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The Integration of G-Invariant Functions and the Geometry of the Faddeev-Popov Procedure

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Abstract. Various versions of the Fubini theorem on the principal fibre bundle are derived. The formal generalizations of these theorems are used as a basic tool for the investigation of the geometrical setting of the Faddeev-Popov procedure in the cases of pure Yang-Mills theories, spinless particle and Polyakov's bosonic string.

1. Introduction

In general the gauge theories can be divided into two classes: those which are invariant under true gauge group transformations and those which are invariant under infinitesimal transformations whose commutators do not necessarily close and involve field-dependent structure constants. There exists a general approach to functional quantization of both types of gauge theories developed along Hamiltonian lines by Fradkin and his collaborators [1] and recently reviewed by Henneaux [2]. This approach is based on a functional integral over paths in the canonical phase space suitably extended by additional ghost degrees of freedom. Due to the so-called Fradkin theorem [1] such an integral is formally equivalent to the phase-space functional integral over physical (independent) degrees of freedom of a constrained system [3]. On the other hand in the case of the first type of gauge theories there is a covariant functional approach introduced by Faddeev and Popov in the case of Yang-Mills theories [4]. One starts with the functional integral over configurations of fields and then by the so-called Faddeev-Popov trick extracts the volume of the gauge group from this integral. The functional integral obtained in this way leads to the perturbative expansion with correct Feynman diagrams [5]. Soon after the discovery of the Gribov ambiguity [6] it was recognised that its origin is closely related to the global geometry of the space of fields [7]. It is known that under some assumptions about an action of the gauge group the space of connections $\mathscr{C}$ has a structure of a principal fibre bundle over the orbit space with the group $\mathscr{G}$ of local gauge transformations as a structure group [8, 9]. In this bundle there exists a connection determined by a natural

$\mathscr{G}$-invariant riemannian metric on $\mathscr{C}$. By means of this connection one can construct an induced metric on the orbit space. Babelon and Viallet [10] have shown that it is possible to provide the geometrical interpretation of the Faddeev-Popov determinant in terms of this induced riemannian structure. Their interpretation is based on a heuristic approach to the Feynman integral in which functional measures are introduced by means of volume ∞-forms (which are of course completely formal objects) related to the riemannian structures on the space of fields. More recently simmilar ideas appeared in a series of papers by Fujikawa [11], where it was shown that the anomalies can better understood as arising from the transformation properties of the functional measure. The geometrical approach sketched above plays an especially important role in Polyakov's quantization of the string [12]. In the case of $d<26$ it leads to the Liouville theory [13] and at the critical dimension it can be used to determine the multiloop amplitudes [14–17]. Therefore it seems to be important to understand the geometrical setting of the functional quantization of the first type of gauge theories as well as the geometry of the Faddeev-Popov procedure. In order to achieve this goal we must have at hand some unified and unambiguous method of dealing with the functional integral which takes into account the riemannian structures on the spaces of fields and the global geometrical properties of these spaces. In our previous paper [18] such a method based on the geometrical ideas mentioned above has been proposed and applied to provide the geometrical interpretation of the Faddeev-Popov determinant in the case of Polyakov's bosonic string in $d<26$. The aim of this paper is to develop this approach using a more general framework in a wider class of theories. It allows us to clarify some aspects of the Faddeev-Popov procedure in Polyakov's theory omitted in [18]. Moreover it makes possible the formulation of a conjecture about the general scheme of the covariant quantization of the first type gauge theories. It is important to verify this conjecture in the case of the euclidean gravity. Discussion of this point is far beyond the scope of this paper and will not be included.

Let us stress some basic points of the present approach. It is clear that we need some kind of measures on infinite-dimensional manifolds. The well defined Gaussian measures on abstracts Wiener manifolds [19] are however insufficient in many cases for quantum field theory. So in physics the commonly used approach to the functional integration is the heuristic one based on the analogy with the finite dimensional case. Let us note that the approach recently introduced by Polchinski [20, 21] in which the measure on infinite-dimensional manifold is implicitly defined by means of a Gaussian measure in the tangent space at each point is also based on this analogy. The basic tool in all present discussions of the Faddeev-Popov procedure in the case of Polyakov's string [13–15, 21–23] is the change of variables in a functional measure and therefore it has local character. Extending the ideas of Schwarz [24] and Babelon and Viallet [10] we propose to proceed a little further and use instead of the change of variables a formal generalization of various versions of the Fubini theorem on manifolds.

After this brief motivation of methods used in this paper let us summarize its content. In Sect. 2 the integration of a G-invariant function on a trivial finite-dimensional fibre bundle is considered. Three types of integrals are introduced and some theorems are formulated which allow us to express these integrals in terms of the global gauge slice and induced riemannian metric. In order to make this paper self-contained the proofs of the theorems are outlined in the appendix. In Sect. 3 our geometrical method is presented in the most familiar case of the Yang-Mills theory, and the results of Babelon and Viallet [10] are obtained. In Sect. 4 the covariant functional quantization of the scalar spinless particle is considered. In this case appropriate functional integrals can be evaluated and one can establish the correct form of the functional representation of the particle propagator. It is important because it provides some hints how to quantize the bosonic string. In Sect. 5 our geometrical method is applied to the detailed discussion of the Faddeev-Popov procedure in Polyakov's theory of the bosonic string. The results of [18] are completed by discussion of the measure on the Teichmuller space in the case of multiloop amplitude (genus>1) at $d<26$ and at the critical dimension $d=26$. The results of this section agree with those obtained recently by Polchinski's method in [20–22]. In Sect. 6, taking into consideration the results of previous sections, the general scheme of the covariant quantization of the first type of gauge theories is formulated and some open problems are listed.

2. Integration of G-Invariant Functions

In this section we will consider the integration of a G-invariant function on a finite-dimensional trivial principal fibre bundle. After formal generalizations to the infinite-dimensional case the results presented here provide the geometrical interpretation of the Faddeev-Popov procedure. Let us note that in many cases (pure Yang-Mills theory [7, 8], Polyakov's string and euclidean gravity [25]) the quotient of the space of fields by the group of local gauge transformations is a nontrivial principal fibre bundle. The most convenient (and in fact the unique) way to parametrize the orbit space in gauge theories is to use the gauge fixing conditions which can be interpreted as local sections of an appropriate fibre bundle. Therefore in the infinite-dimensional case one can consider by means of the Faddeev-Popov trick the integration only on trivializations of some fibration, and our restriction to trivial bundles is justified. All theorems presented here result from the Fubini theorem on a manifold [26]. The proofs of these theorems are inessential for our considerations and their sketches are presented in the appendix.

We use the following objects: G is an n-dimensional compact Lie group, $P(U, G, \pi')$ is the trivial principal fibre bundle over U with structure group G and the projection $\pi': P \to U \approx P/G$, $R_a: p \in P \to p \cdot a \in P$ $(a \in G)$ is the right action of G on P. We consideer a G-invariant riemannian metric g on P $(\forall a \in G: R_a^* g = g)$. All objects considered here are assumed to be sufficiently smooth.

One can construct a connection on P related to the metric g and given by the splitting [8, 10]:

$$T_p P = V_p \oplus V_p^\perp , \tag{2.1}$$

where $V_p^\perp$ is the horizontal subspace at the point $p \in P$ defined as the orthogonal (with respect to the metric g) complement of the vertical subspace $V_p \equiv T_p \pi'^{-1}(p) \subset T_p P$ tangent to the fiber $\pi'^{-1}(p)$ at the point p. In order to find the explicit expression of the connection form α of the connection defined above, let us introduce the family of mappings $\{\tau_p\}_{p\in P}$:

$$\tau_p : G' = T_eG \to V_p\,, \qquad \tau_p \equiv \beta_{p*}|_e\,, \tag{2.2}$$

where

$$\beta_p : G \to \pi'^{-1}(p)\,, \qquad \beta_p(a) \equiv R_a p = p \cdot a \tag{2.3}$$

(e is the neutral element of G and G' is the Lie algebra of G). Introducing some positively defined nondegenerate inner product $\tilde{h}$ in G' one can define the family of adjoint operators $\{\tau_p^+\}_{p\in P}$:

$$\tau_p^+ : T_pP \to G'\,, \qquad \tilde{h}(\delta a, \tau_p^+ \delta p) = g_p(\tau_p \delta a, \delta p)\,, \tag{2.4}$$

where $\delta a \in G'$, $\delta p \in T_pP$. It is easy to check that the form α defined at every point $p \in P$ by:

$$\alpha_p \equiv (\tau_p^+ \tau_p)^{-1} \tau_p^+$$

is a connection form on P and $\ker \alpha_p = V_p^\perp$. Let us note that α is independent on the inner product $\tilde{h}$ used in the definition of τ_p^+. Starting with a G-invariant riemannian structure g on P one can construct the riemannian metric $\tilde{g}$ on the space of orbits U. Suppose $\delta u, \delta u' \in T_uU$, p is a point in the fiber over u ($p \in \pi'^{-1}(u)$) and $\delta\tilde{u}, \delta\tilde{u}'$ are horizontal lifts (with respect to the connection α) of $\delta u, \delta u'$ at p:

$$\pi'_* \delta\tilde{u} = \delta u\,, \qquad \pi'_* \delta\tilde{u}' = \delta u'\,,$$
$$\alpha_p \delta\tilde{u} = 0\,, \qquad \alpha_p \delta\tilde{u}' = 0\,.$$

We define the metric $\tilde{g}$ by:

$$\tilde{g}_u(\delta u, \delta u') \equiv g_p(\delta\tilde{u}, \delta\tilde{u}')\,. \tag{2.5}$$

From the G-invariance of the metric and properties of the horizontal lifts it follows that the right-hand side of (2.5) is independent of the choice of p in $\pi'^{-1}(u)$.

Now let us proceed to the discussion of integration of G-invariant functions f on P ($\forall a \in G : R_a^* f = f$). Such a function is constant along the fibers of P and can be considered as a function on the orbit space U. For the metric g on P we have the following natural definition of integral of f over U:

$$I_1[f] \equiv \int_U f d\omega^{\tilde{g}}\,, \tag{2.6}$$

where $d\omega^{\tilde{g}}$ denotes the riemannian volume element related to the metric $\tilde{g}$ defined above. Our aim is to mimic the Faddeev-Popov procedure in the finite-dimensional case, so we must express the integral $I_1[f]$ in terms of some global section $\sigma : U \to P$ of P and the metric g on P. In order to do it, let us introduce some additional constructions.

