are the vertices of a triangle in $PSL_2\mathbb{Z}(T)$, then $\langle u,v\rangle = \langle v,w\rangle = \langle w,v\rangle = -1$. The projection of edges of triangles in $PSL_2\mathbb{Z}(T)$ to $\mathbb{D}$ gives the usual $PSL_2\mathbb{Z}$-invariant tesselation of $\mathbb{D}$.

Now, the center $\tilde{\Gamma}_m$ of $\mathring{\mathscr{C}}(\Delta)$ arises from $1 \equiv \Lambda \in \mathbb{R}_+^4$, and we recall the construction of $\mathscr{B} \subset L^+$ from Λ given in Theorem 2.1. In fact, $\mathscr{B}$ is exactly the set of vertices of triangles in $PSL_2\mathbb{Z}(T)$, and the construction furthermore determines a representation of $\Gamma_m = \phi\tilde{\Gamma}_m$ as a group of motions preserving $\mathscr{B}$ setwise and mapping triangles in $PSL_2\mathbb{Z}(T)$ to triangles in $PSL_2\mathbb{Z}(T)$. The proposition follows. $\square$

The argument indicates the fact that the proof of Theorem 2.1 is constructive in the sense that if $\Lambda \in \mathbb{R}_+^4$, for Δ an i.t. of F_g^s, and $(\Delta, \Lambda) = \tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$, then one can algorithmically compute a matrix group corresponding to $\phi\tilde{\Gamma}_m \in \mathscr{T}_g^s$; of course, one can also compute the decoration $\mathscr{B} \subset L^+$ from Λ. For instance, taking the triangle T above together with its reflection in the plane $x_2 = 0$ as a fundamental domain for the action of $\pi_1(F_1^1)$, one computes that the corresponding matrix group is generated by $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

We wish to extend this to a construction of (decorated and marked) matrix groups from the λ-length data $\{\lambda(e;\tilde{\Gamma}_m): e \in \Delta'\}$ in case Δ' is an i.c.d. of F_g^s; our approach also leads to the notion of the "center" of $\mathring{\mathscr{C}}(\Delta')$. We begin with some definitions.

Let P denote an oriented convex Euclidean n-gon in the plane for $n \geqq 3$. The orientation of P will be used to enumerate the edges in their clockwise order starting from some fixed vertex. We say P is *r-cyclic* if P inscribes in a circle of radius r, and we say P is simply *cyclic* if it is r-cyclic for some $r > 0$. A cyclic polygon is *on-center* if its interior intersects every diameter of the circumscribing circle; otherwise P is *off-center*. An on-center cyclic polygon has a unique edge which shares an endpoint (or perhaps coincides) with an otherwise disjoint diameter; this edge is said to be *long*.

If a, b, $c \in \mathbb{R}_+$ satisfy all three strict triangle inequalities, then we define

$$\Sigma(a,b,c) = \sqrt{(a+b-c)(a+c-b)(b+c-a)(a+b+c)}/abc\,.$$

Lemma 6.1. *Suppose that P is a triangle with edges (in order) of lengths a, b, c. P is 1-cyclic if and only if $\Sigma(a, b, c) = 1$. Furthermore, P is off-center with first edge long if and only if $a^2 \geqq b^2 + c^2$.*

Proof. The classical formula of Heron gives a relation between the semi-perimeter $s = \frac{1}{2}(a+b+c)$ and the area A of P, namely

$$A = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

It is an exercise to verify that $A = abc/4r$, where r is the radius of the circle which circumscribes P. Equating the two expressions for A and solving for $\Sigma = \frac{1}{r}$ proves the first part. The second part follows from the law of cosines. □

Theorem 6.2. *Suppose that $(b_1, \ldots, b_n)$, $n \geqq 3$, is a tuple of positive numbers satisfying the strict triangle inequalities*

$$b_i < \sum_{j \neq i} b_j, \qquad \textit{for } i = 1, \ldots, n\,.$$

There exists a unique $r > 0$ and a unique (up to congruence) r-cyclic polygon P whose edge lengths (in order starting from some fixed vertex of P) are $b_1, \ldots, b_n$.

Remark. The uniqueness part of the theorem is a version of Cauchy's Theorem on rigidity of convex surfaces for cyclic planar polygons. The existence part further gives the tuple $(b_i)_1^n$ as a complete modulus for congruence classes of cyclic polygons.

Proof. Suppose without loss that $b_1 \geqq b_i$, $i = 1, \ldots, n$. The idea of the proof is to inscribe in a circle $C_r \subset \mathbb{R}^2$ of radius $r \geqq \frac{1}{2} b_1$ a broken arc whose component line segments (i.e., chords of C_r) have respective lengths $b_1, \ldots, b_n$. We then let r vary and apply the Mean Value Theorem to prove existence. In fact, if $r > \frac{1}{2} b_1$, there are two ways to inscribe the first arc (of length b_1) in C_r so that the arc has $(r, 0) \in \mathbb{R}^2$ as an endpoint: the two possibilities correspond to the on- and off-center cases.

In fact, we proceed somewhat more analytically and let

$$\beta_i(r) = \sin^{-1}(b_i/2r)\,, \qquad i = 1, \ldots, n\,,$$

be half the angle subtended by a chord of length b_i in C_r; each $\beta_i(r)$ is a strictly monotone decreasing function of r. We define

$$\mu(r) = \pi - \beta_1(r)\,, \qquad \nu(r) = \beta_1(r)\,.$$

There is thus an r-cyclic n-gon realizing the tuple $(b_i)_1^n$ if and only if

$$\omega(r) = \sum_{i=2}^{n} \beta_i(r) \in \{\mu(r), \nu(r)\}\,,$$

where $\omega(r) = \mu(r)$ gives an on-center and $\omega(r) = \nu(r)$ gives an off-center polygon, respectively. Meanwhile, $\beta = \mu(r)$ and $\beta = \nu(r)$ are the upper and lower sheets of $r = \frac{1}{2} b_1 \csc \beta$, respectively.

