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Conditional Transformation of Drift Formula and Potential Theory for $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$

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Abstract. Using conditional Brownian motion and the transformation of drift formula (of Cameron–Martin, Girsarov, Maruyama) we give integral conditions on a vector field b which imply the harmonic measures and Green functions for $\frac{1}{2}\Delta$ and $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$ on a bounded Lipschitz domain D are equivalent. By equivalent we mean there exist two-sided inequalities with constants depending only on b and D. This enables one to conclude the potential theory for $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$ on D and $\frac{1}{2}\Delta$ on D are the same.

1. Introduction

The purpose of this paper is to study the operator $L = \frac{1}{2}\Delta + b(\cdot)\cdot\nabla$ on a domain D. We shall impose integral conditions on b which allow b to have singularities and D will be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geqq 2$. Under our conditions on b and D, there will be two-sided inequalities $c^{-1}G(x,y) \leqq G_L(x,y) \leqq cG(x,y)$ and $c^{-1}w^x(dz) \leqq w_L^x(dz) \leqq cw^x(dz)$ between Green functions and harmonic measures for L and $\frac{1}{2}\Delta$ on D. The approach is probabilistic and follows closely Cranston, Fabes and Zhao (1986). The ideas here can trace their history to the works of Chung (1985), Falkner (1983) and Zhao (1983, 1984).

There are two main differences between treating $\frac{1}{2}\Delta + q$ as in the first work mentioned above and L in the present work. The first is in the use of the transformation of drift formula (sometimes called the Cameron–Martin–Girsanov formula which was also studied by Maruyama (1954)) instead of the Feynman–Kac formula. The second difference is in the use of the stochastic process version of the John–Nirenberg Theorem (see Delacherie–Meyer (1980)) instead of Khasminski's Lemma. In the previous works q was taken in the Kato class K_d^{loc} (see the proof for Corollary 3.14). The condition that arises from our techniques is that $|b|^2 \in K_d^{\mathrm{loc}}$ and $|b| \in K_{d+1}^{\mathrm{loc}}$.

The two-sided inequalities between Green functions and harmonic measures enable one to obtain potential theoretic results for L which are known to hold for $\frac{1}{2}\Delta$ and we give several though by no means all consequences that may be derived in this manner. We also give analytic conditions on b in the case when D is a $C^{1,1}$ domain.

Similar, though not overlapping, Green function estimates have been obtained in Hueber, Sieveking (1982) and Stampacchia (1965).

2. Preliminaries and Notation

Throughout we shall assume D is some bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with Green function $G(x, y)$ and Martin kernel functions $K(x, z)$, $z \in \partial D$. Our results also hold in $d \geq 3$ if D is assumed to be a bounded N.T.A. domain as studied in Jerison, Kenig (1981). When $d = 2$ one only need assume $m(D) < \infty$ and then one must discuss $K(x, z)$ for z in the minimal Martin boundary of D, but this provides no difficulty. For the reader unfamiliar with N.T.A. domains we assume the slightly stronger condition that ∂D is Lipschitz.

Let (W, P^x, F_t) be Brownian motion killed on exiting D and let $p(t, x, y)$ be its transition density. We shall need conditional Brownian motion as introduced in Doob (1957). Namely, if $z \in \partial D$, set

$$p^z(t, x, y) = \frac{p(t, x, y) K(y, z)}{K(x, z)},$$

or when $z \in D$, set

$$p^z(t, x, y) = \frac{p(t, x, y) G(y, z)}{G(x, z)}.$$

These transition densities give rise to measures P_z^x on continuous paths corresponding to Brownian motion conditioned to exit D at z. Another handy description is to consider the process defined by $dY_t = dW_t + (\nabla K(Y_t, z)/K(Y_t, z))dt$, when $z \in \partial D$, and we replace $K(y, z)$ with $G(y, z)$ when $z \in D$. Then the processes (Y, P^x) and (W, P_z^x) are the same.

Suppose also that b is a Borel measurable vector field on D. Under the assumptions which we make below we are guaranteed the existence of a unique solution X_t to

$$dX_t = dW_t + b(X_t)dt, \quad X_0 = x.$$

Then X is the diffusion with generator $L = \frac{1}{2}\Delta + b(\cdot)\cdot\nabla$. Define $M_t = \int_0^t b(W_s)dW_s$, $\tau_D = \inf\{t > 0 \colon W_t \notin D\}$, and assume $\sup_{x \in D} E^x[\langle M \rangle(\tau_D)] < \infty$, where $\langle M \rangle_t = \int_0^t |b(W_s)|^2 ds$. The transformation of drift formula tells us that if we define $N_t = \exp\{M_t - \frac{1}{2}\langle M \rangle_t\}$ and the measure Q^x by $dQ^x/dP^x|_{F_t} = N_t$ on $\{t < \tau_D\}$, then (W, Q^x) and (X, P^x) are different descriptions of the same process, even up to the exit time τ_D. In particular, $P^x(X_{\tau_D} \in dz) = P^x(N_{\tau_D}; W_{\tau_D} \in dz)$, or in the briefer notation of the analysts $\omega_L^x(dz) = H(x, z)\omega^x(dz)$, where $\omega_L^x(dz)$ is L-harmonic measure on D, $\omega^x(dz)$ is harmonic measure on D and $H(x, z) = E_z^x[N(\tau_D)]$. Our main result, Theorem 3.1, states that under appropriate conditions on b there is a positive constant c such that $c^{-1} \leqq H(x, y) \leqq c$ for all x, $y \in \bar{D}$.