It is convenient to consider the isomorphic fibration

$$P(\Sigma = \sigma(U), G, \pi = \sigma \circ \pi')$$

instead of $P(U, G, \pi')$ so we can treat the base space Σ as a submanifold of the bundle P. Let us consider the following family of submanifolds of P $\{\Sigma_a\}_{a\in G}$,

$\Sigma_a \equiv R_a \Sigma = \Sigma \cdot a$. At every point $p = u \cdot a \in P$ ($u \in \Sigma$, $a \in G$) we define the orthogonal decomposition:

$$T_pP = W_p \oplus W_p^{\perp}, \tag{2.7}$$

where $W_p \equiv T_p\Sigma_a \subset T_pP$ is the subspace tangent to the submanifold Σ_a at the point $p = u \cdot a \in \Sigma_a$ and $W_p^{\perp}$ is the orthogonal complement of W_p with respect to the metric g. We have a pair of projection operators related to the splitting (2.7):

$$\Pi_p^W : T_pP \to W_p, \qquad \Pi_p^{W^{\perp}} : T_pP \to W_p^{\perp}. \tag{2.8}$$

Let us consider the family of operators $\{\Delta_p\}_{p \in P}$:

$$\Delta_p : G' \to W_p^{\perp}, \qquad \Delta_p \equiv \Pi_p^{W^{\perp}} \circ \tau_p \tag{2.9}$$

[see (2.2) for definition of τ_p] and the family of adjoint operators $\{\Delta_p^+\}_{p \in P}$:

$$\Delta_p^+ : T_pP \to G',$$

defined by

$$\tilde{h}(\delta a, \Delta_p^+ \delta p) = g_p(\Delta_p \delta a, \delta p) \tag{2.10}$$

for every $\delta a \in G'$, $\delta p \in T_pP$, where $\tilde{h}$ is the inner product used in the definition of τ_p^+ in (2.4).

We have the following theorem for $I_1[f]$:

Theorem 1.

$$I_1[f] = \int_{\Sigma} f \cdot \frac{(\det \Delta_u^+ \Delta_u)^{1/2}}{(\det \tau_u^+ \tau_u)^{1/2}} \, d\omega^{\Sigma}, \tag{2.11}$$

where $d\omega^{\Sigma}$ denotes the riemannian volume element related to the induced metric g^{Σ} on the submanifold $\Sigma \subset P$ and operators $\tau_u, \tau_u^+ (\Delta_u, \Delta_u^+)$ are defined in (2.2), (2.4) *(respectively in* (2.9) *and* (2.10)*).*

Let us note that the quotient:

$$\frac{\det \Delta_p^+ \Delta_p}{\det \tau_p^+ \tau_p}$$

is a G-invariant function on P and can be considered as a function on Σ. Moreover it is independent of the inner product $\tilde{h}$ used in the definition of τ_u^+, Δ_u^+.

Now let us introduce another type of integral of the G-invariant function f on P:

$$I_2[f] \equiv \frac{\int_P f d\omega^g}{\int_G d\omega^h}, \tag{2.12}$$

where $d\omega^g$ denotes the riemannian volume element related to the metric g on P and $d\omega^n$ is the volume element related to some fixed right-invariant metric h on G. For $I_2[f]$ the following theorem is true:

Theorem 2.

$$I_2[f] = \int_{\Sigma} f (\det \Delta_u^+ \Delta_u)^{1/2} \mathrm{d}\omega^{\Sigma}, \tag{2.13}$$

where $d\omega^{\Sigma}$, *as before, denotes the volume element related to the induced metric* g^{Σ} *on* Σ *and* Δ_u *is the operator introduced in* (2.9). *The operator* Δ_u^+ *in the right-hand side of* (2.13) *is now defined by:*

$$h_e(\delta a, \Delta_p^+ \delta p) = g_p(\Delta_p \delta a, \delta p) \qquad \delta a \in G', \ \delta p \in T_p P, \tag{2.14}$$

where $h_e \equiv h|_{T_e G}$ *and* h *is the right-invariant metric on G used in the definition of* I_2.

In the end of this section we define one more integral $I_3[f]$ of the G-invariant function of P, which is a special generalization of the integral $I_2[f]$. The definition of I_3 requires an additional object on P, namely such a family $\{h^p\}_{p\in P}$ of right-invariant metrices on G that:

$$\tilde{f}(p) \equiv \int_G d\omega^{h^p}$$

is G-invariant function on P. Because any invariant metric on G is determined by its value in $T_e G = G'$, one can treat $\{h^p\}_{p\in P}$ as a family $\{h_e^p\}_{p\in P}$ of inner products in G' such that for a given base in G' $\det h_e^p$ is a G-invariant function on P. We define

$$I_3[f] \equiv \int_P f\left(\int_G d\omega^{h^p}\right)^{-1} d\omega^g . \tag{2.15}$$

We have the following theorem [18]:

Theorem 3.

$$I_3[f] = \int_{\Sigma} f(\det \Delta_u^+ \Delta_u)^{1/2} d\omega^{\Sigma}, \tag{2.16}$$

where the definition of Δ_p^+ *the inner product* $h_e^p \equiv h^p|_{T_e G}$ *in* G' *is used. Another symbols in* (2.16) *have the same meaning as in Theorem 2.*

3. The Yang-Mills Theory

The riemannian geometry of the configuration space of pure Yang-Mills theories and geometrical interpretation of the Faddeev-Popov determinant in noncovariant functional integral have been discussed in detail in [10]. Let us only recall that the functional integral on the configuration space (i.e. the functional integral obtained from the phase space functional integral after integration over momenta) corresponds to the integral of type I_1 introduced in the previous section. In other words the Faddeev-Popov determinant is related to the natural metric on the orbit space, which in this case is the quotient of the space of irreducible connections on the principal fibration over three-dimensional space by the group of local, time-independent gauge transformations. In this paper we are mainly interested in the functional integral for gauge theories in the Lagrange formulation, so in this section we will present from our point of view only those results of [10] which concern the covariant functional integral of pure Yang-Mills theory. As we shall see the euclidean version of this integral can be treated as a formal generalization of the integral of type I_2 for the infinite-dimensional case. The results of the infinite-dimensional geometry of the space of connections and the gauge group are rather well known and we refer to [8, 9] for details. In the following the notations of [8] will be used.

Let us consider the infinite-dimensional principal fibre bundle

$$\bar{\mathscr{C}} \to \bar{\mathscr{C}}/\bar{\mathscr{G}},$$

where $\bar{\mathscr{C}}$ denotes the space of all irreducible connections on the principal fibration P over $M = S^4$ with the structure group G and $\bar{\mathscr{G}} \equiv \mathscr{G}/Z$ is the quotient of the group $\mathscr{G} \equiv \mathrm{Aut}_v(P)$ of vertical automorphisms of P by its centre Z. We define the space $\Omega^q(M, \mathrm{ad}P)$ as the space of q-forms on M with values in the bundle $\mathrm{ad}P \equiv P \times_{\mathrm{ad}} G'$ with the Lie algebra of G as a standard fibre and with the adjoint action G on G'. Let us introduce the cup operator:

$$\curlywedge : \Omega^q(M, \mathrm{ad}P) \times \Omega^{q'}(M, \mathrm{ad}P) \to \Omega^{q+q'}(M)$$

by means of the exterior product for ordinary forms on M and some fixed invariant inner product in G'. For a given riemannian metric on M one can construct the natural inner product $(\,|\,)_q$ in $\Omega^q(M, \mathrm{ad}P)$ by:

$$(\varphi|\varphi')_q \equiv \int_M \varphi \curlywedge * \varphi',$$

where $\varphi, \varphi' \in \Omega^q(M, \mathrm{ad}P)$ and $*$ denotes the Hodge operator. The space $\mathscr{C}$ of all connections on P is an affine space modelled on $\Omega^1(M, \mathrm{ad}P)$, therefore for every $A \in \mathscr{C}$, $\mathscr{T}_A\mathscr{C} = \Omega^1(M, \mathrm{ad}P)$, and the inner product $(\,|\,)_1$ provides the $\mathscr{G}$-invariant riemannian structure $G(\,|\,)$ on $\mathscr{C}$:

$$G_A(\,|\,) : \mathscr{T}_A\mathscr{C} \times \mathscr{T}_A\mathscr{C} \to \mathbb{R},$$

$$G_A(\,|\,) \equiv (\,|\,)_1 .$$

Because $\bar{\mathscr{C}}$ is an open submanifold of $\mathscr{C}$, $G(\,|\,)$ can be treated as a riemannian structure on $\bar{\mathscr{C}}$. In order to construct a right-invariant riemannian structure on $\mathscr{G}$ it is sufficient to define an inner product in the Lie algebra $\mathscr{G}' \equiv \mathscr{T}_{\mathrm{id}}\mathscr{G}$ of $\mathscr{G}$. In physical applications $\mathscr{G}' \equiv \bar{\mathscr{G}}' = \Omega^0(M, \mathrm{ad}P)$, and a natural choice is the following one:

$$H_{\mathrm{id}}(\,|\,) : \mathscr{T}_{\mathrm{id}}\bar{\mathscr{G}} \times \mathscr{T}_{\mathrm{id}}\bar{\mathscr{G}} \to \mathbb{R},$$

$$H_{\mathrm{id}}(\,|\,) \equiv (\,|\,)_0 .$$

Due to the Gribov ambiguity [6, 7] one can consider only local sections of the fibrations $\bar{\mathscr{C}} \to \bar{\mathscr{C}}/\bar{\mathscr{G}}$ near some given background connection A_0. Let us consider the so-called background gauge condition:

$$d^*_{A_0}(A - A_0) = 0, \tag{3.1}$$

where $d^*_A : \Omega^1(M, \mathrm{ad}P) \to \Omega^0(M, \mathrm{ad}P)$ is the adjoint operator to the covariant derivative d_A restricted to the space $\Omega^0(M, \mathrm{ad}P)$ and defined by:

$$(d_A\psi, \varphi)_1 = (\psi, d^*_A\varphi)_0$$

for every $\psi \in \Omega^0(M, \mathrm{ad}P)$, $\varphi \in \Omega^1(M, \mathrm{ad}P)$. It can be shown [8, 9] that for every $A \in \bar{\mathscr{C}}$ there exists an open subset $\mathscr{U}(A) \subset \bar{\mathscr{C}}/\bar{\mathscr{G}}$ of the orbit space such that $\Pi(A) \in \mathscr{U}(A)$ and the following subset of $\bar{\mathscr{C}}$:

$$\mathscr{Q}(A) \equiv \{A' \in \bar{\mathscr{C}} : \Pi(A') \in \mathscr{U}(A),\ d^*_A(A' - A) = 0\}$$

does not contain two gauge equivalent connections, and therefore (3.1) defines a local section of the fibration $\mathscr{C}\to\mathscr{C}/\mathscr{G}$ over $\mathscr{U}(A)$. (Π in the above definition denotes the projection in the fibration $\mathscr{C}\to\mathscr{C}/\mathscr{G}$.) Then proceeding to a discussion of the Faddeev-Popov procedure we will consider the following trivial principal fibre bundle:

$$\mathscr{C}(A_0)\equiv\Pi^{-1}(\mathscr{U}(A))\to\mathscr{C}(A_0)/\mathscr{G}=\mathscr{U}(A_0)$$

instead of $\mathscr{C}\to\mathscr{C}/\mathscr{G}$.

