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# String Quantization on Group Manifolds and the Holomorphic Geometry of Diff $S^1/S^1$ *

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**Abstract.** The recent results by Bowick and Rajeev on the relation of the geometry of Diff $S^1/S^1$ and string quantization in $\mathbb{R}^{d,1}$ are extended to a string moving on a group manifold. A new derivation of the curvature formula $(-\frac{26}{12}m^3+\frac{1}{6}m)\delta_{n,-m}$ for the canonical holomorphic line bundle over Diff $S^1/S^1$ is given which clarifies the relation of that bundle with the complex line bundles over infinite-dimensional Grassmannians, studied by Pressley and Segal.

## I. Introduction

Recently Frenkel, Garland and Zuckerman have formulated the conditions for the consistency of string theory in the flat background $\mathbb{R}^{d,1}$ as conditions for Lie algebra cohomology for the Virasoro algebra, with coefficients in the Fock space of the string, [FGZ]. The results of Bowick and Rajeev in the Kähler geometry of the complexified tangent bundle of Diff $S^1/S^1$ can be seen as a step toward globalizing the algebraic approach in [FGZ], i.e. replacing Lie algebra cohomology by group cohomology. In this paper we shall carry out the program of [BR] in the case of a string on a group manifold.

Let $G$ be a simple compact Lie group and $LG$ the space of smooth loops in $G$, which is a group under point-wise multiplication of maps $S^1\to G$. In string theory, the space $LG$ can be considered either as the configuration space of a closed string moving in the manifold $G$ or as the phase space of an open string. Namely, let $g(\tau,\sigma)$ be an open string parametrized by the time $\tau\in\mathbb{R}$ and the string coordinate $\sigma\in[0,\pi]$ with the boundary conditions $g'(\tau,0)=g'(\tau,\pi)=0$; here $g'=\dfrac{dg}{d\sigma}$ and $\dot g=\dfrac{dg}{d\tau}$. One can then introduce a new coordinate $h(\tau,\sigma)$ by

$$h(\tau,\sigma)=\exp[(g^{-1}\dot g)(\tau,\sigma)+(g^{-1}g')(\tau,\sigma)]\,,\qquad 0\leqq\sigma\leqq\pi$$
$$h(\tau,\sigma)=\exp[(g^{-1}\dot g)(\tau,-\sigma)-(g^{-1}g')(\tau,-\sigma)]\,,\qquad -\pi\leqq\sigma\leqq 0\,.$$

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For each $\tau \in \mathbb{R}$ the map $\sigma \mapsto h(\tau, \sigma)$ is an element of $LG$. Conversely, $h(\tau, \sigma)$ together with initial values $g(\tau_0, \sigma)$ determine the map $g$.

Our point of view to string quantization is as follows. There is a set of natural line bundles $E^k$ over $LG$, parametrized by $k \in \mathbb{Z}$, which have a natural connection and curvature. The curvature form in $LG$ is

$$\Omega(X, Y) = \frac{\theta^2}{4\pi} \int_{S^1} \operatorname{tr} X dY, \tag{1.1}$$

where tr is the trace in the adjoint representation of the Lie algebra $g$ of $G$ and $\theta^2 = (\text{length})^2$ of the longest root of $g$. The tangent vectors of $LG$ have been identified as loops $X, Y: S^1 \to g$. Furthermore, there is a natural metric on $LG$ and we can define the covariant Laplace operator $\Delta$ in $LG$. We shall think of the string as a point particle moving in $LG$ and the field $\Omega$ as a generalized magnetic monopole field. The most simple quantum mechanical system corresponding to this picture is the one described by the Schrödinger equation

