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One-Dimensional Random Ising Systems with Interaction Decay $r^{-(1+\varepsilon)}$: A Convergent Cluster Expansion

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Abstract. We consider a one-dimensional random Ising model with Hamiltonian

$$H=\sum_{i\neq j}\frac{J_{ij}}{|i-j|^{1+\varepsilon}}s_is_j+h\sum_i s_i \ ,$$

where $\varepsilon>0$ and J_{ij} are independent, identically distributed random variables with distribution $dF(x)$ such that

i) $\int x\,dF(x)=0$,

ii) $\int e^{tx}dF(x)<\infty \quad \forall t\in\mathbb{R}$.

We construct a cluster expansion for the free energy and the Gibbs expectations of local observables. This expansion is convergent almost surely at every temperature. In this way we obtain that the free energy and the Gibbs expectations of local observables are C^∞ functions of the temperature and of the magnetic field h. Moreover we can estimate the decay of truncated correlation functions. In particular for every $\varepsilon'>0$ there exists a random variable $c(\omega)$, finite almost everywhere, such that

$$|\langle s_0s_j\rangle_H-\langle s_0\rangle_H\langle s_j\rangle_H|\leqq\frac{c(\omega)}{|j|^{1+\varepsilon-\varepsilon'}} \ ,$$

where $\langle\ \rangle_H$ denotes the Gibbs average with respect to the Hamiltonian H.

1. Introduction, Definitions and Results

In [4] a one-dimensional Ising spin system with random interactions decaying like $1/r^{1+\varepsilon}$ was considered. A weak version of uniqueness of Gibbs state was proven there for such a system by showing that at every temperature the expectation of an

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observable localized far away from the boundary has a weak dependence from the boundary conditions (see [6–8, 12, 13] for previous related results). Two different fixed (i.e. non-random) boundary conditions give rise with probability one to the same Gibbs state, but one cannot exclude in principle non-uniqueness and even breaking of spin-flip symmetry if boundary conditions dependent on the realization of the coupling are allowed.

In this paper we strengthen the result of [4] and prove that no phase transition of any order occurs for these systems in the sense that the free energy is a.s. a C^∞ function of the thermodynamic parameters. This result answers a question that was not settled even at a heuristic level (see [1]). The analogous results for systems with interaction decay $1/r^{3/2+\varepsilon}$ was obtained in [6] by exhibiting a cluster expansion that converges a.e.

The estimates needed to prove the almost sure convergence are much more delicate here than in [6], since in the hypotheses of this paper the supremum of the interaction among two contiguous half-lines is not finite a.e. If we divide the volume into blocks, we can think of the spin configuration in a block as a single "spin" and we can endow the space of spin configurations in a block with a "free measure" given by the finite volume Gibbs state corresponding to the internal interactions. Then one can prove that the interaction between two such "spins" is bounded uniformly in the block size apart from some bad sets whose "free measure" can be uniformly estimated. The situation is reminiscent of superstable unbounded spin systems (see [15]) with the difference that here the bad part of the space is not given once and for all but is determined by the realization of the random interaction.

The method here, like in [6], is basically the following: we partition the lattice $\mathbb{Z}$ into blocks, then we apply a "decimation" procedure over alternating blocks and show that at any temperature the system is weakly coupled. The size of the blocks increases with the distance from the origin and the size of the block containing the origin depends on the random interaction. In this way one only obtains convergence for real values of the parameters, C^∞ properties and decay of correlations, but not analyticity which in fact is not expected to hold due to the possible existence of Griffith's like singularity.

The present paper heavily relies on [4] from which we take notation and results.

Let us now define the model and state the main results.

Given $\Lambda\subset\mathbb{Z}$ the configuration space in Λ is the set $\mathcal{S}_\Lambda=\{-1,1\}^\Lambda$. For any unordered pair i,j, $i\neq j$, we introduce a random variable J_{ij} taking values in $\mathbb{R}$. The variables J_{ij}'s are independent, identically distributed with distribution $dF(x)$. We shall denote by Ω the probability space on which the J_{ij}'s are defined and by $\mathbb{P}$ and $\mathbb{E}$ respectively the probability measure on Ω and the expectation with respect to $\mathbb{P}$. As customary, usually we shall not explicitly write the dependence of random quantities on the point $\omega\in\Omega$.

We make the following assumptions on the distribution $dF(x)$:

$$\begin{aligned} &\text{i)} && \int x\,dF(x)=0\ , \\ &\text{ii)} && \forall t\in\mathbb{R}:\int \exp\,(tx)dF(x)<\infty\ . \end{aligned} \tag{1.1}$$

Given Λ finite, $\Lambda\subset\mathbb{Z}$ the Hamiltonian H_Λ is the random function on $\mathcal{S}_\Lambda$ defined by

$$H_\Lambda(s) = -\sum_{\substack{i,j\in\Lambda \\ i\neq j}} \frac{J_{ij}}{|i-j|^{1+\varepsilon}} s_i s_j - h\sum_{i\in\Lambda} s_i \ , \tag{1.2}$$

where $\varepsilon > 0$ and h is a real constant called magnetic field (in the Hamiltonian considered in [4] the magnetic field was not present but the results of that paper extend to this case without changes). The Gibbs measure in Λ at inverse temperature β is the probability measure on $\mathscr{S}_\Lambda$ defined by the average $\langle\cdot\rangle^\Lambda_{\omega,\beta,h}$ on functions f on $\mathscr{S}_\Lambda$:

$$\langle f\rangle^\Lambda_{\omega,\beta,h} = \sum_{s\in\mathscr{S}_\Lambda} f(s)\exp\left[-\beta H_\Lambda(s)\right]/Z^\Lambda_{\omega,\beta,h} \ , \tag{1.3}$$

where $Z^\Lambda_{\omega,\beta,h}$ is the normalizing factor and the dependence on ω is through the random realization of the interaction. The free energy per site in the volume Λ, $F_\Lambda(\omega,\beta,h)$, is given by:

$$F_\Lambda(\omega,\beta,h) = \frac{1}{|\Lambda|}\log Z^\Lambda_{\omega,\beta,h} \ . \tag{1.4}$$

Let $\mathscr{E}(\mathbb{R}^2)$ be the space of real C^∞ functions of two variables with the topology of uniform convergence of derivatives of any order on compact sets. Then the main results of this paper are contained in the following two theorems.

1.1. Theorem. *Let f be a cylindrical function on $\mathscr{S}_\mathbb{Z}$ (*i.e.* a function that depends only on the values of the configuration on a finite set of sites). Then for every increasing sequence of intervals Λ_n such that $\bigcup_n \Lambda_n = \mathbb{Z}$ for* a.e. *ω the following limit exists in $\mathscr{E}(\mathbb{R}^2)$,*

$$\langle f\rangle_{\omega,\beta,h} = \lim_{n\to\infty} \langle f\rangle^{\Lambda_n}_{\omega,\beta,h} \ . \tag{1.5}$$

Moreover

$$\mathbb{E}(\langle f\rangle_{\cdot,\beta,h}) = \lim_{n\to\infty} \mathbb{E}(\langle f\rangle^{\Lambda_n}_{\cdot,\beta,h}) \ , \tag{1.6}$$

also in $\mathscr{E}(\mathbb{R}^2)$.

1.2. Theorem. *For every increasing sequence of intervals Λ_n such that $\bigcup_n \Lambda_n = \mathbb{Z}$, there exists almost everywhere in $\mathscr{E}(\mathbb{R}^2)$ the limit:*

$$F(\beta,h) = \lim_{n\to\infty} \mathbb{E}(F_\Lambda(\cdot,\beta,h)) \ .$$

Moreover $F_\Lambda(\omega,\beta,h)$ converges almost everywhere to $F(\beta,h)$ in $\mathscr{E}(\mathbb{R}^2)$.

1.3. Remark. For simplicity we have stated our results for Gibbs states with zero boundary conditions. As it will be clear from the proofs, they continue to hold if one imposes fixed (i.e. non-random) boundary conditions. Two different boundary conditions give rise almost everywhere to the same limits $\langle f\rangle_{\omega,\beta,h}$, $F(\beta,h)$ for the expectation of a local observable and the free energy respectively.

Our cluster expansion allows us to easily get estimates on the truncated correlation functions of the model. The result is summarized in the following theorem.

1.4. Theorem. *For every $\varepsilon'>0$ there exists a random variable $c(\omega)$, finite almost everywhere, such that*

$$|\langle s_0 s_j\rangle - \langle s_0\rangle\langle s_j\rangle| \leqq \frac{c(\omega)}{|j|^{1+\varepsilon-\varepsilon'}} .$$

In Sect. 2 we construct the cluster expansion. In Sect. 3 we prove its convergence. In Sect. 4 we prove Theorems 1.1 and 1.2. The proof of Theorem 1.4, which is a quite natural consequence of our cluster expansion, is contained in the final Appendix.

2. The Polymer Expansion

We want to evaluate the m-th derivative with respect to β or h of

$$\langle f\rangle^{\Lambda}_{\beta,h} = \frac{Z^{\Lambda}_{\beta,h}(f)}{Z^{\Lambda}_{\beta,h}} , \tag{2.1}$$

where

$$Z^{\Lambda}_{\beta,h}(f) = \sum_{s\in\mathscr{S}_\Lambda} f(s)\exp(-\beta H_\Lambda(s) , \tag{2.2}$$

$$Z^{\Lambda}_{\beta,h} = Z^{\Lambda}_{\beta,h}(1) = \sum_{s\in\mathscr{S}_\Lambda} \exp(-\beta H_\Lambda(s) , \tag{2.3}$$

and f is a cylindrical function with support Δ centered at the origin.

We start introducing a partition of $\mathbb{Z}$ into blocks whose size increases with a power law with the distance from the origin:

$$\mathbb{Z} = \bigcup_{n=-\infty}^{+\infty} Q_n .$$

Q_0 is centered at the origin and $|Q_0| = q_0$, where q_0 is an odd integer. Q_1 is defined as

$$Q_1 = \left\{ t\in\mathbb{Z} : \frac{q_0-1}{2} + 1 \leqq t \leqq \frac{q_0-1}{2} + q(1) \right\}$$

and, for $n\geqq 2$:

$$Q(n) = \left\{ t\in\mathbb{Z} : \frac{q_0-1}{2} + 1 + \sum_{j=1}^{n-1} q(j) \leqq t \leqq \frac{q_0-1}{2} + \sum_{j=1}^{n} q(j) \right\} \tag{2.4}$$

with $q(j) = [q_0 j^\lambda]$, where $\lambda>0$ will be suitably chosen later.

For $n<0$ we define $Q_n = -Q_{-n}$.