Now we consider the following functional integral:

$$I=\Big(\int_{\mathscr{G}} d\Omega^H\Big)^{-1}\int_{\mathscr{C}(A_0)} e^{-S[A]}d\Omega^G,$$

where $d\Omega^H(d\Omega^G)$ denotes the formal volume "∞-form" related to the riemannian structure $H(\,|\,)$ on $\mathscr{G}$ [respectively $G(\,|\,)$ on $\mathscr{C}(A_0)$] and $S[A]$ is the euclidean Yang-Mills action. Of course such an expression is completely formal and we must find another more tractable (at least perturbatively) form of I. We will show that the resolution of this problem provided by the Faddeev-Popov trick is related to Theorem 2 of previous section. In fact the integral I can be seen as an infinite-dimensional counterpart of the integral I_2 in the left-hand side of Eq. (2.13). In order to find an appropriate counterpart of the right-hand side of (2.13) we must find an infinite-dimensional counterpart of the operators Δ_u, Δ_u^+ [see (2.9) and (2.14)]. We have the following correspondence of relevant objects:

finite-dimensional case	infinite-dimensional case
P	$\mathscr{C}(A_0)$
$f(p)$	$e^{-S[A]}$
$u\in\Sigma$	$A\in\mathscr{Q}(A_0)$
W_u	$\mathscr{W}_A\equiv\{\varphi\in\Omega^1: d^*_{A_0}\varphi=0\}$
$W_u^\perp$	$\mathscr{W}_A^\perp\equiv\{\varphi\in\Omega^1:\varphi=d_{A_0}\psi,\psi\in\Omega^0\}$
$\tau_u: G'\to T_uP$	$d_A:\Omega^0\to\Omega^1$
$\Pi_u^{W^\perp}: T_uP\to W_u^\perp$	$d_{A_0}(d^*_{A_0}d_{A_0})^{-1}d^*_{A_0}:\Omega^1\to\mathscr{W}_A^\perp$
$\Delta_u: G'\to W_u^\perp$	$d_{A_0}(d^*_{A_0}d_{A_0})^{-1}d^*_{A_0}d_A:\Omega^0\to\mathscr{W}_A^\perp$
$\Delta_u^+: T_uP\to G'$	$d^*_Ad_{A_0}(d^*_{A_0}d_{A_0})^{-1}d^*_{A_0}:\Omega^1\to\Omega^0$,

where an abbreviated notation $\Omega^q(M,\operatorname{ad}P)=\Omega^q$ was used.

It can be easily seen that the infinite-dimensional counterpart of the right-hand side of (2.13) has the form:

$$\int_{\mathscr{Q}(A_0)} e^{-S[A]}\frac{\det d^*_Ad_{A_0}}{(\det d^*_{A_0}d_{A_0})^{1/2}}d\Omega^{\mathscr{Q}},$$

where $d\Omega^{\mathscr{Q}}$ is the volume "∞-form" related to the induced riemannian structure on $\mathscr{Q}(A_0)$. Finally, using the functional δ-function with appropriate Jacobian one can proceed to the more familiar form of I:

$$I=\int_{\mathscr{C}(A_0)}\delta(d^*_{A_0}(A-A_0))\det d^*_Ad_{A_0}e^{-S[A]}d\Omega^G.$$

Let us note that because the riemannian structure $G(|)$ is constant on $\overline{\mathscr{C}}$ the measure $d\Omega^G$ can be treated as a formal "Lebesgue measure" on $\overline{\mathscr{C}}$ and we have full agreement with usual expressions [5].

4. The Spinless Scalar Particle

The Faddeev-Popov procedure in the covariant functional quantization of the free relativistic particle has been considered in the context of Polyakov's string in [23] and more recently (and more carefully) in [21]. The aim of this section is to compare three possible definitions of the particle propagator based on formal generalizations of the integrals introduced in Sect. 2. As we will see, the definition related to the integral I_3 is the correct one. In this case our approach reproduces the derivation and results of [21].

Let us consider the following (euclidean) action [27]:

$$S[x,e] \equiv \tfrac{1}{2}\int_0^1 \left(\frac{\dot{x}^2}{e} + m_0^2 e\right) dt, \tag{4.1}$$

where $x(t) = \{x^\mu(t)\}_{\mu=1}^d$ denotes an embedding of the interval $L \equiv [0,1]$ into the Euclidean space $\mathbb{R}^d$ and describes the position of the particle, $e = e(t)dt$ is the "einbein" of the metric e^2 on the interval L and $x_*^{-1}e$ describes a distance scale on the path of the particle. This action is reparametrization invariant, so the group of diffeomorphisms $\mathscr{D}_L$ of the interval L plays the role of the gauge group.

Let us start our discussion with some geometrical preliminaries. We will consider the following fibration:

$$\mathscr{M}_L \times \mathscr{E}_x^y \to \mathscr{M}_L \times \mathscr{E}_x^y / \mathscr{D}_L, \tag{4.2}$$

where $\mathscr{E}_x^y$ denotes the space of all embeddings $\{x^\mu(t)\}_{\mu=1}^d$ of L into $\mathbb{R}^d$ with the property that $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu (x, y \in \mathbb{R}^d)$; $\mathscr{M}_L$ is the space of all "einbeins" on L and the right action of $\mathscr{D}_L$ on $\mathscr{M}_L \times \mathscr{E}_x^y$ is defined by pull-back $R_f(e,x) \equiv (f^*e, f^*x)$, $f \in \mathscr{D}_L$. Resorting to Sobolev spaces, it is possible to give a Hilbert manifold structure to $\mathscr{D}_L, \mathscr{E}_x^y, \mathscr{M}_L$ and prove that (4.2) is the principal fibre bundle. All relevant results can be found in [8] (and references therein). Here we mention only that although $\mathscr{D}_L$ is not a Lie group, nevertheless it is a Hilbert manifold modelled on the space $H_0(TL)$ of vector fields on L vanishing at the ends of L. With this structure $\mathscr{D}_L$ is a topological group under composition of maps. The left action $L_f f' \equiv f \circ f'$ is continuous (but not smooth) and the right action $R_f f' \equiv f' \circ f$ is smooth for any $f \in \mathscr{D}_L$. One can construct the right-invariant riemannian (weak) structure $H(|)$ on $\mathscr{D}_L$ by defining an inner product in the tangent space $\mathscr{T}_{\mathrm{id}}\mathscr{D}_L = H_0(TL)$ at the identity diffeomorphisms:

$$H_{\mathrm{id}}(|): H_0(TL) \times H_0(TL) \to \mathbb{R}.$$

If we have some metric $g = e^2$ on L, the most natural parametrization-invariant inner product on $H_0(TL)$ is the following one:

$$H_{\mathrm{id}}^e(\delta t|\delta t') \equiv \int_L e^3 \delta t \delta t' dt, \tag{4.3}$$

where $\delta t, \delta t' \in H_0(TL)$. The metric defined by this product we will denote by $H^e(\,|\,)$. The space $\mathcal{M}_L$ is the open subset of the linear space $\Omega^1(L)$ of one-forms on L, hence it is an infinite-dimensional manifold modelled on $\Omega^1(L)$. The space $\mathcal{E}_x^y$ is an affine space modelled on the space $\mathcal{E}_0 \equiv \overset{n}{\times}\, \Omega_0^0(L)$, where $\Omega_0^0(L)$ is the space of real functions on L vanishing at the ends of the interval L. One can construct natural riemannian structures on $\mathcal{M}_L$ and $\mathcal{E}_x^y$ as follows:

$$M_e(\delta e|\delta e') \equiv \int_L \frac{\delta e \delta e'}{e} dt\,, \tag{4.4}$$

$$E_x^e(\delta x|\delta x') \equiv \int_L e \delta x \delta x' dt\,, \tag{4.5}$$

where $\delta e, \delta e' \in \mathcal{T}_e \mathcal{M}_L = \Omega^1(L)$; $e \in \mathcal{M}_L$; $\delta x, \delta x' \in \mathcal{T}_x \mathcal{E}_x^y = \mathcal{E}_0$, $x \in \mathcal{E}_x^y$. The above structures define a cartesian product riemannian structure $G(\,|\,)$ on $\mathcal{M}_L \times \mathcal{E}_x^y$ which is right-invariant due to reparametrization invariance of (4.4) and (4.5). Now let us consider the problem of existence of global sections of the principal fibration (4.2). It is clear that $(\mathcal{M}_L \times \mathcal{E}_x^y)/\mathcal{D}_L = (\mathcal{M}_L/\mathcal{D}_L) \times \mathcal{E}_x^y$, hence we are in fact interested in sections of the principal fibration:

$$\mathcal{M}_L \to \mathcal{M}_L/\mathcal{D}_L\,.$$

Fortunately this fibration is trivial and $\mathcal{M}_L/\mathcal{D}_L$ is isomorphic to the positive real half axis $\mathbb{R}_+$. This follows from the fact [23] that given a cart of the interval $L=[0,1]$ and given an einbein $e \in \mathcal{M}_L$, one can always find a diffeomorphism $f \in \mathcal{D}_L$ such that in this cart $f^*e = c dt$, where c is a positive constant determined by:

$$c = \int_L e(t) dt\,. \tag{4.6}$$

Equation (4.6) can be seen as a condition determining the fiber of the bundle $\mathcal{M}_L \to \mathcal{M}_L/\mathcal{D}_L$ containing the einbein e. It is easy to verify that for every $\bar{e} \in \mathcal{M}_L$ the submanifold,

$$\Sigma_{\bar{e}} \equiv \{e \in \mathcal{M}_L : e = \lambda \bar{e},\ \lambda > 0,\ \lambda \in \mathbb{R}\}\,,$$

intersects every orbit exactly in one point and can be used as a global gauge slice (there is no Gribov ambiguity).

Now let us proceed to quantization. The relevant object is the euclidean propagator $P(x,y)$ defined, according to the general idea of a covariant functional approach to gauge theories, as a sum over all paths in $\mathbb{R}^d$ with internal metric and with fixed endpoints $x, y \in \mathbb{R}^d$. In order to give a meaning to the above formal definition we must first define a measure on the orbit space and then express $P(x,y)$ in terms of parametrizations (x,e) of one-dimensional riemannian submanifolds of $\mathbb{R}^d$ and some gauge fixing condition. As follows from the considerations of Sect. 2 this can be achieved at least by three different ways. From the geometrical point of view the most natural definition is that corresponding to the integral I_1 [see (2.6)] namely:

$$P_1(x,y) \equiv \int_{\mathcal{M}_L \times \mathcal{E}_x^y/\mathcal{D}_L} e^{-s} d\Omega^{\tilde{G}}\,,$$

where $d\Omega^{\tilde{G}}$ denotes the formal volume element related to the metric $\tilde{G}(\,|\,)$ on $\mathcal{M}_L \times \mathcal{E}_x^y/\mathcal{D}_L$ defined by analogy with the metric $\tilde{g}$ considered in Sect. 2. On the other hand the results in the Yang-Mills theory described in Sect. 3 suggest that an appropriate definition is the following one:

$$P_2(x,y) \equiv \left(\int_{\mathscr{D}_L} d\Omega^{H^{e_0}}\right)^{-1} \times \int_{\mathscr{M}_L \times \mathscr{E}_x^y} e^{-s} d\Omega^G ,$$

where the formal measure $d\Omega^{H^{e_0}}$ is related to the metric $H^{e_0}(|)$ on $\mathscr{D}_L$ for some $e_0 \in \mathscr{M}_L$ chosen for the normalization of the volume of the gauge group and $d\Omega^G$ is related to the metric $G(|)$ on $\mathscr{M}_L \times \mathscr{E}_x^y$. One can also consider the third possibility [see definition (2.15) of I_3] introducing the family $\{H^e\}_{e \in \mathscr{M}_L}$ of riemannian structures on $\mathscr{D}_L$:

$$P_3(x,y) \equiv \int_{\mathscr{M}_L \times \mathscr{E}_x^y} \left(\int_{\mathscr{D}_L} d\Omega^{H^e}\right)^{-1} e^{-s} d\Omega^G . \tag{4.7}$$

For each of the "propagators" defined above one can apply the Faddeev-Popov procedure using appropriate generalizations of theorems presented in Sect. 2. The nice feature of the model under consideration is that due to the very simple structure of the orbit space and quadratic dependence of the action on the x variable one can explicitly evaluate all the above "propagators." It turns out that only $P_3(x,y)$ is a well known euclidean propagator so we now proceed to the discussion of the integral (4.7). The results of P_1 and P_2 will be only mentioned in the end of this section.