The lower bound does not hold on general domains D even for $b(\cdot) \equiv b$, though $|b|$ may be arbitrarily small. In Cranston, McConnell (1983) an example of a bounded domain $D \subseteq \mathbb{R}^3$ was given for which $P_z^x(\tau_D = \infty) = 1$ occurred for a particular boundary point z, the domain was not N.T.A. For this b, D and z,

$$E_z^x\left[\exp\left\{\int_0^{\tau_D} b \cdot dW_s - \tfrac{1}{2}\int_0^{\tau_D} |b|^2 ds\right\}\right] = E_z^x \exp\{b\cdot(z-x) - \tfrac{1}{2}|b|^2 \tau_D\}.$$
$$= 0.$$

On the other hand, the upper bound $H(x,y) \leqslant c$ cannot hold even on nice domains without some assumption on b. Let, for example, $D = \{z : |z| < 1\} \subseteq \mathbb{C}$ and $K(\cdot, z)$ be the Poisson kernel with pole at z and normalized so that $K(0,z) = 1$, setting $b(x) = \nabla K(x,z)/K(x,z)$, we have $H(,z) \equiv \infty$. To see this observe that $\log K(W_t, z) = \int_0^t \nabla K/K(W_s, z) dW_s - \frac{1}{2}\int_0^t |\nabla K/K|^2 \quad (W_s, z) ds$. Thus, if $\tau_r = \inf\{t > 0 : K(W_t, z) = r\}$, then $H(x,z) = \lim_{r\uparrow\infty} E_z^x K(W_{\tau_r}, z) = \infty$.

We shall use the John–Nirenberg inequality which may be found in Delacherie, Meyer (1980).

In the present context if $Z_t = M_{t\wedge\tau} - \frac{1}{2}\langle M\rangle_{t\wedge\tau}$, where τ is a stopping time, set $Z^* = \sup_t |Z_t|$. The John–Nirenberg inequality tells us that if for any pair of stopping times $S \leqq T$ we have $E_y^x[|Z_T - Z_S| \| F_S] \leqq c$ a.s., then in fact $E_y^x \exp\{\lambda Z^*\} \leqslant (1 - 4\lambda c)^{-1}$. In particular, we would have $E_y^x N_\tau \leqslant (1-4c)^{-1}$ provided $c < \frac{1}{4}$.

3. Main Results

We shall need the following assumptions on b and D:

$$D \text{ is a bounded Lipschitz domain,} \tag{3.1.1}$$

$$\sup_{x\in D} \int_D |b(y)|^2 G(x,y) dy < \infty, \tag{3.1.2}$$

$|b(\cdot)|^2$ is uniformly integrable with respect to the measures

$$\mu(x,z;dy) = \begin{cases} G(x,z)^{-1} G(x,y) G(y,z) dy, & \text{for } (x,z) \in D \times D, \\ K(x,z)^{-1} G(x,y) K(y,z) dy, & \text{for } (x,z) \in D \times \partial D. \end{cases} \tag{3.1.3}$$

$|b(\cdot)|$ is uniformly integrable with respect to the measures

$$\nu(x,z;dy) = \begin{cases} G(x,z)^{-1} G(x,y) \nabla G(y,z) dy, & z \in D, \\ K(x,z)^{-1} G(x,y) \nabla K(y,z) dy, & z \in \partial D. \end{cases} \tag{3.1.4}$$

The condition (3.1.1) is a rather mild boundary condition which insures $P_z^x(\tau_D < \infty) = 1$ for $x \in D$, $z \in \partial D$. For this purpose N.T.A. would suffice for $d \geqq 3$, see Cranston (1985). In the case $d = 2$, $m(D) < \infty$ is enough where m is Lebesgue by Cranston and McConnell (1983).