We introduce now a partition of $\mathbb{Z}$ into alternating A and B blocks:

$$\mathbb{Z} = \bigcup_{n=-\infty}^{+\infty} (A_n \cup B_n) ,$$

where

$$A_n = Q_{2n} \qquad B_n = Q_{2n-1} .$$

We use the following notation for the spin configurations in the blocks

$$\gamma_n = s_{Q_n}, \qquad \alpha_n = s_{A_n}, \qquad \beta_n = s_{B_n} .$$

We suppose that our spin system is enclosed in a volume Λ, exactly partitioned into A and B blocks but not necessarily symmetric with respect to the origin

$$\Lambda=\Lambda_{p,p'}=\left(\bigcup_{j=-p}^{p'} A_j\right)\cup\left(\bigcup_{j=-p+1}^{p'} B_j\right)=\bigcup_{j=-2p}^{2p'} Q_j .$$

For $k,k'\in\mathbb{Z}$, $k\neq k'$ we define:

$$W_{Q_k,Q_{k'}}(\gamma_k,\gamma_{k'})=\sum_{\substack{i\in Q_k\\ j\in Q_{k'}}}\frac{J_{ij}s_is_j}{|i-j|^{1+\varepsilon}} .$$

For $k,i,j\in\mathbb{Z}$, $i\neq j$, we define

$$\begin{aligned}\bar{H}_{\beta,h}(\alpha_k,\beta_{k+1},\alpha_{k+1})&=-\beta[H_{B_{k+1}}(\beta_{k+1})+W_{B_{k+1},A_k}(\beta_{k+1},\alpha_k)\\&\quad+W_{B_{k+1},A_{k+1}}(\beta_{k+1},\alpha_{k+1})] ,\\ \bar{W}_\beta(\gamma_i,\gamma_j)&=-\beta W_{Q_i,Q_j}(\gamma_i,\gamma_j) ,\\ \bar{H}_{\beta,h}(\gamma_k)&=-\beta H_{Q_k}(\gamma_k) .\end{aligned}\tag{2.6}$$

In the following we are going to transform our spin system into a gas of polymers whose only interaction is hardcore exclusion. In the next section we shall show that the activities of the polymers are small with high probability. In the present section we only perform algebraic manipulations without giving any estimate.

Let the partition function $Z_{B_{k+1}}^{\alpha_k,\alpha_{k+1}}$ be defined by

$$Z_{B_{k+1}}^{\alpha_k,\alpha_{k+1}}=\sum_{\beta_{k+1}\in\mathscr{S}_{B_{k+1}}}\exp\,(\bar{H}(\alpha_k,\beta_{k+1},\alpha_{k+1})) .\tag{2.7}$$

$Z_{B_{k+1}}^{\alpha_k,\alpha_{k+1}}$ has with large probability factorization properties that are expressed in Proposition 3.1 in terms of some functions v_k, $\bar{v}_k$ on $\mathscr{S}_{A_k}$.

We shall treat the partition function $Z_{\beta,h}^{\Lambda}$: only trivial changes are needed to study $Z_{\beta,h}^{\Lambda}(f)$. It will turn out to be convenient to divide $Z_{\beta,h}^{\Lambda}$ by a suitable normalization factor.

To simplify the notation we shall omit everywhere the subscripts β and h and we shall write

$$\sum_{\alpha_k},\ \sum_{\beta_k}\quad\text{for}\quad\sum_{\alpha_k\in\mathscr{S}_{A_k}},\ \sum_{\beta_k\in\mathscr{S}_{B_k}}$$

respectively.

We call

$$\lambda_k=\sum_{s\in\mathscr{S}_{Q_k}}\exp\,(-\beta H_{Q_k}(s)\tag{2.8}$$

the partition function relative to the block Q_k with "zero boundary conditions." Let $\bar{v}_k(\alpha_k)$, $v_k(\alpha_k)$ be the functions whose existence is stated in Proposition 3.1 of the next section.

We set

$$d_k=\frac{1}{\lambda_{2k}}\left[\sum_{\alpha_k}\bar{v}_k(\alpha_k)v_k(\alpha_k)\exp\,(\bar{H}(\alpha_k))\right]\tag{2.9}$$

and

$$\mathcal{N}=\prod_{j=-2p}^{2p'}\lambda_j\prod_{j=-p}^{p'}d_j\ . \tag{2.10}$$

The partition function is given by

$$Z^{\Lambda}=\sum_{\substack{\{\alpha_k\},\{\beta_k\}\\ A_k,B_k\subset\Lambda}}\exp\left[\sum_{k=-p}^{p'-1}(\bar{H}(\alpha_k)+\bar{H}(\alpha_k,\beta_{k+1},\alpha_{k+1}))+\bar{H}(\alpha_k)\right]\prod_{\{Q_i Q_j\}\in\mathscr{F}}\exp\bar{W}(\gamma_i,\gamma_j)\ , \tag{2.11}$$

where $\mathscr{F}$ is the set of all pairs of blocks $Q_i, Q_j: Q_i, Q_j\subset\Lambda$, and $|i-j|\geqq 2$.

We can write

$$\prod_{\{Q_i,Q_j\}\in\mathscr{F}}\exp\bar{W}(\gamma_i,\gamma_j)=\sum_{F\subset\mathscr{F}}\prod_{\{Q_i,Q_j\}\in F}(\exp\bar{W}(\gamma_i,\gamma_j)-1)$$

with the convention that

$$\prod_{\{Q_i,Q_j\}\in\emptyset}(\exp\bar{W}(\gamma_i,\gamma_j)-1)=1\ .$$

Given $F\subset\mathscr{F}$ we define $\tilde{F}$ the "support of F" as

$$\tilde{F}=\bigcup_{\substack{\{Q_i,Q_j\}\in F\\ \text{for some } j}}Q_i\cup\left(\bigcup_{\substack{\{B_k,Q_j\}\in F\\ \text{or }\{B_{k+1},Q_j\}\in F\text{ for some } j}}A_k\right)\ . \tag{2.12}$$

Namely $\tilde{F}$ contains the A and B blocks belonging to some pairs in F and, moreover, all A blocks adjacent to a B block belonging to some pair in F.

We have:

$$\frac{Z}{\mathcal{N}}=\frac{1}{\mathcal{N}}\sum_{F\subset\mathscr{F}}\sum_{\substack{\{\alpha_k\},A_k\subset\Lambda\\ \{\beta_k\},B_k\subset\tilde{F}}}\exp\left(\sum_{k=-p}^{p'}\bar{H}(\alpha_k)\right)\cdot\exp\left(\sum_{k\cdot B_k\subset\tilde{F}}\bar{H}(\alpha_k,\beta_{k+1},\alpha_{k+1})\right)\prod_{k:B_{k+1}\not\subset\tilde{F}}Z^{\alpha_k\alpha_{k+1}}_{B_{k+1}}\cdot\prod_{\{Q_i,Q_j\}\in F}(\exp\bar{W}(\gamma_i,\gamma_j)-1)\ ,$$

where $Z^{\alpha_k\alpha_{k+1}}_{B_{k+1}}$ has been defined in Eq. (2.7) and the empty set is included in the sum over F.

Given F we set, for $s_{\tilde{F}}\in\mathscr{S}_{\tilde{F}}$,

$$\hat{H}_{\tilde{F}}(s_{\tilde{F}})=\sum_{Q_k\subset\tilde{F}}\bar{H}(\gamma_k)+\sum_{\substack{Q_k,Q_{k'}\subset\tilde{F}\\ |k-k'|=1}}\bar{W}(\gamma_k,\gamma_{k'})\ , \tag{2.14}$$

namely $\hat{H}_{\tilde{F}}$ is the Hamiltonian relative to the (generally disconnected) set $\tilde{F}$ in which we have dropped all the interactions among non-contiguous blocks. Now recalling the definition of $\mathcal{N}$, we have

$$\frac{Z}{\mathcal{N}}=\left(\prod_{j=-p}^{p'} d_j\right)^{-1} \sum_{F\subset\mathcal{F}} \sum_{\substack{\{\alpha_k\},A_k\subset\Lambda\\ \{\beta_k\},B_k\subset\tilde{F}}} \left(\prod_{k:A_k\not\subset\tilde{F}} \lambda_{2k}^{-1}\exp\bar{H}(\alpha_k)\right)$$

$$\cdot\prod_{k:B_k\not\subset\tilde{F}}\left(\left(\frac{Z_{B_{k+1}}^{\alpha_k,\alpha_{k+1}}}{\lambda_{2k+1}}-\bar{v}_k(\alpha_k)v_{k+1}(\alpha_{k+1})\right)+\bar{v}_k(\alpha_k)v_{k+1}(\alpha_{k+1})\right)$$

$$\cdot\exp\hat{H}_{\tilde{F}}(s_{\tilde{F}})\left(\prod_{j:Q_j\subset\tilde{F}}\lambda_j\right)^{-1}\prod_{\{Q_i,Q_j\}\in F}(\exp\bar{W}(\gamma_i,\gamma_j)-1)\ . \tag{2.15}$$

We recall that the functions $\bar{v}_k$ and v_k are characterized in Proposition 3.1 of the next section.

Given $\tilde{F}$ we call $\mathcal{B}_{\tilde{F}}$ the union of B-blocks in Λ that do not belong to $\tilde{F}$. For any non-empty subset Γ of $\mathcal{B}_{\tilde{F}}$ we call

$$\mathfrak{A}_{\tilde{F},\Gamma}=\left(\bigcup_{\substack{k:A_k\subset\tilde{F}\\ B_k\not\subset\Gamma\cup\tilde{F}}}A_k\right)\cup\left(\bigcup_{\substack{k:B_{k+1}\subset\Gamma\\ B_k\not\subset\Gamma\cup\tilde{F}}}A_k\right),$$

$$\bar{\mathfrak{A}}_{\tilde{F},\Gamma}=\left(\bigcup_{\substack{k:A_k\subset\tilde{F}\\ B_{k+1}\not\subset\Gamma\cup\tilde{F}}}A_k\right)\cup\left(\bigcup_{\substack{k:B_k\subset\Gamma\\ B_{k+1}\not\subset\Gamma\cup\tilde{F}}}A_k\right).$$

