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Fermions and Octonions

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Abstract. We analyse further the algebraic structure of dependent fermions, namely ones interrelated by the vertex operator construction. They are associated with special sorts of lattice systems which are introduced and discussed. The explicit evaluation of the relevant cocycles leads to the result that the operator product expansion of the fermions is related in a precise way to one or another of the division algebras given by complex numbers, quaternions or octonions. The latter case is seen to be realised in the light cone formalism of superstring theory.

1. Introduction

It has emerged that free massless Majorana fermions in a two-dimensional spacetime play an important role in various problems in theoretical physics controlled by conformal symmetry. In the fermionic string theory [1], multiplets of such fields occur which transform vectorially either under the Lorentz group $SO(9,1)$ or its light cone restriction $SO(8)$. In the heterotic string theory [2] there are fermion fields forming linear or non-linear representations of the gauge group [3]. In statistical physics, such fermions give a convenient description of certain two-dimensional models [4]. The Ising model can be discussed in terms of one such field [5]. Fermion fields also arise in the theory of solitons [6, 7].

The simplest situation occurs when the various real component fermi fields ψ_i are "independent" in that they anticommute when their suffices differ. This statement can be expressed via the operator product expansion as

$$\psi_i(z)\psi_j(\zeta)=\frac{z\delta_{ij}}{z-\zeta}+\text{regular}\ ,\qquad |z|>|\zeta|\ , \tag{1.1}$$

using notation common in string theory (see the review [8]). This framework is insufficiently general to encompass all situations of physical interest. For example, when Green and Schwarz [9] reformulated superstring theory [10] so as to render manifest the space-time supersymmetry, at least in the light cone gauge, they introduced fermi fields carrying an $SO(8)$ spinor index. Although satisfying (1.1) amongst themselves, these fields neither commuted nor anticommuted with the familiar Ramond-Neveu-Schwarz fields mentioned above as carrying a vector index. Thus the spinorial fields are not "independent" of the vector fields but "dependent" on them. Similar dependent fermi fields arise in the fermionic description of the $E_8 \times E_8$ gauge group of the heterotic string [3] and in statistical physics when "spin operators" occur [4].

The vertex operator construction of complex fermi fields, originally due to Skyrme [6, 11], furnishes a concrete way of understanding the possibilities and in particular the nature of this dependence. The clearest formulation is in terms of the Fubini-Veneziano field which describes the motions of a free bosonic string theory [12]:

$$Q^j(z) = q^j - ip^j \ln z + i \sum_{n \neq 0} \frac{\alpha_n^j}{n} z^{-n} , \tag{1.2}$$

yielding the "vertex operator",

$$\psi_\omega(z) = z^{\omega^2/2} : e^{i\omega \cdot Q(z)} : c_\omega , \tag{1.3}$$

which satisfies the hermiticity property

$$\psi_\omega(z) = \psi_{-\omega}(1/z^*) . \tag{1.4}$$

The vertex operator (1.3) provides a complex fermi field when [13, 14]

$$\omega^2 = 1 . \tag{1.5}$$

The real and imaginary parts of (1.3) yield two independent fermi fields satisfying (1.1). c_ω is the generalised "Klein transformation" [15] constructed out of p^j in (1.2) needed in general to correct certain signs.

In order to obtain $2r$ independent real fermi fields one considers r mutually perpendicular unit vectors $\omega_1, \omega_2, \dots, \omega_r$ and their negatives. These $2r$ unit vectors form the short roots of the Lie algebra $so(2r+1) = B_r$, while sums of perpendicular pairs form the long roots.

Two complex "dependent" fermions are each obtained from expression (1.3) with ω given by unit vectors ω_1 and ω_2. By the standard product identity of vertex operators [14] we find

$$\psi_{\omega_1}(z_1)\psi_{\omega_2}(z_2) = \varepsilon(\omega_1, \omega_2)(1 - z_2/z_1)^{\omega_1 \cdot \omega_2} [\psi_{\omega_1+\omega_2}(z_1) + O(z_1 - z_2)] , \quad z_1 > z_2 , \tag{1.6}$$

where ε denotes a cocycle (a complex number of unit modulus depending on ω_1 and ω_2). As z_1 tends to z_2, this vanishes if $\omega_1 \cdot \omega_2 > 0$, but if $\omega_1 \cdot \omega_2 < 0$ it defines a new field $\psi_{\omega_1+\omega_2}$. If this new field is to be fermionic, $\omega_1 + \omega_2$ must be a unit vector (1.5). Thus

$$\omega_1 \cdot \omega_2 = -1/2 . \tag{1.7}$$

It is easy to see that the six fermion fields $\psi_{\pm\omega_1}$, $\psi_{\pm\omega_2}$ and $\psi_{\pm(\omega_1+\omega_2)}$ form a closed set under the operator product expansion (1.6).

One can now seek to add fermi fields independent of ψ_{ω_1} but not of ψ_{ω_2}. By the above discussion such a field will be associated with a unit vector orthogonal to ω_1 but not ω_2. Repeating this we build up what we shall call an "orbit" of perpendicular unit vectors and their negatives including ω_1 with associated independent fermi fields. The same can be done for ω_2 and so on.

In a previous paper [16], of which this can be regarded as a continuation, we saw how to construct such "orbits" as subsets of the short roots of a simple Lie algebra g. g could have short roots of unit length and long roots of length $\sqrt{2}$, i.e. could be B_r, C_r or F_4. Alternatively g could have all roots of equal length, all regarded as short, i.e. of unit length. These constructions were used to construct level one representations of $\hat{B}_r$, $\hat{C}_r$, and $\hat{F}_4$, the associated affine Kac-Moody algebras [16, 17].

Part of the purpose of this paper is first to redevelop the results in a more systematic way, starting with the concept of a "Z_2 lattice pair" explained in Sect. 2. Families of independent fermions correspond to "orbits" of unit vectors. If there is more than one orbit each orbit can possess only 2, 4 or 8 elements suggesting a connection with the division algebras formed by complex numbers, quaternions and octonions respectively.

The main result of this paper is the construction of a relation between the operator product (1.6) and the product laws of these algebras, as explained in Sect. 5. The principal difficulty in this demonstration is the evaluation of the cocycle $\varepsilon(\omega_1, \omega_2)$ occurring in (1.6) and is solved in Sect. 4 making use of a special symmetric "gauge choice" for the cocycles, proven in Appendix B. This evaluation depends in turn on general results about the structure of the orbits and their interrelationships found in Sect. 3.

Thus we have related three apparently different concepts, dependent fermions, Z_2 lattice pairs and division algebras. There are also hints, mentioned in the conclusion, Sect. 6, of the relevance of Jordan algebras.

We have already mentioned the example of the superstring, when orbits with 8 elements enter. The exploitation of our discovery of the role played by octonions must surely further clarify our understanding of that theory which is the most attractive candidate for unifying particle interactions.

2. Z_2 Lattice Pairs and Dependent Fermions

Dependent fermions appeared naturally in the recently found construction of level one representations of affine untwisted Kac-Moody algebras $\hat{g}$ for which g is simple and has roots of two distinct lengths in the ratio $\sqrt{2}:1$, i.e. B_r, C_r or F_4 [16, 17]. Two sorts of dependent fermion could occur, ones associated with what we called "orbits" (certain subsets of short roots) and ones associated with the individual elements of these orbits. A unifying point of view was found for these constructions based on the following concept which we adopt here as our starting point [16]. We call $\Lambda_0 \subset \Lambda$ a "Z_2 lattice" pair if

$$\Lambda_0 \quad \text{and} \quad \sqrt{2}\Lambda \quad \text{are even lattices} \ ; \tag{2.1a}$$

$$\Lambda \subset \Lambda_0^* \quad \text{(the dual of } \Lambda_0) \ ; \tag{2.1b}$$

$$\Lambda/\Lambda_0 \quad \text{is isomorphic to } (Z_2)^n \text{ for some } n \ . \tag{2.1c}$$

It is easy to find examples of lattice pairs $\Lambda_0 \subset \Lambda$ satisfying any two of the three conditions (2.1) but not the third; thus the conditions are independent. Condition (2.1a) implies that Λ_0 and $\sqrt{2}\Lambda$ are integral lattices but Λ need not be.

We denote by Λ_a, $(0 \leqq a \leqq 2^n - 1)$, the 2^n cosets of Λ by Λ_0, implied by (2.1c). We can write

$$\Lambda_a = \lambda_a + \Lambda_0 \ , \tag{2.2}$$

where λ_a is a fixed element of Λ_a. It follows from (2.1a) and (2.1b) that λ_a^2 is an integer which is either odd or even independently of the choice of the representative λ_a. Accordingly we say that Λ_a is "odd" or "even".

Λ_a is itself a lattice only if $a=0$ but $\Lambda_0 \cup \Lambda_a$ is always a lattice and furthermore integral. [Condition (2.1c) implies that $\Lambda_0 + \Lambda_0 \subset \Lambda_0$, $\Lambda_0 + \Lambda_a \subset \Lambda_a$ and $\Lambda_a + \Lambda_a \subset \Lambda_0$; thus $\Lambda_0 \cup \Lambda_a$ is a lattice while integrality follows from (2.1a) and (2.1b).]

There are two Lie algebras associated with a Z_2 lattice pair. We denote by $\Lambda^{(1)}$ the set of vectors of length 1 in Λ and by $\Lambda_0^{(2)}$ those of length $\sqrt{2}$ in Λ_0:

$$\Lambda^{(1)} = \{\alpha \in \Lambda,\ \alpha^2 = 1\} \ ; \qquad \Lambda_0^{(2)} = \{\alpha \in \Lambda_0, \alpha^2 = 2\} \ . \tag{2.3}$$

Using properties (2.1) it is easy to verify that the sets $\Lambda^{(1)} \cup \Lambda_0^{(2)}$ and $\Lambda_0^{(2)}$ each satisfy the axioms of a root system. They therefore define Lie algebras $g(\Lambda, \Lambda_0)$ and $g(\Lambda_0)$ respectively, both of rank equal to the dimension of Λ, $\dim \Lambda$. Further

$$g(\Lambda_0) \cong g(\Lambda, \Lambda_0)_L \subset g(\Lambda, \Lambda_0) \ , \tag{2.4}$$

where g_L denotes the subalgebra of g of the same rank defined by the long roots of g. If $\Lambda_0^{(2)}$ is empty, $g(\Lambda_0) \cong u(1)^{\dim \Lambda}$, while if $\Lambda^{(1)}$ is empty, $g(\Lambda_0) \cong g(\Lambda, \Lambda_0)$. Z_2 lattice pairs exist such that $g(\Lambda, \Lambda_0)$ is any given simple Lie algebra except G_2 [16].