$$\Delta \psi = i \frac{\partial}{\partial \tau} \psi, \tag{1.2}$$

where $\psi$ is a section of the line bundle. However, the Laplacian $\Delta$ in the infinite-dimensional space $LG$ is a priori ill-defined. It comes well-defined when we specify *a complex structure* on $E^k$ and restrict $\psi$ to be in the space of holomorphic sections. In fact, $\Delta$ is just the generator $L_0$ of rotations in the Virasoro algebra. Now our system (1.2) is well-defined but it is not invariant under the reparametrization group $\operatorname{Diff} S^1$, because the complex structure of $E^k$ is not. To recover reparametrization invariance, we have to introduce a "ghost." Geometrically, this means that we have to extend the system to consist of sections of a vector bundle $\bar{B}$ over $\operatorname{Diff} S^1/S^1$ with fiber $\cong \Gamma^{k,\lambda}$, a subspace of $\Gamma(E^k)$, the space of sections of $E^k$. We have divided by $S^1$ since the complex structure will be invariant under rotations. Elements of $\operatorname{Diff} S^1/S^1$ parametrize the different complex structures in $E^k$, connected by $\operatorname{Diff} S^1$ action. The existence of a $\operatorname{Diff} S^1$ invariant vacuum vector in $\bar{B}$ can be reformulated as the *vanishing of the curvature* of $\bar{B}$, leading to the familiar condition $26 = k \cdot \dim g/(k + \kappa(g))$, where $\kappa$ is the dual Coxeter number of $g$, [GeW].

A mathematically interesting by-product of the present paper is a new derivation of the curvature formula $(-\frac{26}{12}m^3 + \frac{1}{6}m)\delta_{n,-m}$ for the canonical holomorphic line bundle over $\operatorname{Diff} S^1/S^1$. This formula was computed by Bowick and Rajeev from the Kähler geometry of the tangent bundle of $\operatorname{Diff} S^1/S^1$, [BR], whereas we shall obtain the same result by embedding $\operatorname{Diff} S^1/S^1$ in a certain infinite-dimensional Grassmannian manifold whose geometry has been studied by Pressley and Segal, [PS]. The curvature in the former is the pull-back of the curvature of a certain canonical line bundle over the latter manifold.

## II. Quantum Mechanics on $LG$

We shall first shortly describe the geometry of the canonical $S^1$ bundle $\hat{L}G$ over $LG = \{f : S^1 \to G | f \text{ smooth}\}$, when $G$ is a simple compact Lie group. Let $DG = \{f : D \to G | f \text{ smooth}\}$, $D \subset \mathbb{C}$ is the unit disk and $\mathscr{G} = \{f : DG | f|_{\partial D} = 1\}$. Both $DG$

and $\mathscr{G}$ are groups under point-wise multiplication; $\mathscr{G} \subset DG$ is a normal subgroup and obviously $LG \cong DG/\mathscr{G}$. For $f \in DG$ and $g \in \mathscr{G}$ we define

$$\omega(f, g) = \frac{\theta^2}{16\pi^2} \int_D \operatorname{tr} f^{-1} df \wedge dg g^{-1} - \frac{\theta^2}{48\pi^2} \int_B \operatorname{tr}(g^{-1} dg)^3, \tag{2.1}$$

where $\theta$ is as in the introduction, $B = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| \leqq 1\}$ and $g$ has been extended to $B$ as follows: Since $g = 1$ on the boundary $S^1 = \partial D$, we can think of $g$ as a mapping $g: S^2 \to G$ (the boundary of $D$ is identified as the north pole of $S^2$). From $\pi_2 G = 0$, it follows that there is a smooth extension $g: B \to G$, to the inside of $S^2$. However, there is no natural way to choose the extension. One can show that the value of $\exp 2\pi i C(g)$, where $C(g)$ is the second term in (2.1), does not depend on the extension, [W]. We shall denote the first term (in the right-hand side) by $\gamma(f, g)$.

Consider the group $DG \times S^1$ with the multiplication

$$(f, \lambda) \cdot (f', \lambda') = (ff', \lambda\lambda' \exp 2\pi i \gamma(f, f')). \tag{2.2}$$

One can embed $\mathscr{G}$ as a normal subgroup in $DG \times S^1$ using the homomorphism $\varphi(g) = (g, \exp 2\pi i C(g))$, and $\hat{L}G = DG \times S^1/\varphi(\mathscr{G})$ is then a central extension by $S^1$ of $LG$, [M1].