By expanding the product

$$\prod_{k:B_k\not\subset\tilde{F}}\left(\left(\frac{Z_{B_{k+1}}^{\alpha_k\alpha_{k+1}}}{\lambda_{2k+1}}-\bar{v}_k(\alpha_k)v_{k+1}(\alpha_{k+1})\right)+\bar{v}_k(\alpha_k)v_{k+1}(\alpha_{k+1})\right),$$

and, recalling the definition (2.9), we get

$$\frac{Z}{\mathcal{N}}=\left(\prod_{j=-p}^{p'} d_j\right)^{-1}\sum_{F\subset\mathcal{F}}\sum_{\Gamma\subset\mathcal{B}_{\tilde{F}}}\prod_{k\cdot B_k\cup B_{k+1}\subset\mathcal{B}_{\tilde{F}}\setminus\Gamma}$$

$$\cdot\sum_{\substack{\{\alpha_k\}:A_k\subset\Gamma\cup\tilde{F}\\ \{\beta_k\}:B_k\subset\tilde{F}}}\prod_{B_{k+1}\subset\Gamma}\left(\frac{Z_{B_{k+1}}^{\alpha_k\alpha_{k+1}}}{\lambda_{2k+1}}-\bar{v}_k(\alpha_k)v_{k+1}(\alpha_{k+1})\right)$$

$$\cdot\prod_{\substack{k:A_k\subset\tilde{F}\\ (B_k\cup B_{k+1})\cap\Gamma\neq\emptyset}}\frac{\exp\bar{H}(\alpha_k)}{\lambda_{2k}}\prod_{k:A_k\subset\mathfrak{A}_{\tilde{F},\Gamma}}v_k(\alpha_k)\prod_{k:A_k\subset\bar{\mathfrak{A}}_{\tilde{F},\Gamma}}\bar{v}_k(\alpha_k)$$

$$\cdot\exp\hat{H}_{\tilde{F}}(s_{\tilde{F}})\left(\prod_{j:Q_j\subset\tilde{F}}\lambda_j\right)^{-1}\prod_{\{Q_i,Q_j\}\in F}(\exp\bar{W}(\gamma_i,\gamma_j)-1)\ . \tag{2.16}$$

Now, looking at the expression (2.16) we distinguish two kinds of "bonds":
1) $\ell=\{Q_i,Q_j\}$ for $\{Q_i,Q_j\}\in F$ to which we associate the factor

$$\Phi_\ell(\gamma_i,\gamma_j)=\exp\ \bar{W}(\gamma_i,\gamma_j)-1\ . \tag{2.17}$$

2) $U=\{A_k,B_{k+1},A_{k+1}\}$ for $B_{k+1}\in\Gamma$ to which we associate the factor

$$\Psi_U(\alpha_k,\alpha_{k+1})=\frac{Z_{B_{k+1}}^{\alpha_k,\alpha_{k+1}}}{\lambda_{2k+1}}-\bar{v}_k(\alpha_k)v_{k+1}(\alpha_{k+1})\ . \tag{2.18}$$

A bond $U=\{A_k, B_{k+1}, A_{k+1}\}$ is said to be compatible with a bond $\ell=\{Q_i, Q_j\}$ if $B_{k+1}\neq Q_i$, $B_{k+1}\neq Q_j$. Given $\ell=\{Q_i, Q_j\}$ we call support of ℓ the set

$$\tilde{\ell}=Q_i\cup Q_j\cup\left(\bigcup_{k:(B_{k+1}\cup B_k)\cap(Q_i\cup Q_j)\neq\emptyset} A_k\right) .$$

More explicitly

1) for $\ell=\{A_k, A_m\}$, $\tilde{\ell}=A_k\cup A_m$,
2) for $\ell=\{A_k, B_m\}$ $\tilde{\ell}=A_k\cup B_m\cup A_{m-1}\cup A_m$,
3) for $\ell=\{B_k, B_m\}$ $\tilde{\ell}=A_{k-1}\cup B_k\cup A_k\cup A_{m-1}\cup B_m\cup A_m$.

For $U=\{A_k, B_{k+1}, A_{k+1}\}$ the support $\tilde{U}$ is simply $\tilde{U}=A_k\cup A_{k+1}$. Two bonds are connected if their supports have a non-empty intersection.

A polymer R is a maximal connected set of compatible bonds

$$R=\{U_1,\ldots, U_k, \ell_1,\ldots, \ell_m\} .$$

The support $\tilde{R}$ of R is given by

$$\tilde{R}=\left(\bigcup_{U\in R}\tilde{U}\right)\cup\left(\bigcup_{\ell\in R}\tilde{\ell}\right) . \tag{2.19}$$

The support of a polymer decomposes as the union of intervals that can be either isolated A blocks or intervals I of the form

$$I=I(k,\ell)=A_k\cup B_{k+1}\cup A_{k+1}\cup\ldots\cup B_{k+\ell}\cup A_{k+\ell} .$$

We set

$$\hat{Z}_I=\sum_{s\in\mathscr{S}_I}\exp\hat{H}_I(s) , \tag{2.20}$$

where $\hat{H}$ is defined as in Eq. (2.14).

We define the "extended support" $\hat{R}$ of a polymer R as the set

$$\hat{R}=\tilde{R}\cup\left(\bigcup_{k:U=\{A_k, B_{k+1}, A_{k+1}\}\in R} B_{k+1}\right) . \tag{2.21}$$

We define

$$\mathfrak{A}_R=\bigcup_{\substack{k:A_k\subset\tilde{R}\\ B_k\not\subset\hat{R}}} A_k ,$$

$$\bar{\mathfrak{A}}_R=\bigcup_{\substack{k:A_k\subset\tilde{R}\\ B_{k+1}\not\subset\hat{R}}} A_k .$$

We define a probability measure associated to a polymer R by the average

$$\langle f\rangle_{\tilde{R}}=\sum_{s\in\mathscr{S}_{\tilde{R}}}\prod_{\substack{I\text{ connected}\\ \text{component of }\tilde{R}}}\frac{\exp\hat{H}_I(s_I)}{\hat{Z}_I}\prod_{k:A_k\subset\tilde{R}}\frac{\exp H(\alpha_k)}{\lambda_{2k}} f(s) . \tag{2.22}$$

We can write

$$\frac{Z^{\Lambda}_{\beta,b}}{\mathscr{N}}\equiv\Xi(\Lambda,\beta,h)=1+\sum_{n\geq 1}\sum_{\substack{R_1,\ldots,R_n\\ \hat{R}_i\subset\Lambda,\hat{R}_i\cap\hat{R}_j=\emptyset}}\prod_{i=1}^{n}\zeta_{R_i} , \tag{2.23}$$

where, for $R=\{U_1,\dots,U_k,\ell_1,\dots,\ell_m\}$

$$\zeta_R=\left\langle\prod_{U\in R}\Psi_U\prod_{\ell\in U}\Phi_\ell\prod_{k:A_k\subset\mathfrak{A}_R}v_k\prod_{k:A_k\subset\mathfrak{A}_R}\bar{v}_k\right\rangle_{\tilde{R}}\vartheta_R\ ,\tag{2.24}$$

where

$$\vartheta_R=\prod_{k\cdot A_k\subset\tilde{R}}\frac{1}{d_k}\prod_{\substack{I\text{ connected}\\ \text{component of }\tilde{R}}}\left(\frac{\hat{Z}_I}{\prod_{k:Q_k\subset I}\lambda_k}\right).\tag{2.25}$$

We call $\mathscr{R}_\Lambda(q_0,\lambda)$ the set of all polymers R with $\hat{R}\subset\Lambda$ and $\mathscr{R}_{\mathbb{Z}}(q_0,\lambda)$ the set of all polymers R with $\hat{R}$ finite subset of $\mathbb{Z}$, where we have put in evidence the dependence on the size q_0 of the central block and on the rate λ of increase of the blocks. The last dependence will be omitted when it is not important.

We want to distinguish the β and h dependence of $Z^\Lambda_{\beta,h}$ from that of the quantities λ_k, $\hat{Z}_I$, v_k, $\bar{v}_k$ appearing in Ξ which are introduced artificially in our polymer expansion. Therefore we introduce the polymer system with partition function

$$\Xi(\Lambda,\beta',\bar{\beta},h',\bar{h})\equiv 1+\sum_{n\geqq 1}\sum_{\substack{R_1,\dots,R_n\subset\mathscr{R}_\Lambda(q_0,\lambda)\\ \hat{R}_i\cap\hat{R}_j=\emptyset}}\prod_{i=1}^n\zeta_{R_i}(\beta',\bar{\beta},h',\bar{h})\ ,\tag{2.26}$$

where $\zeta_R(\beta',\bar{\beta},h',\bar{h})$ is defined as in Eqs. (2.24), (2.25) but Ψ_U, Φ_ℓ, $\hat{H}_I$, $\bar{H}(\alpha_k)$ are evaluated for $\beta=\beta'$, $h=h'$, whereas v_k, $\bar{v}_k$, λ_k, $\hat{Z}_I$ are evaluated for $\beta=\beta'$, $h=h'$. The quantities we are interested in can be expressed in terms of expectations of local observables [see Eq. (4.9) below]. It is evident that these last quantities do not depend on $\bar{\beta}$ and $\bar{h}$. Therefore, to prove our results, we shall only need to take derivatives with respect to β', h' for $\beta'=\bar{\beta}=\beta$, $h'=\bar{h}$.

3. Probability Estimates

We start with a proposition that can be obtained by a simple adaptation of the methods of [4].

3.1. Proposition. *If q_0 is sufficiently large and $\eta_k=|B_k|^{-\frac{\varrho}{4}}$ for a suitable $\varrho>0$ there exists a positive constant c_1 and two positive functions $\bar{v}_k$ on $\mathscr{S}_{A_k}$ and v_{k+1} on $\mathscr{S}_{A_{k+1}}$ such that for every $\alpha\in\mathscr{S}_{A_k}$, $\alpha'\in\mathscr{S}_{A_{k+1}}$,*

$$\begin{aligned}&\text{(i)}\quad |\bar{v}_k(\alpha)|\leqq\exp\left(c_1(\log|B_{k+1}|)^{3/4}\right)\ ,\\ &\text{(ii)}\quad |v_{k+1}(\alpha')|\leqq\exp\left(c_1(\log|B_{k+1}|)^{3/4}\right)\ ,\\ &\text{(iii)}\quad \mathbb{P}\left(\left|\frac{Z^{\alpha,\alpha'}_{B_k+1}}{\lambda_{2k+1}}-\bar{v}_k(\alpha)v_{k+1}(\alpha')\right|>\eta_{k+1}\right)\leqq\exp\left(-c_1(\log|B_{k+1}|)^{3/2}\right)\ .\end{aligned}\tag{3.1}$$