Let us note that due to the "product" structure of the fibration (4.2) and the "diagonal" property of the metric $G(|)$ on $\mathscr{M}_L \times \mathscr{E}_x^y$ the integral (4.7) can be rewritten in the following form:

$$P_3(x,y) = \int_{\mathscr{M}_L} \left(\int_{\mathscr{D}_L} d\Omega^{H^e}\right)^{-1} \left(\int_{\mathscr{E}_x^y} e^{-s} d\Omega^{E^e}\right) d\Omega^M ,$$

where the formal measure $d\Omega^{E^e}(d\Omega^M)$ is related to the riemannian structure $E^e(|)$ on $\mathscr{E}_x^y$ (respectively $M(|)$ on $\mathscr{M}_L$). The integral over $\mathscr{E}_x^y$ is Gaussian so can be formally performed:

$$\begin{aligned}\int_{\mathscr{E}_x^y} e^{-s} d\Omega^{E^e} &= e^{-\frac{1}{2}\left[\frac{(x-y)^2}{v} + m_0^2 v\right]} \int_{\mathscr{E}_0} e^{-\frac{1}{2}\int_L x^\mu (\mathscr{L}_e x^\mu) e dt} d\Omega^{E^e} \\ &= e^{-\frac{1}{2}\left[\frac{(x-y)^2}{v} + m_0^2 v\right]} (\det \mathscr{L}_e)^{-d/2} ,\end{aligned}$$

where $v \equiv \int_L e$ and $\mathscr{L}_e$ denotes the one-dimensional Laplace-Beltrami operator:

$$\mathscr{L}_e \equiv -\frac{1}{e}\frac{d}{dt}\frac{1}{e}\frac{d}{dt},$$

acting on the space $\Omega_0^0(L)$.

Now we apply the appropriate formal generalization of Theorem 3 to the integral:

$$P_3(x-y) \equiv P_3(x,y) = \int_{\mathscr{M}_L} \left(\int_{\mathscr{D}_L} d\Omega^{H^e}\right)^{-1} e^{-\frac{1}{2}\left[\frac{(x-y)^2}{v} + m_0^2 v\right]} (\det \mathscr{L}_e)^{-d/2} d\Omega^M .$$

In order to do it we must find the infinite-dimensional counterpart of the right-hand side of Eq. (2.16). We have the following correspondence between relevant geometrical objects:

finite-dimensional case	infinite-dimensional case	
$P \to P/G$	$\mathcal{M}_L \to \mathcal{M}_L/\mathcal{D}_L$	
$f(p)$	$e^{-\frac{1}{2}\frac{(x-y)^2}{v}+m^2 v}\,(\det \mathscr{L}_e)^{-d/2}$	
$\{h^p\}_{p\in P}$	$\{H^e\}_{e\in \mathcal{M}_L}$	
$u \in \Sigma$	$e=\lambda\bar{e}\in\Sigma_{\bar{e}}$	
W_u	$\mathcal{W}_e \equiv \{\delta e\in\Omega^1(L):\ \delta e=\delta\lambda\bar{e},\ \delta\lambda\in\mathbb{R}\}$	
$W_u^{\perp}$	$\mathcal{W}_e^{\perp} \equiv \left\{\delta e\in\Omega^1(L):\ \int_L \delta e=0\right\}$	
$\tau_u : G' \to T_u P$	$\frac{d}{dt}(e\,\cdot): H_0(TL)\to\Omega^1(L)$	
$\Delta_u : G' \to W_u^{\perp}$	$\frac{d}{dt}(e\,\cdot): H_0(TL)\to\mathcal{W}_e^{\perp}$	
$\Delta_u^+ : T_u P \to G'$	$\frac{1}{e^2}\frac{d}{dt}\left(\frac{1}{e}\,\cdot\right): \Omega^1(L)\to H_0(TL)$	
$\Delta_u^+\Delta_u : G' \to G'$	$\frac{1}{e}\mathscr{L}_e e : H_0(TL)\to H_0(TL)\,.$	(4.8)

Let us note that now in the infinite-dimensional case the gauge slice is orthogonal to the fibers, therefore $\mathcal{W}_e^{\perp}$ is the space tangent to the fiber containing e and the counterparts of operators τ_u and Δ_u are identical. It is easy to see that the operator $\frac{1}{e}\mathscr{L}_e e$ acting on $H_0(TL)$ has the same spectrum as the operator $\mathscr{L}_e$ acting on $\Omega_0^0(L)$, so we have:

$$\det\frac{1}{e}\mathscr{L}_e e=\det\mathscr{L}_e\,,$$

and from Theorem 3 of Sect. 2 and the correspondence (4.8) the following form of the integral (4.7) can be derived:

$$P_3(x-y)=\int_{\Sigma_{\bar{e}}} e^{-\frac{1}{2}\left[\frac{(x-y)^2}{\lambda\bar{v}}-m_0^2\lambda\bar{v}\right]}(\det\mathscr{L}_{\lambda\bar{e}})^{\frac{1}{2}-\frac{d}{2}}d\Omega^{\Sigma_{\bar{e}}}\,,$$

where $\bar{v}=\int_L \bar{e}$ and $d\Omega^{\Sigma_{\bar{e}}}$ denotes the formal riemannian measure related to the metric induced on $\Sigma_{\bar{e}}\subset\mathcal{M}_L$. Let us evaluate this metric at the point $\lambda\bar{e}\in\Sigma_{\bar{e}}$. For $\delta\lambda\bar{e}$, $\delta\lambda'\bar{e}\in\mathscr{T}_{\lambda\bar{e}}\Sigma_{\bar{e}}$ we have:

$$M_{\lambda\bar{e}}(\delta\lambda\bar{e}|\delta\lambda'\bar{e})=\frac{\delta\lambda\delta\lambda'}{\lambda}\bar{v}\,.$$

Changing the variable of integration we arrive at the following expression:

$$P_3(x-y)=\int_0^{\infty} d\lambda\bar{v}(\lambda\bar{v})^{-1/2}e^{-\frac{1}{2}\left[\frac{(x-y)^2}{\lambda\bar{v}}-m_0^2\lambda\bar{v}\right]}(\det\mathscr{L}_{\lambda\bar{e}})^{\frac{1}{2}-\frac{d}{2}}\,. \tag{4.9}$$

The last task is to regularize the determinant of $\mathscr{L}_{\lambda\bar{e}}$. Using the ζ-function regularization [29] we have [21, 23]:

$$\log \det_R(\mathscr{L}_{\lambda\bar{e}}) = \log \lambda\bar{v} + \log 2 . \tag{4.10}$$

Inserting (4.10) into expression (4.9) we have:

$$P_3(x-y) = c_3' \int_0^\infty d\lambda \lambda^{-\frac{d}{2}} e^{-\frac{1}{2}\left[\frac{(x-y)^2}{\lambda} + m_0^2\lambda\right]} .$$

Taking the Fourier transformation and changing the order of integration it is easy to obtain:

$$\hat{P}_3(k) = c_3(k^2 + m_0^2)^{-1} ,$$

where c_3 is an irrelevant constant.

A similar consideration can be performed for P_1 and P_2 with the following results:

$$\hat{P}_1(k) = c_1(k^2 + m_0^2)^{-1/2} ,$$

$$\hat{P}_2(k) = c_2(k^2 + m_0^2)^{-1/4} .$$

One can see that the correct geometrical interpretation of the Faddeev-Popov procedure is provided by formal generalization of the definition of the integral I_3 and Theorem 3 of Sect. 2. The other definitions lead to nonlocal theories and can not be considered as a "quantization" of the relativistic particle.

5. Polyakov's Bosonic String

Let us briefly recall the basic concepts of Polyakov's method for calculating of averages of functionals defined over surfaces [12]. Such averages can be generally expressed in the symbolic form as follows:

$$\int_{\mathscr{G}} F e^{-s} d\Omega ,$$

where $\mathscr{G}$ denotes some space of surfaces embedded in $\mathbb{R}^d$ and endowed with an intrinsic riemannian metric. S is some action defined over such surfaces and integration in general involves summation over topologies. Depending on physical applications [13–17, 20–22] various spaces of surfaces and various actions are used. In this section, in order to avoid inessential (to understand the geometry of the Faddeev-Popov procedure) boundary terms, we will consider the following functional integral:

$$Z_h = \int_{\mathscr{G}_h} e^{-s} d\Omega , \tag{5.1}$$

where integration goes over the space $\mathscr{G}_h$ of all closed surfaces with $h = 0, 1, \ldots$ handles. Such an integral can be seen as an h-loop vacuum amplitude for a closed bosonic string [14–17]. A surface $s \in \mathscr{G}_h$ can be described by its parametrization (g, x), where $x = (x^1, \ldots, x^d)$ is the embedding of some fixed two-dimensional orientable manifold M_h of genus h, and g is a riemannian metric on M_h. We have a natural right action of the group $\mathscr{D}_h$ of orientation preserving diffeomorphisms of M_h on the space of parametrizations $\mathscr{P}_h \equiv \mathscr{M}_h \times \mathscr{E}_h$ ($\mathscr{M}_h$ denotes the space of metrics on M_h and $\mathscr{E}_h$ – the space of embeddings of M_h into $\mathbb{R}^d$) defined for $f \in \mathscr{D}_h$ as follows:

$$R_f(g, x) \equiv (f^*g, f^*x).$$

It is easy to see that for any $f \in \mathscr{D}_h$, parametrizations (g, x) and (f^*g, f^*x) describe the same surface in $\mathbb{R}^d$ with its intrinsic metric $(x^{-1})^*g$. Therefore we can write:

$$\mathscr{G}_h = (\mathscr{M}_h \times \mathscr{E}_h)/\mathscr{D}_h = \mathscr{M}_h/\mathscr{D}_h \times \mathscr{E}_h.$$

In terms of parametrizations one can define the action for surfaces from $\mathscr{G}_h$ introducing a $\mathscr{D}_h$-invariant functional on $\mathscr{P}_h$. The action used by Polyakov had been previously proposed by Brink et al. [27]:

$$s[g, x] \equiv \tfrac{1}{2} \int_{M_h} \sqrt{g}\, d^2 z g^{ab} \partial_a x^\mu \partial_b x^\mu, \tag{5.2}$$

where $x \in \mathscr{E}_h$, $g \in \mathscr{M}_h$. Let us note that using the principle of renormalizability one can derive the most general action containing two additional terms [30]:

$$\int_{M_h} \sqrt{g}\, d^2 z, \quad \frac{1}{4\pi} \int_{M_h} \sqrt{g}\, R d^2 z.$$

However, the first term on the classical level leads to an inconsistent equation of motion [30]. The second term is a topological invariant called the Euler characteristic, and being a total derivative it does not contribute to the equation of motion. The classical action (5.2) is invariant under global rotations and translations in the Euclidean space $\mathbb{R}^d$, under diffeomorphisms of M_h and under Weyl transformations of the metric:

$$g_{ab}(z) \to e^{2\varphi(z)} g_{ab}(z). \tag{5.3}$$

The conformal deformation of metric (5.3) is parametrized by the scalar function $\varphi: M_h \to \mathbb{R}$ and the abelian group of these transformations is isomorphic to the additive group $\mathscr{W}_h$ of all scalar real functions on M_h.

