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Erratum

The Distributional Borel Summability and the Large Coupling Φ^4 Lattice Fields

E. Caliceti, V. Grecchi and M. Maioli

Dipartimento di Matematica, Università di Modena, I-41100 Modena, Italy

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Abstract. We complete and correct some proofs of an earlier paper on distributional Borel summability and we add an application which can be useful in the discussion of semiclassical problems.

1. Introduction

In this paper we give some required remarks for a better understanding of Theorem 1 and Proposition 2 of our previous paper [3]. Moreover we suggest an application giving an insight on the discussion of general methods for constructing semiclassical solutions. Since this note is a complement of [3], we use the same notations without repeating definitions and statements.

2. Remarks on References [3]

1) The method of distributional Borel summability described in [3], p. 164, makes use of a Borel transform which in general is not necessarily a distribution, but more precisely belongs to the wider class of hyperfunctions (see [5]). However, in the criterion given in Theorem 1 of [3] we actually restrict ourselves to a class of Borel transforms which are distributions. This justifies the name given to this kind of sum under our assumptions.

2) We can also consider sums of different types, that is of the form $f_\mu(z) = \mu\Phi(z) + (1-\mu)\bar{\Phi}(\bar{z})$, $0 \leqq \mu \leqq 1$. In particular $f_\mu(z)$ becomes the "upper sum" and the "lower sum" of the given asymptotic expansion for $\mu = 1$ and $\mu = 0$, respectively. If the criterion, given in [3] for the distributional Borel sum (which corresponds to the case $\mu = \frac{1}{2}$), applies to a certain series, then it guarantees the existence of all these different types of sums. As a matter of fact the criterion more directly refers to the "upper sum" $\Phi(z)$.

3) In order to justify the last statement at the bottom of p. 166 of [3], we want to

show that we can change the path of the integral $R_N^{(2)}(t+i\eta t)$ in (2.6) from $\Gamma_{\tau/N}$ to $\Gamma_{\tau/N,\varepsilon}$, when $\varepsilon = \arctan(\eta)$ as indicated in the line after (2.10). The point is to estimate the integral on an arc $\gamma_\delta = \{z = \delta e^{i\theta}/\theta_1 \leqq \theta \leqq \theta_2\}$, where $\theta_1 = \arccos(\delta N/\tau)$ and $\theta_2 = \varepsilon - \arccos(\delta N \cos(\varepsilon)/\tau)$ are determined so that the endpoints of the arc belong to $\Gamma_{\tau/N}$ and $\Gamma_{\tau/N,\varepsilon}$ respectively. In particular we need to prove that

$$I_\delta = (2\pi i)^{-1} \int_{\gamma_\delta} \exp((t+i\eta t)/z)(\Phi(z) - \bar{\Phi}(\bar{z}))z^{-1}\,dz \to 0 \quad \text{as} \quad \delta \to 0.$$

First of all note that $\forall z \in \gamma_\delta$, the term $\exp((t+i\eta t)/z) = \exp(|t+i\eta t|\cos(\varepsilon-\theta)/\delta)$ is bounded by the value taken on at $\theta = \theta_2$, where it can be estimated by $\exp(Nt(1+\eta^2)^{1/2}/\tau)$, as in (2.11)–(2.12) of [3]. On the other hand, from (2.2) we have $|\Phi(z) - \bar{\Phi}(\bar{z})| \leqq 2c_0\delta c(\theta+\pi/2)$ and, with the choice $c(\theta+\pi/2) = (\theta+\pi/2)^{-1}$ made at the beginning of the proof of Theorem 1 of [3], this implies

$$\begin{aligned}|I_\delta| &\leqq \delta\pi^{-1}c_0 \exp(Nt(1+\eta^2)^{1/2}/\tau) \int_{\theta_1}^{\theta_2} (\pi/2+\theta)^{-1}\,d\theta \\ &= \pi^{-1}c_0\delta \exp(Nt(1+\eta^2)^{1/2}/\tau)(\ln(\theta_2+\pi/2) \\ &\quad - \ln(\theta_1+\pi/2)) = O(\delta\ln(\delta)), \quad \text{as } \delta \to 0.\end{aligned}$$

4) The proof of Proposition 2 of [3] needs the following corrected version.