Proof. The proof can be obtained by applying the method of [4] Sect. 4 to the volume B_k with boundary conditions α and α', i.e. by dividing B_k into $2N_k+1$ blocks $\bar{A}^{(k)}_{-N_k},\dots,\bar{A}^{(k)}_{N_k}$, where $N_k=[|B_k|^\gamma]$, cutting the non-nearest neighbour block interactions and applying the transfer matrix result of Proposition A.1 of [4]. The constants ϱ, γ are required to verify

$$0<\varrho<(2\varepsilon(1-\gamma)-\gamma)/10\ . \quad \square \tag{3.2}$$

By using Proposition 3.1 we can have an estimate for the expectation of the p-th power of $|\Psi_U(\alpha,\alpha')|$. Indeed if $U=\{A_k, A_{k+1}\}$,

$$\begin{aligned}\mathbb{E}(|\Psi_U(\alpha,\alpha')|^p) &= \mathbb{E}(|\Psi_U(\alpha,\alpha')|^p \chi_{\Psi_U(\alpha,\alpha')\leqq\mu_k}) + \mathbb{E}(|\Psi_U(\alpha,\alpha')|^p \chi_{\Psi_U(\alpha,\alpha')>\eta_k})\\ &\leqq \mu_k^p + \mathbb{E}(|\Psi_U(\alpha,\alpha')|^{2p})^{1/2}\,\mathbb{P}(|\Psi_U(\alpha,\alpha')|>\mu_k)^{1/2}\\ &\leqq |B_{k+1}|^{-\frac{\varrho p}{4}} + (e^{c_2 4p^2} + e^{4pc_1(\log|B_{k+1}|)^{3/4}})^{1/2}\\ &\quad \cdot \exp\left(-\frac{c_1}{2}(\log|B_{k+1}|)^{3/2}\right),\end{aligned} \tag{3.3}$$

where

$$\chi_{a\leqq b}(\omega)=\begin{cases}1 & \text{if } a\leqq b\\ 0 & \text{if } a>b\end{cases}.$$

The last estimate is obtained by

$$\begin{aligned}\mathbb{E}(|\Psi_U(\alpha,\alpha')|^{2p}) &\leqq \mathbb{E}\left(\left(\frac{Z_{B_{k+1}}^{\alpha,\alpha'}}{\lambda_{2k+1}}\right)^{2p}\right) + \mathbb{E}((\bar{v}_k(\alpha)v_{k+1}(\alpha'))^{2p})\\ &\leqq \langle \mathbb{E}(\exp(2p(W_{A_k,B_{k+1}}(\alpha,\beta_{k+1}) + W_{B_{k+1},A_{k+1}}(\beta_{k+1},\alpha'))))\rangle_{H_{B_{k+1}}}\\ &\quad + e^{4p(\log|B_{k+1}|)^{3/4}},\end{aligned} \tag{3.4}$$

where at the last step we have used Jensen's inequality and the bounds (3.1) i), (3.1) ii). The expectation on the right-hand side of (3.4) can be evaluated so that we obtain (3.3) for a suitable constant c_2.

We want now to estimate the expectation of $|\Phi_\ell(\gamma_i,\gamma_j)|^p$ with respect to the interactions uniformly in the spin configurations γ_i and γ_j. For this we first evaluate the variances of the random variables $\bar{W}(\gamma_i,\gamma_j)$.

We have

$$\begin{aligned}\mathbb{E}(\bar{W}(\gamma_i,\gamma_j)^2) &\leqq c_0 \sum_{\substack{k\in Q_i\\ \ell\in Q_j}} |k-\ell|^{-2(1+\varepsilon)}\\ &\leqq c_0|Q_i|\,|Q_j| \Big/ \left(\sum_{m=i+1}^{j-1} q_0(|m|+1)^\lambda\right)^{2(1+\varepsilon)},\end{aligned} \tag{3.5}$$

where we have assumed $i<j$. It is easy to check that the right-hand side of (3.5) can be bounded for every choice of i,j by

$$\begin{aligned}&c_0 4^{2(1+\varepsilon)\lambda} q_0^{-2\varepsilon} \max(|i|,|j|)^{-2\lambda\varepsilon}|i-j|^{-2(1+\varepsilon)}\\ &= c_3 q_0^{-2\varepsilon}\max(|i|,|j|)^{-2\lambda\varepsilon}|i-j|^{-2(1+\varepsilon)} \equiv V_\ell\ .\end{aligned} \tag{3.6}$$

Let us now define for $\ell=\{Q_i,Q_j\}$, $\eta_\ell = V_\ell^{1/2}|\log V_\ell|$.

We have

$$\mathbb{E}(|\Phi_\ell(\gamma_i,\gamma_j)|^p) = \mathbb{E}(|\Phi_\ell|^p\chi_{|\Phi_\ell|\leqq 2\eta_\ell}) + \mathbb{E}(|\Phi_\ell|^p\chi_{|\Phi_\ell|>2\eta_\ell})\ . \tag{3.7}$$

If q_0 is sufficiently large,

$$\mathbb{P}(|\Phi_\ell| > 2\eta_\ell) \leqq \mathbb{P}(|\bar{W}(\gamma_i, \gamma_j)| > \eta_\ell) \leqq \exp\left(-\frac{(\log V_\ell)^2}{2}\right) \tag{3.8}$$

and

$$\mathbb{E}(|\Phi_\ell|^p) \leqq 2^p V_\ell^{p/2} (\log V_\ell)^p + e^{2c_4 V_\ell p^2 - \frac{(\log V_\ell)^2}{4}} . \tag{3.9}$$

Ratio of Normalizing Constants. In order to estimate the activities of the polymers we first consider the factors ϑ_R that appear in (2.24) and are defined in (2.25). The estimates (3.1) i), ii) imply:

$$d_k \leqq \exp\left(c_1((\log |B_k|)^{3/4} + (\log |B_{k-1}|)^{3/4})\right) . \tag{3.10}$$

We consider now for a given interval I the probability

$$\mathbb{P}\left(\frac{\hat{Z}_I}{\prod\limits_{k:Q_k \subset I} \lambda_k} \geqq \exp\left(\sum_{k:Q_k \subset I} (\log |Q_k|)^{3/4}\right)\right) . \tag{3.11}$$

The probability in (3.11) can be estimated by noticing that if $I = I(k,l) = A_k \cup B_{k+1} \cup A_{k+1} \cup \ldots \cup A_{k+\ell}$,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\hat{Z}_I}{\prod\limits_{k:Q_k \subset I} \lambda_k}\right)^p\right] &= \mathbb{E}\left(\left\langle \exp\left(\sum_{j=k}^{k+\ell-1} (\bar{W}(\alpha_j, \beta_{j+1}) + \bar{W}(\beta_{j+1}, \alpha_{j+1}))\right)\right\rangle_{H_I^0}^p\right) \\
&\leqq \left\langle \mathbb{E}\left(\exp p \sum_{j=k}^{k+\ell-1} (\bar{W}(\alpha_j, \beta_{j+1}) + \bar{W}(\beta_{j+1}, \alpha_{j+1}))\right)\right\rangle_{H_I^0} \\
&\leqq e^{2c_2 p^2 \ell} ,
\end{aligned} \tag{3.12}$$

where

$$H_I^0 = \sum_{k:Q_k \subset I} \bar{H}(\gamma_k) .$$

The probability in (3.11) is therefore less than

$$e^{2c_2 p^2 \ell} \exp\left(-p \sum_{k:Q_k \subset I} (\log |Q_k|)^{3/4}\right) . \tag{3.13}$$

If $m = \min\{|j| : k \leqq j \leqq k + \ell\}$ and we choose $p = (\log |Q_m|)^{3/4}/4c_2$, we see that (3.11) is less than

$$\exp\left(-\frac{(\log |Q_m|)^{3/2}}{8c_2}\right) \exp\left(-\frac{|2\ell - 1) 8 \log |Q_0|)^{3/2}}{8c_2}\right) . \tag{3.14}$$

By using the estimate (3.14) and adding over all intervals $I(k,l)$ we have if q_0 is sufficiently large,

$$\mathbb{P}\left(\bigcup_{k,\ell \geqq 0} \left(\frac{\hat{Z}_{I(k,\ell)}}{\prod\limits_{j:Q_j \subset I(k,\ell)} \lambda_j} \geqq \exp\left(\sum_{j:Q_j \subset I(k,\ell)} (\log |Q_j|)^{3/4}\right)\right)\right) \leqq e^{-c_5 (\log q_0)^{3/2}} \tag{3.15}$$

for a positive constant c_5. Therefore we have that outside a set of interactions whose probability is less than the right-hand side of (3.15) we have that

$$\vartheta_R \leqq \exp\left((2c_1+1) \sum_{Q_j \subset \tilde{R}} (\log |Q_j|)^{3/4}\right). \tag{3.16}$$

Estimates for the Activity of a Polymer. We start from the formula (2.24) and by using the bound (3.1) i), ii) and (3.16) we have

$$|\zeta_R| \leqq \exp\left((4c_1+1) \sum_{Q_k \subset \tilde{R}} (\log |Q_k|)^{3/4}\right) \left\langle \prod_{U \in R} |\Psi_U| \prod_{\ell \in R} |\Phi_\ell| \right\rangle_{\tilde{R}}. \tag{3.17}$$

For $U=\{A_k, B_{k+1}, A_{k+1}\}$ or $\ell=\{Q_i, Q_j\}$ we define t_U and t_ℓ by

$$t_U = |B_{k+1}|^{-\varrho/8}, \qquad t_\ell = |V_\ell|^{1-\vartheta}, \tag{3.18}$$

where $0<\vartheta<1/2$ [see (3.6) for the definition of V_ℓ]. We want to estimate

$$\mathbb{P}\left(\left\langle \prod_{U \in R} |\Psi_U| \prod_{\ell \in R} |\Phi_\ell| \right\rangle_{\tilde{R}} > \bar{t}_R\right), \tag{3.19}$$

where

$$\bar{t}_R = \prod_{U \in R} t_U \prod_{\ell \in R} t_\ell. \tag{3.20}$$

The probability in (3.19) is equal to

$$\mathbb{P}\left(g_{\bar{t}_R}\left(\left\langle \prod_{U \in R} |\Psi_U| \prod_{\ell \in R} |\Phi_\ell| \right\rangle_{\tilde{R}}\right) > \bar{t}_R\right), \tag{3.21}$$

where $g_\eta(x)$ is the function

$$g_\eta(x) = 2(|x| - \eta/2)\chi_{|x| \geqq \eta/2}. \tag{3.22}$$

Since $g_{\bar{t}_R}$ is a convex function, (3.21), by Jensen's inequality, is less than or equal to

$$\mathbb{P}\left(\left\langle g_{\bar{t}_R}\left(\prod_{U \in R} |\Psi_U| \prod_{\ell \in R} |\Phi_\ell|\right)\right\rangle_{\tilde{R}} > \bar{t}_R\right). \tag{3.23}$$

By Markov-Chebychev inequality and Fubini's theorem (3.23) can be bounded by

$$\frac{1}{\bar{t}_R}\left\langle \mathbb{E}\left(g_{\bar{t}_R}\left(\prod_{U \in R} |\Psi_U| \prod_{\ell \in R} |\Phi_\ell|\right)\right)\right\rangle_{\tilde{R}}. \tag{3.24}$$