In (3.1.3) and (3.1.4) we mean that a function f is uniformly integrable with respect to a family of measures $\{\mu_\lambda\}_{\lambda\in\Lambda}$ if given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $m(C) < \delta$ then $\int_C |f(y)|^{\mu}{}_\lambda(dy) < \varepsilon$ for all $\lambda \in \Lambda$. Our main result is

Theorem 3.1. *Suppose b and D satisfy (3.1.1)–(3.1.4). Then there is a positive constant c, depending on b and D such that*

$$c^{-1} \leqq H(x,y) \leqq c \quad \text{for all } x, y \in \bar{D}. \tag{3.1.5}$$

Remark. If we do not assume ∂D is Lipschitz (actually nontangentially accessible suffices) the measure P_y^x may not exist for x, $y \in \partial D$, see Salisbury (1985) for the existence of P_y^x. In $d=2$ if only $m(D) < \infty$ is assumed then the conclusion (3.1.5) holds for all $(x, y) \in \bar{D} \times \bar{D} \backslash S$, where S is a of $\omega^x \times \omega^x$ measure zero, again by results of Salisbury (1985).

Proof. We shall give the proof only for the case $x \in D$, $z \in \partial D$. The proof follows Chung (1985) and in the present case we use the transformation of drift formula and John–Nirenberg inequality in place of the Feynman–Kac formula and Khasminski's lemma. The reader is referred to Cranston, Fabes, Zhao (1986) for the remaining cases and should be able to make the appropriate modifications. We begin with the following lemma from Chung (1985).

Lemma 3.2. *Given* $\frac{1}{4} > \varepsilon > 0$, *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that if C is an open connected subset of D with* $\partial D \subset \partial C$ *and* $m(C) < \delta$, *then*

$$\sup_{\substack{x \in C \\ z \in \partial D}} E_z^x[|M(\tau_C) - \langle M \rangle_{\tau_C}|] < \varepsilon. \tag{3.2.1}$$

and

$$\sup_{\substack{x \in C \\ z \partial D}} E_z^x[N(\tau_C)] \leqq \frac{1}{1-4\varepsilon}. \tag{3.2.2}$$

Proof. By the strong Markov property and the John–Nirenberg inequality, the estimate (3.2.1) implies (3.2.2). Thus we estimate, for $z \in \partial D$ we have

$$\begin{aligned} E_z^x|M(\tau_C)| &= E_z^x \left| \int_0^{\tau_C} b(W_s) dW_s \right| \\ &\leqq E^x \left| \int_0^{\tau_C} b(Y_s) dW_s \right| + E^x \left| \int_0^{\tau_C} b(Y_s) \frac{\nabla K(Y_s, z)}{K(Y_s, z)} ds \right| \\ &\leqq \left[E^x \int_0^{\tau_C} |b(Y_s)|^2 ds \right]^{1/2} + \int_C \frac{G(x,y)|b(y)||\nabla K(y,z)|}{K(x,z)} dy \\ &\leqq \left[\int_C \frac{G(x,y)|b(y)|^2 K(y,z)}{K(x,z)} dy \right]^{1/2} \\ &\quad + \int_C \frac{G(x,y)|b(y)||\nabla K(y,z)|}{K(x,z)} dy. \end{aligned}$$

Now invoking assumptions (3.1.3) and (3.1.4) we can select $\delta = \delta(\varepsilon) > 0$, so that $E_z^x|M(\tau_C)| < \varepsilon/2$ whenever $m(C) < \delta$ (m denotes Lebesgue measure). Also, in a similar manner

$$E_z^x \langle M \rangle_{\tau_C} = \tfrac{1}{2} E_z^x \int_0^{\tau_C} |b(W_s)|^2 ds \leqq \tfrac{1}{2} \int_C \frac{G(x,y)|b(y)|^2 K(y,z)}{K(x,z)} dy.$$

Thus, by (3.1.3), there is a δ, possibly smaller than the previous δ, such that $E_z^x\langle M\rangle_{\tau_C} < \varepsilon/2$ whenever $m(C) < \delta$. Combining the estimates for $E_z^x M(\tau_C)$ and $E_z^x\langle M\rangle_{\tau_C}$ gives (3.2.1).

We now fix a connected subdomain $C \subset D$ with $\partial D \subset \partial C$ and $m(C) < \delta(\frac{1}{8})$, so that $E_z^x[|M(\tau_C)| + \langle M\rangle_{\tau_C}] \leqq \frac{1}{8}$ and consequently $\sup_{z\in\partial D}^{x\in C} E_z^x[N(\tau_C)] \leqq 2$. Set $D_1 = D\backslash C$ and let D_2 be a subdomain of D such that $\bar{D}_1 \subset D_2$. Then as in Chung (1985) we have

Lemma 3.3. *There exist positive constants c_1 and c_2 depending on D and b such that*

$$c_1 \leqq E_z^x\{N(\tau_D); \tau_D = \tau_C\} \leqq c_2, \quad x\in\partial D_2, z\in\partial D.$$