To prove existence, notice that $\omega(r)$ ($\mu(r)$, $\nu(r)$, respectively) is a strictly monotone decreasing (increasing, decreasing, respectively) function of r, and

$$\lim_{r\to\infty} \nu(r)/\omega(r) = b_1 \Big/ \sum_{i=2}^{n} b_i < 1\,,$$

$$\lim_{r\to\infty} \mu(r)/\omega(r) = \infty\,.$$

Thus, for r large

$$\nu(r) < \omega(r) < \mu(r)\,;$$

on the other hand, for $r = \frac{1}{2}b_1$, either

$$\omega(\tfrac{1}{2}b_1) \geqq \frac{\pi}{2} = \mu(\tfrac{1}{2}b_1) = \nu(\tfrac{1}{2}b_1)$$

[and we conclude that $\omega(r_0) = \mu(r_0)$ for some unique $r_0 \geqq \frac{1}{2}b_1$], or perhaps

$$\omega(\tfrac{1}{2}b_1) < \frac{\pi}{2}$$

[and we conclude that $\omega(r_1) = \nu(r_1)$ for at least one $r_1 \geqq \frac{1}{2}b_1$]. This completes the proof of existence.

To prove uniqueness, we first claim that $\nu(r) - \omega(r)$ is a strictly monotone decreasing function whenever it is non-negative. To this end, compute

$$\frac{d\beta_i}{dr} = -\frac{1}{r}\tan\beta_i\,.$$

Thus,

$$\frac{d}{dr}\left(\nu(r) - \omega(r)\right) = \frac{1}{r}\left(\sum_{i=2}^{n} \tan\beta_i - \tan\beta_1\right).$$

However, an easy induction proves that if $\theta_2 + \ldots + \theta_m \leqq \theta_1 \leqq \frac{\pi}{2}$, $\theta_i > 0$, for $i = 1, \ldots, m$, $m \geqq 3$, then

$$\tan\theta_2 + \ldots + \tan\theta_m < \tan\theta_1\,.$$

The claim follows.

A final application of the Mean Value Theorem shows that $\omega(r)$ equals one of $\mu(r)$, $\nu(r)$ exactly once, as desired. □

If $(b_1, \ldots, b_n)$ is a tuple satisfying all strict triangle inequalities, then we define the *scaling function* $\Sigma(b_1, \ldots, b_n)$ to be the reciprocal of the radius of the circle which circumscribes the cyclic polygon with edge lengths $(b_1, \ldots, b_n)$. Σ is a smooth homogeneous function (of degree one) which is invariant under cyclic permutation of its arguments. The expression for Σ when $n = 3$ is given before Lemma 6.1. When $n = 4$, one can explicitly write down an (unpleasant) expression for Σ using the result for $n = 3$ and Ptolemy's Theorem. For $n \geqq 5$, it seems difficult to write down Σ explicitly.

Let P be the cyclic polygon realizing the tuple $(b_i)_1^n$, and suppose that P has corresponding edges $(e_i)_1^n$. We associate a sign to (P, e_i), $i = 1, \ldots, n$, by

$$\varepsilon(P, e_i) = \begin{cases} -1, & \text{if } P \text{ is off-center with } e_i \text{ long} \\ 0, & \text{if } e_i \text{ is a diameter of the circumscribing circle} \\ 1, & \text{else} \end{cases}$$

Consider the triangle $T \subset P$ with edges e_j, e_{j+1}, e_*, where e_* is a diagonal of P (and the subscript j is cyclic). If e_* has length μ, then

$$\varepsilon(P, e_j) = \operatorname{sgn}(\mu^2 + b_{j+1}^2 - b_j^2)$$

by Lemma 6.1 [where $\operatorname{sgn}(0) = 0$ by convention].

Now, suppose that Δ' is an i.c.d. of F_g^s and $\tilde{\Gamma}_m \in \tilde{\mathcal{T}}_g^s$. There is then an induced

$$\Lambda_{\tilde{\Gamma}_m} \in \mathbb{R}_+^{\Delta'},$$

where $\Lambda_{\tilde{\Gamma}_m}(e) = \lambda(e_j; \tilde{\Gamma}_m)$.

Theorem 6.3. *If Δ is an* i.c.d. *of F_g^s and $\tilde{\Gamma}_m \in \mathring{\mathcal{C}}(\Delta)$, then $\Lambda_{\tilde{\Gamma}_m}$ uniquely determines $\tilde{\Gamma}_m$. Furthermore, $\mathring{\mathcal{C}}(\Delta)$ is parametrized by all $\Lambda \in \mathbb{R}_+^{\Delta}$ so that the following conditions hold.*

(i) *If R is a component of $F_g^s - \Delta$ with (consecutive) edges $(e_i)_1^n \subset \Delta$, then*

$$\Lambda(e_i) < \sum_{j \neq i} \Lambda(e_j), \qquad \textit{for } i = 1, \ldots, n.$$

(ii) *Suppose that $e \in \Delta$ separates a n-gon component N of $F_g^s - \Delta$ with consecutive edges $(f_i)_1^n$ from an m-gon component M of $F_g^s - \Delta$ with consecutive edges $(e_j)_1^m$. Then*

$$\begin{aligned} 0 < \varepsilon(N, e)\, & \Sigma(\Lambda(f_1), \ldots, \Lambda(f_n)) \\ & + \varepsilon(M, e)\, \Sigma(\Lambda(e_1), \ldots, \Lambda(e_m)). \end{aligned}$$

Proof. We begin with necessity of the conditions and suppose that $\Lambda = \Lambda_{\tilde{\Gamma}_m} \in \mathbb{R}_+^{\Delta}$ for some $\tilde{\Gamma}_m \in \mathring{\mathcal{C}}(\Delta)$. If R is as in condition (i), then R lifts to a face Φ of the hull of $\mathcal{B}$, where $\tilde{\Gamma}_m = \Gamma_m \times \mathcal{B}$. Since the plane of Φ is elliptic by Proposition 4.4, the ordinary triangle inequality holds in this plane in the induced structure. It follows easily that condition (i) is necessary.