The Lie algebra $\hat{g}$ of $\hat{L}G$ is the Kac-Moody algebra associated to $g$. As a vector space $\hat{g}$ is the direct sum of the loop algebra $Lg$ and of the center $\mathbb{R}$. Let $pr_c$ be the projection onto the center in $\hat{g}$ and denote $A = -i\,\mathrm{pr}_c g^{-1} dg$, where $g^{-1} dg$ is the Maurer-Cartan one-form on $\hat{L}G$. The pull-back of the form $A$ with respect to the canonical projection $\pi: DG \times S^1 \to \hat{L}G$ is

$$(\pi^* A)(X, a) = a - \frac{\theta^2}{8\pi} \int_D \operatorname{tr} f^{-1} df \wedge dX, \tag{2.3}$$

where $(X, a)$ is a tangent vector at the point $(f, \lambda) \in DG \times S^1$. The exterior derivative of $A$ is

$$(dA)(X, Y) = \frac{\theta^2}{4\pi} \int_{S^1} \operatorname{tr} X dY. \tag{2.4}$$

We denote $\Omega = dA$. The form $A$ is invariant under the right action of $S^1$ in $\hat{L}G$ (and in fact invariant under the right action of any element of $\hat{L}G$) and the value of $A$ for a vertical tangent vector $(0, a)$ is equal to $a$; it follows that $A$ is a connection form in the principal bundle $\hat{L}G$, $\Omega$ being the curvature form.

Let $E^k$ be the complex line bundle associated to $\hat{L}G$ by the representation $\lambda \mapsto \lambda^k$ of $S^1$ in $\mathbb{C}$, $k \in \mathbb{Z}$. The curvature of $E^k$ is $k\Omega$. The Schrödinger wave function of a string propagating on the group manifold $G$ is an element in the space $\Gamma(E^k)$ of sections of the line bundle $E^k$. Let $\{T^1, \ldots, T^N\}$ be an orthonormal basis of $g$. The vectors $T_n^a = T^a e^{in\varphi}$ form a basis in the loop algebra $Lg (1 \leqq a \leqq N, n \in \mathbb{Z})$ with the orthogonality relations $\langle T_n^a, T_m^b \rangle = \delta^{ab} \delta_{n,-m}$. Elements in the Lie algebra of $\hat{L}G$ correspond to left-invariant vector fields on the group manifold $\hat{L}G$ in the usual way, so the vector $T_n^a$ form a basis for complex left-invariant vector fields. We denote by $\nabla_n^a$ the covariant derivative acting on $\Gamma(E^k)$, in the direction of the vector field $T_n^a$. We define the Schrödinger operator of the string to be the covariant Laplacian

$$\Delta = \frac{1}{\theta^2(k+\kappa)} \sum_{a,n} :\nabla_{-n}^a \nabla_n^a: = \frac{1}{\theta^2(k+\kappa)} \sum_{a=1}^{N} \left( \nabla_0^a \nabla_0^a + 2 \sum_{n=1}^{\infty} \nabla_{-n}^a \nabla_n^a \right), \tag{2.5}$$

where $\kappa = \kappa(g)$ is the dual Coxeter number, [GO]. There are two differences when compared to a Laplacian on a finite-dimensional group manifold. First, $\Delta$ does not commute with the group action; in fact, $\Delta$ is the generator $L_0$ in the Virasoro algebra defined by the Sugawara construction

$$L_n = \frac{1}{\theta^2(k+\kappa)} : \sum_{a,m} \nabla^a_{-m} \nabla^a_{n+m} : , \tag{2.6}$$

since the covariant derivatives close the Kac-Moody algebra

$$[\nabla^a_n, \nabla^b_m] = \lambda^{ab}_c \nabla^c_{n+m} + k \cdot \frac{\theta^2}{2} n \delta^{ab} \delta_{n,-m}, \tag{2.7}$$

where the $\lambda$'s are the structure constants of $g$,

$$[T^a, T^b] = \lambda^{ab}_c T^c . \tag{2.8}$$

The invariant Casimir operator is obtained from $\Delta$ by extending the Lie algebra $\hat{g}$ by the derivation $[d, T^a_n] = n T^n_a$ and defining $c_2 = \Delta + d$, [K]. The second difference is that the action of $\Delta$ on an element $\psi \in \Gamma(E^k)$ is not necessarily well-defined (the infinite sum may diverge). However, one can restrict $\Delta$ to certain subspaces of $\Gamma(E^k)$ which carry an irreducible representation of $\hat{g}$ and in which $\Delta$ is well-defined, [PS]. The subspaces we shall consider consist of holomorphic sections in a line bundle over $LG/T$, where $T \subset G$ is a maximal torus. We shall give here a somewhat different description of the holomorphic structure than in [PS].