Without loss of generality we may assume $a(\pi/2) = 1$. By hypothesis, setting $g(z) := d(z)\exp(1/\sigma e z)$, $|g(z)| \leqq \exp(1/\sigma e r)$ if $\operatorname{Re} z^{-1} = r^{-1}$ and $|g(z)| = O(\exp(1/\sigma e|z|))$ as $|z| \to 0$ uniformly for $\operatorname{Re} z^{-1} > r^{-1}$. Now we claim that $|g(z)| \leqq e$, for $0 < z < r$. Indeed, for $n = 1, 2, \ldots$ we can consider the interval $[1/\sigma e(n+1), 1/\sigma e n]$ and there use the n-th estimate: $|d(z)| \leqq (\sigma n z)^n$ obtained from the hypothesis in the case $\varepsilon = \pi/2$. In any such interval, i.e. for $z = 1/\sigma e(n+\delta)$, $0 \leqq \delta \leqq 1$, $|g(z)| \leqq (n/(n+\delta))^n \exp(\delta) \leqq e$. This bound extends to the interval $(1/\sigma e, r]$, if $(\sigma e)^{-1} < r$ by monotonicity of $\exp(1/\sigma e z)$ for $z > 0$, and this proves the above claim. By symmetry we can restrict ourselves to the case $\operatorname{Im} z \geqq 0$. Then the function $\Omega(w) := g([(iw)^{1/2} + r^{-1}]^{-1})$ is analytic for $\operatorname{Re} w > 0$, it is uniformly $O(\exp(|w|^{1/2}))$ as $|w| \to \infty$ and it is bounded for $\operatorname{Re}(w) = 0$. Thus, by a Phragmen-Lindelof theorem ([2], Theorem 1.4.1, p. 3), $\Omega(w)$ is bounded uniformly for $\operatorname{Re}(w) \geqq 0$. Hence for $\operatorname{Re}(z^{-1}) \geqq r^{-1}$, $|\arg(z)| = -\varepsilon + \pi/2$, $n \geqq 1$,

$$|d(z)| \leqq L|\exp(1/\sigma e z)| \leqq L(\sigma e|z|)^n n!(\sin\varepsilon)^{-n} \leqq \sigma^n(M\varepsilon^{-1})^n n!|z|^n,$$

for some L, $M > 0$.

3. Application of the Method

As an application of the method, we mention that the distributional Borel summability of order $\alpha = 1/6$ applies also to the expansion, for large argument, of the Airy function $Bi(x)$. It follows that the expansion presented in the best-known handbooks, such as Abramowitz–Stegun [1] (p. 449, formula 10.4.63),

$$Bi(z) \approx \pi^{-1/2} z^{-1/4} e^{\beta}\left(1 + \sum_{k=1}^{\infty} c_k \beta^{-k}\right) \quad \text{as } z \to \infty,$$

where $\beta = 2z^{3/2}/3$, $c_k = \Gamma(3k+1/2)/(54^k k!\Gamma(k+1/2))$, can be regarded not only as a weak connection by asymptotism, but also as a one-to-one relationship by distributional Borel summability of power series. It can thus be used for the justification of the traditional JWKB method ([6], p. 2524), in order to connect the solutions obtained in regions which are separated by turning points of the classical motion. In particular we can overcome the difficulty of Borel summability of semiclassical expansions on the Stokes lines as we prove in the linear potential case ([7], p. 252-(6.35), [6]) where the solutions are just Airy functions. Indeed we consider the function

$$f(v) = (\pi^{1/2} z^{1/4} e^{-\beta} Bi(z) - 1)/v = (\Phi(v) + \bar{\Phi}(\bar{v}))/2,$$

where

$$\Phi(v) = (2\pi^{1/2}(ze^{2i\pi/3})^{1/4} e^{-\beta} Ai(ze^{2i\pi/3}) - 1)/v$$

$$= \Gamma(5/6)^{-1} 6v^{-1} \int_0^\infty \exp(-(t/v)^6)(t/v)^5 t^{-1}((1 - i0 - t^6/2)^{-1/6} - 1)\,dt$$

with $v^{-6} = \beta = (2/3)z^{3/2}$ (see [1] formulas 10.4.6, 10.4.26 and [4], p. 19, 7.3(17)). Thus it turns out, by direct inspection, that $f(v)$ is the distributional Borel sum of order $\alpha = 1/6$ (see [3], Theorem 4) of the series

$$\sum_{k=1}^{\infty} c_k v^{6k-1} \quad \text{for} \quad \mathrm{Re}(v^{-6}) > R^{-6}, \quad \text{i.e.}\ \mathrm{Re}(\beta) > R^{-1}, \quad \forall R > 0.$$

Remarks. i) It is important to notice that, strictly speaking, the Borel sum is defined on the positive real axis and is extended by analyticity to the region $\mathrm{Re}(v^{-1/\alpha}) > R^{-1/\alpha}$, where α is the order of the Borel sum. This possibility of extension is typical of Laplace transforms in $v^{-1/\alpha}$.

ii) In this example we see that $\Phi(v)$ (and $\bar{\Phi}(\bar{v})$) is the ordinary Borel sum of its expansion in complex directions: for $0 < \arg(v) < \pi/3$. Thus, for v on the positive real axis, $\Phi(v)$ is the limit as $\varepsilon \to 0^+$ of the Borel sum of the series

$$\sum_{k=1}^{\infty} (e^{i\varepsilon(6k-1)} c_k) v^{6k-1} \quad \text{(and similarly } \bar{\Phi}(\bar{v}) \quad \text{as } \varepsilon \to 0^-).$$

Of course it is not always true that on the real axis $\Phi(v)$ is such a limit of Borel sums as well as it is not always true that such a limit of Borel sums, if it exists, defines a function of the type "upper sum." Of course, if both the limit from above and the "upper sum" exist, then they coincide.

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