To finish the proof of necessity, suppose that $e \in \Delta$ is as in the statement of condition (ii); we may assume that $e = e_1 = f_1$. Consider triangles $T \subset N$ and $S \subset M$ with edges $(e_1 f_2, f_*)$ and $(e_1 e_2, e_*)$, respectively. T and S lift to triangles in $\mathbb{M}$ which lie in adjacent faces of the hull of $\mathcal{B}$. As in the proof of necessity in Theorem 5.1, we must have

$$\begin{aligned} 0 < \Lambda(e_2)\, \Lambda(e_*)\, & (\Lambda^2(f_2) + \Lambda^2(f_*) - \Lambda^2(e)) \\ & + \Lambda(f_2)\, \Lambda(f_*)\, (\Lambda^2(e_2) + \Lambda^2(e_*) - \Lambda^2(e)). \end{aligned}$$

It requires only arithmetic to check that this is equivalent to

$$\begin{aligned} 0 < \varepsilon(T, e)\, & \Sigma^2(\Lambda(e), \Lambda(f_2), \Lambda(f_*)) \\ & + \varepsilon(S, e)\, \Sigma^2(\Lambda(e), \Lambda(e_2), \Lambda(e_*)). \end{aligned}$$

Finally, we have

$$\Sigma(\Lambda(e), \Lambda(e_2), \Lambda(e_*)) = \Sigma(\Lambda(e_2))_1^m,$$

$$\Sigma(\Lambda(e), \Lambda(f_2), \Lambda(f_*)) = \Sigma(\Lambda(f_i))_1^n,$$

and

$$\varepsilon(T, e) = \varepsilon(N, e), \qquad \varepsilon(S, e) = \varepsilon(M, e).$$

The first pair of equalities holds because if the vertices of a Euclidean triangle τ are in fact (some of) the vertices of an r-cyclic polygon P, then τ is r-cyclic as well. The second pair of equalities holds because if e is a common edge of τ and P, then $\varepsilon(\tau, e) = -1$ if and only if τ is off-center with e long; this, in turn, is equivalent to the condition that P is off-center with e long. Necessity of the conditions is therefore established.

To see that $\Lambda_{\tilde{\Gamma}_m}$ uniquely determines $\tilde{\Gamma}_m$ if $\tilde{\Gamma}_m \in \mathring{\mathscr{C}}(\Delta)$, suppose that R is an n-gon as in condition (i). R lifts to a face Φ of the hull of $\mathscr{B}$ in $\mathbb{M}$, where $\tilde{\Gamma}_m = \Gamma_m \times \mathscr{B}$. Since Φ lies in an elliptic plane and L^+ intersects this plane in a round circle in the induced structure (see Sect. 1), Φ is a cyclic n-gon with edge lengths $(\Lambda_{\tilde{\Gamma}_m}(e_i))_1^n$ in the induced structure. This n-gon is uniquely determined (up to congruence) by $\Lambda_{\tilde{\Gamma}_m}$ as in Theorem 6.2. Thus, $\Lambda_{\tilde{\Gamma}_m}$ uniquely determines $\tilde{\Gamma}_m \in \mathring{\mathscr{C}}(\Delta)$.

To prove the conditions are sufficient, suppose that $\Lambda \in \mathbb{R}_+^\Delta$ satisfies the conditions and extend Δ to an i.t. $\Delta' \supset \Delta$ of F_g^s. Δ' induces a triangulation of each component of $F_g^s - \Delta$. We extend Λ to $\Lambda' \in \mathbb{R}_+^{\Delta'}$ in the natural way: suppose that R is a component of $F_g^s - \Delta$ and $e \in \Delta' - \Delta$ is contained in R; $\Lambda|_{\partial R}$ uniquely determines a cyclic Euclidean polygon by Theorem 6.2, and we define $\Lambda'(e)$ to be the Euclidean length of the diagonal of this polygon corresponding to e. The assignment $\Lambda' \in \mathbb{R}_+^{\Delta'}$ determines $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$ by Theorem 2.1, and $\Lambda_{\tilde{\Gamma}_m}$ restricts to Λ on Δ, as desired. □

Remarks. 1) Since the function Σ is not explicitly known, we are *not* able to algorithmically construct matrix groups from elements of $\mathbb{R}_+^\Delta$ for Δ an arbitrary i.c.d. of F_g^s. However, since the scaling function Σ is known for quadrilaterals, all elements of codimension-one or -zero (and many high-codimension) cells *are* amenable to algorithmic construction.

2) If Δ is an i.c.d. of F_g^s so that there is $1 \neq \varphi \in MC_g^s$ with $\varphi(e) = e$ for all $e \in \Delta$, then we say Δ is "hyperelliptic." It is easy to see that "most" i.c.d.'s are *not* hyperelliptic: indeed, a hyperelliptic i.c.d. Δ has the property that either $F_g^s - \Delta$ is connected or consists of exactly two n-gons, for some $n \geqq 3$; for instance, any i.c.d. of F_1^1 is hyperelliptic. In case Δ is not hyperelliptic, it follows from the theorem and Corollary 3.3 that $\{\tilde{\Gamma}_m \in \mathring{\mathscr{C}}(\Delta)$: there is no $1 \neq \varphi \in MC_g^s$ with $\varphi_*(\tilde{\Gamma}_m) = \tilde{\Gamma}_m\}$ is a set of full measure (with respect to Lebesgue measure on λ-lengths) in $\mathring{\mathscr{C}}(\Delta)$.