Proof. Exactly as in Chung (1985) it holds that, for some positive constant c depending on D alone, $P_z^y[\tau_C = \tau_D] \geqslant c$ for $y\in\partial D_2$, $z\in\partial D$. Thus by Jensen's inequality,

$$\begin{aligned} E_z^x[N(\tau_D)|\tau_C = \tau_D] &\geqq \exp\left[E_z^x[-|M(\tau_D) - \tfrac{1}{2}\langle M\rangle_{\tau_D}||\tau_C = \tau_D]\right\} \\ &\geqq \exp\left\{-\frac{1}{C} E_z^x|M(\tau_C) - \tfrac{1}{2}\langle M\rangle_{\tau_C}|\right\}. \\ &\geqq e^{-\varepsilon/c}. \end{aligned}$$

On the other hand,

$$E_z^x[N(\tau_D)|\tau_C = \tau_D] \leqq \frac{1}{c} E_z^x[N(\tau_C)] \leqq \frac{1}{c}\cdot\frac{1}{1-4\varepsilon}. \tag{3.2.2}$$

This proves the lemma.

We now complete the proof of Theorem 3.1. Define the sequences of stopping times starting with $T_0 \equiv 0$,

$$\begin{aligned} T_{2n-1} &= T_{2n-2} + \tau_{D_2}\circ\theta_{T_{2n-2}}, \\ T_{2n} &= T_{2n-1} + \tau_C\circ\theta_{T_{2n-1}}, \quad n \geqq 1. \end{aligned}$$

Then since $P_z^x(\tau_D < \infty) = 1$, we have a.s., $T_{2n} = \tau_D$ for some finite n. Thus, by the strong Markov property,

$$\begin{aligned} H(x,z) &= \sum_{n=1}^{\infty} E_z^x\{N(\tau_D) \colon T_{2n} = \tau_D\} \\ &= \sum_{n=1}^{\infty} E_z^x\{N(T_{2n-1}) E_z^{W(T_{2n-1})}[N(\tau_D); \tau_C = \tau_D]; T_{2n-2} < \tau_D]. \end{aligned}$$

Noticing that when $T_{2n-2} < \tau_D$ we have $W(T_{2n-1})\in\partial D_2$, it follows from Lemma 3.3 that

$$c_1 \sum_{n=1}^{\infty} E_z^x\{N(T_{2n-1}); T_{2n-2} < \tau_D\} \leqq H(x,z) \leqq c_2 \sum_{n=1}^{\infty} E_z^x\{N(T_{2n-1}); T_{2n-2} < \tau_D\}. \tag{3.3.1}$$

The definition of P_z^x gives the following expression for the n^{th} term in (3.3.1),

$$E_z^x\{N(T_{2n-1}); T_{2n-2} < \tau_D\} = K(x,z)^{-1} E^x[K(W(T_{2n-1}),z)N(T_{2n-1}); T_{2n-2} < \tau_D\}. \tag{3.3.2}$$

Suppose first that $x \in \bar{D}_2$. On $\{T_{2n-2} < \tau_D\}$, $W(T_{2n-1}) \in \bar{D}_2$. The function $K(x,z)$ is jointly continuous and bounded from zero on $\bar{D}_2 \times \partial D$ so for some positive constant c we have

$$\frac{K(y,z)}{K(x,z)} \leqq c \frac{K(y,z')}{K(x,z')}, \quad x, y \in \bar{D}_2, z, z' \in \partial D.$$

Consequently, from (3.3.2)

$$\sup_{z \in \partial D} E_z^x\{N(T_{2n-1}); T_{2n-2} < \tau_D\} \leqq c \inf_{z \in \partial D} E_z^x\{N(T_{2n-1}); T_{2n-2} < \tau_D\}.$$

Furthermore, since $1 = E^x[N(\tau_D)] = \int_{\partial D} H(x,z)\omega^x(dz)$,

$$\inf_{z \in \partial D} H(x,z) \leqq 1 \leqq \sup_{z \in \partial D} H(x,z). \tag{3.3.3}$$

Thus, from (3.3.1), (3.3.2) and (3.3.3)

$$\sup_{\substack{z \in \partial D \\ x \in D_2}} H(x,z) \leqq c.$$

When $x \in D \setminus \bar{D}_2$, write

$$\begin{aligned} H(x,z) &= E_z^x\{N(\tau_C); \tau_C = \tau_D\} + E_z^x\{H(X_{\tau_C}, z); \tau_C < \tau_D\} \\ &\leqq c_2 + \sup_{\substack{x \in \bar{D}_2 \\ z \in \partial D}} H(x,z) \\ &\leqq c_2 + c, \end{aligned}$$

and the proof of the upper bound is complete. The lower bound follows using Jensen's inequality.