We want now to find an estimate for the expectation in (3.24) uniform in the spin configuration. This can be easily obtained by applying inequalities (3.3) and (3.9). Indeed

$$g_{\bar{t}_R}(x) \leqq 2|x|\chi_{|x| \geqq \bar{t}_R/2}. \tag{3.25}$$

So by applying Schwarz inequality and again Markov-Chebychev inequality we get,

$$\mathbb{E}\left(g_{\bar{t}_R}\left(\prod_{U\in R}|\Psi_U|\prod_{\ell\in R}|\Phi_\ell|\right)\right)$$
$$\leqq 2^{\frac{p+1}{2}}\bar{t}_R^{-p/2}\mathbb{E}\left(\prod_{U\in R}|\Psi_U|^2\prod_{\ell\in R}|\Phi_\ell|^2\right)^{1/2}\mathbb{E}\left(\prod_{U\in R}|\Psi_U|^p\prod_{\ell\in R}|\Phi_\ell|^p\right)^{1/2}$$
$$=2^{\frac{p+1}{2}}\prod_{U\in R}\frac{\mathbb{E}(|\Psi_U|^2)^{1/2}\mathbb{E}(|\Psi_U|^p)^{1/2}}{t_U^{p/2}}\prod_{\ell\in R}\frac{\mathbb{E}(|\Phi_\ell|^2)^{1/2}\mathbb{E}(|\Phi_\ell|^p)^{1/2}}{t_\ell^{p/2}}$$

for an arbitrary $p>0$. It follows from (3.3) that given p if q_0 is sufficiently large and $U=\{A_k, B_{k+1}, A_{k+1}\}$,

$$\frac{\mathbb{E}(|\Psi_U|^2)^{1/2}\mathbb{E}(|\Psi_U|^p)^{1/2}}{t_U^{p/2}}\leqq|B_{k+1}|^{-p\varrho/16} . \tag{3.27}$$

Similarly we obtain from (3.9) that given p if $\ell=\{Q_i, Q_j\}$

$$\frac{\mathbb{E}(|\Phi_\ell|^2)^{1/2}\mathbb{E}(|\Phi_\ell|^p)^{1/2}}{t_\ell^{p/2}}\leqq q_0^{-\varepsilon p\vartheta/4}\frac{\max(|i|,|j|)^{-p\delta\varepsilon\vartheta/2}}{|i-j|^{(1+\varepsilon)p\vartheta/2}} \tag{3.28}$$

if q_0 is sufficiently large.

The previous estimates, for p sufficiently large, allow us to apply the Borel-Cantelli lemma so that, from Eqs. (3.17), (3.19), (3.21), (3.24), (3.27), (3.28) we get

3.2. Proposition. *Consider the system of polymers defined in Eqs.* (2.23), (2.24), (2.25). *For* $R\in\mathscr{R}_{\mathbb{Z}}(q_0,\lambda)$, *let*

$$\tilde{\zeta}_R=\exp\left((4c_1+1)\sum_{Q_k\subset\tilde{R}}(\log|Q_k|)^{3/4}\right)\prod_{U\in R}t_U\prod_{\ell\in R}t_\ell ,$$

where ϑ *is such that* $(1+\varepsilon)(1-\vartheta)>1$ *and* c_1, t_U, t_ℓ *are defined in Eqs.* (3.1), (3.6), (3.18). *Then for any* $\bar{p}>0$ *there exists* $\bar{q}$ *such that if* $q_0>\bar{q}$,

$$\mathbb{P}(\exists R\in\mathscr{R}_{\mathbb{Z}}(q_0,\lambda):|\zeta_R|>\tilde{\zeta}_R)\leqq q_0^{-\bar{p}} . \tag{3.29}$$

Now let

$$D(\omega)=\{q_0:|\zeta_R|<\tilde{\zeta}_R\,\forall R\in\mathscr{R}_{\mathbb{Z}}(q_0,\lambda)\} .$$

We define for every $\omega\in\Omega$, $q_0(\omega)$ as

$$q_0(\omega)=\begin{cases}\inf\{q:q'\in D(\omega)\text{ for every }q'\geqq q\}\\ \qquad\text{if this set is non-empty}\\ \infty\quad\text{otherwise}\end{cases} \tag{3.30}$$

From Proposition 3.2 we have that for every $\bar{p}$ there exists $\bar{\bar{q}}$ such that for $k>\bar{\bar{q}}$,

$$\mathbb{P}(q_0(\omega)>k)\leqq k^{-\bar{p}} , \tag{3.31}$$

In particular the set

$$\Omega_0=\{\omega\in\Omega\,|\,q_0(\omega)<\infty\} \tag{3.32}$$

is such that

$$\mathbb{P}(\Omega_0)=1 . \tag{3.33}$$

We can now prove the following

3.3. Proposition. *There exists a constant K such that for* $q \geqq \max\{K, q_0(\omega)\}$ *the partition function* $\Xi(\Lambda, \beta, h)$ *of the system of polymers defined by Eqs.* (2.23), (2.24), (2.25),

$$\Xi(\Lambda, \beta, h) = \exp \sum_{n \geqq 1} \sum_{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda)} \varphi^T(R_1, \ldots, R_n) \prod_{i=1}^{n} \zeta_{R_i} \tag{3.34}$$

with

$$\varphi^T(R_1, \ldots, R_n) = \frac{1}{n!} \sum_{\mathscr{G} \in \mathbb{G}_n(R_1, \ldots, R_n)} (-1)^{\#\{\text{edges in } \mathscr{G}\}}, \tag{3.35}$$

where $\mathbb{G}_n(R_1, \ldots, R_n)$ *is the set of connected graphs with n vertices and edges* $\{i, j\}$ *corresponding to pairs* R_i, R_j *such that* $\hat{R}_i \cap \hat{R}_j \neq \emptyset$. *We set the sum equal to zero if* $\mathbb{G}_n$ *is empty and to* 1 *if* $n = 1$.

Proof. If we take in (3.18) ϑ such that $(1+\varepsilon)(1-\vartheta) > 1$, then it is easy to get (see for example the proof of Lemma 2.3 of [6] or Lemma 2.1 of [5]) that there exists a constant K_1 such that

$$\sup_{j \in \mathbb{Z}} \sum_{\substack{R \in \mathscr{R}_{\mathbb{Z}}(q, \lambda) \\ Q_j \subset \hat{R}}} \tilde{\zeta}_R \leqq q^{-\min(\varepsilon(1-\vartheta)/2, \varrho/16)} \tag{3.36}$$

if $q > K_1$.

The result follows then for K sufficiently large from general arguments of the theory of cluster expansion for polymer systems (see [9, 2, 14]). □

4. Proofs of Theorems 1.1 and 1.2

The strategy of the proofs of Theorems 1.1 and 1.2 is similar to that for the analogous theorems in [6], but since here the polymer expansion is quite different we give the complete argument to be self-contained.

We shall make use of the following lemmas that will be proven at the end of this section.

4.1. Lemma. *Let* $\bar{\beta} = \beta' = \beta$, $\bar{h} = h' = h$. *For all positive integers* k_1, k_2 *there exists constants* M, $\bar{\lambda}$ *such that for* $\lambda < \bar{\lambda}$,

$$\sum_{n \geqq 1} \sum_{R_1, \ldots, R_n \in \mathscr{R}_{\mathbb{Z}}(q, \lambda)} \left| \varphi^T(R_1, \ldots, R_n) D^{k_1, k_2} \prod_{i=1}^{n} \zeta_{R_i}(\beta', \bar{\beta}, h', \bar{h}) \right| \leqq C_{k_1, k_2} q^{k_1 + k_2} \tag{4.1}$$

for all $q \geqq \max(q_0(\omega), M)$, *where* $q_0(\omega)$ *is a random constant satisfying the estimate* (3.31) *(it is not necessarily equal to the constant defined there) and in particular* $q_0(\omega) < \infty$ a.e. *and where, given a function* $f(\beta', \bar{\beta}, h', \bar{h})$, *we have put*

$$D^{k_1, k_2} f(\beta_1, \beta_2, h_1, h_2) = \frac{\partial^{k_1 + k_2}}{\partial \beta_1^{k_1} \partial h_2^{k_2}} f(\beta_1, \beta_2, h_1, h_2) \tag{4.2}$$

(C_{k_1, k_2} *is a suitable positive constant).*

4.2. Lemma. *For all positive integers* k_1, k_2 *there exist* $\bar{\lambda} > 0$, $M > 0$ *such that for* $\lambda < \bar{\lambda}$ *and if* $\Lambda = \Lambda_{p, p'} = \bigcup_{j=-2p}^{2p'} Q_j$, $\bar{\Lambda} = \Lambda_{\bar{p}, \bar{p}'} = \bigcup_{j=-2\bar{p}}^{2\bar{p}} Q_j$, *with* $0 < p \leqq \bar{p}$, $0 < p' \leqq \bar{p}'$,

$$\sum_{n\geqq 1} \sum_{\substack{R_1,\ldots,R_n\in\mathscr{R}_{\bar{\Lambda}}(q,\lambda) \\ R_j\notin\mathscr{R}_\Lambda(q,\lambda) \text{ for some } j \\ 1\leqq j\leqq n, Q_0\subset \hat{R}_1\cup\ldots\cup\hat{R}_n}} \left|\varphi^T(R_1,\ldots,R_n)D^{k_1\ k_2}\prod_{i=1}^n \zeta(\beta',\bar{\beta},h',\bar{h})\right|$$

$$\leqq C_{k_1,k_2} q^{k_1+k_2} \operatorname{dist}(Q_0,\partial\Lambda)^{-\varepsilon/2} \tag{4.3}$$

for $q\geqq \max(q_0(\omega), M)$, *where again* $q_0(\omega)$ *is a random constant satisfying the estimate* (3.31) *and* ε *is the exponent in* (1.2) *and* C_{k_1,k_2} *is a suitable positive constant.*

Now we are able to give bounds on the derivatives of the Gibbs averages of cylindrical functions.