The definition we shall adapt is a simple generalization of the holomorphic structure in line bundles over the unit sphere $S^2 = SU(2)/U(1)$. For each $k \in \mathbb{Z}$ we can define a line bundle over $S^2$ such that the space $\Gamma^k$ of sections consists of functions $\psi : SU(2) \to \mathbb{C}$ such that $\psi(gh) = h^{-k}\psi(g)$, where $g \cdot h$ denotes the right action of an element $h \in U(1)$ through the matrix representation $h \mapsto \mathrm{diag}(h, h^{-1})$. If $\mathcal{D}^j_{m_1 m_2}(g)$ denotes the matrix element $\langle jm_1 | D(g) | jm_2 \rangle$ in an irreducible representation of SU(2) [spin $j$, $m$ is the eigenvalue of $U(1)$ generator], then $\Gamma^k$ is spanned by the functions

$$\mathcal{D}^j_{m,-k} (j = |k|, |k|+1, \ldots; m = -j, -j+1, \ldots, j).$$

The holomorphic sections can be characterized as those which satisfy the differential equation $L^r_+ \psi = 0$, where $L_+$ is the generator of $SU(2)$ which raises the eigenvalue $m$ and "$r$" refers to the right action of $SU(2)$ on itself. Thus, for $k \leqq 0$ the space of holomorphic sections is spanned by the functions $\mathcal{D}^j_{mj}$ with $j = -k$ and for $k > 0$ there are no non-zero holomorphic sections. Furthermore (for $k \leqq 0$), the space of holomorphic sections carries an irreducible representation of the group $SU(2)$.

A section of the bundle $E^k$ over $LG$ can be thought of as a map $\psi : \hat{L}G \to \mathbb{C}$ such that $\psi(gh) = h^{-k}\psi(g)$ for $g \in \hat{L}G$ and $h$ in the center $S^1$ of $\hat{L}G$. Let $k$ be *positive* and $\lambda$ an integral anti-dominant weight of $(G, T)$ (i.e. $\lambda$ is the lowest weight in an irreducible finite-dimensional representation of $G$). We can define a line bundle $E^{k,\lambda}$ over $LG/T$ such that the space of sections $\Gamma(E^{k,\lambda})$ consists of vectors $\psi \in \Gamma(E^k)$ for which

$$\psi(gt) = \lambda(t)^{-1} \psi(g), \tag{2.9}$$

for $t \in T$ and $g \in \hat{L}G$. We have defined the covariant derivatives $\nabla_n^a$ through the left action of $LG$ on itself. Similarly, we define the operators $\partial_n^a$ using the right action of $LG$. *A section $\psi$ of $E^{k,\lambda}$ is said to be holomorphic if*

(i) $$\partial_n^a \psi = 0 \quad \forall n > 0 ,$$

(ii) $$\sum_a \alpha_a \partial_0^a \psi = 0 ,$$

whenever $\sum \alpha_a T^a$ is in the subspace of $g$ corresponding to the positive roots. Pressley and Segal showed that the representation $R$ of $\hat{L}G$ in $\Gamma_h(E^{k,\lambda})$ ( = the space of holomorphic sections), given by $(R(g_0)\psi)(g) = \psi(g_0^{-1}g)$, is *irreducible and unitarizable with lowest weight* $(\lambda, k)$. [PS, Chap. 11]. Since $\Delta$ is well-defined by the Sugawara construction, we have a perfectly well-defined quantum mechanical system in $\Gamma_h(E^{k,\lambda})$ described by the Schrödinger equation $\Delta\psi = i\frac{\partial}{\partial r}\psi$. However, *from the point of string theory this is not satisfactory*, since Diff $S^1$ is not a symmetry group of the equation. In the next section we shall make the necessary modifications to make the system invariant under Diff $S^1$.