Before we proceed further let us recall some geometrical preliminaries about the functional spaces under consideration. Let us denote by $H(T_2)$ the linear space of symmetric tensor fields of type (0, 2) on the manifold M_h. By resorting to Sobolev spaces in the usual way, one can give the Hilbert space structure to $H(T_2)$. The space of metrices $\mathscr{M}_h$ is the open cone in $H(T_2)$, so $\mathscr{M}_h$ can be considered as a Hilbert manifold modelled on the space $H(T_2)$ [8, 28]. On $\mathscr{M}_h$ we have a natural riemannian (weak) structure $M(\,|\,)$ defined by:

$$M_g(\,|\,): H(T_2) \times H(T_2) \to \mathbb{R},$$

$$M_g(\delta g, \delta g') \equiv \int \sqrt{g}\, d^2 z [\tfrac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{cb} - g^{ab}g^{cd}) + c g^{ab} g^{cd}] \delta g_{ab} \delta g'_{cd},$$

where $\delta g, \delta g' \in \mathscr{T}_g \mathscr{M}_h = H(T_2)$, $g \in \mathscr{M}_h$. This metric structure is natural and unique as far as we are looking for a $\mathscr{D}_h$-invariant expression which is local and does not involve derivations of the metric (so-called ultralocality) and does not involve arbitrary functions. Let us note that the above metric structure is not Weyl invariant.

The space $\mathscr{E}_h = \overset{d}{\times}\, \Omega^0(M_h)$ is linear, so there is no problem with the Hilbert manifold structure and the natural (in the above sense) $\mathscr{D}_h$-invariant weak riemannian structure $E^g(\,|\,)$ on $\mathscr{E}_h$ is defined in the tangent space $\mathscr{T} \times \mathscr{E}_h$ at the point $x \in \mathscr{E}_h$ by:

$$E^g_x(\,|\,): \mathscr{E}_h \times \mathscr{E}_h \to \mathbb{R}\,,$$
$$E^g_x(\delta x|\delta x') \equiv \int_{M_h} \sqrt{g}\, d^2z \delta x^\mu \delta x'^\mu\,,$$

where $\delta x, \delta x' \in \mathscr{T}_x\mathscr{E}_h = \mathscr{E}_h$. By the cartesian product construction we have a natural $\mathscr{D}_h$-invariant structure $G(\,|\,)$ on $\mathscr{P}_h$:

$$G_{(g,x)}(\delta g, \delta x|\delta g', \delta x') \equiv M_g(\delta g, \delta g') + E_g(\delta x, \delta x')\,,$$

where $(\delta g, \delta x)$, $(\delta g', \delta x') \in \mathscr{T}_{(g,x)}(\mathscr{M}_h \times \mathscr{E}_h) = H(T_2) \times \mathscr{E}_h$.

The Hilbert manifold structure of $\mathscr{D}_h$ has the same properties as in the case of $\mathscr{D}_L$ briefly described in the previous section [8, 28]. $\mathscr{D}_h$ is a Hilbert manifold modelled on the space $H(TM_h)$ of vector fields on M_h, and the natural right-invariant riemannian structure $H^g(\,|\,)$ on $\mathscr{D}_h$ is determined by its value in the tangent space at the identity diffeomorphism:

$$\begin{aligned} &H^s_{\mathrm{id}}(\,|\,): H(TM_h) \times H(TM_h) \to \mathbb{R}\,,\\ &H^g_{\mathrm{id}}(\delta f|\delta f') \equiv \int_{M_h} \sqrt{g}\, d^2z g_{ab} \delta f^a \delta f'^b\,, \end{aligned} \tag{5.4}$$

where $\delta f, \delta f' \in \mathscr{T}_{\mathrm{id}}\mathscr{D}_h = H(TM_h)$.

The group $\mathscr{W}_h$ of conformal deformations of the metric has an obvious Hilbert space structure with a natural invariant riemannian structure $W^g(\,|\,)$ on $\mathscr{W}_h$ defined for $\varphi \in \mathscr{W}_h$ by:

$$W^g_\varphi(\,|\,): \Omega^0(M_h) \times \Omega^0(M_h) \to \mathbb{R}\,,$$
$$W^g_\varphi(\delta\varphi|\delta\varphi') \equiv \int_{M_h} \sqrt{g}\, d^2z \delta\varphi \delta\varphi'\,,$$

where $\delta\varphi, \delta\varphi' \in \mathscr{T}_\varphi \mathscr{W}_h = \Omega^0(M_h)$, $\varphi \in \mathscr{W}_h$, $g \in \mathscr{M}_h$.

Finally we consider the semi-direct product $\mathscr{D}_h \odot \mathscr{W}_h$. The right action of the element $(f, \varphi) \in \mathscr{D}_h \odot \mathscr{W}_h$ on $\mathscr{P}_h$ is defined by:

$$R_{(f,\varphi)}(g, x) \equiv (e^\varphi f^* g, f^* x)\,.$$

One can introduce the right-invariant riemannian structure in $\mathscr{D}_h \odot \mathscr{W}_h$ by defining its value at identity $(\mathrm{id}, 0) \in \mathscr{D}_h \odot \mathscr{W}_h$:

$$V^g_{(\mathrm{id},0)}(\delta f, \delta\varphi|\delta f', \delta\varphi') \equiv \int_{M_h} \sqrt{g}\, d^2z g_{ab} \delta f^a \delta f'^b + \int_{M_h} \sqrt{g}\, d^2z \delta\varphi \delta\varphi'\,. \tag{5.5}$$

Let us note that we prefer to work with $\mathscr{D}_h \odot \mathscr{W}_h$ instead of $\mathscr{W}_h \odot \mathscr{D}_h$, because in the last case the extension of (5.5) to the whole group by the right action leads to the "nondiagonal" metric. In the case of $\mathscr{D}_h \odot \mathscr{W}_h$ such an extension leads to:

$$V^g_{(f,\varphi)}(\delta f, \delta\varphi|\delta f', \delta\varphi') = \int_{M_h} \sqrt{g}\, d^2z g_{ab} \delta f \circ f^{-1} \delta f' \circ f^{-1} + \int_{M_h} \sqrt{g}\, d^2z \delta\varphi \circ f^{-1} \delta\varphi' \circ f^{-1}$$

for any $(f, \varphi) \in \mathscr{D}_h \odot \mathscr{W}_h$ and $(\delta f, \delta\varphi), (\delta f', \delta\psi') \in \mathscr{T}_{(f,\varphi)}\mathscr{D}_h \odot \mathscr{W}_h$.

Now let us proceed to Polyakov's definition of the functional integral (5.1). As follows from the symmetries of the classical action (5.2) the gauge group of the string theory under consideration must be identified with $\mathscr{D}_h \odot \mathscr{W}_h$. So the physical configurations of the string are described by the quotient $\mathscr{P}_h / \mathscr{D}_h \odot \mathscr{W}_h$. In order to eliminate the redundant degrees of freedom one can try to apply one of the schemes described in Sect. 2. At this point, however, we face troubles. All schemes considered in Sect. 2 required some gauge-invariant metrics on the original field space, while in our case the most natural riemannian structure on $\mathscr{P}_h$ is not Weyl invariant. Moreover, any other ultralocal $\mathscr{D}_h \odot \mathscr{W}_h$-invariant structures requires introduction into the theory of some arbitrary length scale which is physically unacceptable. It led Polyakov to the consideration of the theory described by the action (5.2) as a gauge theory of the diffeomorphisms group. On the other hand the action (5.2) is a straightforward generalization of the action (4.1) of a spinless particle, so it is natural to define Z_h by analogy with the definition (4.7) of the particle propagator:

$$Z_h \equiv \int_{\mathscr{P}_h} \left(\int_{\mathscr{D}_h} d\Omega^{H^g} \right)^{-1} e^{-S[g,x]} d\Omega^G , \tag{5.6}$$

where $d\Omega^{H^g} (d\Omega^G)$ is the formal volume element related to the riemannian structure $H^g(\,|\,)$ on $\mathscr{D}_h$ [respectively $G(\,|\,)$ on $\mathscr{P}_h$] and $\{H^g\}_{g \in \mathscr{M}_h}$ is the family of riemannian structures defined in (5.4). The restriction of the gauge group $\mathscr{D}_h \odot \mathscr{W}_h$ to $\mathscr{D}_h$ has important consequences. This means that physical configurations are described by the quotient $\mathscr{P}_h / \mathscr{D}_h$ and the conformal factor φ is treated as the dynamical variable. However the integral (5.6) is well defined only when the dynamics of the conformal factor is non-trivial. It can happen only as the result of the anomaly because φ is not present in the action (5.2). It was shown by Polyakov [12] that for $d<26$ the conformal anomaly really appears and φ acquires a non-trivial dynamics. For $d \geqq 26$ the integral Z_h diverges. More recently the covariant path integral for the BDHP string has been reconsidered at critical dimension by several authors [14–17, 20–22]. The following definition suggested by the true gauge group $\mathscr{D}_h \odot \mathscr{W}_h$ and by the expression of the point particle propagator (4.7) has been proposed:

$$Z_h^{26} \equiv \int_{\mathscr{P}_h} \left(\int_{\mathscr{D}_h \odot \mathscr{W}_h} d\Omega^{V^g} \right)^{-1} e^{-S[g,x]} d\Omega^G , \tag{5.7}$$

where $d\Omega^{V^g}$ is related to the metric structure $V^g(\,|\,)$ on $\mathscr{D}_h \odot \mathscr{W}_h$. It is clear that for $d<26$ this integral formally vanishes due to an infinite volume of the Weyl group in the denominator. In $d=26$, however, one can hope that this integral leads to a correct expression, as in the critical dimension the local conformal anomaly disappears. It was shown further [22] that there is no nonlocal anomaly either and the integration over the conformal factor is perfectly cancelled by the volume of the Weyl group.

In the rest of this section we will apply our geometrical formulation of the Faddeev-Popov procedure to the integrals Z_h and Z_h^{26} for $h \geqq 2$. The cases of sphere S^2 and torus T^2 require a slight generalization of the scheme described in Sect. 2 and some special results concerning the geometry of the quotients $\mathscr{M}_0 / \mathscr{D}_0$ and $\mathscr{M}_1 / \mathscr{D}_1$, and will not be discussed here.