Since the dependent fermions are to be related to the points of $\Lambda^{(1)}$, we ignore the possibility that it be empty and divide it between the cosets Λ_a by defining

$$\Omega_a = \Lambda^{(1)} \cap \Lambda_a \ , \qquad a > 0 \ . \tag{2.5}$$

If Ω_a is not empty we call it an "orbit". By (2.1c), $\Lambda_a = -\Lambda_a$, and so, by (2.3),

$$\Omega_a = -\Omega_a \ . \tag{2.6}$$

So any point of Ω_a can be paired with its negative. Thus Ω_a comprises pairs of opposite unit vectors; and these are mutually orthogonal (because if ω and $\omega' \in \Omega_a$, $\omega \cdot \omega' \in Z$, as $\Lambda_0 \cup \Lambda_a$ is integral, and being unit vectors, $\omega \cdot \omega' = 0$ or ± 1 with $\omega = \pm \omega'$ in the latter case).

We shall henceforth assume that $g(\Lambda, \Lambda_0)$ is simple (and that $\Lambda^{(1)}$ is non-empty). Then, we show, the orbits are isomorphic and contain two elements if and only if $\Lambda_0^{(2)}$ is empty.

For, if $\Lambda_0^{(2)}$ is non-empty, $g(\Lambda_0)$ has a Weyl group which is an invariant subgroup of that of $g(\Lambda_0, \Lambda)$ by (2.4) [16]. These orbits are indeed the orbits of the short roots of $g(\Lambda, \Lambda_0)$ under the action of the Weyl group of $g(\Lambda_0)$ and necessarily have four or more elements.

If $\Lambda_0^{(2)}$ is empty, each orbit can only have two elements. For if not, we can find two orthogonal elements whose sum is automatically in $\Lambda_0^{(2)}$, contradicting the supposition that it is empty.

Because each orbit consists of $|\Omega|$ unit vectors occurring in orthogonal pairs, we can construct $|\Omega|$ independent real fermion fields for each orbit by the vertex operator construction (1.3). Whether or not fermions defined on different orbits are dependent or independent depends on the following relationship between the orbits. If Ω_a and Ω_b are distinct orbits (i.e. non-empty), Λ_a and Λ_b are "odd". If $\Lambda_a+\Lambda_b=\Lambda_c$ is also odd, Ω_c must be an orbit, i.e. non-empty as $2|\omega\cdot\omega'|$ must be odd, and hence one if $\omega\in\Omega_a$ and $\omega'\in\Omega_b$. Hence either $\omega+\omega'$ or $\omega-\omega'$ has length one and lies in Ω_c which cannot then be empty. If $\Lambda_a+\Lambda_b$ is "even" then $\omega\cdot\omega'$ vanishes.

If $g(\Lambda,\Lambda_0)=B_r$, there is just one orbit and $2r$ independent real fermions, a well understood situation which we shall henceforth ignore. We are left with $g(\Lambda,\Lambda_0)=F_4$, C_r or a simply laced algebra all of whose roots are considered as "short", that is of unit length. The number of elements in each orbit, $|\Omega|$, is 8, 4, and 2, respectively, numbers reminiscent of the division algebras formed by octonions, quaternions and complex numbers, respectively. This observation is enhanced by the construction of part of Freudenthal's magic square referring to the multiplication table of the aforementioned division algebras as follows. We say that two Z_2 lattice systems (possibly of different dimensions) "match" if they share a common value of n, (2.1c), so that the cosets can be put into one-to-one correspondence $\Lambda_a\leftrightarrow\Lambda_a'$ and that this correspondence can be chosen so that (a) it respects the $(Z_2)^n$ group properties, (b) it respects the even-oddness properties, and (c) such that Ω_a and Ω_a' are either both orbits or both empty.

Let us consider concrete cases: suppose we define

$$\Lambda=\Lambda_R(g)\ ,\qquad \Lambda_0=\Lambda_R(g_L)\ ,\qquad (g=B_r,C_r,F_4)\ ; \tag{2.7a}$$

$$\Lambda=\Lambda_R(g)/\sqrt{2}\ ,\qquad \Lambda_0=\sqrt{2}\Lambda_R(g)\ ,\qquad (g=A_r,D_r,E_r)\ . \tag{2.7b}$$

Then (Λ,Λ_0) is a Z_2 lattice pair (2.1) in each case with n equalling the number of short simple roots of g, i.e.

$$n=1,r-1,2\quad\text{for}\quad B_r,C_r,F_4\ , \tag{2.8a}$$

$$n=\operatorname{rank} g\equiv r\quad\text{for}\quad A_r,D_r,E_r\ . \tag{2.8b}$$

It is understood that $\Lambda_R(g)$ denotes the root lattice of g with the long roots of g normalised to have length $\sqrt{2}$. Because of the $\sqrt{2}$ divisor in Λ given by (2.7b), the simply laced g are thought of as having only short roots (of unit length).

Of the above Z_2 lattice pairs there are three for $n=2$ which match, namely those with $g=A_2$, C_3, and F_4. For higher n those with $g=A_n$ and C_{n+1} match. Since those for $g=D_r$ or E_r match only themselves we discard them henceforth.

Given a pair of matching Z_2 lattice pairs it is possible to construct an even lattice whose dimension is the sum of the dimensions of the Z_2 lattice pairs:

$$(\Lambda,\Lambda_0)\times(\Lambda',\Lambda_0')=\sum_{a=0}^{2^n-1}\Lambda_a\oplus\Lambda_a'\ . \tag{2.9}$$

The points of length $\sqrt{2}$ define a simply-laced algebra. Thus we can compile a multiplication table of Lie algebras. A simple calculation yields the following:

$n=2)$

	A_2	C_3	F_4
A_2	$A_2 \times A_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

$n=N-1>2)$

	$su(N)$	$sp(N)$
$su(N)$	$su(N) \times su(N)$	$su(2N)$
$sp(N)$	$su(2N)$	$so(4N)$

As pointed out in our previous paper [16], these tables reproduce part of Freudenthal's "magic square" associated with multiplying complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. This tallies with the fact that the orbits have order 2, 4, and 8 for $g=A_n$, C_{n+1}, and F_4 respectively. We shall understand the basis of this coincidence better in what follows and also see how to incorporate the reals.

3. Triality and Orbit Triples

The division algebras furnished by the complex numbers, quaternions and octonions will be realised by assigning fermionic vertex operators to the points of the orbits of order 2, 4 or 8 mentioned above in connection with $g(\Lambda, \Lambda_0)=A_n$, C_{n+1}, and F_4 respectively, and considering what we shall call a "triple of orbits", i.e. three orbits Ω_1, Ω_2, and Ω_3, say, lying in odd cosets related by

$$\Lambda_1+\Lambda_2+\Lambda_3=\Lambda_0 \ . \tag{3.1}$$

(Thus $\Lambda_i+\Lambda_{i+1}=\Lambda_{i+2}$, with i defined mod 3.) The operator product expansion (1.6) for the multiplication of two vertex operators associated with two of these orbits will define a vertex operator associated with the third orbit of the triple. If we can construct a satisfactory identification between the points of the three orbits we shall have effectively defined a multiplication law on each orbit itself, which will turn out to relate to the expected division algebra.

In this section we construct this identification, which we call "triality" and discuss its implications for the further structure of the individual orbits.

We expect this triality, τ, to be a linear map, of order 3, preserving Λ and Λ_0 and cyclically permuting Ω_1, Ω_2, and Ω_3, i.e.

$$\tau(\Lambda)=\Lambda \ , \quad \tau(\Lambda_0)=\Lambda_0 \ , \quad \tau^3=1 \ , \quad \tau(\Omega_i)=\Omega_{i+1} \ , \quad i \bmod 3 \ . \tag{3.2}$$

In particular τ is an automorphism of the root lattice of $g(\Lambda_0)$ but not in its Weyl group. It must correspond, therefore, to an automorphism of the Dynkin diagram of $g(\Lambda_0)$ if that group is not abelian [18]. When $g(\Lambda, \Lambda_0)=F_4$, $g(\Lambda_0)=D_4$ and its Dynkin diagram has a symmetry of order three cyclically permuting the three equal legs. Indeed this is often called triality. When $g(\Lambda, \Lambda_0)=C_r$, $g(\Lambda_0)=(A_1)^r$ and its Dynkin diagram consists of r points, the cyclic permutation of three of which gives again an order three symmetry. If $g(\Lambda, \Lambda_0)=A_r$, $g(\Lambda_0)$ is abelian and so has no Dynkin diagram.

We now give a general construction of τ, showing how it relates to the above diagram automorphisms in Appendix A. Let $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ with the sign of ω_2 chosen so that $\omega_1 \cdot \omega_2$ equals $-1/2$. Then $\omega_3 = -\omega_1 - \omega_2 \in \Omega_3$ and $\omega_i \cdot \omega_j = -1/2$, $i \neq j$. We define $\tau \in W(g(\Lambda, \Lambda_0))$, the Weyl group of $g(\Lambda, \Lambda_0)$, by

$$\tau = \sigma_{\omega_1} \sigma_{\omega_2} , \tag{3.3}$$

where σ_ω is the Weyl reflection in ω. Whenever $\omega \in \Lambda^{(1)}$, σ_ω preserves both Λ and Λ_0 by (2.1). So therefore does τ by (3.3). Hence τ permutes the cosets Λ_a. (In fact it permutes the odd ones amongst themselves and the even ones amongst themselves.) It is easy to check that $\tau(\omega_i) = \omega_{i+1}$ with i defined mod 3. From this follows the last property (3.2). τ is actually a rotation through $2\pi/3$ in the plane containing the ω_i and so has order 3 and satisfies (3.2).