3) It follows from the theorem and Corollary 3.3 that the map

$$\begin{aligned} \tilde{\mathscr{T}}_g^s &\to \{\mathbb{R}_+^\Delta : \Delta \text{ is an i.c.d. of } F_g^s\} \\ \tilde{\Gamma}_m &\mapsto (\Delta(\tilde{\Gamma}_m), \Lambda_{\tilde{\Gamma}_m}) \end{aligned}$$

is a MC_g^s-equivariant embedding.

4) It may be of interest to use the interior angles of cyclic polygons to give coordinates on $\mathring{\mathscr{C}}(\Delta)$, for Δ an i.c.d.

If Δ is an i.c.d. of F_g^s, then we define the *center* $\tilde{\Gamma}_m(\Delta) \in \tilde{\mathscr{T}}_g^s$ of $\mathring{\mathscr{C}}(\Delta)$ to be the point of $\tilde{\mathscr{T}}_g^s$ determined by the assignment $1 \equiv \Lambda \in \mathbb{R}_+^\Delta$; notice that Λ satisfies conditions (i) and (ii) of the theorem, so in fact, $\tilde{\Gamma}_m(\Delta) \in \mathring{\mathscr{C}}(\Delta)$. Just as before, we have

Proposition 6.4. *If Δ is an* i.c.d. *of F_g^s and $\varphi \in MC_g^s$, then*

$$\varphi_* \tilde{\Gamma}_m(\Delta) = \tilde{\Gamma}(\varphi^{-1}\Delta).$$

In particular, the conformal symmetry group of $\tilde{\Gamma}_m(\Delta)$ is isomorphic to the isotropy group of $\mathring{\mathscr{C}}(\Delta)$ in MC_g^s. □

7. The Representation of the Mapping Class Group

Suppose that $a, b, c, d, e \in \Delta$, so that (a, b, e) and (c, d, e) are (distinct) triangles in Δ, and adopt the (usual) notation of Fig. 7.1 i for nearby ends. Consider the operation (called an *elementary move*) on Δ indicated in Fig. 7.1 ii, and let Δ_e denote the resulting i.t..

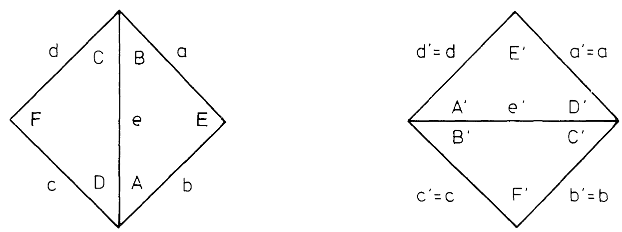

Fig. 7.1 (i) (ii)

We define the *Ptolemy groupoid* of F_g^s to be the groupoid Π_g^s generated by elementary moves. Thus, an element of Π_g^s is an equivalence class of sequence $(\Delta_j)_0^m$ (called a "chain") of i.t.'s so that Δ_{j+1} arises from Δ_j by an elementary move, $j = 0, \ldots, m-1$; two chains $(\Delta_j)_0^m$ and $(\Delta_i')_0^n$ are regarded as equivalent if $\Delta_0 = \Delta_0'$ and $\Delta_m = \Delta_n'$. There is a natural correspondence between arcs of Δ_j and Δ_{j+1}, and hence an induced correspondence between arcs in any Δ_i, Δ_j, $0 \leqq i, j \leqq m$. Notice that there may be a pair $e \in \Delta_j$, $e' \in \Delta_{j+1}$ of corresponding arcs so that e separates triangles of Δ_j, yet e' does not separate triangles of Δ_{j+1}; see Fig. 7.2.

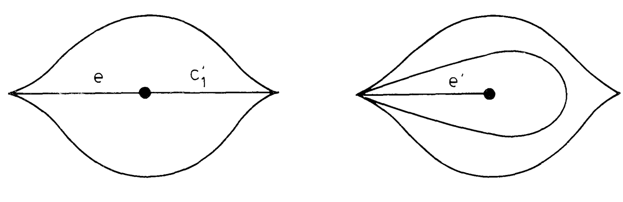

Fig. 7.2

Proposition 7.1. *Π_g^s acts transitively on* i.t.'s *of F_g^s. That is, given* i.t.'s *Δ, Δ' there is a chain $(\Delta_j)_0^m$ of* i.t.'s *with $\Delta = \Delta_0$, $\Delta' = \Delta_m$.*

Proof. Given two i.t.'s Δ and Δ' of F_g^s, consider the corresponding centers (see Sect. 6) $\tilde{\Gamma}_m, \tilde{\Gamma}_m' \in \tilde{\mathscr{T}}_g^s$ respectively. Since $\tilde{\mathscr{T}}_g^s$ is connected, we may join $\tilde{\Gamma}_m$ to $\tilde{\Gamma}_m'$ by a path in $\tilde{\mathscr{T}}_g^s$. By general position, we may choose a path which meets only the codimension-zero and codimension-one faces of our cell-decomposition $\mathscr{C}_g^s$ of $\tilde{\mathscr{T}}_g^s$. Since two top-dimensional cells $\mathscr{C}(\Delta_1)$ and $\mathscr{C}(\Delta_2)$ share a codimension-one face if and only if Δ_1 and Δ_2 differ by an elementary move, the proposition follows. □