Let us recall some relevant geometrical results. The action of $\mathscr{D}_h$ on $\mathscr{M}_h$ is not free, so we will consider the following fibration:

$$\bar{\mathscr{M}}_h \to \bar{\mathscr{M}}_h/\mathscr{D}_h, \tag{5.8}$$

where $\bar{\mathscr{M}}_h$ is the subspace of $\mathscr{M}_h$ consisting of metrics with a trivial isometry group. $\bar{\mathscr{M}}_h$ is the open and dense subset of $\mathscr{M}_h$ and the fibration (5.8) has the structure of a principal fibre bundle [8, 28]. Simple consideration of the homotopy of $\mathscr{D}_h$ shows that the bundle (5.8) can not be trivial [25]. In order to overcome this problem, one can consider the following fibration instead of (5.8):

$$\bar{\mathscr{M}}_h \to \bar{\mathscr{M}}_h/\mathscr{D}_h^0, \tag{5.9}$$

where $\mathscr{D}_h^0$ is the connected component of identity. For $h>1$ there are no topological obstructions to the bundle (5.9) to be trivial [25]. Before we proceed to discussion of gauge fixing, let us consider the following fibration:

$$\bar{\mathscr{M}}_h \to \bar{\mathscr{M}}_h/\mathscr{D}_h^0 \odot \mathscr{W}_h \equiv \mathscr{T}_h. \tag{5.10}$$

The quotient $\mathscr{M}_h/\mathscr{D}_h^0 \odot \mathscr{W}_h$ is called a Teichmuller space for the surface of genus h [31]. It is known that $\mathscr{T}_h$ is an open, convex subset of $\mathbb{R}^{6h-6}$. Let us note that the action of $\mathscr{D}_h^0 \odot \mathscr{W}_h$ on $\bar{\mathscr{M}}_h$ is free for $h>1$. In fact for surfaces of genus $h>1$ the group

$$\mathscr{C}_h^g \equiv \{f \in \mathscr{D}_h : \exists \varphi \in \mathscr{W}_h : f^* g = e^{2\varphi} g\}$$

of so-called conformal diffeomorphisms for a given metric g is discrete (and finite) and $\mathscr{D}_h^0 \cap \mathscr{C}_h^g = \{\mathrm{id}\}$ [31]. A discussion of a principal fibre bundle structure of $\bar{\mathscr{M}}_h \to \mathscr{T}_h$ and its triviality requires some rather advanced mathematical methods and will be omitted in this paper. We assume here that the global section $\imath$ of the fibration (5.10) exists:

$$\imath : \mathscr{T}_h \ni t \to g^t \in \bar{\mathscr{M}}_h.$$

For a given $\imath$ one can construct the global section σ of the fibration (5.9) defining the gauge slice Σ_i by:

$$\Sigma_i \equiv \{g \in \mathscr{M}_h : g = e^{\varphi} g^t,\ t \in \mathscr{T}_h,\ \varphi \in \mathscr{W}_h\},$$

and

$$\sigma : \bar{\mathscr{M}}_h/\mathscr{D}_h^0 \ni u \to g = \sigma(u) \in \bar{\mathscr{M}}_h,$$

$$\{g\} = \Sigma_i \cap \Pi^{-1}(u),$$

where Π denotes the projection of $\bar{\mathscr{M}}_h \to \bar{\mathscr{M}}_h/\mathscr{D}_h^0$.

Let us consider the integral (5.6). Due to the "diagonal" structure of the metric $G(|)$ on $\mathscr{P}_h$, the integration over $\mathscr{E}_h$ can be performed. In order to do it we must separate the constant pieces of x. One can introduce the following orthogonal decomposition:

$$\mathscr{E}_h = \mathbb{R}^d \oplus \tilde{\mathscr{E}}_h,$$

where $\tilde{\mathscr{E}}_h$ contains those $\tilde{x}$ which are orthogonal to constants:

$$\int_{M_n} \sqrt{g}\, d^2 z \tilde{x}^\mu = 0, \qquad \mu = 1, \dots, d.$$

Using an appropriate generalization of the Fubini theorem (rather trivial in this case) we have:

$$\int_{\mathscr{E}_h} e^{-\frac{1}{2}\int_{M_h} \sqrt{g} d^3 z g^{ab} \partial_a x^\mu \partial_b x^\mu} d\Omega^{E^g} = \int_{\mathbb{R}^d} \left(\int_{M_h} \sqrt{g}\, d^2 z \right)^{d/2} \prod_{\mu=1}^{d} dx^\mu \times (\det \mathscr{L}_g)^{-d/2},$$

where $\mathscr{L}_g$ denotes the Laplace-Beltrami operator:

$$\mathscr{L}_g \equiv -\frac{1}{\sqrt{g}}\partial_a \sqrt{g}\, g^{ab}\partial_b$$

acting on the space of scalar real functions on M_h (symbol det′ for determinants means that zero-modes of $\mathscr{L}_g$ are omitted). The finite-dimensional divergent integral in the above formula can be regularized by putting the system in a box of dimension L [22], then:

$$Z_h = L^d \int_{\mathscr{M}_h} \left(\int_{\mathscr{D}_h^0} d\Omega^{H^g}\right)^{-1} \left(\frac{\det' \mathscr{L}_g}{\int_{M_h} \sqrt{g}\, d^2 z}\right)^{-d/2} d\Omega^M .$$

As before we will apply the appropriately generalized Theorem 3 of Sect. 2. Let us introduce the following subspaces of the space $\mathscr{T}_g\bar{\mathscr{M}}_h$ tangent to $\bar{\mathscr{M}}_h$ at the point $g = e^{\varphi} g^t \in \Sigma_i$ (see Fig. 1):

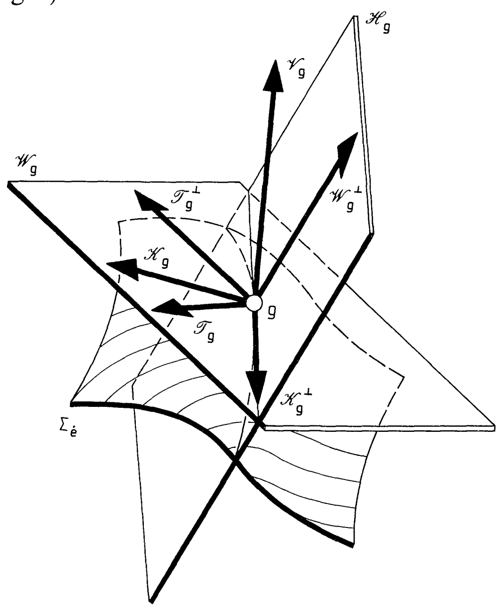

Fig. 1

$$\mathscr{W}_g \equiv \mathscr{T}_g \Sigma_i \subset \mathscr{T}_g \bar{\mathscr{M}}_h ,$$

$$\mathscr{W}_g = \left\{ \delta g \in H(T_2) \colon \delta g = \delta\varphi e^{\varphi} g^t + \delta t^k e^{\varphi} \frac{\partial}{\partial t^k} g^t ,\ \delta\varphi \in \Omega^0(M_h),\ \delta t \in \mathbb{R}^{6h-6} \right\},$$

$$\mathscr{K}_g \equiv \{ \delta g \in H(T_2) \colon \delta g = \delta\varphi e^{\varphi} g^t ,\ \delta\varphi \in \Omega^0(M_h) \} ,$$

$$\mathscr{H}_g \equiv \{ \delta g \in H(T_2) \colon g^{tab} \delta g_{ab} = 0 \} ,$$

$$\mathscr{T}_g \equiv \left\{ \delta g \in H(T_2) \colon \delta g = \delta t^k e^{\varphi} \frac{\partial}{\partial t^k} g^t ,\ \delta t \in \mathbb{R}^{6h-6} \right\},$$

$$\mathscr{V}_g \equiv \{ \delta g \in H(T_2) \colon \delta g_{ab} = \nabla_a(\delta V_b) + \nabla_b(\delta V_a),\ \delta V \in T(M_h) \} ,$$

(∇_g denotes the covariant derivative defined by $e^{\varphi}g^t$). We have the following orthogonal decompositions:

$$\mathcal{T}_g\bar{\mathcal{M}}_h=\mathcal{W}_g\oplus\mathcal{W}_g^{\perp}, \tag{5.11}$$

$$\mathcal{T}_g\bar{\mathcal{M}}_h=\mathcal{V}_g\oplus\mathcal{V}_g^{\perp},$$

$$\mathcal{T}_g\bar{\mathcal{M}}_h=\mathcal{K}_g\oplus\mathcal{H}_g, \tag{5.12}$$

$$\mathcal{W}_g=\mathcal{K}_g\oplus\mathcal{K}_g^{\perp},$$

$$\mathcal{W}_g=\mathcal{T}_g\oplus\mathcal{T}_g^{\perp},$$

$$\mathcal{H}_g=\mathcal{K}_g^{\perp}\oplus\mathcal{W}_g^{\perp}, \tag{5.13}$$

where $\mathcal{W}_g^{\perp}(\mathcal{V}_g^{\perp})$ denotes the orthogonal complement of $\mathcal{W}_g$ (respectively $\mathcal{V}_g$) in $\mathcal{T}_g\bar{\mathcal{M}}_h$, and $\mathcal{K}_g^{\perp}(\mathcal{T}_g^{\perp})$ denotes the orthogonal complement of $\mathcal{K}_g$ (respectively $\mathcal{T}_g$) in $\mathcal{W}_g$ (on $\mathcal{W}_g$ we have an induced scalar product). Let us consider the projection operators:

$$\Pi_g^{\mathcal{W}^{\perp}}:\mathcal{T}_g\bar{\mathcal{M}}_h\to\mathcal{W}_g^{\perp},$$

$$\tilde{\Pi}_g^{\mathcal{W}^{\perp}}:\mathcal{H}_g\to\mathcal{W}_g^{\perp},$$

$$\Pi_g^{\mathcal{H}}:\mathcal{T}_g\bar{\mathcal{M}}_h\to\mathcal{H}_g,$$

related to the splitting (5.11), (5.13) and (5.12) respectively. We have the obvious relation:

$$\Pi^{\mathcal{W}^{\perp}}=\tilde{\Pi}^{\mathcal{W}^{\perp}}\circ\Pi^{\mathcal{H}}.$$

The infinite-dimensional counterpart $T_g(\tilde{P}_g)$ of the linear operator τ_p (respectively Δ_p) considered in Sect. 2 has in the present case the following form:

$$T_g:H(TM_h)\to H(T_2),$$

$$(T_g\delta V)_{ab}\equiv(\nabla_a g_{bc}+\nabla_b g_{ac})\delta V^c,$$

$$\tilde{P}_g:H(TM_n)\to\mathcal{W}_g^{\perp},$$

$$\tilde{P}_g\equiv\Pi_g^{\mathcal{W}^{\perp}}\circ T_g=\tilde{\Pi}_g^{\mathcal{W}^{\perp}}\circ\Pi_g^{\mathcal{H}}\circ T_g=\tilde{\Pi}_g^{\mathcal{W}^{\perp}}\circ P_g,$$

where P_g denotes the operator introduced by Alvarez [13]:

$$P_g:H(TM_h)\to\mathcal{H}_g,$$

$$(P_g\delta V)_{ab}\equiv(g_{ac}\nabla_b+g_{bc}\nabla_a-g_{ab}\nabla_c)\delta V^c,$$

where $g=e^{\varphi}g^t$ and ∇ is the covariant derivative defined by g. Let us consider the orthogonal splitting:

$$\mathcal{H}_g=\operatorname{Im}P_g\oplus\operatorname{Ker}P_g^+, \tag{5.14}$$

where $P_g^+:\mathcal{H}_g\to H(TM_h)$ is defined by:

$$G_g(\delta g|P_g\delta V)=H_{\mathrm{id}}^g(P_g^+\delta g|\delta V)$$

for all $\delta g\in\mathcal{H}_g$, $\delta V\in H(TM_h)$. One can construct the projection operators related to this decomposition:

$$\Pi^p:\mathcal{H}_g\to\operatorname{Im}P_g,$$

$$\Pi^{p^+}:\mathcal{H}_g\to\operatorname{Ker}P_g^+.$$

It is easy to verify that the adjoint operator $\tilde{P}_g^+ : \mathscr{H}_g \to H(TM_h)$ defined by:

$$G_g(\delta g|\tilde{P}_g \delta V) = H^g_{\mathrm{id}}(\tilde{P}_g^+ \delta g|\delta V)$$

for all $\delta g \in \mathscr{H}_g$, $\delta V \in H(TM_h)$ fulfills the relation:

$$\tilde{P}_g^+ = P_g^+ \circ \Pi_g^p ,$$

and

$$\det \tilde{P}_g^+ \tilde{P}_g = \det \mathscr{P}_g^p \det \mathscr{P}_g^{w^\perp} \det P_g^+ P_g ,$$

where $\mathscr{P}_g^p \equiv \Pi^p|_{\mathscr{W}_g^\perp}$, $\mathscr{P}_g^{\mathscr{W}^\perp} \equiv \tilde{\Pi}_p^{\mathscr{W}^\perp}|_{\mathrm{Im}\,P_g}$. Let us note that determinants of $\mathscr{P}^p$ and $\mathscr{P}^{\mathscr{W}^\perp}$ in the formula above are evaluated in an orthonormal basis of spaces $\mathscr{W}_g^\perp$ and $\mathrm{Im}\,P_g$. Using a formal generalization of the first part of the lemma of the Appendix in the case of the decompositions (5.13) and (5.14) we have:

$$\det \mathscr{P}_g^p = \det \mathscr{P}_g^{\mathscr{W}^\perp} = \det[G_g(\delta\tilde{\psi}_i|\delta\tilde{\chi}_j)] ,$$

where $\{\delta\tilde{\psi}_i\}_{i=1}^{6h-6}$ ($\{\delta\tilde{\chi}_j\}_{j=1}^{6h-6}$) denotes the orthonormal basis in $\mathrm{Ker}\,P_g^+$ (respectively in $\mathscr{K}_g^\perp$). For any arbitrary (not necessarily orthonormal) basis $\{\delta\psi_i\}_{i=1}^{6h-6}$ of $\mathrm{Ker}\,P_g^+$ we have:

$$\det \mathscr{P}_g^p = \frac{\det[G_g(\delta\psi_i|\delta\tilde{\chi}_j)]}{(\det H(P^+))^{1/2}} ,$$

where H is the matrix introduced by Alvarez [13]:

$$H(P^+)_{ij} \equiv G_g(\delta\psi_i|\delta\psi_j) .$$

Resuming we have the following correspondence:

finite-dimensional case	infinite-dimensional case
$P \to P/G$	$\bar{\mathscr{M}}_h \to \bar{\mathscr{M}}_h/\mathscr{D}_h^0$
$f(p)$	$\left(\dfrac{\det' \mathscr{L}_g}{\int \sqrt{g}\, d^2 z}\right)^{-d/2}$
$u \in \Sigma$	$g = e^{\varphi} g^t \in \Sigma_\iota$
$\{h^p\}_{p \in P}$	$\{H^g\}_{g \in \bar{\mathscr{M}}_h}$
$T_u P = W_u \oplus W_u^\perp$	$\mathscr{T}_g \bar{\mathscr{M}}_h = \mathscr{W}_g \oplus \mathscr{W}_g^\perp$
$\tau_u : G' \to T_u P$	$T_g : H(TM_h) \to H(T_2)$
$\Delta_u : G' \to W_u^\perp$	$\tilde{P}_g = \tilde{\Pi}_g^{\mathscr{W}^\perp} \circ P_g : H(TM_h) \to \mathscr{W}_g^\perp$
$\Delta_u^+ : T_u P \to G'$	$\tilde{P}_g^+ = P_g^+ \circ \Pi_g^p : \mathscr{H}_g \to H(TM_h)$
$\det \Delta_u^+ \Delta_u$	$\det \tilde{P}_g^+ \tilde{P}_g = \det P_g^+ P_g \dfrac{(\det G_g(\delta\psi_i|\delta\tilde{\chi}_j))^2}{\det H(P_g^+)}$.

Using the formal generalization of Theorem 3, the following form of the integral Z_h can be derived:

$$Z_h = L^d \int_{\Sigma_\iota} \det G_g(\delta\psi_i|\delta\tilde{\chi}_j) \left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} \left(\frac{\det' \mathscr{L}_g}{\int_{M_h} \sqrt{g}\, d^2 z}\right)^{-d/2} d\Omega^{\Sigma_\iota} ,$$

where $d\Omega^{\Sigma_\iota}$ denotes the formal volume element on Σ_ι related to the metric induced on Σ_ι.

The next step is to express the integration over Σ_i in terms of integration over a conformal factor φ and Teichmuller parameters t^k. It can be achieved by applying an appropriate formal extension of the Fubini theorem in the case of fibration:

$$\Pi:\Sigma_i \to \mathscr{T}_h .$$

In order to do it we will construct the formal volume form on Σ_i of the form:

$$\Pi^*(de^1 \wedge \ldots \wedge de^{6h-6}) \wedge d\tilde{\Omega},$$

where $de^1 \wedge \ldots \wedge de^{6h-6}$ is the euclidean volume form on $\mathbb{R}^{6h-6}$ and $d\tilde{\Omega}$ is a formal form on Σ_i such that:

$$d\tilde{\Omega}|_{\Pi^{-1}(t)} = d\Omega^{\Sigma_i}|_{\Pi^{-1}(t)} . \tag{5.15}$$

Let us consider two bases of $\mathscr{T}_g\Sigma_i$, the orthonormal one:

$$\{\delta\tilde{\chi}_j\}_{j=1}^{6h-6} \cup \{\delta\tilde{\varphi}_k\}_{k=1}^{\infty},$$

$$\delta\tilde{\chi}_j \in \mathscr{K}_g^{\perp}, \quad (j=1,\ldots,6h-6),$$

$$\delta\tilde{\varphi}_k \in \mathscr{K}_g, \quad (k=1,2,\ldots),$$

and the following one (in general not orthonormal):

$$\{\delta\chi_j\}_{j=1}^{6h-6} \cup \{\delta\tilde{\varphi}_k\}_{k=1}^{\infty},$$

$$\delta\chi_j \equiv i_*\delta e_j = e^{\varphi}\frac{\partial}{\partial t^j} g^t \in \mathscr{T}_g .$$

For dual bases $\{d\tilde{\chi}^j\}_{j=1}^{6h-6} \cup \{d\tilde{\varphi}^i\}_{i=1}^{\infty}$, $\{d\chi^j\}_{j=1}^{6h-6} \cup \{d\varphi^i\}_{i=1}^{\infty}$, we formally have:

$$\begin{aligned} d\Omega^{\Sigma_i} & \bigwedge_{j=1}^{6h-6} d\tilde{\chi}^j \wedge \bigwedge_{k=1}^{\infty} d\tilde{\varphi}^k \\ &= \det G_g(\delta\tilde{\chi}_j|\delta\chi_i) \bigwedge_{j=1}^{6h-6} d\chi^j \wedge \bigwedge_{k=1}^{\infty} d\varphi^k \\ &= \det G_g(\delta\tilde{\chi}_j|\delta\chi_i)\Pi^*(de^1 \wedge \ldots \wedge de^{6h-6}) \wedge d\tilde{\Omega} . \end{aligned} \tag{5.16}$$

Let us note that in general $d\tilde{\varphi}^k \neq d\varphi^k$, nevertheless the relation (5.15) is valid. Now from (5.16) and the Fubini theorem the following expression of Z_h can be derived:

$$\begin{aligned} Z_h &= L^d \int_{\mathscr{T}_h} d^{6h-6}t \int_{\Pi^{-1}(t)} d\tilde{\Omega} \det G_g(\delta\psi_i|\delta\tilde{\chi}_j) \\ &\quad \times \det G_g(\delta\tilde{\chi}_j|\delta\chi_k)\left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} \left(\frac{\det' \mathscr{L}_g}{\int\limits_{M_h} \sqrt{g}\, d^2z}\right)^{-d/2} \\ &= L^d \int_{\mathscr{T}_h} d^{6h-6}t \int_{\Pi^{-1}(t)} d\tilde{\Omega} \det G_g(\delta\psi_i|\delta\bar{\chi}_k)\left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} \left(\frac{\det' \mathscr{L}_g}{\int\limits_{M_h} \sqrt{g}\, d^2z}\right)^{-d/2}, \end{aligned} \tag{5.17}$$

where

$$(\delta\bar{\chi}_j)_{ab} \equiv e^{\varphi}\left(\frac{\partial}{\partial t^j} g^t_{ab} - \tfrac{1}{2} g^t_{ab} g^{tcd} \frac{\partial}{\partial t^j} g^t_{cd}\right).$$

It is easy to check by straightforward calculation that $G_g(\delta\psi_i|\delta\bar{\chi}_j)$ is independent of the conformal factor φ. The formal volume form $d\tilde{\Omega}$ in the formula (5.17) can be considered as one related to the induced riemannian structure on $\Pi^{-1}(t)$. Changing the variable of integration $\varphi \to e^{\varphi} g^t$ we finally have:

$$Z_h = L^d \int_{\mathscr{T}_h} d^{6h-6} t \det G_g(\delta\psi_i|\delta\bar{\chi}_i) \int_{\mathscr{W}_h} d\Omega^{\tilde{W}} \left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} \left(\frac{\det' \mathscr{L}_g}{\int_{M_h} \sqrt{g}\, d^2 z}\right)^{-d/2}, \quad (5.18)$$

where $d\Omega^{\tilde{W}}$ denotes the volume element related to the nonconstant riemannian structure $\tilde{W}(\,|\,)$ on $\mathscr{W}_h$:

$$\tilde{W}_{\varphi}(\delta\varphi, \delta\varphi') \equiv \int_{M_h} e^{\varphi} \sqrt{g^t}\, d^2 z\, \delta\varphi\, \delta\varphi' \quad (\delta\varphi, \delta\varphi' \in \Omega^0(M_h)).$$

The formula (5.18) has been derived for the first time by Alvarez [13] for the special case of gauges $\imath: \mathscr{T}_h \to \mathscr{M}_h$ for which $\delta\chi_j \in \operatorname{Ker} P_g^+$, and more recently in the general case by Moore and Nelson [22].

Let us comment on the problem of overcounting of physical configurations. In fact in the above discussion the full gauge group $\mathscr{D}_h$ has been replaced by $\mathscr{D}_h^0$ which leads to the residual gauge invariance under the group $\Gamma_h \equiv \mathscr{D}_h/\mathscr{D}_h^0$. Γ_h is the so-called homeotopy or mapping class group and is the properly discontinuous group of the Teichmuller space [14]. The quotient $\mathscr{T}_h/\Gamma_h$ is isomorphic with the moduli space of riemannian surface of genus h, defined as a space of all complex structures on M_h. The problem of this overcounting can be overcome by restricting the integral over Teichmuller parameters in (5.18) to the fundamental domain $[\Gamma_h] \subset \mathscr{T}_h$ of Γ_h in $\mathscr{T}_h$. This is possible because the singular points of $[\Gamma_h]$ form a set of zero measure and the intergrand of (5.18) has modular invariance.