Theorem 3.4. *Assume (3.1.1)–(3.1.4) hold. Then*

$$\omega_L^x(dz) = H(x,z)\omega^x(dz), x \in D, z \in \partial D, \tag{3.4.1}$$

$$G_L(x,y) = H(x,y)G(x,y), x, y \in D, \tag{3.4.2}$$

and there exists a positive constant c depending on b and D such that

$$c^{-1}\omega^x(dz) \leqq \omega_L^x(dz) \leqq c\omega^x(dz), \tag{3.4.3}$$

$$c^{-1}G(x,y) \leqq G_L(x,y) \leqq cG(x,y). \tag{3.4.4}$$

Proof. Formula (3.4.1) is a restatement of the transformation of drift formula. The consequences (3.4.3) and (3.4.4) follow immediately from (3.4.1), (3.4.2) and Theorem (3.1). Thus we only need prove (3.4.2). First observe, by applying L to both sides, that

$$G_L(x,y) = G(x,y) + \int_D G_L(x,\omega)b(\omega)\nabla G(\omega,y)d\omega. \tag{3.4.5}$$

Next notice that $dN_t = N_t dM_t$, $N_0 = 1$, so

$$\begin{aligned}
H(x,y) &= 1 + E_y^x \int_0^{\tau_D} N_s dM_s \\
&= 1 + E_y^x \int_0^{\tau_D} N_s b(W_s) dW_s \\
&= 1 + E^x \int_0^{\tau_D} N_s^Y b(Y_s) dY_s, \quad \text{(where by } N^Y \text{ we mean substitute } Y \text{ for } W \text{ in the definition of } N) \\
&= 1 + E^x \int_0^{\tau_D} N_s^Y b(Y_s) \frac{\nabla G(Y_s, y)}{G(Y_s, y)} ds \\
&= 1 + E_y^x \int_0^{\tau_D} N_s b(W_s) \frac{\nabla G(W_s, y)}{G(W_s, y)} ds \\
&= 1 + \int_0^{\infty} E_y^x \left[N_s b(W_s) \frac{\nabla G(W_s, y); s < \tau_D}{G(W_s, y)} \right] ds \\
&= 1 + \int_0^{\infty} E^x \left[N_s b(W_s) \frac{\nabla G(W_s, y); s < \tau_D}{G(x, y)} \right] ds \\
&= 1 + G(x,y)^{-1} \int_D G_L(x,w) b(w) \nabla G(w,y) dw, \text{ or}
\end{aligned}$$

$$H(x,y) = 1 + G(x,y)^{-1} \int_0 G_L(x,w) b(w) \nabla G(w,y) dw. \tag{3.4.6}$$

Comparing Eqs. (3.4.5) and (3.4.6) yields $G_L(x,y) = H(x,y)G(x,y)$ as desired.

We now give consequences of Theorems 3.1 and 3.4 all of which assume (3.1.1)–(3.1.4) hold.

Theorem 3.5. (*Strong Harnack*). *Given δ, $0 < \delta < 1$ there exists a positive constant $c = c(\delta, b, D)$ such that if $B(x,r) \subset D$ and u is a positive solution of $Lu = 0$ in D, then*

$$u(y) \leqq cu(z) \quad \textit{for } y, z \in B(x, \delta r).$$

Proof. Denote by H_r the H for $B(x,r)$. Then (3.1.5) holds for H_r with the same c used for H. Also denote by $\omega_{L,r}$ and ω_r the harmonic measures on $B(x,r)$ for L and $\frac{1}{2}\Delta$. Then

$$\begin{aligned}
u(y) &= \int_{\partial B(x,r)} u(w) w_{L,r}^y(dw) \\
&= \int_{\partial B(x,r)} u(w) H_r(y,w) w_\tau^y(dw) \\
&\leqq c \int_{\partial B(x,r)} u(w) w_r^y(dw), \text{ by (3.1.5)}, \\
&\leqq c \int_{\partial B(x,r)} u(w) w_r^z(dw), \text{ by Harnack for } \tfrac{1}{2}\Delta, \\
&\leqq c \int_{\partial B(x,r)} u(w) H_r(w,z) w_r^z(dw), \text{ by (3.1.5)}, \\
&= cu(z).
\end{aligned}$$

Theorem 3.6. (*Boundary Harnack*). *Given $\delta, 0<\delta<1$ there exists a positive constant $c=c(\delta,b,D)$ such that whenever u and v are positive L-harmonic functions in D vanishing continuously on $\partial D\cap B(z,r)$ for some $z\in\partial D$, then*

$$\frac{u}{v}(x)\leqq c\frac{u}{v}(y),\ x,y\in B(z,\delta r)\cap D.$$

Proof. Select a Lipschitz subdomain $D_r\subset D$ with the same Lipschitz character as D and with $D\cap B(z,r)\supset D_r\supset D\cap B(z,\delta r)$. Let H_r be the H for D_r, and again H_r satisfies (3.1.5) without a change in the c there. Similarly, denote by w_r the harmonic measure for D_r. Thus for x, $y\in D\cap B(z,\delta r)$,

$$\begin{aligned}\frac{u}{v}(x)&=\frac{\int_{\partial D_r} u(w)H_r(x,w)w_r^x(dw)}{\int_{\partial D_r} v(w)H_r(x,w)w_r^x(dw)}\\ &\leqq c\frac{\int_{\partial D_r} u(w)w_r^x(dw)}{\int_{\partial D_r} v(w)w_r^x(dw)}\\ &\leqq c\frac{\int_{\partial D_r} u(w)w_r^y(dw)}{\int_{\partial D_r} v(w)w_r^y(dw)},\ \text{by Boundary Harnack for } \tfrac{1}{2}\Delta\\ &\leqq c\frac{u}{v}(y),\end{aligned}$$

by reinsertion of $H_r(y,w)$ in numerator and denominator at the expense of another constant.