Lemma 7.2. *Given $e \in \Delta$ separating triangles of Δ, adopt the notation above for Δ_e. If $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$, let*

$$\lambda_a = \lambda(a; \tilde{\Gamma}_m), \ldots, \lambda_{e'} = \lambda(e'; \tilde{\Gamma}_m),$$
$$\alpha = h(A; \tilde{\Gamma}_m), \ldots, \varphi' = h(F'; \tilde{\Gamma}_m)$$

denote the corresponding λ- and h-lengths. Then

$$\lambda_e \lambda_{e'} = \lambda_a \lambda_c + \lambda_b \lambda_d$$

and

$$\varepsilon' = \beta + \gamma, \qquad \varphi' = \alpha + \delta,$$

$$\alpha' = \frac{\varphi}{\varphi'}\,\alpha, \qquad \beta' = \frac{\varphi}{\varepsilon'}\,\beta,$$

$$\gamma' = \frac{\varepsilon}{\varepsilon'}\,\gamma, \qquad \delta' = \frac{\varepsilon}{\varphi'}\,\delta.$$

Proof. The first assertion is simply a restatement of the "ideal Ptolemy theorem" Proposition 2.6a. Furthermore, Corollary 3.4 has the immediate consequences

$$\varphi' = \alpha + \delta \qquad \varepsilon' = \beta + \gamma$$

$$\gamma' + \delta' = \varepsilon \qquad \alpha' + \beta' = \varphi,$$

and by definition

$$\alpha'/\gamma' = \beta'/\delta' \qquad \alpha'/\beta' = \gamma'/\delta'.$$

One easily derives the asserted formulas by direct computation. □

Proposition 7.3. *Π_g^s admits a faithful rational representation. That is, if Δ, Δ are* i.t.'s *of F_g^s, then there exists a q-tuple of homogeneous (of degree one) integral rational maps*

$$R_{\Delta,\Delta'}\colon \mathbb{R}_+^{\Delta} \to \mathbb{R}_+^{\Delta'}$$

so that

(i) *For all* i.t.'s *Δ_1, Δ_2, Δ_3 of F_g^s*

$$R_{\Delta_1,\Delta_3} = R_{\Delta_2,\Delta_3} \circ R_{\Delta_1,\Delta_2}.$$

(ii) *$R_{\Delta,\Delta'}$ is the identity if and only if $\Delta = \Delta'$.*

Proof. Suppose that Δ is an i.t. of F_g^s and consider $\Delta' = \Delta_e$ for some $e \in \Delta$. If $\Lambda \in \mathbb{R}_+^{\Delta}$, then Lemma 7.2 describes the rational computation of $\Lambda' \in \mathbb{R}_+^{\Delta'}$, so that (Δ, Λ) and (Δ', Λ') represent the same point of $\tilde{\mathscr{T}}_g^s$; say $\Lambda' = R_{\Delta,\Delta'}(\Lambda)$.

Now, given arbitary i.t.'s Δ, Δ' of F_g^s choose a chain $(\Delta_j)_0^m$ so that $\Delta_0 = \Delta$ and $\Delta_m = \Delta'$, and define

$$R_{\Delta,\Delta'} = R_{\Delta_{m-1},\Delta_m} \circ \cdots \circ R_{\Delta_1,\Delta_2} \circ R_{\Delta_0,\Delta_1};$$

the independence of $R_{\Delta,\Delta'}$ on the choice of chain and property (ii) each follows directly from the fact that λ-lengths give coordinates on $\tilde{\mathscr{T}}_g^s$; property (i) holds by definition. □

Corollary 7.4. *Fix an* i.t. *Δ of F_g^s. The natural action of MC_g^s on λ-length coordinates with respect to Δ is by rational maps.*

Proof. As in Corollary 3.3, if $\varphi \in MC_g^s$, then φ^{-1} induces a correspondence between Δ and $\varphi^{-1}\Delta$, and hence a natural map $\mathbb{R}_+^{\Delta} \to \mathbb{R}_+^{\varphi^{-1}(\Delta)}$. Composing with $R_{\varphi(\Delta),\Delta}^{-1}$ gives our rational representation of $\varphi \in MC_g^s$. □

Remarks. 1) As in [Mo], one might think of MC_g^s as a subgroup of finite-index in Π_g^s.

2) Since only the projective class of λ-lengths is needed to compute the underlying conformal types (see Remark 1 after Theorem 3.1), we can clear denominators in the tuple of rational maps to obtain a faithful representation of MC_g^s as a tuple of integral polynomials.

3) An interesting and important problem is to discover computable conjugacy invariants of a mapping class from its representing polynomials.

4) Of course, the second part of Lemma 7.1 gives the action of MC_g^s on h-length parameters for $\tilde{\mathscr{T}}_g^s$. Since h-lengths are ($\mathbb{R}$-valued) $\mathbb{C}$-analytic functions of strand coordinates on $\tilde{\mathscr{T}}_g^s$ (see the end of Sect. 3), we derive a faithful representation of the action of MC_g^s on strand coordinates as a group of analytic motions of $\mathbb{C}^{2q}$ preserving the variety determined by the coupling equations.