We shall want τ to satisfy an additional property to (3.2), namely that it leave fixed a Weyl chamber, C, say, of $g(\Lambda_0)$ when that algebra is not abelian, i.e.

$$\tau(C) = C . \tag{3.4}$$

We now show that τ given by (3.3) can be modified to satisfy both (3.2) and (3.4). If τ does not already satisfy (3.4), let σ be the unique element of $W(g(\Lambda_0))$ for which $\sigma(C) = \tau^{-1}(C)$. Then $\tau' = \tau\sigma$ obviously satisfies (3.2) and (3.4) except possibly the condition $(\tau')^3 = 1$. Since $\tau W(g(\Lambda_0))\tau^{-1} = W(g(\Lambda_0))$, $(\tau')^3 = \tau^3 \sigma' = \sigma' \in W(g(\Lambda_0))$. But, by (3.4), $\sigma'(C) = C$, which implies that $\sigma' = 1$, thereby establishing the result. We henceforth call τ', τ and will show that it can also be written in the form (3.3) for suitable ω_1 and ω_2.

Let us choose C, the Weyl chamber of $g(\Lambda_0)$ fixed by τ, (3.4), as the positive one. Then if ϱ_L is the half sum of positive roots of $g(\Lambda_0)$,

$$\varrho_L = \sum_{\alpha > 0} \alpha/2 , \tag{3.5}$$

it is left invariant by τ, because of (3.4); that is,

$$\tau(\varrho_L) = \varrho_L . \tag{3.6}$$

We call $x \cdot \varrho_L$ the height of x, generalising the usual notion for the root system of $g(\Lambda_0)$:

$$\text{height } x = \varrho_L \cdot x . \tag{3.7}$$

By (3.6), x and $\tau(x)$ have the same height.

We now prove the result: each of the orbits Ω_1, Ω_2, and Ω_3 consist of two elements of zero height, summing to zero, and one and only one element of each height $\pm 1, \pm 2, \ldots, (|\Omega| - 2)/2$. Further, if $\omega \in \Omega_1$,

$$\omega \cdot \tau(\omega) = -1/2 \quad \text{if } \omega \text{ has zero height,} \tag{3.8a}$$

$$= 1/2 \quad \text{if } \omega \text{ has non-zero height.} \tag{3.8b}$$

The proof is in stages; first, two distinct elements of one orbit Ω_i have the same height (3.7) only if that height is zero and they sum to zero. For, if $\omega, \omega' \in \Omega_i$, $\omega \neq \omega'$, either $\omega \cdot \omega' = 0$ or $\omega + \omega' = 0$ by a previous result. In the former case, $\omega - \omega' \in \Lambda_0^{(2)}$, and so is a root of $g(\Lambda_0)$ whose height in the usual sense of $g(\Lambda_0)$, $\varrho_L \cdot (\omega - \omega')$, vanishes if height ω equals height ω'. As $g(\Lambda_0)$ has no roots of zero height, we must have $\omega + \omega' = 0$ which implies the common height is zero.

Now, if $\omega \in \Omega_1$, $\omega \cdot \tau(\omega) = 1/2$ or $-1/2$ only. But $[\omega + \tau(\omega) + \tau^2(\omega)]^2 = 6(\omega \cdot \tau(\omega) + 1/2)$. So if $\omega \cdot \tau(\omega) = -1/2$,

$$\omega + \tau(\omega) + \tau^2(\omega) = 0 \ , \tag{3.9}$$

which implies that ω has zero height as each term has the same height.

If $\omega \cdot \tau(\omega) = 1/2$, $\omega - \tau(\omega) \in \Omega_3$ and has height zero (thereby establishing that Ω_1 certainly has elements of zero height, $\pm e$, say). Thus

$$\omega - \tau(\omega) = \pm \tau^2(e) \ . \tag{3.10}$$

If $\omega = e$, this contradicts $\tau^3 = 1$, showing that ω has non-zero height, thereby establishing (3.8).

We now see that the elements of Ω_i must have integer heights since roots of $g(\Lambda_0)$ do. For, if $e \in \Omega_1$ has zero height and $\omega \in \Omega_1$ has non-zero height, $e \cdot \omega = 0$, and so $e + \omega \in \Lambda_0^{(2)}$ and is a root of $g(\Lambda_0)$ with the same height as ω. Also, if α is a simple root of $g(\Lambda_0)$ (and so has unit height) and is not orthogonal to e, $e \cdot \alpha = \pm 1$. So $\sigma_\alpha(e) = e \mp \alpha$ is an element of Ω_1 with height ∓ 1. Repeating this argument we complete the proof of the announced result.

As a consequence there is a striking geometrical structure to the orbits Ω_1, Ω_2, and Ω_3, common to the three values of $|\Omega|$. If we henceforth let e denote one of the two elements of Ω_1 with zero height, we see that e, $\tau(e)$ and $\tau^2(e)$ are coplanar, (3.9), and make angles $2\pi/3$ with each other. If $f \in \Omega_1$ has non-zero height, f, $\tau(f)$ and $\tau^2(f)$ are not coplanar but they subtend the same angles $2\pi/6$ with each other irrespective of their common height.

In particular, τ is given by (3.3) with $\omega_1 = e$, $\omega_2 = \tau(e)$. These results are crucial to obtaining the division algebra structures. Besides (3.9) satisfied by e and (3.10), there are other linear relations between Ω_1, Ω_2, and Ω_3 arising when $|\Omega| = 8$, and hence needed for the octonion analysis.

In this case there are 8 vectors in Ω_1, denoted $\pm e$, $\pm f_1$, $\pm f_2$, and $\pm f_3$ with the height of f_n being n. As $f_3 + \tau(f_1)$ has height 4, $f_3 \cdot \tau(f_1) = 1/2$, since if it was $-1/2$ we would have an element of Ω_3 with height 4, contradicting the structural result above. Therefore $f_3 - \tau(f_1) \in \Omega_3 = \tau^2(\Omega_1)$, and has height 2. Hence

$$f_3 - \tau(f_1) = \tau^2(f_2) \ . \tag{3.11}$$

Likewise

$$f_3 - \tau(f_2) = \tau^2(f_1) \ , \tag{3.12}$$

again using the "height conserving" property.

Choosing $\omega = f_n$ in (3.10),

$$f_n - \tau(f_n) = \xi_n \tau^2(e) \ , \quad n = 1, 2, 3 \ , \tag{3.13}$$

where ξ_1, ξ_2, and ξ_3 take values ± 1. We choose the sign of e so that $\xi_1 = 1$. Applying

$1-\tau$ to (3.11) and using (3.13) we obtain $\xi_3\tau^2(e)-e=\xi_2\tau(e)$. Comparing with (3.9) satisfied by $\omega=e$, we obtain

$$\xi_1=\xi_2=-\xi_3=1 \ . \tag{3.14}$$

This completes the list of linear relations.

In Appendix A we give explicit constructions of Ω_1, Ω_2, and Ω_3 for the three cases.

4. Evaluation of Cocycles Occurring in the Fermionic OPE

We assign vertex operators

$$\psi_\omega(z)=\sqrt{z}:e^{i\omega\cdot Q(z)}:c_\omega \tag{4.1}$$

to each element ω of Ω_1, Ω_2, and Ω_3. c_ω is a generalised "Klein transformation" [15]. Since $\Omega_a=-\Omega_a$, (2.6), there is also a vertex operator $\psi_{-\omega}$ which we take to be hermitian conjugate to ψ_ω:

$$\psi_{-\omega}(z)=\psi_\omega(1/z^*)^\dagger \ . \tag{4.2}$$

The presence of c_ω ensures that the ψ_ω, $\omega\in\Omega_1$, say, define $|\Omega|$ real independent fermion fields (i.e. mutually anticommuting) (see [8]). We have shown [16] that c_ω can be constructed for any $\omega\in\Lambda$ satisfying:

$$e^{i\alpha\cdot q}c_\alpha e^{i\beta\cdot q}c_\beta=S(\alpha,\beta)e^{i\beta\cdot q}c_\beta e^{i\alpha\cdot q}c_\alpha=\varepsilon(\alpha,\beta)e^{i(\alpha+\beta)\cdot q}c_{\alpha+\beta} \ , \tag{4.3}$$

for any $\alpha,\beta\in\Lambda$. $S(\alpha,\beta)$ is called the "symmetry factor" and $\varepsilon(\alpha,\beta)$ the "cocycle". Both are of unit modulus, and possibly complex. The associativity of the multiplication in (4.3) implies certain consistency conditions to be satisfied by the symmetry factors and cocycles;

$$S(\alpha,\alpha)=1 \ , \qquad S(\alpha,\beta)S(\beta,\alpha)=1 \ , \tag{4.4a}$$

$$S(\alpha,\beta+\gamma)=S(\alpha,\beta)S(\alpha,\gamma) \ , \tag{4.4b}$$

$$\varepsilon(\alpha,\beta)=S(\alpha,\beta)\varepsilon(\beta,\alpha) \ , \tag{4.5a}$$

$$\varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma)=\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) \ . \tag{4.5b}$$

Once $S(\alpha,\beta)$ is assigned for all $\alpha,\beta\in\Lambda$ the cocycle is unique up to a "gauge freedom" involving a discrete gauge transformation at each point of Λ:

$$\varepsilon(\alpha,\beta)\to\varepsilon'(\alpha,\beta)=\varepsilon(\alpha,\beta)\eta(\alpha)\eta(\beta)/\eta(\alpha+\beta) \ , \tag{4.6}$$

where $\eta(\alpha)$ is a phase factor. We can use this gauge freedom to ensure that:

$$\varepsilon(\alpha,0)=\varepsilon(0,\alpha)=\varepsilon(\alpha,-\alpha)=1 \ , \tag{4.7}$$

thereby guaranteeing (4.2). We then find, quite generally, from the cocycle condition (4.5b), that the cocycle satisfies the cyclic symmetry property

$$\varepsilon(\alpha,\beta)=\varepsilon(\beta,\gamma)=\varepsilon(\gamma,\alpha) \quad \text{when} \quad \alpha+\beta+\gamma=0 \ , \tag{4.8}$$

and the hermiticity property

$$\varepsilon(\alpha, \beta)^* = \varepsilon(-\beta, -\alpha) \ . \tag{4.9}$$

For any lattice Λ satisfying (2.1) a consistent assignment of symmetry factors was given in our previous paper [16]. If α and β are in the same lattice $\Lambda_0 \cup \Lambda_a$,

$$S(\alpha, \beta) = (-1)^{\alpha\cdot\beta + \alpha^2\beta^2} \ , \quad \alpha, \beta \in \Lambda_0 \cup \Lambda_a \ , \quad a = 1, 2, \ldots 2^{n-1} \ . \tag{4.10}$$

This ensures that the vertex operators (4.1), $\omega \in \Omega_a$, indeed define $|\Omega|$ independent real fermion fields.