## III. Reparametrization Invariance

From our construction of the central extension $\hat{L}G$ of $LG$ it follows immediately that a section $\psi$ of $E^k$ can be thought of as a function $\psi : DG \to \mathbb{C}$ such that

$$\psi(fg) = e^{-k \cdot 2\pi i \omega(f,g)} \psi(f) , \tag{3.1}$$

where $f \in DG$ and $g \in \mathcal{G}$. A diffeomorphism $h : S^1 \to S^1$ can be extended to $\tilde{h} : D \to D$ as $\tilde{h}(\varphi, r) = (h(\varphi), r)$; $0 \leqq \varphi \leqq 2\pi, 0 \leqq r \leqq 1$. There is a natural action of $h$ on $\psi$ given by $(h \cdot \psi)(f) = \psi(f \circ \tilde{h})$. In fact, the right-hand side does not depend on the extension $\tilde{h}$ of $h$, as can be seen from (3.1) using the invariance of $\omega$ under the group Diff $S^2$; a diffeomorphism of $S^2$ is identified as a diffeomorphism of $D$ which is the identity mapping on the boundary. To define the operator $\Delta$ we have needed (i) an inner product in $Lg$; (ii) the complex structure defined by the splitting $Lg = H_+ \oplus H_-$ to positive and negative Fourier modes ( = the normal ordering prescription in (2.5)). These two structures are invariant exactly under the rotation subgroup $S^1 \subset$ Diff $S^1$; any other diffeomorphism mixes the positive frequency operators $\nabla_n^a$, $n > 0$, with the negative frequency operators. To recover reparametrization invariance one can proceed as in [BR] in the case of a string in a flat space. We introduce a vector bundle $B$ over the manifold $M = \text{Diff}\, S^1/S^1$ with the fiber $B_x$ at each $x \in M$ being isomorphic with the vector space $\Gamma^{k,\lambda} = \Gamma_h(E^{k,\lambda})$. The space $M$ is contractible, [H], so the bundle $B$ is necessarily isomorphic with $M \times \Gamma^{k,\lambda}$ and thus the sections of $B$ are just vector valued functions on $M$. Points of $M$ represent complex geometries on $E^{k,\lambda}$ obtained by acting with Diff $S^1$ on the initial splitting $Lg = H_+ \oplus H_-$ and the inner product in $Lg$. The action of Diff $S^1$ moves $\Gamma^{k,\lambda}$ in the space $\Gamma(E^k)$. Using the triviality of the bundle $B$ we can adopt the viewpoint that the space $\Gamma^{k,\lambda}$ is kept fixed but we are moving the operator $\Delta$ in the space $\Gamma^{k,\lambda}$. From the results of Goodman and Wallach, [GW], it follows that there is a unitary projective representation $\mathcal{D}$ of the group Diff $S^1$ in the lowest weight representation $\Gamma^{k,\lambda}$ of the Kac-Moody algebra $\hat{L}g$ such that

$$\mathcal{D}(h) X \mathcal{D}(h)^{-1} = h \cdot X , \tag{3.2}$$

where $X \mapsto h \cdot X$ is the natural action of $\mathrm{Diff}\, S^1$ on the elements of the loop algebra $Lg, (h \cdot X)(\varphi) = X(h^{-1}(\varphi))$. Infinitesimally, $\mathscr{D}$ is just the Sugawara representation of the Virasoro algebra. Since $\Delta = L_0$ is also defined by the Sugawara construction, $\Delta$ has automatically the expected commutation relations with the representation $\mathscr{D}$; infinitesimally $[L_n, \Delta] = nL_n$. We recall the commutation relations of the Virasoro algebra, [GO],

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m}, \tag{3.3}$$

where $c = k \dim g/(k+\kappa)$.

The projective action of $\mathrm{Diff}\, S^1$ in the space $\Gamma(B)$ is

$$(h \cdot \psi)(h_1) = \mathscr{D}(h)\psi(h^{-1}h_1), \tag{3.4}$$

where $h_1 \in M$ and $\psi: M \to \bar{L}^{k,\lambda}$ is a section of $B$. It will be also useful to think of the sections of the bundle $B$ as functions $\psi: \mathrm{Diff}\, S^1 \to \bar{L}^{k,\lambda}$ such that $\psi(hs) = s^{+\alpha}\psi(h)$ for $s \in S^1$, where $\alpha$ is the lowest eigenvalue of $L_0$ in the space $\Gamma^{k,\lambda}$. This is equivalent to thinking of the sections as functions $M \to \bar{L}^{k,\lambda}$, since the fibering $\mathrm{Diff}\, S^1 \to M$ is trivial.