4.3. Proposition. *Given* k_1, k_2 *and* $\lambda<\bar{\lambda}$, *where* $\bar{\lambda}$ *has been defined in Lemmas 4.1 and 4.2, for every interval* Δ *with center at the origin,* f *cylindrical function with support in* Δ *and*

$$\Lambda=\Lambda_{p,p'}=\bigcup_{j=-2p}^{2p'} Q_j\ ,\qquad \Delta\subset Q_0\ ,$$

$$\left|\frac{\partial^{k_1+k_2}}{\partial\beta^{k_1}\partial h^{k_2}}\langle f\rangle^\Lambda_{\beta,h}\right|\leqq \bar{c}_{k_1,k_2}\max(|\Delta|, q_0(\omega)^{k_1+k_2}\|f\|\ , \tag{4.4}$$

where $\|f\|=\sup_{s\in\mathscr{S}_\Delta}|f(s)|$, $q_0(\omega)$ *satisfies the estimate* (3.31) *and* $\bar{c}_{k_1,k_2}$ *is a suitable constant.*

Proof. From Eq. (2.1) we get

$$\langle f\rangle^\Lambda_{\beta,h}=\exp\left(\sum_{n\geqq 1}\sum_{\substack{R_1,\ldots,R_n\in\mathscr{R}_\Lambda(q,\lambda)\\ Q_0\subset\hat{R}_1\cup\ldots\cup\hat{R}_n}} \varphi^T(R_1,\ldots,R_n)\left(\prod_{i=1}^n \zeta^{(f)}_{R_1}-\prod_{i=1}^n\zeta_{R_i}\right)\right)\ , \tag{4.5}$$

where $\zeta^{(f)}_R$ is defined like ζ_R [see Eqs. (2.22), (2.24), (2.25)] with $\exp(-\beta H_{Q_0}(\gamma_0))$ replaced by $f(\gamma_0)\exp(-\beta H_{Q_0}(\gamma_0))$. Of course $\zeta^{(f)}_R=\zeta_R$ if $Q_0\not\subset\hat{R}$. The result is then obtained by applying Lemma 4.1. □

Proof of Theorem 1.1. We shall show that for every two positive integers k_1, k_2, given any cylindrical function f we have almost surely

$$\frac{\partial^{k_1+k_2}}{\partial\beta^{k_1}\partial h^{k_2}}(\langle f\rangle^{\bar{\Lambda}}_{\beta,h}-\langle f\rangle^\Lambda_{\beta,h}) \underset{\Lambda\to\mathbb{Z}}{\to} 0 \tag{4.6}$$

uniformly for β, h in bounded sets and uniformly in $\bar{\Lambda}\supset\Lambda$. We choose the origin of $\mathbb{Z}$ in the support Δ_b of f. We shall assume that Q_0 is so large that Q_0 contais Δ_f.

We consider $\Lambda, \bar{\Lambda}$ of the form

$$\Lambda=\bigcup_{j=-2p}^{2p'} Q_j\ ,\qquad \bar{\Lambda}=\bigcup_{j=-2\bar{p}}^{2\bar{p}'} Q_j$$

with $\bar{p}\geqq p$, $\bar{p}'\geqq p'$.

We can write

$$\langle f\rangle^{\bar{\Lambda}}_{\beta,h}-\langle f\rangle^\Lambda_{\beta,h}=\langle f\rangle^\Lambda_{\beta,h}(\exp(\tau_f)-1)-\langle f\rangle^{\bar{\Lambda}}_{\beta,h}(\exp(\tau)-1)\ , \tag{4.7}$$

where, for q as in Proposition 3.3,

$$\tau = \sum_{n \geqq 1} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_{\bar{\Lambda}}(q, \lambda) \\ Q_0 \subset \hat{R}_1 \cup \ldots \cup \hat{R}_n}} \varphi^T(R_1, \ldots, R_n) \prod_{i=1}^{n} \zeta_{R_i}$$

$$- \sum_{n \geqq 1} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_{\bar{\Lambda}}(q, \lambda) \\ Q_0 \subset \hat{R}_1 \cup \ldots \cup \hat{R}_n}} \varphi^T(R_1, \ldots, R_n) \prod_{i=1}^{n} \zeta_{R_i} ,$$

$$\tau_f = \sum_{n \geqq 1} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_{\bar{\Lambda}}(q, \lambda) \\ Q_0 \subset \hat{R}_1 \cup \ldots \cup \hat{R}_n}} \varphi^T(R_1, \ldots, R_n) \prod_{i=1}^{n} \zeta_{R_i}^{(f)}$$

$$- \sum_{n \geqq 1} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_{\bar{\Lambda}}(q, \lambda) \\ Q_0 \subset \hat{R}_1 \cup \ldots \cup \hat{R}_n}} \varphi^T(R_1, \ldots, R_n) \prod_{i=1}^{n} \zeta_{R_i}^{(f)} .$$

It follows from Lemma 4.2 and Proposition 4.3 that given two integers k_1, k_2 there exists a constant $\bar{C}_{k_1,k_2}$ such that

$$\frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} (\langle f \rangle_{\beta,h}^{\bar{\Lambda}} - \langle f \rangle_{\beta,h}^{\Lambda}) \leqq \bar{C}_{k_1,k_2} \max (|\Delta|, q_0(\omega))^{k_1+k_2} \|f\| \, |\Lambda|^{-\varepsilon/2} . \quad (4.8)$$

The theorem is so proven in the special case $\Lambda = \Lambda_{p,p'}$, $\bar{\Lambda} = \bar{\Lambda}_{\bar{p},\bar{p}'}$. The general case can be dealt with by taking care of the fact that the first and the last block need not belong to the canonical partition. We skip the exposition of this part of the argument, as it requires only trivial changes.

To prove the second part of Theorem 1.1, it is sufficient to remark that the uniform bound given by Eq. (4.4) is in $L^1(\Omega, \mathbb{P})$ as a consequence of the probability estimate of Eq. (3.31); the result follows then from the dominated convergence theorem. □

Proof of Theorem 1.2. For every finite interval Λ we can write

$$\begin{aligned} \frac{\partial}{\partial h} F_\Lambda(\beta, h) &= \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \langle s_i \rangle_{\beta,h}^{\Lambda} , \\ \frac{\partial}{\partial \beta} F_\Lambda(\beta, h) &= \frac{1}{|\Lambda|} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{J_{ij} \langle s_i s_j \rangle_{\beta,h}^{\Lambda}}{|i-j|^{1+\varepsilon}} . \end{aligned} \quad (4.9)$$

Then to prove the theorem it is sufficient to show that for every $k_1, k_2 \geqq 0$ almost surely and uniformly for β and h varying in finite intervals

$$\exists \lim_{\Lambda \to \mathbb{Z}} \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \left(\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \langle s_i \rangle_{\beta,h}^{\Lambda} \right) = \lim_{\Lambda \to \mathbb{Z}} \mathbb{E} \left[\frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \left(\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \langle s_i \rangle_{\beta,h}^{\Lambda} \right) \right] , \quad (4.10)$$

$$\exists \lim_{\Lambda \to \mathbb{Z}} \frac{\partial^{k_1}}{\partial \beta^{k_1}} \left(\frac{1}{|\Lambda|} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{J_{ij} \langle s_i s_j \rangle_{\beta,h}^{\Lambda}}{|i-j|^{1+\varepsilon}} \right) = \lim_{\Lambda \to \mathbb{Z}} \mathbb{E} \left(\frac{\partial^{k_1}}{\partial \beta^{k_1}} \left(\frac{1}{|\Lambda|} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{J_{ij}}{|i-j|^{1+\varepsilon}} \langle s_i s_j \rangle_{\beta,h}^{\Lambda} \right) \right) .$$

We define $\Delta_\Lambda, \Lambda_0$ by

$$\begin{aligned} \Delta_\Lambda &= \{i \in \Lambda / \text{dist}\,(i, \Lambda^c) \leqq |\Lambda|^{1/2}\} \ , \\ \Lambda_0 &= \Lambda \backslash \Delta_\Lambda \ . \end{aligned} \tag{4.11}$$

We have

$$\begin{aligned} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_i \rangle_{\beta,h}^{\Lambda} = & \frac{1}{|\Lambda|} \sum_{i \in \Delta_\Lambda} \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_i \rangle_{\beta,h}^{\Lambda} \\ & + \frac{1}{|\Lambda|} \sum_{i \in \Lambda_0} \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_i \rangle_{\beta,h}^{\mathbb{Z}} \\ & + \frac{1}{|\Lambda|} \sum_{i \in \Lambda_0} \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} (\langle s_i \rangle_{\beta,h}^{\Lambda} - \langle s_i \rangle_{\beta,h}^{\mathbb{Z}}) \ , \end{aligned} \tag{4.12}$$

where

$$\langle s_i \rangle_{\beta,h}^{\mathbb{Z}} = \lim_{\Lambda \uparrow \mathbb{Z}} \langle s_i \rangle_{\beta,h}^{\Lambda} \ . \tag{4.13}$$

Given $i \in \Lambda$ we can make the construction of blocks centered at i instead of the origin.

In force of Eq. (4.4) and of the probability estimate (3.31) we have that for an arbitrary $\bar{p}$ for Λ sufficiently large

$$\mathbb{P}\left(\exists i \in \Delta_\Lambda : \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_i \rangle_{\beta,h}^{\Lambda} \geqq |\Lambda|^{1/4}\right) \leqq |\Delta_\Lambda| \, |\Lambda|^{-\bar{p}} \ . \tag{4.14}$$

Similarly, using Eq. (4.8) and the probability estimate (3.31) we get that for every $\bar{p} > 0$ for $|\Lambda|$ large enough

$$\mathbb{P}\left(\exists i \in \Lambda_0 : \left| \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} (\langle s_i \rangle_{\beta,h}^{\Lambda} - \langle s_i \rangle_{\beta,h}^{\mathbb{Z}}) \right| > |\Lambda|^{-\varepsilon/4}\right) \leqq |\Lambda_0| \, |\Lambda|^{-\bar{p}} \ . \tag{4.15}$$

By the probability estimates (4.14) and (4.15), using Borel-Cantelli Lemma we are left to prove that almost surely

$$\lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} \sum_{i \in \Lambda_0} \frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_i \rangle_{\beta,h}^{\mathbb{Z}} = \mathbb{E}\left(\frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_0 \rangle_{\beta,h}^{\mathbb{Z}} \right) \ , \tag{4.16}$$

and this is a consequence of Birckhoff's theorem, since by Proposition 4.3 and the estimate (3.31) we know that

$$\frac{\partial^{k_1+k_2}}{\partial \beta^{k_1} \partial h^{k_2}} \langle s_0 \rangle_{\beta,h}^{\mathbb{Z}} \quad \text{is in} \quad L^1(\Omega, \mathbb{P}) \ .$$

This concludes the proof when we deal with the first possibility of (4.10). The case of the k_1-th derivative with respect to the inverse temperature β can be treated along the same lines. Indeed if we look at the expression

$$\frac{1}{|\Lambda|} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{J_{ij}}{|i-j|^{1+\varepsilon}} \frac{\partial^{k_1}}{\partial \beta^{k_1}} \langle s_i s_j \rangle_{\beta,h}^{\Lambda} \ ,$$

we see that the denominator $|i-j|^{1+\varepsilon}$ controls the behaviour at large distances and for small $|i-j|$ we are back to the previous situation. □

Proof of Lemmas 4.1 *and* 4.2. For simplicity of notation we consider the case when k_1 or $k_2=0$ and write D^k for $D^{k,0}$ or $D^{0,k}$. We have

$$D^k \sum_{n\geqq 1} \sum_{\substack{R_1,\dots,R_n\in\mathscr{R}_{\mathbb{Z}}(q_0,\lambda)\\ Q_0\subset \hat{R}_1\cup\dots\cup\hat{R}_n}} \varphi^T(R_1,\dots,R_n)\prod_{i=1}^n \zeta_{R_i}(\beta',\bar\beta,h',\bar h)$$

$$=\sum_{n\geqq 1} \sum_{\substack{R_1,\dots,R_n\\ Q_0\subset \hat{R}\cup\dots\cup\hat{R}_n}} \varphi^T(R_1,\dots,R_n)\sum_{\substack{k_1,\dots,k_n\\ \Sigma k_i=k}} \frac{k!}{k_1!\dots k_n!}\prod_{i=1}^n D^{k_i}\zeta_{R_i}(\beta',\bar\beta,h',\bar h)\ . \tag{4.17}$$