If α and β lie in distinct cosets Λ_a and Λ_b, $a \neq b \neq 0$, it follows from (4.4), (2.1), and (4.10) that

$$S(\alpha, \beta)^2 = S(\alpha, 2\beta) = (-1)^{2\alpha\cdot\beta} \ . \tag{4.11}$$

Thus if one or all three of $\Lambda_a, \Lambda_b, \Lambda_a + \Lambda_b$ are odd, $S(\alpha, \beta)$ is imaginary. In particular, $S(\alpha, \beta)$ is imaginary if α and β lie in two distinct elements of the triple (3.1).

The operator product expansion (1.6) for $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, say, has a singularity like $(z_1 - z_2)^{-1/2}$ if $\omega_1 + \omega_2 \in \Omega_3$ and its coefficient is proportional to $\psi_{\omega_1 + \omega_2}$. This defines the multiplication law we want to study and part of the structure constant is the cocycle $\varepsilon(\omega_1, \omega_2)$. This section is devoted to showing that there exists a particularly convenient gauge choice (4.6) in which we can evaluate all the cocycles entering the fermion operator product expansion.

The basic idea involved in this choice is to observe that the symmetry factors inherit certain symmetry properties of the lattice Λ, namely

$$S(-\alpha, -\beta) = S(\alpha, \beta) \ ; \quad \alpha, \beta \in \Lambda \ ; \tag{4.12a}$$

$$S(\tau(\alpha), \tau(\beta)) = S(\alpha, \beta) \ , \quad \alpha, \beta \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \ , \tag{4.12b}$$

by virtue of (4.4), where τ is the triality of the previous section. It is shown in Appendix B that it is possible to find a gauge (4.6) in which the cocycles also inherit the same symmetries in addition to (4.7),

$$\varepsilon(-\alpha, -\beta) = \varepsilon(\alpha, \beta) \ , \quad \alpha, \beta \in \Lambda \ ; \tag{4.13a}$$

$$\varepsilon(\tau(\alpha), \tau(\beta)) = \varepsilon(\alpha, \beta) \ , \quad \alpha, \beta \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \ . \tag{4.13b}$$

Property (4.12a) follows immediately from (4.4) while (4.12b) follows from (4.10) and the cyclic symmetry property of S corresponding to (4.8).

Let us note an immediate consequence of the gauge choice (4.13a) with (4.9) and (4.5a):

$$\varepsilon(\alpha, \beta)^2 = S(\alpha, \beta) \ . \tag{4.14}$$

This means that the remaining gauge ambiguity (4.6) in the cocycle is a sign which can be further chosen to satisfy (4.13b), as shown in Appendix B. It also shows that when Λ satisfies (2.1) the values taken by the cocycle in the gauge (4.13a) are ± 1, $\pm i$ and $(\pm 1 \pm i)/\sqrt{2}$.

The cocycles needed for the operator product expansion of fermions are of the form $\varepsilon(\omega, \tau(\omega'))$, where ω, ω', and ω'' are all elements of Ω_1 and

$$\omega + \tau(\omega') + \tau^2(\omega'') = 0 \ . \tag{4.15}$$

The linear relation (4.15) implies further linear relations between Ω_1, Ω_2, and Ω_3 (possibly the same), namely

$$\omega' + \tau(\omega'') + \tau^2(\omega) = 0 \ , \qquad \omega'' + \tau(\omega) + \tau^2(\omega') = 0 \ . \tag{4.16}$$

By the cyclic symmetry property (4.8) and the invariant gauge choice (4.13) we have

$$\begin{aligned} &\varepsilon(\omega, \tau(\omega')) = \varepsilon(\omega', \tau(\omega'')) = \varepsilon(\omega'', \tau(\omega)) \\ &\qquad = \varepsilon(-\omega, -\tau(\omega')) = \varepsilon(-\omega', -\tau(\omega'')) = \varepsilon(-\omega'', -\tau(\omega)) \ . \end{aligned} \tag{4.17}$$

Thus six of the cocycles we are interested in are equal.

We have already found all linear relations of this type in the previous section. For example, we had from (3.9)

$$e + \tau(e) + \tau^2(e) = 0 \ . \tag{4.18}$$

Hence, by (4.17)

$$\varepsilon(e, \tau(e)) = \varepsilon(-e, -\tau(e)) = \varepsilon_0 \ , \tag{4.19}$$

say, as (4.18) is τ invariant, If $|\Omega| \geqq 4$, (3.13) and (4.17) yield

$$\begin{aligned} &\varepsilon(f_n, -\tau(f_n)) = \varepsilon(-f_n, -\xi_n\tau(e)) = \varepsilon(-\xi_n e, \tau(f_n)) \\ &\qquad = \varepsilon(-f_n, \tau(f_n)) = \varepsilon(f_n, \xi_n\tau(e)) = \varepsilon(\xi_n e, -\tau(f_n)) = \varepsilon_n \ , \end{aligned} \tag{4.20}$$

say. Of course $n=1$ if $|\Omega|=4$ and 1, 2 or 3 if $|\Omega|=8$. We now relate (4.19) and (4.20) by showing that

$$\varepsilon_n = -\varepsilon_0 \ , \qquad n=1 \quad \text{or} \quad n=1, 2 \text{ or } 3 \ . \tag{4.21}$$

In the cocycle condition (4.5b) insert $\alpha = \xi_n e$, $\beta = \xi_n\tau(e)$ and $\gamma = f_n$ so that $\alpha + \beta = -\xi_n\tau^2(e)$ and $\beta + \gamma = \tau^2(f_n)$ to get, using (4.13b), (4.19), and (4.20),

$$\varepsilon_0\varepsilon_n = \varepsilon(\xi_n e, \tau^2(f_n))\,\varepsilon(\xi_n\tau(e), f_n) = \varepsilon(\xi_n\tau(e), f_n)^2$$

which, by (4.5a),

$$= S(\xi_n\tau(e), f_n)^2\,\varepsilon_n^2 = S(\xi_n\tau(e), 2f_n)\,\varepsilon_n^2 \ .$$

Finally, by (4.11), this equals $-\varepsilon_n^2$, which establishes (4.21).

Thus, if $|\Omega| = 2$ or 4 there is only one independent cocycle in the fermion operator product expansion between Ω_1 and Ω_2. When $|\Omega|=8$, so that octonions will be involved, there are two further linear relations, namely (3.11) and (3.12). By the same method we find one further independent cocycle, ε', associated with them so that

$$\begin{aligned} \varepsilon' &= \varepsilon(f_1, \tau(f_2)) = \varepsilon(f_2, -\tau(f_3)) = \varepsilon(-f_3, \tau(f_1)) \\ &= \varepsilon(-f_1, -\tau(f_2)) = \varepsilon(-f_2, \tau(f_3)) = \varepsilon(f_3, -\tau(f_1)) \\ &= -\varepsilon(f_2, \tau(f_1)) = -\varepsilon(f_1, -\tau(f_3)) = -\varepsilon(-f_3, \tau(f_2)) \\ &= -\varepsilon(-f_2, -\tau(f_1)) = -\varepsilon(-f_1, \tau(f_3)) = -\varepsilon(f_3, -\tau(f_2)) \ . \end{aligned} \tag{4.22}$$

That there are relations between the cocycles as above could have been anticipated by observing that the ratios of such cocycles are invariant with respect to the gauge transformations (4.6) when $\eta(\alpha)^2=1$ by virtue of the gauge choice (4.13).

5. Division Algebras and the Fermionic Operator Product Expansion

We consider the division algebras formed by the complex numbers, quaternions and octonions respectively [19]. Elements are real linear combinations of the unit, u_0, and the imaginary units u_1 (complex numbers), u_1, u_2, and u_4 (quaternions) and u_1, u_2, u_3, u_4, u_5, u_6, and u_7 (octonions). Thus

$$z=u_0 a_0+\sum_I u_\alpha a_\alpha \ , \tag{5.1}$$

where I denotes the relevant set of imaginary units and the coefficients a are real numbers. This reality property is preserved under multiplication defined by

$$u_0^2=u_0 \ , \qquad u_\alpha u_\beta=-u_0\delta_{\alpha\beta}+f_{\alpha\beta\gamma}u_\gamma \ , \tag{5.2a}$$

$$u_0 u_\alpha=u_\alpha u_0=u_\alpha \ , \tag{5.2b}$$

since the structure constants are real. $f_{\alpha\beta\gamma}$ is totally antisymmetric and real. Its non-zero elements are specified by

$$f_{\alpha,\alpha+1,\alpha+3}=1 \ . \tag{5.3}$$

Thus, for complex numbers, f vanishes, while for quaternions, it is the $su(2)$ structure constant. The multiplication defined by (5.2) and (5.3) fails to be associative only for octonions.

Given the reality of the coefficients a there is a complex conjugation operation defined by

$$\bar{z}=u_0 a_0-\sum_I u_\alpha a_\alpha \ , \tag{5.4}$$

according to which the imaginary units justify their name.