**Theorem 3.1.** *There are no non-zero* $\mathrm{Diff}\, S^1$ *invariant vectors in* $\Gamma(B)$.

*Proof.* The complex vector field $l_n = ie^{in\varphi}\frac{d}{d\varphi}$ on the circle is acting through

$$\varrho(l_n) \cdot \psi = \mathscr{L}_n \psi + L_n \psi \tag{3.5}$$

in $\Gamma(B)$; here $\mathscr{L}_n$ denotes the Lie derivative acting on a function, corresponding to the generator $l_n$ of $\mathrm{Diff}\, S^1$. Since

$$[l_n, l_m] = (n-m)l_{n+m}, \tag{3.6}$$

we get

$$[\varrho(l_n), \varrho(l_m)] = (n-m)\varrho(l_{n+m}) + \frac{c}{12} n(n^2-1)\delta_{n,-m}. \tag{3.7}$$

It follows that the only vector satisfying $\varrho(l_n)\psi = 0\ \forall n \in \mathbb{Z}$ is $\psi = 0$. □

*Remark.* The above result can be interpreted in terms of the geometry of the bundle $B$. The formula (3.5) defines a connection in the bundle: the covariant derivative in the direction of the vector field on $M$ generated by the left action of $l_n$ is given by the right-hand side of (3.5). The curvature of the connection is the two-form

$$\mathrm{curvature}(l_n, l_m) = [\varrho(l_n), \varrho(l_m)] - \varrho([l_n, l_m]) = \frac{c}{12} n(n^2-1)\delta_{n,-m}. \tag{3.8}$$

The non-existence of the $\mathrm{Diff}\, S^1$ invariant vacuum vector in $B$ can now be traced to the *non-vanishing of the curvature of* $B$.

The curvature (3.8) is related also to Berry's phase. Namely, let $B^0$ be line bundle over $M$ such that the fiber $B^0_h$ at a point $h \bmod S^1$ in $M$ is spanned by the vector $\mathscr{D}(h)\vartheta_0$ in $\Gamma^{k,\lambda}$, where $\vartheta_0$ is the lowest weight vector. We have a family $\Delta(h) = \mathscr{D}(h)\Delta\mathscr{D}(h^{-1})$ of Hamiltonians parametrized by elements of $M$. The multiplicity of the lowest eigenvalue $\alpha$ of $L_0$ is one and $L_0\vartheta_0 = \alpha\vartheta_0$. Then $\alpha$ is also an eigenvalue for each $\Delta(h)$ and the corresponding eigenspace is $B^0_h$. Using the section $\psi(h) = \mathscr{D}(h)\vartheta_0$ we can compute the connection and curvature of $B^0$, with the help of the general formula in [S]. The value of the vector potential to the direction of the left-invariant vector field $l_n$ is

$$A(l_n)=\left\langle \psi(h), \frac{d}{dt}\psi(he^{tl_n})|_{t=0}\right\rangle = \left\langle \vartheta_0, \mathscr{D}(h)^{-1}\frac{d}{dt}\mathscr{D}(he^{tl_n})\vartheta_0|_{t=0}\right\rangle$$

$$= \frac{d}{dt}\langle \vartheta_0, e^{\varepsilon(h,\exp tl_n)}\mathscr{D}(e^{tl_n})\vartheta_0\rangle|_{t=0} = \langle \vartheta_0, L_n\vartheta_0\rangle + \frac{d}{dt}\varepsilon(h, \exp tl_n)|_{t=0}, \tag{3.9}$$

where $\varepsilon$ is the projective factor,

$$\mathscr{D}(h_1)\mathscr{D}(h_2)=\mathscr{D}(h_1h_2)e^{\varepsilon(h_1,h_2)}. \tag{3.10}$$