Example. Let c_1, c_2, c_3, c_4 be the ideal arcs in F_1^1 corresponding to the meridian, longitude, one-one, and one-(minus one) curves, respectively, and let Δ be the ideal triangulation of F_1^1 corresponding to $\{c_i\}_1^3$. Let τ_m and τ_ℓ denote the right Dehn twists along the meridian and longitude, respectively. We have

$$\begin{aligned} \tau_m^{-1}(c_1) &= c_1\,, \\ \tau_m^{-1}(c_3) &= c_2\,, \\ \tau_m^{-1}(c_2) &= c_4\,. \end{aligned}$$

If $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_1^1$ and $\lambda_i = \lambda(c_i; \tilde{\Gamma}_m)$, $i = 1, \ldots, 4$, then

$$\lambda_4 = \lambda_3^{-1}(\lambda_1^2 + \lambda_2^2)\,.$$

It follows that the action of τ_m on the coordinates $(\lambda_i)_1^3$ for $\tilde{\mathscr{T}}_1^1$ with respect to Δ is given by

$$\tau_m\colon (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, \lambda_3^{-1}(\lambda_1^2 + \lambda_2^2), \lambda_2)\,.$$

One similarly computes that

$$\begin{aligned} \tau_m^{-1}\colon\ (\lambda_1, \lambda_2, \lambda_3) &\mapsto (\lambda_1, \lambda_3, \lambda_2^{-1}(\lambda_1^2 + \lambda_3^2))\,, \\ \tau_\ell\ \ \colon\qquad\qquad\qquad &\mapsto (\lambda_3, \lambda_2, \lambda_1^{-1}(\lambda_2^2 + \lambda_3^2))\,, \\ \tau_\ell^{-1}\colon\qquad\qquad\qquad &\mapsto (\lambda_3^{-1}(\lambda_1^2 + \lambda_2^2), \lambda_2, \lambda_1)\,, \end{aligned}$$

describing a faithful rational representation of MC_1^1. The underlying projective-linear structure of our representation of $MC_1^1 \approx PSL_2\mathbb{Z}$ is not understood.

We comment briefly on the action of MC_1^1 on the set $\{\tilde{\Gamma}_m(\Delta)\colon \Delta$ is an ideal triangulation of $F_1^1\}$. Recall Markov's diophantine equation

$$m^2 + m_1^2 - \mathrm{m}_2^2 = 3\,m m_1 m_2\,,$$

which arises in diophantine approximation (see [Ca]). As a quadratic in m, one computes the other root to be

$$m' = 3\,m_1 m_2 - m = m^{-1}(m_1^2 - \mathrm{m}_2^2)\,.$$

It is well-known that any Markov triple (i.e., diophantine solution to Markov's equation) arises from the Markov triple (1,1,1) by a finite sequence of transformations of the form $(m, m_1, m_2) \mapsto (m', m_1, m_2)$. Comparison of the previous equation with our formula for the action of MC_1^1 shows that the set of Markov triples is exactly the set of coordinates on $\tilde{\mathscr{T}}_1^1$ so that the corresponding point is of the form $\tilde{\Gamma}_m(\Delta)$ for Δ an ideal triangulation of F_1^1.

Remarks. 1) There is a quadratic form classically associated with a Markov triple. It turns out that the quadratic form corresponding to $\tilde{\Gamma}(\Delta')$ for Δ' an ideal triangulation of F_1^1 is an eigenvector of a certain hyperbolic transformation in the group of Möbius transformations underlying $\tilde{\Gamma}_m(\Delta')$.

2) Further examples of such representations are pursued in the addendum.

Addendum. Surfaces with Distinguished Cusps and the Braid Groups

Given the surface F_g^s with $s \geqq 1$, we choose a cusp x of F_g^s once and for all. We say an ideal arc c in F_g^s *is based at* x if c runs from x to x. The λ-length of c depends on $\Gamma_m \in \mathscr{T}_g^s$ and a choice of orbit of parabolic fixed points corresponding to x alone, and so ratios of λ-lengths of ideal arcs based at x depend only on Γ_m. We denote the λ-length of c with respect to some fixed choice of horocycle about x by $\lambda(c; \Gamma_m)$. An *ideal triangulation* Δ_x *of* F_g^s *based at* x is (the isotopy class of) a maximal family of disjointly embedded essential ideal arcs based at x. Components of $F_g^s - \Delta_x$ are either triangles or once-punctured mono-gons. There are $(s-1)$ components of the latter type and $q = 6g - 5 + s$ components of Δ_x itself.

Theorem A.1. *Fix an ideal triangulation Δ_x of F_g^s based at x. λ-lengths of edges of Δ_x give projective coordinates on $\mathscr{T}_g^s$. That is, given a projective tuple in $\mathbb{R}_+^q$, there is a unique $\Gamma_m \in \mathscr{T}_g^s$ realizing the tuple as λ-lengths on ideal arcs in Δ_x.*

Proof. Δ_x extends in a unique way to an ideal triangulation Δ of F_g^s by adjoining to Δ_x one ideal arc in each of the punctured mono-gons. The λ-lengths of the ideal arcs in $\Delta - \Delta_x$ (relative to some choice of orbit of parabolic fixed point for cusps of F_g^s) can be chosen independently of the other λ-lengths and can be ignored. (The λ-lengths on $\Delta - \Delta_x$ serve only to fix the orbits for cusps other than x.) The argument of Theorem 3.1 applies to prove the theorem. □

Remark. Suppose that $z = 1/\sqrt{2}(1,1,0) \in L^+$ is a point in the orbit corresponding to the distinguished cusp x, a condition we can always arrange by conjugating Γ_m inside $SO^+(1,2)$ and re-scaling. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2\mathbb{R}$ corresponds to $\gamma \in \Gamma_m$ with $\gamma(z) = w \neq z$, then

$$\langle z, w\rangle = \langle z, \gamma(z)\rangle = 1/2\,\langle (1,1,0), (a^2+b^2, a^2-b^2, 2\mathrm{ab})\rangle = -b^2.$$

Thus, if $c \subset F_g^s$ is the ideal arc arising from the geodesic in $\mathbb{D}$ running from $\bar{z}$ to $\bar{w}$, then $\lambda(c; \Gamma_m) = |b|$.

λ-lengths of ideal arcs based at x are natural for the action of the subgroup $MC_g^s(x)$ of MC_g^s which leaves x invariant.