Because f is antisymmetric, we see from (5.2) that

$$\overline{z_1 z_2}=\bar{z}_2\bar{z}_1 \ . \tag{5.5}$$

Our aim is to demonstrate a sense in which these properties (5.1)–(5.5) can be realised by the fermionic vertex operators (4.1) defined on the three orbits Ω_1, Ω_2, and Ω_3 with $|\Omega|=2, 4$ or 8. The first step is to identify the reality property preserved by the operator product expansion, $\omega_1\in\Omega_1$, $\omega_2\in\Omega_2$,

$$\begin{aligned}\psi_{\omega_1}(z)\psi_{\omega_2}(\zeta)&=\frac{\varepsilon(\omega_1,\omega_2)}{\sqrt{(1-\zeta/z)}}\,[\psi_{\omega_1+\omega_2}(z)+O(z-\zeta)] \ , &&\text{if}\quad \omega_1+\omega_2\in\Omega_3\\ &=\frac{O(z-\zeta)}{\sqrt{(1-\zeta/z)}} &&\text{if}\quad \omega_1+\omega_2\notin\Omega_3 \ .\end{aligned} \tag{5.6}$$

The snag is that the cocycle $\varepsilon(\omega_1, \omega_2)$ is, as we have seen, necessarily complex. The most general linear combination of fermionic vertex operators associated with the orbit Ω_1 is

$$\sum_{\omega \in \Omega_1} A_1(\omega)\psi_\omega(z) \ , \tag{5.7}$$

where the coefficients $A_1(\omega)$ are complex numbers. The reality condition analogous to the reality of the coefficients a in (5.1) is

$$A_1(\omega)^* = A_1(-\omega) \ . \tag{5.8}$$

This makes sense because $\Omega_1 = -\Omega_1$ by (2.6). By (5.6), the most singular part of the operator product expansion of two expressions (5.7) defined on Ω_1 and Ω_2 is given by

$$\left[\sum_{\omega_1 \in \Omega_1} A_1(\omega_1)\psi_{\omega_1}(z)\right]\left[\sum_{\omega_2 \in \Omega_2} A_2(\omega_2)\psi_{\omega_2}(\zeta)\right]$$
$$= \frac{e^{i\phi}}{\sqrt{(1-\zeta/z)}}\left[\sum_{\substack{\omega_1 \in \Omega_1 \\ \omega_2 \in \Omega_2 \\ \omega_1+\omega_2 \in \Omega_3}} e^{-i\phi}\varepsilon(\omega_1, \omega_2)A_1(\omega_1)A_2(\omega_2)\psi_{\omega_1+\omega_2}(z) + O(z-\zeta)\right] . \tag{5.9}$$

Thus the operator product expansion has defined a third expression of the type (5.7) on Ω_3 and we have to check that there is a choice of the phase ϕ introduced in (5.9) such that the reality condition is preserved. Comparing (5.7) and (5.9) the new coefficients inside the square brackets on the right of (5.9) are

$$A_3(\omega_3) = e^{-i\phi} \sum_{\substack{\omega_1 \in \Omega_1 \\ \omega_2 \in \Omega_2 \\ \omega_1+\omega_2 = \omega_3}} \varepsilon(\omega_1, \omega_2)A_1(\omega_1)A_2(\omega_2) \ . \tag{5.10}$$

Complex conjugating this expression (5.10), assuming the reality condition (5.8) for A_1 and A_2, we find

$$A_3(-\omega_3)^* = e^{i\phi} \sum_{\substack{\omega_1 \in \Omega_1 \\ \omega_2 \in \Omega_2 \\ \omega_1+\omega_2 = \omega_3}} \varepsilon(-\omega_1, -\omega_2)^* A_1(\omega_1)A_2(\omega_2) \ .$$

This equals $A_3(\omega_3)$, (5.10), providing

$$e^{-i\phi}\varepsilon(\omega_1, \omega_2) = e^{i\phi}\varepsilon(-\omega_1, -\omega_2)^* = e^{i\phi}\varepsilon(\omega_2, \omega_1) \ , \tag{5.11}$$

using (4.9). This has to hold for all ω_1 and ω_2 occurring in the sum (5.10). But, by (4.5a), (5.11) equals

$$e^{i\phi}S(\omega_2, \omega_1)\varepsilon(\omega_1, \omega_2) \ ,$$

so that the requirement is, using (4.4), that

$$e^{2i\phi} = S(\omega_1, \omega_2) \ . \tag{5.12}$$

This can be satisfied as $S(\omega_1, \omega_2)$ is indeed independent of the choice of ω_1 and ω_2 in the sum (5.10), by the properties (4.4) and (4.10). Thus, if we choose the phase ϕ in

(5.10) to be given by (5.12), the product defined by (5.10) does preserve the reality condition (5.8).

So far we have taken no advantage of the symmetric gauge choice (4.13) for the cocycles. We shall adopt this choice (4.13) henceforth and note that, by virtue of (4.14) and (5.12), the quantities (5.11) take values $+1$ or -1 only. We therefore define

$$\eta(\omega_1, \omega_2) = e^{-i\phi}\varepsilon(\omega_1, \omega_2) \;, \tag{5.13}$$

taking values ± 1 when $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $\omega_1 + \omega_2 \in \Omega_3$.

It is now useful to make some changes in notation. We shall define a new product, denoted "$\circ$", as the most singular part of the operator product expansion (5.6) or (5.9), but incorporating the phase $e^{-i\phi}$. Thus, for $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, (5.6) now reads

$$\psi_{\omega_1} \circ \psi_{\omega_2} = \eta(\omega_1, \omega_2)\psi_{\omega_1+\omega_2} \;, \quad \text{if} \quad \omega_1 + \omega_2 \in \Omega_3 \;, \tag{5.14a}$$

$$= 0 \qquad \text{otherwise} \;. \tag{5.14b}$$

We further redefine $\psi_\omega = \psi(\omega)$ in the left-hand factor of the $\circ$ product and $\psi_\omega = \psi(\tau^{-1}(\omega))$ in the right-hand factor of the $\circ$ product. Finally, in the overall result on the right-hand side of (5.14) we define $\psi_\omega = \psi(\tau^{-2}(\omega))$. Thus we reexpress (5.14) in terms of vectors ω, ω', and ω'' all in the same orbit Ω_1:

$$\psi(\omega) \circ \psi(\omega') = \eta(\omega, \tau(\omega'))\psi(-\omega'') \;, \quad \text{if} \quad \omega + \tau(\omega') \in \Omega_3 \tag{5.15a}$$

$$= 0 \qquad \text{if} \quad \omega + \tau(\omega') \notin \Omega_3 \;, \tag{5.15b}$$

ω'' is defined by the linear relation (4.15).

The quantities ψ are not real according to our definition (5.8), and so we introduce real linear combinations of them:

$$V_+(\omega) = \psi(\omega) + \psi(-\omega) = V_+(-\omega) \;, \tag{5.16a}$$

$$V_-(\omega) = i(\psi(\omega) - \psi(-\omega)) = -V_-(-\omega) \;, \tag{5.16b}$$

essentially twice the cosine and sine of $\omega \cdot Q$ respectively.

Whether Ω_1 has 2, 4 or 8 elements, it possesses two of zero height, $\pm e$. Using the linear relation (4.18), $e + \tau(e) + \tau^2(e) = 0$, and choosing the phase, ϕ, so that

$$\eta_0 \equiv e^{-i\phi}\varepsilon_0 = 1 \;, \tag{5.17}$$

we find, from (5.15) and (5.16) that

$$V_+(e) \circ V_+(e) = V_+(e) \;, \tag{5.18a}$$

$$V_+(e) \circ V_-(e) = V_-(e) \circ V_+(e) = -V_-(e) \;, \tag{5.18b}$$

$$V_-(e) \circ V_-(e) = -V_+(e) \;. \tag{5.18c}$$

Since this is a two dimensional algebra, we expect it to relate to the algebra of complex numbers generated by u_0 and u_1 satisfying (5.1), (5.2), and (5.3). Equation (5.18a) indicates that $V_+(e)$ is a unit element, and (5.18c) that $V_-(e)$ is an imaginary unit. This suggests that we identify

$$V_+(e) \leftrightarrow u_0 \ , \tag{5.19a}$$

$$V_-(e) \leftrightarrow u_1 \ . \tag{5.19b}$$

However there is then a wrong sign on the right-hand side of (5.18b). As this is the only occurrence of the imaginary unit on the right-hand side of (5.18) this sign can be rectified by introducing the complex conjugation (5.4). Thus we find that with the correspondence (5.19)

$$u_\lambda \circ u_\mu = \overline{u_\lambda u_\mu} \ , \tag{5.20}$$

at least for λ, $\mu = 0, 1$. This relation between the operator product "$\circ$" and the division algebra product (5.2) will be our main result when we have extended the correspondence (5.19) to all elements of Ω when there are four or eight of them.

The three products in (5.18b) and (5.18c) are related in that the structure constant is cyclically symmetric in the sense that if $a \circ b = \eta c$, then $b \circ c = \eta a$ and $c \circ a = \eta b$. This is a general feature of the "$\circ$" product as we now check by considering the general linear relation (4.15) and its corollaries (4.16).

If $\eta = e^{-i\phi}\varepsilon(\omega, \tau(\omega'))$, we find, using (5.15) and (5.16)

$$V_+(\omega) \circ V_+(\omega') = \eta V_1(\omega'') \ , \tag{5.21a}$$

$$V_+(\omega') \circ V_+(\omega'') = \eta V_+(\omega) \ , \tag{5.21b}$$

$$V_+(\omega'') \circ V_+(\omega) = \eta V_+(\omega') \ . \tag{5.21c}$$

In addition we also find

$$V_-(\omega) \circ V_-(\omega') = -\eta V_+(\omega'') \ , \tag{5.22a}$$

$$V_-(\omega') \circ V_+(\omega'') = -\eta V_-(\omega) \ , \tag{5.22b}$$

$$V_+(\omega'') \circ V_-(\omega) = -\eta V_-(\omega') \ . \tag{5.22c}$$

Further

$$V_+(\omega) \circ V_-(\omega') = -\eta V_-(\omega'') \ , \tag{5.23a}$$

$$V_-(\omega') \circ V_-(\omega'') = -\eta V_+(\omega) \ , \tag{5.23b}$$

$$V_-(\omega'') \circ V_+(\omega) = -\eta V_-(\omega') \ . \tag{5.23c}$$

Finally,

$$V_-(\omega) \circ V_+(\omega') = -\eta V_-(\omega'') \ , \tag{5.24a}$$

$$V_+(\omega') \circ V_-(\omega'') = -\eta V_-(\omega) \ , \tag{5.24b}$$

$$V_-(\omega'') \circ V_-(\omega) = -\eta V_+(\omega') \ . \tag{5.24c}$$

This exhausts the possible "$\circ$" products associated with the linear relation (4.15). We see that they fall into groups of three related by the cyclic symmetry mentioned above. These results will facilitate our verification that the operator product denoted "$\circ$" relates to the division algebras (5.2) via (5.20).