In fact, we have considered the vector potential as a one-form on Diff $S^1$ (and not on $M$), using the representation of the sections of $B^0$ as equivariant functions on Diff $S^1$. The curvature is

$$\operatorname{curv}(l_n, l_m)=l_m \cdot A(l_n)-l_m\cdot A(l_n)-A([l_n, l_m])=\left(\frac{c}{12}n(n^2-1)+2n\alpha\right)\delta_{n,-m}, \tag{3.11}$$

the term $2n\alpha\delta_{n,-m}$ coming from the term $\langle\vartheta_0, L_n\vartheta_0\rangle=\alpha\delta_{n,0}$ in (3.9). The curvature (3.11) is equivalent to (3.8) in the sense that after the redefinition $L_0'=L_0-\frac{\alpha}{2}$ the two forms will be equal. Thus we can say that *the non-existence of the* Diff $S^1$ *invariant vacuum in B is related to the non-zero Berry's phase* (3.11) in the line bundle $B^0$ for the family $\Delta(h)$ of Hamiltonians.

Next we shall introduce a ghost field such that the new system will have an invariant vacuum in the case $26=k\dim g/(k+\kappa)$.

To start with, we shall give a *new derivation of the curvature of the canonical holomorphic line bundle over M*. Following Pressley and Segal, [PS], consider a direct sum of Hilbert spaces $H=H_+\oplus H_-$ (with $\dim H_\pm=\infty$) and the subgroup $GL_1$ of the connected component of the general linear group $GL(H)$ consisting of operators

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \tag{3.12}$$

such that both $b$ and $c$ are Hilbert-Schmidt operators, $\operatorname{tr}(b^\dagger b)<\infty$ and $\operatorname{tr}(c^\dagger c)<\infty$. The group $GL_1$ has a non-trivial central extension which can be described as follows, [PS]. Let $Q$ consist of all triples $(g,q,\lambda)\in GL_1\times GL(H_+)\times\mathbb{C}^\times$ such that $aq^{-1}-1$ is of trace-class; $Q$ inherits a group structure from its constituents and the subgroup $N=\{(1,q,\det q)|q\in GL(H_+),\ q-1 \text{ trace class}\}$ is normal. The central extension is $\widehat{GL_1}=Q/N$. The central projection of the Maurer-Cartan one-form on $\widehat{GL_1}$ defines a connection in the principal $\mathbb{C}^\times$-bundle $\widehat{GL_1}\to GL_1$ which has the curvature

$$\operatorname{curv}(\delta_1 g, \delta_2 g)=\operatorname{tr}(\delta_1 b\delta_2 c-\delta_2 b\delta_1 c), \tag{3.13}$$

where

$$\delta_i g=\begin{pmatrix}\delta_i a & \delta_i b\\ \delta_i c & \delta_i d\end{pmatrix}, \quad i=1,2, \tag{3.14}$$

are tangent vectors at $g\in GL_1$, [M2]. Since (3.13) does not depend on the diagonal blocks, we may consider its restriction to the unitary subgroup $U(H)$ as a two-form on the Grassmannian manifold

$$Gr_1=U(H)\cap GL_1/U(H_+)\times U(H_-). \tag{3.15}$$

Let now $H$ be the completion of the space of smooth vector fields on $S^1$ with respect to the $L^2$ inner product and $H_+$ (respectively $H_-$) the subspace spanned by Fourier components with positive (respectively non-positive) index. The group $\mathrm{Diff}\, S^1$ acts unitarily on $H$ by

$$(h \cdot X)(\varphi) = |g'(\varphi)|^{1/2} X(g(\varphi)), \tag{3.16}$$

where $g$ is the inverse of $h: S^1 \to S^1$. From the discussion in [PS, Sect. 6.8] it follows that (3.16) gives a homomorphism $\mathrm{Diff}\, S^1 \to GL_1$. However, this map is not continuous. Instead, the composite map

$$\mathrm{Diff}\, S^1 \to U(H) \cap GL_1 \to U(H) \cap GL_1 / U(H_+) \times U(H_-) = Gr_1$$

is continuous and even smooth with respect to the standard Frechet topology of $\mathrm{Diff}\, S^1$, [H]. The circle $S^1 \subset \mathrm{Diff}\, S^1$ is mapped to one point in $Gr_1$, but $\mathrm{Diff}\, S^1/S^1 \to Gr_1$ is one-to-one. We shall compute the curvature of $M$ as a pullback with respect to the embedding $M \to Gr_1$.