Warning: The following result is not true in $d=2$ if we only assume $m(D)<\infty$.

Theorem 3.7. (*Martin Boundary*). *The minimal Martin boundary for L on D is the Euclidean boundary. Every positive L-harmonic function u on D has the unique representation*

$$u(x)=\int_{\partial D} K_L(x,z)\mu(dz)$$

for some unique Borel measure μ on ∂D and $K_L(x,z)=\lim_{y\to z}G_L(x,y)/G_L(x_0,y)$, $z\in\partial D$, is a Hölder continuous function of $z\in\partial D$. In addition,

$$K_L(x,z)=\frac{H(x,z)}{H(x_0,z)}K(x,z) \tag{3.7.1}$$

and there exists a positive constant $c=c(b,D)$ such that

$$c^{-1}K(x,z)\leqq K_L(x,z)\leqq cK(x,z). \tag{3.7.2}$$

Proof. This follows from Theorem 3.6 and our previous results using the techniques of Jerison, Kenig (1981).

Our next result says positive solutions to the two problems $Du=0$, $Lv=0$ vanish at the same rate at the boundary.

Theorem 3.8. *Suppose $W \subseteq \mathbb{R}^d$ is open and $K \subset W$ is compact. If u and v are positive harmonic and L-harmonic functions, respectively, on D both of which vanish continuously on $\partial D \cap W$, then u/v extends to a continuous function on $\partial D \cap K$.*

Proof. We require a lemma:

Lemma 3.9. *The function $H(x,\cdot)$ is continuous on $\bar{D}$ for each $x \in D$.*

Proof. Since $G(w,y)/G(x_0,y) \to K(w,z)$ as $y \to z \in \partial D$ uniformly on compacts it follows that $\nabla G(w,y)/G(x_0,y) \to \nabla K(w,z)$ on compacts. Thus for a.e. w, $\nabla G(w,y)/G(x_0,y) \to \nabla K(w,z)$. Now by (3.4.7) and the uniform integrability assumption (3.1.4), we have, taking the limit under the integral,

$$\lim_{y\to z} H(x,y) = 1 + \int_D \frac{G_L(x,w)b(w)\nabla K(w,z)}{K(x,z)}\,dw = H(x,z).$$

For continuity in D we only need the continuity of $\nabla G(w,y)/G(x,y)$ for $y \in D \setminus \{w\}$ and (3.1.4).

Proof (Theorem 3.8). Using Lemma 3.9 for $z \in K \cap \partial D, x \notin K$,

$$\lim_{y\to z} \frac{u}{v}(y) = \lim_{y\to z} \left[\frac{u(y)}{G(x,y)}\right]\left[\frac{G_L(x,y)}{v(y)}\right]\left[\frac{G(x,y)}{G_L(x,y)}\right],$$

and the limit of each factor exists, the first two by the Boundary Harnack and weak maximum principles for $\frac{1}{2}\Delta$ and L, the third by Lemma 3.9, and formula (3.4.2).

Theorem 3.10. *All points on ∂D are regular for the Dirichlet problem for L.*

Proof. A result of Brelot states that minimal Martin boundary points are regular for the Dirichlet problem. Theorem 3.7 says ∂D is equal to the minimal Martin boundary.

We can also prove Fatou's nontangential limit result for positive solutions of $Lu = 0$. Let $D = B(0,1)$ be the unit ball in $\mathbb{R}^d$ and for σ, $0 < \sigma < 1$, take $\Gamma_\sigma(z)$ to be the convex hull of $\{z\} \subset \partial D$ and $B(0,\sigma)$. Define $u^*(z) = \max_{x \in \Gamma_\sigma(z)} u(x)$ and for a positive finite Borel measure v on ∂D, set

$$\mathcal{M}_\mu(v)(z) = \sup_{r>0} \frac{v(\nabla(z,r))}{\mu(\nabla(z,r))},$$

where μ is a fixed positive finite Borel measure on ∂D and for $z \in \partial D$, $\nabla(z,r) = \partial D \cap B(z,r)$. Then we have

Theorem 3.11. *Suppose v is a finite positive Borel measure on $\partial B(0,1)$ and $u(x) = \int_{\partial B(0,1)} K_L(x,z)v(dz)$. Then there exists a positive $c = c(b)$ such that*

$$u^*(z) \leq c\mathcal{M}_{\omega_L^0}(v)(z), \quad z \in \partial B(0,1). \tag{3.11.1}$$

Furthermore, the nontangential limits $\lim_{y\in\Gamma_\sigma(z)}^{y\to z} u(y)$ *exist* a.s. *for each* $0 < \sigma < 1$.