We now replace the general linear relation (4.15) by the special cases (3.13) and (3.14) involving elements $\pm f_n$ of Ω_1 with height $\pm n$, and e. Equations (5.21) and

(5.22) yield respectively

$$V_+(f_n)\circ V_+(f_n) = -V_+(e) \quad \text{and cyclic permutations} \;, \tag{5.25}$$

$$V_-(f_n)\circ V_-(f_n) = -V_+(e) \quad \text{and cyclic permutations} \;. \tag{5.26}$$

The structure constant in all these equations is $e^{-i\phi}\varepsilon_n$, which by (4.21) and our choice of ϕ, is equal to $-e^{-i\phi}\varepsilon_0 = -1$, as quoted. Thus $V_+(f_n)$ and $V_-(f_n)$ both have square $-V_+(e)$ and hence behave like imaginary units. Equations (5.23) and (5.24) further yield

$$V_+(f_n)\circ V_-(f_n) = -\xi_n V_-(e) \quad \text{and cyclic permutations} \;, \tag{5.27}$$

$$V_-(f_n)\circ V_+(f_n) = -\xi_n V_-(e) \quad \text{and cyclic permutations} \;, \tag{5.28}$$

where ξ_n is the sign specified by (3.14). Thus the structure constant for $V_+(f_n)$, $V_-(f_n)$ and $V_-(e)$ is totally antisymmetric. If we identify

$$u_2 \leftrightarrow V_-(f_1) \;, \qquad u_4 \leftrightarrow V_+(f_1) \;, \tag{5.29}$$

$$u_5 \leftrightarrow V_-(f_2) \;, \qquad u_6 \leftrightarrow V_+(f_2) \;, \qquad u_3 \leftrightarrow V_+(f_3) \;, \qquad u_7 \leftrightarrow V_-(f_3) \;, \tag{5.30}$$

and extend (5.20), we find that (u_0, u_1, u_2, u_4), (u_0, u_1, u_3, u_7) and (u_0, u_1, u_5, u_6) each furnish sets of quaternions as $f_{124} = f_{137} = f_{156} = 1$, in agreement with (5.3). In particular, if the orbit Ω_1 possesses four elements, they must be $\pm e$ and $\pm f_1$ and we have, by (5.29), recovered the quaternions (u_0, u_1, u_2, u_4).

When Ω_1 possesses 8 elements, there remain two more linear relations (3.11) and (3.12), with associated cocycles (4.22). Applying (5.21), (5.22), (5.23), and (5.24) and the identifications (5.29) and (5.30) to (3.11) we find that the following structure constants are cyclically symmetric and take the values below:

$$f_{346} = f_{457} = f_{267} = f_{235} = -e^{-i\phi}\varepsilon' = -\eta' \;.$$

Repeating the analysis for (3.12), we find that these structure constants are totally antisymmetric. We have enough gauge freedom (4.6) left to choose $-\eta' = 1$, so that (5.3) is satisfied. Thus the fermionic vertex operators relate to octonions via (5.20) and the correspondence (5.19), (5.29), and (5.30).

6. Discussion and Conclusion

We have considered the fermionic vertex operators associated with a triple of orbits as defined in Sects. 2 and 3 and have shown that there is a relation between the most singular part of their operator product expansion which defines a product denoted "$\circ$", (5.15), and the division algebras whose dimension is the order $|\Omega|$ of these orbits. This connection is expressed via

$$z_1 \circ z_2 = \overline{z_1 z_2} \;, \tag{6.1}$$

and the identifications (5.19), (5.29), and (5.30). Equation (6.1) is our main result and most of the work of this paper concerned the determination of the cocycles arising on the left-hand side and the demonstration that they correctly related to the structure constants of the division algebras via (6.1).

We want to make several comments of a mathematical nature about the result (6.1). The elements z of a division algebra possess a norm $\|z\|$ which is the same for z and $\bar{z}$. It follows from (6.1) that the quantities (5.7) possess a norm which is multiplicative under (5.10). This is

$$\sqrt{\frac{1}{2}\sum_{\omega\in\Omega} A(\omega)^2}\ . \qquad (6.2)$$

The multiplicative property of (6.2) can be checked directly before deriving (6.1), but we omit this analysis here. Notice that the "$\circ$" algebra never possesses a unit element nor is it alternative. The lack of unit element follows from (6.1) and the fact that the division algebras possess a unit element which is real.

The results just discussed depend on the reality condition (5.8). If it is relaxed, we presumably obtain a split version of the "$\circ$" algebra. Then non-zero elements can multiply to zero, witness (5.14b).

The product rule "$\circ$" in (6.1) arises naturally in another context which therefore appears to have many links with the present work, namely the theory of Jordan algebras [19]. Let $J_N(\mathbb{K})$ denote the algebra of $N\times N$ hermitian matrices with entries which are real ($\mathbb{K}=\mathbb{R}$), complex ($\mathbb{K}=\mathbb{C}$), quaternionic ($\mathbb{K}=\mathbb{H}$), or octonionic ($\mathbb{K}=\mathbb{O}$). Hermitian conjugation means transposition followed by (5.4). With respect to the product $A\times B=(AB+BA)/2$, the $J_N(\mathbb{K})$ are Jordan algebras unless $\mathbb{K}=\mathbb{O}$ and $N>3$. Consider as an example the exceptional Jordan algebra $J_3(\mathbb{O})$ and the multiplication rules for the following special elements A, B, and C (which each form 8-dimensional vector spaces):

$$A=2\begin{pmatrix}0&0&\bar{a}\\0&0&0\\a&0&0\end{pmatrix},\qquad B=2\begin{pmatrix}0&b&0\\\bar{b}&0&0\\0&0&0\end{pmatrix},\qquad C=2\begin{pmatrix}0&0&0\\0&0&c\\0&\bar{c}&0\end{pmatrix}. \qquad (6.3)$$

Then

$$\begin{aligned} A\times B=C &\quad \text{if} \quad c=\bar{b}\bar{a}=\overline{ab}=a\circ b\ ,\\ C\times A=B &\quad \text{if} \quad b=\bar{a}\bar{c}=\overline{ca}=c\circ a\ ,\\ B\times C=A &\quad \text{if} \quad a=\bar{c}\bar{b}=\overline{bc}=b\circ c\ .\end{aligned}$$

Moreover the derivation algebra of $J_3(\mathbb{O})$ is F_4 whose orbits of short roots were the starting point for our fermionic vertex operators related to octonions.

Equation (6.3) can be extended to the other cases. Freudenthal's magic square is usually constructed by means of these Jordan algebras [20] and it turns out to be a general feature that N, the dimension of the matrix in the Jordan algebra, equals $n+1$ where n is defined by (2.1).

So far we have omitted mention of the division algebra formed by the reals but we now see that it is generated by the vertex operator $V_+(e)$ proportional to $:\cos e\cdot Q:$ corresponding to the unit element. This is actually the real fermion previously associated with each orbit in the general construction of level one representations of non-simply laced algebras [16, 17]. With it we can add one further row and column to the magic squares of Sect. 2 to construct the full square, as in [16].

Finally we want to mention that our octonion result has an important physical application in the formulation of the superstring theory of particle interactions. The fermionic vertex operators related to octonions are associated with the short roots of F_4 and fall into three orbits under the action of the Weyl group of D_4, the subalgebra of F_4 defined by its long roots. $D_4 = so(8)$ is the residue of the Lorentz invariance group of the superstring in the light cone gauge. In superstring theory the fermionic vertex operators are familiar and important constructs. For points of the orbit constituting vector weights of D_4 they are Ramond/Neveu-Schwarz fields. For one of the other two orbits, comprising spinor or conjugate spinor weights, they are the fermion emission-absorption vertices [3]. Thus the algebra of these quantities which is essential to the evaluation of superstring scattering amplitudes appears to be related to the algebra of octonions or, by a previous comment, to the exceptional Jordan algebra $J_3(\mathbb{O})$. This is another idea we wish to pursue further.

It has long been a dream that octonions or exceptional Jordan algebras should play a role in fundamental physical theories. One context proposed was the theory of quark confinement [21] but it now seems that the more natural arena is, as we have suggested, the theory of superstrings.

Appendix A

Here we give a concrete treatment of the "orbit triples" and "triality" maps, τ, of Sect. 3 for the lattices $\Lambda = \Lambda_R(A_n)/\sqrt{2}$, $\Lambda_R(C_{n+1})$ and $\Lambda_R(F_4)$. Except for $\Lambda_R(A_2)/\sqrt{2}$ which has no non-zero τ-invariant points, we construct bases for these lattices which are permuted by τ, a result needed in Appendix B and used below to establish that the τ invariance of the symmetry factor on $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, (4.13b), can be extended to all of Λ by a judicious choice of some of the undetermined signs in the symmetry factor.

When $g = A_n$, $(n \geqq 2)$, is regarded as having short roots only, the simple roots are

$$\alpha_j = (e_j - e_{j+1})/\sqrt{2}\ , \quad j = 1 \ldots n\ , \tag{A1}$$

and generate $\Lambda_R(A_n)/\sqrt{2}$. Since $\Lambda_0^{(2)}$ is empty, $g_L = u(1)^n$, so that ϱ_L vanishes. The orbits each contain two elements and we take as a "triple" of orbits:

$$\Omega_3 = \{\pm(e_1 - e_2)/\sqrt{2}\}\ , \quad \Omega_1 = \{\pm(e_2 - e_3)/\sqrt{2}\}$$

and

$$\Omega_2 = \{\pm(e_3 - e_1)/\sqrt{2}\}\ . \tag{A2}$$

If we choose

$$e = -(e_1 - e_2)/\sqrt{2}\ , \quad \tau(e) = -(e_2 - e_3)/\sqrt{2}\ , \tag{A3}$$

we find that $\tau = \sigma_e \sigma_{\tau(e)}$ maps $e_1 \to e_2 \to e_3 \to e_1$ and leaves fixed e_j, $j \geqq 4$.