To compute the curvature we need the infinitesimal action of $\mathrm{Diff}\, S^1$ in $H$; but from (3.16) it follows that this is precisely the adjoint action of the algebra of vector fields on itself. Using the basis in $H$ given by the generators $l_n$, we can compute the matrix representing $l_p$,

$$(l_p)_{nm} = [l_p, l_m]_n = (p-m)\delta_{n,p+m}. \tag{3.17}$$

Thus the curvature form on $M$ is

$$\begin{aligned} \mathrm{curv}(l_p, l_q) &= \mathrm{tr}[(l_p)_b (l_q)_c - (l_q)_b (l_p)_c] \\ &= \sum_{m \leqq 0} \sum_{n > 0} (l_p)_{nm} (l_q)_{mn} - \sum_{m \leqq 0} \sum_{n > 0} (l_q)_{nm} (l_p)_{mn} \\ &= \left(-\tfrac{26}{12} q^3 + \tfrac{1}{6} q\right) \delta_{p,-q}. \end{aligned} \tag{3.18}$$

This agrees with the results of Bowick and Rajeev, [BR], *including the coefficient* $\frac{1}{6}$!

Let $F$ be the line bundle over $M$ obtained as the pull-back of the canonical line bundle over $Gr_1$. The ghost field in string quantization is now a section of the dual bundle $F^*$. The complete string wave function is a section of the bundle $\bar{B} = F^* \otimes B$ over $M$; note that the fiber of $\bar{B}$ is isomorphic with the fiber $B_x \cong \Gamma^{k,\lambda}$.

**Theorem 3.2.** *The curvature of the bundle* $\bar{B}$ *is*

$$\mathrm{curv}(l_n, l_m) = \left( \frac{c-26}{12} n^3 + \left( \frac{1}{6} - \frac{c}{12} \right) n \right) \delta_{n,m}.$$

*In particular, for* $c = 26$, *after redefining the* $\mathrm{Diff}\, S^1$ *action in* $B$ *by* $L_0' = L_0 + \dfrac{2-c}{24}$, *there is a* $\mathrm{Diff}\, S^1$ *invariant vacuum in* $\bar{B}$ *given by* $\psi(h) = \xi(h) \mathscr{D}(h) \vartheta_0$, *where* $\xi(h)$ *is a phase factor.*

*Proof.* The curvature in the product bundle $\bar{B} = F^* \otimes B$ is the sum of the curvature of $F^*$ and the curvature of $B$; on the other hand, curvature $F^* = -$curvature $F$. Infinitesimally, the $\mathrm{Diff}\, S^1$ action on the sections of $\bar{B}$ is given by the covariant derivatives to the directions of the vector fields $l_n$. Taking account that the base space $M$ is contractibel, the existence of a covariantly constant section is equivalent to the vanishing of the curvature. Let first $\psi(h) = \mathscr{D}(h) \vartheta_0$. The action of the generator $l_n$ of Diff $S^1$ on $\psi$ is given by

$$\begin{aligned}
&\mathscr{L}_n\psi + L_n\psi + V(h; l_n)\psi \\
&= \frac{d}{dt}\mathscr{D}(e^{-tl_n}h)\vartheta_0|_{t=0} + L_n\psi + V(h; l_n)\psi \\
&= \frac{d}{dt}\mathscr{D}(e^{-tl_n})\mathscr{D}(h)e^{\varepsilon(\exp - tl_n, h)}\vartheta_0|_{t=0} + L_n\psi + V(h; l_n)\psi \\
&= \left(-L_n + \frac{d}{dt}\varepsilon(e^{-tl_n}, h)|_{t=0}\right)\mathscr{D}(h)\vartheta_0 + L_n\psi + V(h; l_n)\psi \\
&= \left(V(h; l_n) - \frac{d}{dt}\varepsilon(e^{tl_n}, h)|_{t=0}\right)\psi(h),
\end{aligned} \tag{3.19}$$

where $V(h; l_n)$ is the connection form on the bundle $F^*$ (corresponding to the curvature (3.18)). Thus $\psi$ is covariantly constant up to a phase; using the vanishing of the total curvature we know that it is possible to redefine the phase of $\psi$ such that the new section is covariantly constant. □

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