Proof. Denote by v the harmonic function $v(x) = \int_{\partial B(0,1)} K(x,z)v(dz)$. Then by (3.7.2) there is a two-sided inequality $c^{-1}v^*(z) \leqq u^*(z) \leqq cv^*(z)$. Also, by (3.4.3) $c^{-1}\mathcal{M}_{\omega^0}(v)(z) \leqq \mathcal{M}_{\omega_L^0}(v)(z) \leqq c\mathcal{M}_{\omega^0}(v)(z)$ holds. It is classical that $v^*(z) \leqq c\mathcal{M}_{\omega^0}(v)(z)$, $z \in \partial B(0,1)$ and (3.11.1) follows from this and the two-sided inequalities above. The nontangential limit statement is a consequence of the maximal function inequality (3.11.1).

When ∂D is $C^{1,1}$ we can give analytic conditions under which (3.1.3) and (3.1.4) are satisfied.

Theorem 3.12. *If ∂D is $C^{1,1}$ then there exists a positive constant c, depending on D alone, such that*

$$\frac{G(x,w)G(w,y)}{G(x,y)} \leqq c\{|x-w|^{2-d} + |w-y|^{2-d}\}, \tag{3.12.1}$$

$$\frac{G(x,w)\,\|\nabla G(w,y)\|}{G(x,y)} \leqq c\{|x-w|^{1-d} + |w-y|^{1-d}\}, \tag{3.12.2}$$

$$\frac{G(x,w)\,\|\nabla K(w,z)\|}{K(x,z)} \leqq c\{|x-w|^{1-d} + |w-z|^{1-d}\}. \tag{3.12.3}$$

Proof. The inequality (3.12.1) is actually valid when ∂D is Lipschitz and is given in Cranston, Fabes, Zhao (1986). For (3.12.2) and (3.12.3) we use the following known estimates

$$G(x,w) \leqq \min\left(\frac{1}{|x-w|^{d-2}}, \frac{\delta(x)}{|x-w|^{d-1}}, \frac{\delta(x)\delta(w)}{|x-w|^{d}}\right), \text{ Widman (1967)}, \tag{3.12.4}$$

$$\|\nabla G(w,y)\| \leqq \min\left(\frac{\delta(y)}{|w-y|^{d}}, \frac{1}{|w-y|^{d-1}}\right), \text{Widman (1967)}, \tag{3.12.5}$$

$$\|\nabla K(w,y)\| \leqq \frac{1}{|w-y|^{d}}, \text{ Widman (1967)}, \tag{3.12.6}$$

$$G(x,y) \geqq \min\left(\frac{1}{|x-y|^{d-2}}, \frac{\delta(x)\delta(y)}{|x-y|^{d}}\right), \text{ Zhao (1986)}, \tag{3.12.7}$$

$$K(x,y) \geqq \frac{\delta(x)}{|x-y|^{d}}, \text{ Zhao (1984)}. \tag{3.12.8}$$

We prove only (3.12.3), that for (3.12.2) being similar. The proof is split into two cases.

Case 1. $|x-z| \leqq 2|w-z|$. By (3.12.3), (3.12.5) and (3.12.7),

$$\frac{G(x,w)\,\|\nabla K(w,z)\|}{K(x,z)} \leqq \frac{\delta(x)}{|x-w|^{d-1}}\,\frac{1}{|w-z|^{d}}\,\frac{|x-z|^{d}}{\delta(x)} \leqq \frac{2^d}{|x-w|^{d-1}}.$$

Case 2. $|x-z| > 2|w-z|$. Notice that in this case, since

$$|x-z| \leqq |x-w| + |w-z| \leqq |x-w| + \tfrac{1}{2}|x-z|,$$

it follows that

$$|x-z| \leqq 2|x-w|.$$

Thus, by (3.12.3), (3.12.5) and

$$\frac{G(x,w)\,\|\nabla K(w,z)\|}{K(x,z)} \leqq \frac{\delta(x)}{|x-w|^{d-1}}\,\frac{1}{|w-z|^{d}}\,\frac{|x-z|^{d}}{\delta(x)} \leqq \frac{2^{d}}{|w-z|^{d-1}}.$$

Combining Cases 1 and 2 gives (3.12.2).

Theorems 3.1, 3.2, and 3.12 give the following.