$\Lambda_R(A_2)/\sqrt{2}$ possesses no non-zero points which are τ invariant and so, being of dimension 2, cannot have a basis permuted by τ. On the other hand, for $n \geqq 3$, $\Lambda_R(A_n)/\sqrt{2}$ has a basis $\{u_i\}$ given by

$$\{(e_1 - e_4)/\sqrt{2},\ (e_2 - e_4)/\sqrt{2},\ (e_3 - e_4)/\sqrt{2},\ \alpha_4, \alpha_5, \ldots, \alpha_n\}\ . \tag{A4}$$

τ cyclically permutes u_1, u_2, and u_3 and leaves invariant the remaining u_i. That (A4) is an integer basis will follow from a special case of the argument for $\Lambda_R(C_{n+1})$ below.

The simple roots of $g=C_{n+1}$ are taken as

$$\alpha_1=(e_1-e_2)/\sqrt{2}\ ,\qquad \alpha_2=(e_2-e_3)/\sqrt{2},\ldots,\alpha_n=(e_n-e_{n+1})/\sqrt{2}\ ,$$
$$\alpha_{n+1}=\sqrt{2}e_{n+1}\ . \tag{A5}$$

g_L is $(A_1)^{n+1}$ with simple roots $\{\sqrt{2}e_1,\sqrt{2}e_2,\ldots,\sqrt{2}e_{n+1}\}$ and hence has a Dynkin diagram consisting of $n+1$ unconnected points, three of which τ cyclically permutes as above. Hence also

$$\varrho_L=\sum_{j=1}^{n+1} \mathrm{e}_j/\sqrt{2}\ . \tag{A6}$$

The $n(n+1)/2$ orbits Ω_{ij} each contain four elements, having the form

$$\Omega_{ij}=\{(\pm e_i\pm e_j)/\sqrt{2}\}\ ,\qquad i\neq j\ . \tag{A7}$$

We choose as the triple $\Omega_1\equiv\Omega_{23}$, $\Omega_2\equiv\Omega_{31}$, and $\Omega_3\equiv\Omega_{12}$. Then

$$f_1=(e_1+e_2)/\sqrt{2}\ ,\qquad e=-(e_1-e_2)/\sqrt{2}\ , \tag{A8}$$

in the notation of Sect. 3.

The quantities $\{u_1,u_2,u_3\}$,

$$\{f_1,\tau(f_1),\tau^2(f_1)\}=\{\alpha_1+2\alpha_2+\alpha_3,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}\ , \tag{A9}$$

form an integral basis for $\Lambda_R(C_3)$ as $\{\alpha_i\}$ does and the matrix of coefficients has integral entries and unit determinant.

For $\Lambda_R(C_{n+1})$, $n\geqq 3$, we consider instead $\{u_i\}$ given by

$$\{(e_1-e_4)/\sqrt{2},\ (e_2-e_4)/\sqrt{2},\ (e_3-e_4)/\sqrt{2},\ \alpha_4,\alpha_5,\ldots,\alpha_{n+1}\}\ . \tag{A10}$$

As $u_1=\alpha_1+\alpha_2+\alpha_3$, $u_2=\alpha_2+\alpha_3$, and $u_3=\alpha_3$ and the matrix of coefficients is again integral and of unit determinant, (A10) constitutes an integer basis of $\Lambda_R(C_{n+1})$ in which u_1, u_2, and u_3 are cyclically permuted by τ (as e_1, e_2, and e_3 are) and the remaining u_i are left invariant. This construction also applies to $\Lambda_R(A_n)/\sqrt{2}$ omitting α_{n+1} and u_{n+1}.

The simple roots of F_4 are taken as

$$\alpha_1=e_2-e_3\ ,\qquad \alpha_2=e_3-e_4\ ,\qquad \alpha_3=e_4\quad\text{and}\quad \alpha_4=(e_1-e_2-e_3-e_4)/2\ . \tag{A11}$$

Then $g_L=D_4$ with simple roots e_1-e_2, e_2-e_3, e_3-e_4, and e_3+e_4. The three orbits each contain 8 elements which are weights of D_4 irreps. Thus

$$\Omega_1\equiv\Omega_v=\{\pm e_i\}\ ,\qquad \Omega_2\equiv\Omega_s=\{(\pm e_1\pm e_2\pm e_3\pm e_4)/2\}\ ,$$
$$\Omega_3\equiv\Omega_{\bar{s}}=\{(\pm e_1\pm e_2\pm e_e\pm e_4)/2\}\ , \tag{A12}$$

with an even and odd number of minus signs respectively. We find

$$\varrho_L=3e_1+2e_2+e_3\ , \tag{A13}$$

and hence

$$e = e_4 \ , \quad f_1 = e_3 \ , \quad f_2 = e_2 \ , \quad f_3 = e_1 \ . \tag{A14}$$

Further

$$\begin{aligned} \tau(e_1) &= (e_1 + e_2 + e_3 + e_4)/2 \ , \\ \tau(e_2) &= (e_1 + e_2 - e_3 - e_4)/2 \ , \\ \tau(e_3) &= (e_1 - e_2 + e_3 - e_4)/2 \ , \\ \tau(e_4) &= (-e_1 + e_2 + e_3 - e_4)/2 \ . \end{aligned} \tag{A15}$$

We deduce that τ cyclically permutes the simple roots $e_1 - e_2$, $e_3 - e_4$, and $e_3 + e_4$ of D_4, leaving $e_2 - e_3$ invariant. This corresponds to the order three symmetry of the Dynkin diagram of D_4 permuting the three equal legs.

We now check that $\{u_1, u_2, u_3, u_4\}$ given by

$$\{f_2, \tau(f_2), \ \tau^2(f_2), \ e_2 - e_3\} \tag{A16}$$

provides a basis for $\Lambda = \Lambda_R(F_4)$ which is permuted by τ. We find $u_1 = f_2 = e_2 = \alpha_1 + \alpha_2 + \alpha_3$, $u_2 = \tau(f_2) = \alpha_4 + e_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $u_3 = \tau^2(f_2) = \alpha_4 + e_2 + e_4 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$, and $u_4 = e_2 - e_3 = \alpha_1$. Again the matrix of coefficients has integer entries and unit determinant and the result follows.

In Sect. 4, we showed by general arguments that, for any choice of the $n(n-1)/2$ arbitrary signs in the choice of symmetry factor, that factor was triality invariant on $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. This coincides with Λ if $\Lambda = \Lambda_R(A_2)/\sqrt{2}$, $\Lambda_R(C_3)$ or $\Lambda_R(F_4)$ i.e. $n = 2$. For $\Lambda = \Lambda_R(C_{n+1})$ and $\Lambda_R(A_n)/\sqrt{2}$, $n \geqq 3$, we now show that the extension of the triality symmetry to Λ depends on a judicious choice of the $n(n-1)/2$ signs of the quantities $S(\alpha_i, \alpha_j) = S_{ij}$ when α_i, α_j are distinct short simple roots. As $S(u_1, u_2) = S_{12}S_{13}$, $S(u_2, u_3) = S_{23}$, $S(u_3, u_1) = S_{31}S_{32}$ with $\{u_i\}$ given by the basis (A10) for $\Lambda_R(C_{n+1})$ for $n \geqq 3$, the triality invariance of the symmetry factor requires $S_{13} = -1$, $S_{12} = -S_{23}$. Further, $S(u_1, u_j) = S(u_2, u_j) = S(u_3, u_j)$, $j \geqq 4$, providing $S_{1j} = S_{2j} = 1$, which is automatic when $j = n+1$ and otherwise a legitimate choice. By the multiplicative property (4.4b), S is then τ symmetric on all of Λ with these choices. This argument applies to A_n by omitting α_{n+1} and u_{n+1} from consideration.

Appendix B. Symmetric Cocycles

Suppose that $S(\alpha, \beta)$ is a symmetry factor associated with a lattice Γ, that is S satisfies the conditions (4.4) for $\alpha, \beta \in \Gamma$ and takes values in Z, some finite subgroup of the group of complex numbers of unit modulus. An associated cocycle $\varepsilon(\alpha, \beta)$ is a function defined for α, $\beta \in \Gamma$, taking values in Z and satisfying Eqs. (4.5). Such a cocycle always exists and further is unique up to a "gauge transformation". (A general construction and proof of this is given in Sect. 5 of [16].) Now suppose further that S is symmetric with respect to some automorphism, τ, of Γ of order 3,

$$S(\tau(\alpha), \ \tau(\beta)) = S(\alpha, \beta) \ , \qquad \alpha, \beta \in \Gamma \ , \tag{B1}$$

with $\tau^3 = 1$. We wish to show that it is possible to choose the cocycle ε so that it also possesses this symmetry,

$$\varepsilon(\tau(\alpha), \tau(\beta)) = \varepsilon(\alpha, \beta) \ , \quad \alpha, \beta \in \Gamma \ . \tag{B2}$$

Any cocycle $\varepsilon(\alpha, \beta)$ associated with an S satisfying (B1) has the property that $\varepsilon(\tau(\alpha), \tau(\beta))$ is also a possible cocycle and so related to it by a gauge transformation, u, say,

$$\frac{\varepsilon(\tau(\alpha), \tau(\beta))}{\varepsilon(\alpha, \beta)} = \frac{\mathrm{u}(\alpha)\,\mathrm{u}(\beta)}{\mathrm{u}(\alpha+\beta)} \ . \tag{B3}$$

We seek another gauge transformation (4.6) which changes ε into a symmetric cocycle ε'. It then follows from $\varepsilon'(\tau(\alpha), \tau(\beta)) = \varepsilon'(\alpha, \beta)$, that

$$\phi(\alpha) = \frac{\eta(\tau(\alpha))}{\eta(\alpha)}\, u(\alpha) \tag{B4}$$

is a trivial gauge transformation,

$$\phi(\alpha+\beta) = \phi(\alpha)\phi(\beta) \ . \tag{B5}$$

Now in fact $u(\alpha)$ is ambiguous up to a factor of such a homomorphism $\phi(\alpha)$; that is we could replace $u(\alpha)$ in (B3) by $u(\alpha)/\phi(\alpha)$, and then (B4) would hold with $\phi(\alpha) = 1$,

$$u(\alpha) = \eta(\alpha)/\eta(\tau(\alpha)) \ . \tag{B6}$$

Since $\tau^3 = 1$, the consistency of (B6) demands

$$u(\alpha)u(\tau(\alpha))u(\tau^2(\alpha)) = 1 \ , \quad \alpha \neq \tau(\alpha) \ , \tag{B7a}$$

$$u(\alpha) = 1 \ , \quad \alpha = \tau(\alpha) \ . \tag{B7b}$$

Thus we have established that, if it is possible to symmetrise ε by a gauge transformation η, it must be possible to choose the gauge transformation as in (B3) so that (B7) holds. The converse is also true. If we can take the u in (B3) to satisfy (B7), we can choose $\eta(\alpha_0)$ arbitrarily (in Z) for one α_0 point in each orbit of τ. Such orbits either consist of a single point, or of three distinct points α_0, $\tau(\alpha_0)$, $\tau^2(\alpha_0)$. We then define

$$\eta(\tau(\alpha_0)) = \eta(\alpha_0)/u(\alpha_0) \ , \tag{B8a}$$

$$\eta(\tau^2(\alpha_0)) = \eta(\alpha_0)/u(\alpha_0)u(\tau(\alpha_0)) \ . \tag{B8b}$$

The gauge transformation (4.6) then provides a symmetric cocycle.