From Eqs. (2.24), (2.25) we have

$$D^m\zeta_R(\beta',\bar\beta,h',\bar h)$$
$$=\Big(\prod_{k:Q_k\subset\tilde R}\lambda_k^{-1}\Big)D^m\Big(\sum_{s\in\mathscr{S}_{\tilde R}}\prod_{k:A_k\in\mathfrak{A}_R} v_k(\alpha_k)\prod_{k:A_k\in\mathfrak{A}_R}\bar v_k(\alpha_k)\Big)$$
$$\cdot\Big(\prod_{\substack{I\subset\tilde R\\ \text{conn. comp.}}}\exp\hat H_I(s_I)\prod_{A_k\subset\tilde R}\exp\bar H(\alpha_k)\prod_{U\in R}\Psi_U\prod_{\ell\in R}\Phi_\ell\prod_{A_k\subset\tilde R}d_k^{-1}\Big)\ . \tag{4.18}$$

It is convenient to split $\hat H_I$ into its single components. For $I=A_k\cup B_{k+1}\cup\dots\cup A_{k+\ell}$, we write

$$\hat H_I=\sum_{j=k}^{k+\ell}\bar H(\alpha_j)+\sum_{j=k+1}^{k+\ell}\bar H(\beta_j)+\sum_{j=k}^{k+\ell-1}(\bar W(\alpha_k,\beta_{k+1})+\bar W(\beta_{k+\ell},\alpha_{k+\ell}))\ . \tag{4.19}$$

We set

$$\prod_{I\text{ conn. comp. of }\hat R}\exp\hat H_I(s_I)\prod_{A_k\subset\tilde R}\bar H(\alpha_k)\prod_{U\in R}\Psi_U\prod_{\ell\in R}\Phi_\ell\equiv\prod_{i=1}^t\varrho_i\ , \tag{4.20}$$

where ϱ_i can be one of the following quantities: $\exp\bar H(\alpha_k)$, $\exp\bar H(\beta_k)$, $\exp\bar W(\alpha_k,\beta_{k+1})$, $\exp\bar W(\beta_k,\alpha_k)$, Ψ_U, Φ_ℓ, d_k^{-1} and $t\equiv t(R)\leqq[\#\{Q_j\subset\tilde R\}]^4 n_\Phi(R)$ with $n_\Phi(R)=\#\{\ell:\ell\in R\}$.

We have

$$D^m\prod_{j=1}^{t(R)}\varrho_j=\sum_{m_1,\dots,m_t}\frac{m!}{m_1!\dots m_t!}\prod_{i=1}^{t(R)}D^{m_i}\varrho_i\ . \tag{4.21}$$

Now we shall give estimates on the derivatives of the possible ϱ_i's. We have for $\ell=\{i,j\}$,

$$\frac{\partial}{\partial h}\Phi_\ell=0$$

$$\left|\frac{\partial^m}{\partial\beta'^m}\Phi_\ell\right|=\left|\frac{\partial^m}{\partial\beta'^m}(e^{-\beta'W(\gamma_i,\gamma_j)}-1)\right|=|(-W(\gamma_i,\gamma_j))^m e^{-\beta'W(\gamma_i,\gamma_j)}|$$
$$\leqq\kappa_m|e^{\beta'\kappa_m'|W(\gamma_i,\gamma_j)|}-1|\equiv\tilde\Phi(\gamma_i,\gamma_j) \tag{4.22}$$

for suitable constants κ_m, κ'_m depending only on m. Moreover it is easy to see that

$$\begin{aligned}
&|D^m \exp \bar{H}_{Q_k}(\gamma_k)| \leqq \xi_k^{(m)}(\gamma_k) \exp \bar{H}_{Q_k}(\gamma_k) \ , \\
&|D^m \exp \bar{W}(\gamma_k, \gamma_{k+1})| \leqq \bar{\xi}_k^{(m)}(\gamma_k, \gamma_{k+1}) \exp \bar{W}(\gamma_k, \gamma_{k+1}) \ , \\
&|D^m \Psi_U(\alpha_k, \alpha_{k+1})| \leqq \hat{\xi}_k^{(m)}(\alpha_k, \alpha_{k+1}) \frac{Z_{B_{k+1}}^{\alpha_k, \alpha_{k+1}}}{\lambda_{2k+1}} \ , \\
&|D^m d_k^{-1}| \leqq \bar{\kappa}_m d_k^{-1} \ ,
\end{aligned} \tag{4.23}$$

for some $\xi_k^{(m)}, \bar{\xi}_k^{(m)}, \hat{\xi}_k^{(m)}, \bar{\kappa}_m$ depending on the J's where, given $\delta > 0$, and for $|Q_0|$ sufficiently large,

$$\begin{aligned}
&\mathbb{P}(\exists m \in \mathbb{N} : \sup_{\gamma_k} \xi_k^{(m)} > |Q_k|^{(1+\delta)m}) \leqq \exp(-|Q_k|^{\delta/2}) \ , \\
&\mathbb{P}(\exists m \in \mathbb{N} : \sup_{\gamma_k, \gamma_{k+1}} \bar{\xi}_k^{(m)}(\gamma_k, \gamma_{k+1}) > |Q_k|^{(1+\delta)m}) \leqq \exp(-|Q_k|^{\delta/2}) \ , \\
&\mathbb{P}(\exists m \in \mathbb{N} : \sup_{\alpha_k, \alpha_{k+1}} \hat{\xi}_k^{(m)}(\alpha_k, \alpha_{k+1}) > |B_k|^{(1+\delta)m}) \leqq \exp(-|B_k|^{\delta/2}) \ , \\
&\mathbb{P}(\exists m \in \mathbb{N} : \bar{\kappa}_m > |A_k|^{(1+\delta)m}) \leqq \exp(-|A_k|^{\delta/2}) \ .
\end{aligned} \tag{4.24}$$

For example for $\xi^{(m)}(\gamma_k)$ we can take

$$\xi_k^{(m)}(\gamma_k) = \max \left\{ \left| \sum_{i \in Q_k} s_i \right|^m, \left| \sum_{i \neq j \in Q_k} J_{ij} s_i s_j |i-j|^{-(1+\varepsilon)} \right|^m \right\} ,$$

and the relative probability estimate follows immediately from the assumption (1.1 ii) on the distribution of the J's.

It follows from the probability estimates (4.24) and Borel-Cantelli's lemma that there exists a random variable $\tilde{q}_0(\omega)$ verifying the estimate (3.31) such that if $|Q_0| \geqq \tilde{q}_0(\omega)$ for every $k \in \mathbb{Z}$ and every $m \in \mathbb{N}$,

$$\begin{aligned}
&|D^m \exp \bar{H}_{Q_k}(\gamma_k)| \leqq |Q_k|^{(1+\delta)m} \exp \bar{H}_{Q_k}(\gamma_k) \ , \\
&|D^m \exp \bar{W}(\gamma_k, \gamma_{k+1})| \leqq |Q_k|^{(1+\delta)m} \exp \bar{W}(\gamma_k, \gamma_{k+1}) \ , \\
&|D^m \Psi_U(\alpha_k, \alpha_{k+1})| \leqq |B_k|^{(1+\delta)m} \frac{Z_{B_{k+1}}^{\alpha_k, \alpha_{k+1}}}{\lambda_{2k+1}} \ , \\
&|D^m d_k^{-1}| \leqq |A_k|^{(1+\delta)m} d_k^{-1} \ .
\end{aligned} \tag{4.25}$$

Now, from (4.18), (4.20), (4.22), (4.25), for $\omega \in \bar{\Omega}_0$, if $n_R = \max\{|n| : Q_n \subset \tilde{R}\}$, we have

$$D^m \zeta_R(\beta', \bar{\beta}, h', \bar{h}) \leqq \sum_{m_1 + \ldots + m_{t(R)} = m} \frac{m!}{m_1! \ldots m_t!} \vartheta_R$$
$$\cdot \left\langle \prod_{U \in R} \Psi_U^{\mathbf{m}} \prod_{\ell \in R} \tilde{\Phi}_\ell \prod_{k : A_k \in \mathfrak{A}_R} v_k \prod_{k : A_k \in \mathfrak{A}_R} \bar{v}_k \right\rangle_{\tilde{R}} (q_0 n_R^\lambda)^{m(1+\delta)} \ , \tag{4.26}$$

where $\mathbf{m}=(m_1,\ldots,m_{t(R)})$ and for $U=\{A_k,B_{k+1},A_{k+1}\}$:

$$\Psi_U^{\mathbf{m}}=\begin{cases}|\Psi_U| & \text{if} \quad \Psi_U=\varrho_i \text{ with } m_i\geqq 1\\ \dfrac{1}{(|k|+1)^\lambda}\dfrac{Z_{B_{k+1}}^{\alpha_k,\alpha_{k+1}}}{\lambda_{2k+1}} & \text{otherwise} .\end{cases}$$

Now for any bond b of R we define the positive number n_b in the following way:

$$\text{for} \quad b=U=\{A_k,B_{k+1},A_{k+1}\} \qquad n_b=4^{\lambda(1+\delta)m} ,$$

$$\text{for} \quad b=\ell=\{Q_k,Q_{k'}\} \qquad n_b=(2|k-k'|)^{\lambda(1+\delta)m} .$$

It is easy to check that:

$$(n_R)^{\lambda m(1+\delta)}\leqq\prod_{b\in R} n_b ,$$

and so:

$$D^m\zeta_R(\beta',\bar\beta,h',\bar h)\leqq q_0^{m(1+\delta)}t^m\vartheta_R$$
$$\cdot\sum_{m_1+\ldots+m_t=m}\frac{m!}{m_1!\ldots m_t!}\left\langle\prod_{U\in R}n_U\Psi_U^{\mathbf{m}}\prod_{\ell\in R}n_\ell\tilde\Phi_\ell\prod_{\substack{k\\A_k\in\mathfrak{A}_R}}v_k\prod_{\substack{k\\A_k\in\tilde{\mathfrak{A}}_R}}\bar v_k\right\rangle_{\tilde R} . \tag{4.27}$$