Theorem 3.13. *Suppose ∂D is $C^{1,1}$ and the vector field b on D satisfies both*

$$\limsup_{r\to 0}\ \sup_{x\in D} \int_{|x-y|<r} \frac{|b(y)|^{2}}{|x-y|^{d-2}}\,dy = 0, \tag{3.13.1}$$

and

$$\limsup_{r\to 0}\ \sup_{x\in D} \int_{|x-y|<r} \frac{|b(y)|}{|x-y|^{d-1}}\,dy = 0. \tag{3.13.2}$$

Then there is a constant c such that

$$c^{-1} \leqq H(x,y) \leqq c, \quad x,y\in\bar{D}. \tag{3.13.3}$$

Furthermore, the conclusions of Theorems 3.4 through 3.11 hold.

Corollary 3.14. *If ∂D is $C^{1,1}$ and $|b|\in L^{p}(D,m)$ for some $p>d$, then there is a constant c such that*

$$c^{-1} \leqq H(x,y) \leqq c.$$

Proof. Since $|b|\in L^{p}$ for some $p>d$, it follows that $|b|^{2}\in K_{d}^{\mathrm{loc}}$ (see Simon (1982)) which is equivalent, by the definition of K_{d}^{loc}, to saying that (3.13.1) is satisfied. For the proof that (3.13.2) is satisfied set $q=p/(p-1)$ and observe that $(1-d)q+d>(1-d)d/(d-1)+d=0$. Thus for any $x\in D$,

$$\begin{aligned}\int_{|y-x|\leqq r} \frac{|b(y)|}{|x-y|^{d-1}}\,dy &\leqq \left(\int_{D} |b(y)|^{p}dy\right)^{1/p} \left(\int_{|y-x|\leqq r} |y-x|^{(1-d)q}dy\right)^{1/q}\\ &\leqq \|b\|_{p}\left(c_{d}\int_{0}^{r} t^{(1-d)q+d-1}\,dt\right)^{1/q}\\ &= \|b\|_{p}\,c(d,q)\cdot r^{(1-d)q+d},\end{aligned}$$

and since $(1-d)q+d>0$, we see (3.13.2) holds.

Finally we can apply our results to extend the Conditional Gauge Theorem to operators of the form $L=\frac{1}{2}\Delta+b(\cdot)\cdot\nabla$ on Lipschitz domains D for which (3.1.2)–(3.1.4) hold. More specifically let μ be a Borel measure on D which is in $K_{d}^{\mathrm{loc}}(D)$, namely, $\lim_{r\to 0}\sup_{x\in D}\int_{|x-y|\leqq r} |\mu|(dy)/|x-y|^{d-2}=0$. Let A be the additive functional associated to μ and set $e_{\mu}(\tau_{D})=e^{-A_{\tau_D}}$. More specifically, A satisfies $E^{x}A_{\tau_D}=\int_{D} G_{L}(x,y)\mu(dy)$. Set $M=L-\mu$ and $F(x,y)=E_{y}^{x}e_{\mu}(\tau_{D})$.

Theorem 3.15. *If $F(x_0,y_0)<\infty$ for some $x_0\neq y_0$ $x_0,y_0\in\bar{D}$, then there exists a positive constant c such that*

$$c^{-1} \leqq F(x,y) \leqq c, \quad x,y\in\bar{D}. \tag{3.15.1}$$

Also the following identities and inequalities hold:

$$\omega_M^x(dz) = F(x,z)\omega_L^x(dz) = F(x,z)H(x,z)\omega^x(dz), \tag{3.15.2}$$

$$G_M(x,y) = F(x,y)G_L(x,y) = F(x,y)H(x,y)G(x,y), \tag{3.15.3}$$

$$c^{-1}\omega^x(dz) \leqq \omega_M^x(dz) \leqq c\omega^x(dz), \tag{3.15.4}$$

$$c^{-1}G(x,y) \leqq G_M(x,y) \leqq cG(x,y). \tag{3.15.5}$$

Furthermore, the conclusions of Theorems 3.5 through 3.11 hold for M.

Proof. The conclusions (3.15.4) and (3.15.5) follow from (3.15.1), (3.15.2), (3.15.3), (3.13.3), (3.4.3) and (3.4.4). The inequality (3.15.1) and identities (3.15.2) and (3.15.3) follow exactly as in Cranston, Fabes, Zhao (1986) once one knows the estimate

$$\frac{G_L(x,y)G_L(y,z)}{G_L(x,z)} \leqq c\left\{\frac{1}{|x-y|^{d-2}} + \frac{1}{|y-z|^{d-2}}\right\}$$

holds, where c depends on L and D. But in the above mentioned work it was shown that

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leqq c\left\{\frac{1}{|x-y|^{d-2}} + \frac{1}{|y-z|^{d-2}}\right\},$$

and $c^{-1}G_L(x,y) \leqq G(x,y) \leqq cG_L(x,y)$ by (3.4.4). Thus trivially G_L satisfies a similar estimate and everything follows.

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