To complete our proof, it remains to construct for S a cocycle ε and gauge transformation satisfying (B3) and (B7). A general method of constructing cocycles was given in Ref. 16. We take a basis $\{u_j\}$ for Γ and introduce generalised γ matrices $\{\gamma_j\}$ satisfying

$$\gamma_i\gamma_j = S_{ij}\gamma_j\gamma_i \ , \tag{B9}$$

where $S_{ij} = S(u_i, u_j)$. Then, for $x = \sum x_j u_j \in \Lambda$, we define

$$\gamma_x = \gamma_1^{x_1} \ldots \gamma_n^{x_n} \tag{B10}$$

and obtain a cocycle from the equation,

$$\gamma_x\gamma_y = \varepsilon(x, y)\gamma_{x+y} \ , \tag{B11}$$

satisfying (4.5). Now, in the case where S has the symmetry (B1),

$$\gamma_j \to \beta_j = \gamma_{\tau(u_j)} \tag{B12}$$

defines an automorphism of the γ matrix algebra as then $\beta_i \beta_j = S_{ij} \beta_j \beta_i$. We can use

$$\beta_x = \beta_1^{x_1} \dots \beta_n^{x_n} \tag{B13}$$

to obtain the gauge transformation $u(x)$ of (B3). Because of the isomorphism (B12), we also have

$$\beta_x \beta_y = \varepsilon(x, y) \beta_{x+y} \; . \tag{B14}$$

But

$$\beta_{\tau(x)} = \gamma_{\tau(u_1)}^{\tau(x)_1} \dots \gamma_{\tau(u_n)}^{\tau(x)_n} = u(x) \gamma_x \tag{B15}$$

for some function $u(x)$, which, from (B11) and (B14), satisfies (B3).

It remains to show that we can arrange that the $u(x)$ of (B15) satisfy (B7). Firstly if $v(\alpha) = u(\alpha) u(\tau(\alpha)) u(\tau^2(\alpha))$,

$$\beta_{\tau(\alpha)} \beta_{\tau^2(\alpha)} \beta_\alpha = v(\alpha) \gamma_\alpha \gamma_{\tau(\alpha)} \gamma_{\tau^2(\alpha)} \; . \tag{B16}$$

But the left-hand side of this equation can be written as $\beta_\alpha \beta_{\tau(\alpha)} \beta_{\tau^2(\alpha)}$, because the factor introduced by this reordering is $S(\tau^2(\alpha), \alpha) S(\tau(\alpha), \alpha)$ and

$$S(\tau^2(\alpha), \alpha) = S(\alpha, \tau(\alpha)) = 1/S(\tau(\alpha), \alpha) \; . \tag{B17}$$

Now the isomorphism of the β and γ matrices implies that

$$\beta_y = v(\alpha) \gamma_y \; . \tag{B18}$$

where $y = \alpha + \tau(\alpha) + \tau^2(\alpha)$. Hence $v(\alpha) = u(y)$, where $\tau(y) = y$, and we have reduced (B7a) to (B7b).

This second condition will follow provided that the basis $\{u_j\}$ of Γ that we are using is permuted by τ and we order the basis with elements in the same orbit adjacent. Then, using again (B17), we deduce that $u(\alpha) = 1$ if $\tau(\alpha) = \alpha$. This completes our proof under the assumption that our lattice has a basis permuted by τ, proven in Appendix A for the lattices of interest except for $\Lambda_R(A_2)/\sqrt{2}$, which has no points satisfying (B7b).

The cocycle $\varepsilon(x, y)$ defined by (B11) has the property

$$\varepsilon(x, y) = \varepsilon(-x, -y) \; . \tag{B19}$$

The symmetrised cocycle obtained from (4.6) will inherit this property provided that $\eta(x) = \eta(-x)$ and $\eta(0) = 1$. In our construction we were free to choose η at one point on each orbit. We can ensure $\eta(x) = \eta(-x)$, provided that we correlate the choice on diametrically opposite orbits $\{\alpha_0, \tau(\alpha_0), \tau^2(\alpha_0)\}$ and $\{-\alpha_0, -\tau(\alpha_0), -\tau^2(\alpha_0)\}$ by choosing $\eta(\alpha_0) = \eta(-\alpha_0)$. Finally, we also wish to ensure that the symmetric cocycle ε' satisfies

$$\varepsilon'(x, -x) = 1 \; . \tag{B20}$$

This will follow if

$$\eta(\alpha_0)^2 = \eta(\alpha_0) \eta(-\alpha_0) = 1/\varepsilon(\alpha_0, -\alpha_0) \; . \tag{B21}$$

This may mean that ε' takes values in $Z^{1/2} = \{z : z^2 \in Z\}$ rather than in Z.

For simplicity we have written our argument out for the specific case at hand of an automorphism of order 3. Various aspects of our discussion immediately generalise to automorphisms of arbitrary finite order but others become more subtle.

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References

1. Ramond, P.: Dual theory for free fermions. Phys. Rev. D**3**, 2415 (1971)
   Neveu, A., Schwarz, J.: Factorizable dual model of pions. Nucl. Phys. B**31**, 86 (1971); Quark model of dual pions. Phys. Rev. D**4**, 1109 (1971)
2. Gross, D., Harvey, J., Martinec, E., Rohm, R.: Heterotic string. Phys. Rev. Lett. **54**, 502 (1985), Heterotic string theory (I). The free heterotic string. Nucl. Phys. B**256**, 253 (1985)
3. Goddard, P., Olive, D., Schwimmer, A.: The heterotic string and a fermionic construction of the E_8 Kac-Moody algebra. Phys. Lett. **157**B, 393 (1985)
4. Kadanoff, L., Ceva, H.: Determination of an operator algebra for the two-dimensional Ising model. Phys. Rev. B**3**, 3918 (1970)
5. Onsager, L.: Crystal statistics. I. A two-dimensional model with an order-disorder transition. Phys. Rev. **65**, 117 (1944)
   Kaufmann, B.: Crystal statistics. II. Partition function evaluated by spinor analysis. Phys. Rev. **76**, 1232 (1949)
6. Skyrme, T.H.R.: Particle states of a quantized meson field. Proc. Roy. Soc. A**262**, 237 (1961)
7. Jimbo, M., Miwa, T.: Solitons and infinite dimensional Lie algebras. Publ. RIMS Kyoto Univ. **19**, 943 (1983)
8. Goddard, P., Olive, D.: Kac-Moody and Virasoro algebras in relation to quantum physics. Int. J. Mod. Phys. A**1**, 303 (1986)
9. Green, M., Schwarz, J.: Supersymmetrical dual string theory. Nucl. Phys. B**181**, 502 (1981)
10. Gliozzi, F., Scherk, J., Olive, D.: Supergravity and the spinor dual model. Phys. Lett. **65**B, 282 (1976). Supersymmetry, supergravity theories and the dual spinor model. Nucl. Phys. B**122**, 253 (1977)
11. Mandelstam, S.: Soliton operators for the quantized sine-Gordon equation. Phys. Rev. D**11**, 3026 (1975)
12. Fubini, S., Veneziano, G.: Duality in operator formalism. Nuovo Cim. **67**A, 29 (1970)
13. Frenkel, I.: Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory. J. Funct. Anal. **44**, 259 (1981)
14. Goddard, P., Olive, D.: Vertex operators in mathematics and physics. MSRI Publication No. 3. Berlin, Heidelberg, New York: Springer 1984, p. 51
15. Klein, O.: J. Phys. Radium **9**, 1 (1938)
16. Goddard, P., Nahm, W., Olive, D., Schwimmer, A.: Vertex operators for non-simply laced algebras. Commun. Math. Phys. **107**, 179 (1986)
    See also Olive, D.: The vertex operator construction for non-simply laced Kac-Moody algebras I; and Goddard, P.: The vertex operator construction for non-simply laced Kac-Moody algebras II. In: Topological and geometrical methods in field theory. Hieterinta, J., Westerholm, J. (eds.) Singapore: World Scientific 1986 and in Proceedings of Symposia in Montreal, Bonn and Meudon

17. Bernard, D., Thierry-Mieg, J.: Level one representations of the simple affine Kac-Moody algebras in their homogeneous gradations. Commun. Math. Phys. **111**, 181–246 (1987)
18. Humphreys, J.E.: Introduction to Lie algebras and representation theory. Berlin, Heidelberg, New York: Springer 1972
19. Jacobsen, N.: Basic algebra. I. San Francisco, CA: Freeman 1974, Chap. 7
20. Freudenthal, H.: Lie groups in the foundations of geometry. Adv. Math. **1**, 145 (1964) J. Tits: Proc. K. Ned. Akad. Wet. A**69**, 223 (1966)
21. Gunaydin, M., Gursey, F.: An octonionic representation of the Poincaré group. Lett. Nuovo Cim. **6**, 401 (1973)

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