The number of terms in the sum on the right-hand side of Eq. (4.27) is bounded by $\#\{\ell:Q_\ell\subset\tilde R\}^{4m}n_\Phi(R)^m$, and so there exists a positive constant γ_m such that:

$$D^m\zeta_R(\beta',\bar\beta,h',\bar h)\leqq q_0^{m(1+\delta)}\vartheta_R\gamma_m ,$$
$$\sup_{\mathbf{m}}\left\langle\prod_{U\in R}(2n_U\Psi_U^{\mathbf{m}})\prod_{\ell\in R}(2n_\ell\tilde\Phi_\ell)\prod_{k:A_k\in\mathfrak{A}_R}v_k\prod_{k:A_k\in\tilde{\mathfrak{A}}_R}\bar v_k\right\rangle_{\tilde R} . \tag{4.28}$$

Now by the same methods that have been used in Sect. 2 we can give good estimates, uniform in m, of the quantity

$$\left\langle\prod_{U\in R}(2n_U\Psi_U^{\mathbf{m}})\prod_{\ell\in R}(2n_\ell\tilde\Phi_\ell)\prod_{k:A_k\in\mathfrak{A}_R}v_k\prod_{k:A_k\in\tilde{\mathfrak{A}}_R}\bar v_k\right\rangle_{\tilde R}$$

that hold with high probability, provided the following condition on λ is satisfied:

$$\lambda(1+\delta)m<\varepsilon . \tag{4.29}$$

We fix $\delta=1$ and we choose

$$\bar\lambda=\frac{\varepsilon}{8k} , \tag{4.30}$$

where k is the order of derivation [see Eq. (4.17)] to control the number of terms in the last sum of the right-hand side of Eq. (4.17). We proceed as in the bound (4.28), namely we extract from the activity of each polymer a factor 1/2 and again we bound $(1/2)^n n^k$ by a constant depending only on K. Lemma 4.1 follows from Eq. (4.28) and from an easy adaptation of Propositions 3.1, 3.2. Lemma 4.2 is a corollary of Lemma 4.1 that can be obtained by standard methods of cluster expansion (see for instance [2]). □

Appendix. Proof of Theorem 1.4

We note that if $\varphi(x)=|x|^{-1-\delta}$ for $x\in\mathbb{Z}$ with $\delta>0$, then there is C_1 such that for every $x\in\mathbb{Z}$,

$$\varphi^{*2}(x)=\sum_{y\in\mathbb{Z}}\varphi(x-y)\varphi(y)=\sum_{y:|x-y|\leqq\frac{|x|}{3}}\varphi(x-y)\varphi(y)$$

$$+\sum_{y:|y|\leqq\frac{|x|}{3}}\varphi(x-y)\varphi(y)+\sum_{\substack{y:|x-y|>\frac{|x|}{3}\\ |y|>\frac{|x|}{3}}}\varphi(x-y)\varphi(y)$$

$$\leqq C_1|x|^{-1-\delta}\ , \tag{A.1}$$

so that

$$\varphi^{*k}(x)\leqq C_1^{k-1}|x|^{-1-\delta}\ ,$$

as it is easy to get by estimating the three terms into which we have divided the sum.

Let us now consider the two point truncated correlation functions in a volume $\Lambda=\Lambda_{p,p'}$

$$\langle s_0,s_j\rangle_{\beta,h}^{T,\Lambda}=\langle s_0,s_j\rangle_{\beta,h}^{\Lambda}-\langle s_0\rangle_{\beta,h}^{\Lambda}\langle s_j\rangle_{\beta,h}^{\Lambda}\ . \tag{A.2}$$

We can write

$$\langle s_0,s_j\rangle_{\beta,h}^{T,\Lambda}=\frac{\partial^2}{\partial t_1\partial t_2}\log Z_{h,\beta,t_1,t_2}^{\Lambda}|_{t_1=t_2=0}\ , \tag{A.3}$$

where $Z_{h,\beta,t_1,t_2}^{\Lambda}$ is the partition function in Λ for the Hamiltonian

$$H_{\Lambda}^{t_1,t_2}(s)=H_{\Lambda}(s)+t_1s_0+t_2s_j\ . \tag{A.4}$$

If we perform our cluster expansion for $Z_{h,\beta,t_1,t_2}^{\Lambda}$, we see that if $q\geqq q_0(\omega)$, (A.3) can be expanded as

$$\langle s_0s_j\rangle_{\beta,h}^{T\Lambda}=\sum_{n=1}^{\infty}\sum_{\substack{R_1,\dots,R_n\in\mathscr{R}_{\Lambda}(q,\lambda)\\ \{0,j\}\subset\hat{R}_1\cup\dots\cup\hat{R}_n}}\varphi^T(R_1,\dots,R_n)\bar{\zeta}(R_1)\dots\bar{\zeta}(R_n)\ , \tag{A.5}$$

where $\bar{\zeta}(R)$ is bounded by $\tilde{\zeta}(R)$ defined in Proposition 3.1 for $q\geqq q_0(\omega)$. Given $\varepsilon'>0$ we choose ϑ in Proposition 3.2 so that $(1+\varepsilon)(1-\vartheta)>1+\varepsilon-\varepsilon'/4$ and λ so that

$$(1+\varepsilon-\varepsilon'/2)/(1+\lambda)\geqq(1+\varepsilon-\varepsilon')\ . \tag{A.6}$$

We want to prove that there exists a constant K such that the sum in (A.5) can be bounded if $q\geqq\max\,(k,q_0(\omega))$ by

$$C(q)|j|^{-(1+\varepsilon-\varepsilon')} \tag{A.7}$$

uniformly in j and in Λ.

We start from an estimate for sums of activities $\tilde{\zeta}$. We claim that there exists a constant K_1 such that if $q\geqq\max\,(K_1,q_0(\omega))$, then

$$\sum_{\substack{Q_0\cup Q_m\subset\hat{R}\\ R\in\mathscr{R}_{\mathbb{Z}}(q,\lambda)}}|\tilde{\zeta}(R)|\leqq|m|^{-1-\varepsilon+\varepsilon'/2}\ . \tag{A.8}$$

Indeed, since R is connected, there must be a chain made out of bonds of type U or ℓ connecting Q_0 to Q_m. The contribution of each bond can be bounded for $q \geqq q_0(\omega)$ by the number of blocks between Q_0 and Q_m to the power $-1-\varepsilon+\varepsilon'/2$ times a constant that goes to 0 as q tends to infinity. We first sum over the remainder of the bonds that are connected to the chain and this gives a constant to the number of bonds in the chain. Then we sum over all possible chains, and thanks to the estimate on each bond and to the estimate (A.1) with $\delta=\varepsilon-\varepsilon'/4$, we get the result for $q \geqq q_0(\omega)$ and sufficiently large.

We now consider the sum on the right-hand side of (A.5). In order to get the estimate we apply the method used in Sect. 3 of [2]. Let j belong to the block Q_m then, using the method of [2], (note that here we incorporate in $\varphi^T(R_1, \ldots, R_m)$ the factor $1/n!$ that in [2] is kept separate) we get

$$\sum_{n=1}^{\infty} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda) \\ \{0, j\} \subset \hat{R}_1 \cup \ldots \cup \hat{R}_n}} |\varphi^T(R_1, \ldots, R_n) \bar{\zeta}(R_1) \ldots \bar{\zeta}(R_n)|$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda) \\ Q_0 \cup Q_n \subset \hat{R}_1 \cup \ldots \cup \hat{R}_n}} |\varphi^T(R_1, \ldots, R_n) \bar{\zeta}(R_1) \ldots \bar{\zeta}(R_n)|$$

$$\leqq \sum_{n=1}^{\infty} n \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda) \\ Q_0 \cup Q_m \subset \hat{R}_1}} |\varphi^T(R_1, \ldots, R_n) \bar{\zeta}(R_1) \ldots \bar{\zeta}(R_n)|$$

$$+ \sum_{n=1}^{\infty} n(n-1) \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda) \\ Q_0 \subset \hat{R}_1 \\ Q_m \subset R_2}} |\varphi^T(R_1, \ldots, R_n) \bar{\zeta}(R_1) \ldots \bar{\zeta}(R_n)| \;. \tag{A.9}$$

Let us consider the second term on the right-hand side of (A.9). The first one can be treated in the same way and presents less difficulties.

Following [2] we have

$$\sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda) \\ Q_0 \subset \hat{R}_1, Q_m \subset \hat{R}_2}} |\varphi^T(R_1, \ldots, R_m) \bar{\zeta}(R_1) \ldots \bar{\zeta}(R_n)|$$

$$\leqq \frac{1}{n!} \sum_{t \in T_n} \sum_{\substack{R_1, \ldots, R_n \in \mathscr{R}_\Lambda(q, \lambda) \\ Q_0 \subset \hat{R}_1, Q_m \subset \hat{R}_2 \\ g(R_1, \ldots, R_n) \supset t}} |\bar{\zeta}(R_1) \ldots \bar{\zeta}(R_n)| \;, \tag{A.10}$$

where T_n is the set of the trees on $\{1, \ldots, n\}$ and $g(R_1, \ldots, R_n)$ is the graph of connections of the set of polymers $R_1, \ldots, R_n$.

Given a tree $t \in T$ there is a sequence $i_1, \ldots, i_{k(t)}$ such that $i_1=1, \ldots, i_{k(t)}=2$, and for $j=1, \ldots, k(t)-1$ is a connection of t. Therefore if $g(R_1, \ldots, R_n) \supset t$, there is a sequence of blocks $Q_{\ell_1}, \ldots, Q_{\ell_{k(t)-1}}$ such that $Q_{\ell_1} \subset \hat{R}_{i_1} \cap \hat{R}_{i_2}, \ldots, Q_{\ell_{k(t)-1}} \subset \hat{R}_{i_{k(t)-1}} \cap \hat{R}_{i_{k(t)}}$.

We can choose among such sequences the least one in a given lexicographic order.

Therefore we can bound (A.10) by

$$\frac{1}{n!}\sum_{t\in T_n}\sum_{Q_{\ell_1},\ldots,Q_{\ell_{k(t)-1}}}\sum_{\substack{R_1,\ldots,R_n\in\mathscr{R}_\Lambda(q,\lambda)\\ Q_0\cup Q_{\ell_1}\subset\hat{R}_1=\hat{R}_{i_1}\\ Q_{\ell_1}\cup Q_{\ell_2}\subset\hat{R}_{i_2}\\ Q_{\ell_{k(t)-1}}\cup Q_m\subset\hat{R}_2=\hat{R}_{i_{k(t)}}}}|\bar{\zeta}(R_1)\ldots\bar{\zeta}(R_m)|\ . \tag{A.11}$$

At this point we can follow directly the proof of [2] Sect. 3 so that we don't give the details. Using the remark (A.1), we get that for $q\geqq q_0(\omega)$ and sufficiently large, (A.8) can be bounded by

$$\bar{C}(q)|m|^{-1-\varepsilon+\varepsilon'/2} \tag{A.12}$$

uniformly in Λ. But we have $|j|\leqq\bar{\bar{C}}(q)|m|^{1+\lambda}$ from the definition of m and the choice (A.6) for λ implies the bound (A.7). □

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