Suppose $\Gamma_m \in \mathscr{T}_g^s$. Let $B \subset L^+$ be a choice of Γ-orbit of parabolic fixed point for x, and let C be the convex hull of B in $\mathbb{M}$. The arguments of Sect. 4 go through to show that the faces of ∂C inside L^+ lie in either elliptic or parabolic planes, and in the parabolic case, the face is the convex hull of infinitely many points of B lying in the orbit of a parabolic subgroup of Γ_m. The set of faces is still locally-finite inside L^+. Let $\Delta_x(\Gamma_m)$ be the set of ideal arcs in F_g^s arising from edges of ∂C inside L^+.

Theorem A.2. *If $\Gamma_m \in \mathcal{T}_g^s$, then $\Delta_x(\Gamma_m)$ consists of a finite collection of disjointly embedded ideal arcs based at x so that components of $F_g^s - \Delta_x(\Gamma_m)$ are either simply connected or once-punctured.* □

Such a decomposition $\Delta_x(\Gamma_m)$ is called an *ideal cell decomposition based at x*. As before, we define

$$\mathscr{C}(\Delta_x) = \{\Gamma_m \in \mathcal{T}_g^s\colon \Delta_x(\Gamma_m) \subseteq \Delta_x\};$$

the arguments in Sect. 5 apply to show that each $\mathscr{C}(\Delta_x)$ is a cell. Moreover, Theorem 6.3 holds verbatim, where the conditions (i) and (ii) are interpreted as constraints on projective assignments of positive numbers to the arcs in an ideal cell decomposition based at x.

Theorem A.3. *$\{\mathscr{C}(\Delta_x)$: Δ_x is an ideal cell decomposition based at $x\}$ is a $MC_g^s(x)$-invariant cell decomposition of $\mathcal{T}_g^s$. $\mathscr{C}(\Delta_x)$ is a face of $\mathscr{C}(\Delta_x')$ if and only if $\Delta_x \subset \Delta_x'$.* □

Remark. If Δ_x is an ideal cell decomposition of F_g^s based at x, then we define $\Gamma_m(\Delta_x) \in \mathcal{T}_g^s$ by the assignment $\Lambda \equiv 1$ of λ-lengths to ideal arcs in Δ_x. The group of topological symmetries of (F_g^s, Δ_x) is the group of conformal symmetries of $\Gamma_m(\Delta_x)$.

Theorem A.4. *$MC_g^s(x)$ admits a faithful representation as a group of tuples of rational maps.*

Proof. If Δ_x is an ideal triangulation of F_g^s based at x, extend it to an ideal triangulation Δ of F_g^s as before. We extend a projective assignment Λ of positive real numbers to ideal arcs in Δ_x to a projective assignment on Δ by setting $\Lambda(c') = \Lambda(c)$ if $c' \in \Delta - \Delta_x$ and $\{c, c'\}$ determine a triangle in F_g^s. Transitivity of the elementary moves applies as before to give a sequence of elementary moves relating $\varphi^{-1}(\Delta, \Lambda)$ to (Δ, Λ''), for $\varphi \in MC_g^s(x)$, where Λ'' is a projective rational function of Λ; we finally simply ignore the values of Λ'' on ideal arcs in $\Delta - \Delta_x$. □

A move on ideal triangulations based at x which is useful in this context is the following. If $c_1 \in \Delta_x$ decomposes a once-punctured bi-gon R in F_g^s with $\partial R = \{c_2, c_3\} \subset \Delta_x$ into a triangle and a once-punctured mono-gon, then we let c_1' be the other such ideal arc in R and replace c_1 by c_1' (see Fig. A.1).

Remark. We believe that the elementary move of Sect. 7 together with this new move act transitively on ideal triangulations of F_g^s based at x.

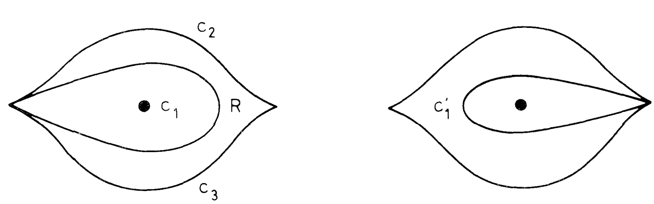

Fig. A1

Lemma A.5. *In the notation above, if $\Gamma_m \in \mathcal{T}_g^s$ and*

$$\lambda_i = \lambda(c_i; \Gamma_m), \qquad i = 1, 2, 3,$$
$$\lambda_1' = \lambda(c_1'; \Gamma_m),$$

then

$$\lambda_1 \lambda_1' = (\lambda_2 + \lambda_3)^2 .$$

Proof. The new move is the result of two elementary moves, and the result follows from two applications of Lemma 7.2. □

Since $MC_0^s(x)$ is isomorphic to the usual braid group on $(s-1)$ strands modulo its center, Theorem A.4 and Lemma A.5 are useful for giving a faithful representation of the braid groups modulo centers. We pursue this in the

Example. Let $\{c_i\}_1^{s-1}$ be a collection of ideal arcs based at x disjointly embedded in F_0^s, $s \geqq 4$, each bounding a once-punctured mono-gon. The complement of these regions in F_g^s is an $(s-1)$-gon R, which we suppose has consecutive edges $(c_1, \ldots, c_{s-1})$. Let $\{d_i\}_1^{s-1}$ be diagonals of R, so that $\{d_i\}$ separates $\{c_i, c_{i+1}\}$ from the other edges. (We regard indices as cyclic, so $c_s = c_1$, etc.)

Fix $\Gamma_m \in \mathcal{T}_0^s$ and define

$$\lambda_i = \lambda(c_i; \Gamma_m),$$
$$\mu_i = \lambda(d_i; \Gamma_m), \qquad i = 1, \ldots, s-1 .$$

Proposition A.6. *In the notation above, the projective class of the tuple $(\lambda_i, \mu_i)_1^{s-1}$ uniquely determines $\Gamma_m \in \mathcal{T}_0^s$.*

Proof. The proof is by induction, and the claim is trivial for $s = 4, 5, 6$, since in these cases $\{c_i, d_i\}_1^{s-1}$ contains an ideal triangulation of F_0^s based at x. Let e_j be the diagonal of R separating $\{c_{j-1}, c_j, c_{j+1}\}$ from the rest of ∂R, $j = 1, \ldots, s-1$. An application of Lemma 7.2 gives

$$v_j = \lambda(e_j; \Gamma) = \lambda_j^{-1}(\mu_{j-1}\mu_j - \lambda_{j-1}\lambda_{j+1}) .$$

Cutting R along d_1 yields (a triangle and) an $(s-2)$-gon S whose edge lengths are given. The diagonals of S required for the induction step are either given or among the $\{e_i\}_1^{s-1}$, and the proposition follows. □

Our approach is to keep track of the over-determined set $(\lambda_i, \mu_i)_1^{s-1}$ of parameters on $\mathcal{T}_0^s$ under the action of $MC_0^s(x)$. Let σ_j denote the half right Dehn twist along the non-trivial curve in F_0^s homotopic to $d_j \cup \{x\}$ in $F_0^s \cup \{x\}$, $j = 1, \ldots, s-1$; the σ_j give generators for $MC_0^s(x)$. The action of $\sigma_j^{\pm 1}$ on $(\lambda_i)_1^{s-1}$ is easily described. Indeed, only λ_j and λ_{j+1} are affected, and

$$\sigma_j^{+1}\colon (\lambda_j, \lambda_{j+1}) \mapsto (\lambda_{j+1}^{-1}(\lambda_j + \mu_j)^2, \lambda_j),$$
$$\sigma_j^{-1}\colon (\lambda_j, \lambda_{j+1}) \mapsto (\lambda_{j+1}, \lambda_j^{-1}(\lambda_{j+1} + \mu_j)^2)$$

by Lemma A.5. Similarly, the action of $\sigma_j^{\pm 1}$ affects only the parameters μ_{j-1} and μ_{j+1} among $\{\mu_j\}_1^{s-1}$. Neither e_j nor e_{j+1} are affected by $\sigma_j^{\pm 1}$, so the effect on μ_{j-1}, μ_{j+1} can be computed from the formula in Proposition A.6:

$$\sigma_j^{+1} : (\mu_{j-1}, \mu_{j+1}) \\ \mapsto \mu_j^{-1}(\nu_j \lambda_{j+1}^{-1}(\lambda_j + \mu_j)^2 + \lambda_{j-1}\lambda_j,\ \lambda_j \nu_j + \lambda_{j+1}^{-1}\lambda_{j+2}(\lambda_j + \mu_j)^2)$$

$$\sigma_j^{-1} : (\mu_{j-1}, \mu_{j+1}) \\ \mapsto \mu_j^{-1}(\nu_j \lambda_{j+1} + \lambda_{j-1}\lambda_j^{-1}(\lambda_{j+1} + \mu_j)^2,\ \nu_{j+1}\lambda_j^{-1}(\lambda_{j+1} + \mu_j)^2 + \lambda_{j+1}\lambda_{j+2}).$$

Remark. It is hoped that a simplification of the representations would arise by specialization of variables and/or a suitable deprojectivization of the coordinates.

Acknowledgements. Insofar as this work spans several years, there are a number of colleagues and institutions to thank. First and foremost, I am indebted to David Epstein. This work had its inception in discussions with him, and his input has been invaluable over the last few years. There have also been helpful discussions at various points with Bob Brooks, Charlie Epstein, Dennis Estes, Bill Goldman, Bernard Morin, Lee Mosher, Bill Thurston, Pekka Tukia, Scott Wolpert and Paul Yang. Furthermore, an unknown referee is thanked for his/her improvement of Theorem 6.2. The author also acknowledges the kind support during 1983–84 of the Institut Mittag-Leffler and the support of the Université Louis Pasteur during 1986.

References

[Ab] Abikoff, W.: The real-analytic theory of Teichmüller space. Lecture Notes in Mathematics, Vol. 820. Berlin, Heidelberg, New York: Springer 1980

[BE] Bowditch, B., Epstein, D.B.A.: Triangulations associated with punctured surfaces. Topology (1987)

[Ca] Cassels, J.W.S.: Rational quadratic forms. New York: Academic Press 1978

[EP] Epstein, D.B.A., Penner, R.C.: Euclidean decompositions of non-compact hyperbolic manifolds. J. Diff. Geom. (1987)

[FS] Friedan, D., Shenker, S.H.: The analytic geometry of two dimensional conformal field theory. Nucl. Phys. B. (1987)

[Ha] Harer, J.: The virtual cohomological dimension of the mapping class group of an oriented surface. Invent. Math **84**, 157–176 (1986)

[HZ] Harer, J., Zagier, D.: The Euler characteristic of the moduli space of curves. Invent. Math. **85**, 457–485 (1986)

[Mo] Mosher, L.: Pseudo-anosovs on punctured surfaces. Princeton University thesis (1983)

[P1] Penner, R.C.: Perturbative series and the moduli space of Riemann surfaces. J. Diff. Geom. (1987)

[P2] Penner, R.C.: The moduli space of punctured surfaces. Proceedings of the Mathematical Aspects of String Theory Conference, University of California, San Diego, World Science Press, 1987

Communicated by A. Jaffe

Received April 3, 1987; in revised form April 27, 1987