Now let us consider the integral Z_h^{26}. Due to the "diagonal" structure of metric $V^g(\,|\,)$ on $\mathscr{D}_h \odot \mathscr{W}_h$, the Fubini theorem yields formal relation:

$$\int_{\mathscr{D}_h \odot \mathscr{W}_h} d\Omega^{V^g} = \int_{\mathscr{D}_h} d\Omega^{H^g} \times \int_{\mathscr{W}_h} d\Omega^{W^g}$$

and

$$Z_h^{26} = \int_{\mathscr{P}_h} \left(\int_{\mathscr{D}_h} d\Omega^{H^g}\right)^{-1} \left(\int_{\mathscr{W}_h} d\Omega^{W^g}\right)^{-1} e^{-S[g,x]} d\Omega^G .$$

The "volume" of the Weyl group is a $\mathscr{D}_h$-invariant functional of g so nearly all calculations can be performed as in the case of Z_n. The result is:

$$Z_h^{26} = L^d \int_{[\Gamma_h]} d^{6h-6} t G_g(\delta\psi_i|\delta\bar{\chi}_j) \int_{\mathscr{W}_h} \left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} \left(\frac{\det' \mathscr{L}_g}{\int_{M_h} \sqrt{g}\, d^2 z}\right)^{-d/2} \times \left(\int_{\mathscr{W}_h} d\Omega^{W^g}\right)\Bigg)^{-1} d\Omega^{\tilde{W}^g} .$$

Using the heat kernel method Alvarez shows [13] that up to φ-independent factors and with an appropriately chosen renormalization constant:

$$\mathrm{P}\left(\frac{\det P_g^+ P_g}{\det H(P_g^+)}\right)^{1/2} \left(\frac{\det' \mathscr{L}_g}{\int_{M_h} \sqrt{g}\, d^2 z}\right)^{-d/2} = \exp\left(-\frac{26-d}{24\pi} S_L[\varphi]\right), \quad (5.19)$$

where $S_L[\varphi]$ denotes the action of Liouville theory. Therefore for $h>1$ and $d=26$ the left-hand side of (5.19) is φ-independent and one can write:

$$Z_h^{26}=L^d \int_{[\Gamma_h]} d^{6h-6}t G_g(\delta\psi_i|\delta\bar{\chi}_j)\left(\frac{\det P_g^+P_g}{\det H(P_g^+)}\right)^{1/2}\left(\frac{\det\mathscr{L}_g}{\int\sqrt{g}\,d^3z}\right)^{-13}$$
$$\times \int_{\mathscr{W}_h}\left(\int_{\mathscr{W}_h} d\Omega^{W^{e^{\varphi}gt}}\right)^{-1} d\Omega^{\tilde{W}^{gt}}.$$

Let us recall that $d\Omega^{\tilde{W}^{gt}}$ is related to the nonconstant metric $\tilde{W}^{g^t}(\,|\,)$:

$$\tilde{W}^{g^t}_{\varphi'}(\delta\varphi|\delta\varphi')\equiv \int_{M_h} e^{\varphi'}\sqrt{g^t}\,\delta\varphi\delta\varphi' d^2z\,,$$

while $d\Omega^{W^g}$ is related to the constant metric $W^g(\,|\,)$:

$$W^{e^{\varphi}g^t}_{\varphi'}(\delta\varphi|\delta\varphi')\equiv \int_{M_h} e^{\varphi}\sqrt{g^t}\,\delta\varphi\delta\varphi' d^2z\,.$$

We have the following formal relations [18]:

$$d\Omega^{\tilde{W}^{gt}}=e^{\frac{1}{4}\int\varphi\delta(0)d^2z}d\Omega^{W^{gt}}\,,$$
$$\int_{\mathscr{W}_h} d\Omega^{W^{e^{\varphi}gt}}=e^{\frac{1}{4}\int\varphi\delta(0)d^2z}\int_{\mathscr{W}_h} d\Omega^{W^g}\,,$$

therefore:

$$\int_{\mathscr{W}_h} d\Omega^{\tilde{W}^{gt}}\left(\int_{\mathscr{W}_h} d\Omega^{W^{e^{\varphi}gt}}\right)^{-1}=1\,,$$

and finally:

$$Z_h^{26}=\int_{[\Gamma_h]} d^{6h-6}t G_g(\delta\psi_i|\delta\bar{\chi}_j)\left(\frac{\det P_g^+P_g}{\det H(P_g^+)}\right)^{1/2}\left(\frac{\det\mathscr{L}_g}{\int_{M_h}\sqrt{g}\,d^2z}\right)^{-13}. \tag{5.20}$$

The expression above coincides with that obtained in [15] and [22].

6. Concluding Remarks

As follows from the above considerations the presented approach provides a convenient method of dealing with functional integrals and gives the geometrical interpretation of the Faddeev-Popov procedure. Moreover the method is sensitive enough to distinguish between three prescriptions for the functional integral over the orbit space given in Sect. 2. The results of Sects. 3 and 4 show that the first definition (related to I_1) is never appropriate and has been considered only for comparision. The second definition (related to I_2) can be considered as a special case of the third one. If the natural right-invariant riemannian structure on the gauge group is field-dependent (as in the Yang-Mills theory) both prescriptions lead to the identical, correct answer. In the case of the point particle the natural right-invariant riemannian structure on the group of diffeomorphisms depend on the metric and these two approaches yield different results. Explicit calculations of Sect. 4 show that only $P_3(x,y)$ is the usual particle propagator; therefore one can expect that the third prescription is the universally correct one. Finally, as has been discussed in Sect. 5, the scheme related to the integral I_3 and Theorem 3 of Sect. 2 applied to Polyakov's string leads to the known expressions of the multi-loop vacuum amplitudes [14–17, 22].

Taking into account the results presented above one can try to formulate the general scheme for covariant quantization of the first class gauge theories. Starting with an euclidean action S we must first choose the appropriate (infinite-dimensional) manifold $\mathscr{M}$ of field configurations and the gauge group $\mathscr{G}$ defined as a group of local symmetry transformations of the action S. The next step is to determine the most natural (ultralocal and not involving the additional structures and parameters) riemannian metric $G(\,|\,)$ on $\mathscr{M}$ and $H(\,|\,)$ on $\mathscr{G}$. The relevant functional integral is then defined by analogy with the finite-dimensional integral I_3 introduced in Sect. 2:

$$Z=\int_{\mathscr{M}}\left(\int_{\mathscr{G}} d\Omega^{H}\right)^{-1} e^{-S} d\Omega^{G}. \tag{6.1}$$

Let us note that in general the metric $H(\,|\,)$ can be field-dependent. The possibility of giving a meaning to the integral (6.1) by the Faddeev-Popov procedure of course depends on the geometry of the action of $\mathscr{G}$ on $\mathscr{M}$. The possible obstructions can be seen as anomalies.

Of course it must be checked that the above scheme is a scheme of quantization. It is far from obvious that the functional integral (6.1) considered for example as a propagator (with the appropriate choice of $\mathscr{M}$) leads to a unitary motion of some quantum system. The possible resolution of this problem is to show the equivalence of the presented approach with the canonical one introduced by Fradkin [1] which has clear physical interpretation and is formally unitary "by definition." This equivalence is obvious in the case of the relativistic particle [33] and well established for the Yang-Mills theories [2]. In the case of the bosonic string the situation is not so clear (for some related points see [32, 34]). Let us note that Polyakov's covariant quantization of the string at $d<26$ is beyond the general scheme described here. As it was discussed in the previous section only a part of the full gauge group is used in the definition of the appropriate functional integral [see (5.6)]. It was shown by Marnelius [35] that in this case Polyakov's theory is equivalent to the canonically quantized suitably modified classical model of the string. As one can expect this modification consists in the introduction of additional degrees of freedom on the classical lavel. This problem is probably more important in the case of euclidean gravity so it seems to be interesting to consider scheme sketched in this section also in this case. Some attempts in this direction have been made in [36] where geometrical interpretation of the Faddeev-Popov determinant was also discussed.

Appendix

The aim of this appendix is to give some justification of theorems formulated in Sect. 2. For simplicity special requirements concerning differentiability and measurability will be omitted. We generally assume that all structures under consideration are sufficiently smooth. Let us note that the integrals considered in Sect. 2 are oriented so all manifolds are assumed to be orientable.

We start with some additional constructions. At every point $p \in P$ we have two orthogonal decompositions (2.1), (2.7):

$$T_pP = W_p \oplus W_p^\perp = V_p \oplus V_p^\perp$$

and two pairs of projection operators related to these splittings:

$$\Pi_p^W : T_pP \to W_p\,, \qquad \Pi_p^{W\perp} : T_pP \to W_p^\perp\,,$$
$$\Pi_p^V : T_pP \to V_p\,, \qquad \Pi_p^{V\perp} : T_pP \to V_p^\perp\,.$$

Let us define:

$$\mathscr{P}_p^V \equiv \Pi_p^V|_{W_p^\perp} : W_p^\perp \to V_p\,,$$
$$\mathscr{P}_p^{V\perp} \equiv \Pi_p^{V\perp}|_{W_p} : W_p \to V_p^\perp\,,$$
$$\mathscr{P}_p^W \equiv \Pi_p^W|_{V_p^\perp} : V_p^\perp \to W_p\,,$$
$$\mathscr{P}_p^{W\perp} \equiv \Pi_p^{W\perp}|_{V_p} : V_p \to W_p^\perp\,.$$

All these operators are bijective which follows from the fact that $\sigma : U \to P$ is a section of P and g is nondegenerate. It is convenient to prove the following:

Lemma.
$$|\det \mathscr{P}_p^V| = |\det \mathscr{P}_p^{V\perp}| = |\det \mathscr{P}_p^W| = |\det \mathscr{P}^{W\perp}|\,, \tag{A.1}$$

where the determinants of all operators are evaluated in the orthonormal (with respect to the metric g) bases in V_p, $V_p^\perp$, W_p, $W_p^\perp$. The absolute value of all determinants in (A.1) *is a G-invariant function on P and*

$$|\det \mathscr{P}_P^{W\perp}| = \frac{(\det \Delta_p^+ \Delta_p)^{1/2}}{(\det \tau_p^+ \tau_p)^{1/2}}\,. \tag{A.2}$$

Proof of the Lemma. Let us choose two orthonormal (with respect to the metric g) bases $\{\delta w_i\}_{i=1}^m$, $\{\delta v_i\}_{i=1}^m$ in T_pP such that:

$$\begin{aligned} &\delta w_i \in W_p^\perp\,, \quad \delta v_i \in V_p \quad \text{for} \quad i = 1, \ldots, n\,, \\ &\delta w_i \in W_p\,, \quad \delta v_i \in V_p^\perp \quad \text{for} \quad i = n+1, \ldots, m\,. \end{aligned} \tag{A.3}$$

Further we construct the basis $\{\delta t_i\}_{i=1}^m$ of T_pP (in general not orthonormal):

$$\begin{aligned} &\delta t_i \equiv \delta v_i \quad \text{for} \quad i = 1, \ldots, n\,, \\ &\delta t_i \equiv \delta w_i \quad \text{for} \quad i = n+1, \ldots, m\,. \end{aligned} \tag{A.4}$$

We have:

$$\delta t_i = \sum_{j=1}^m A_{ij} \delta w_j = \sum_{j=1}^m B_{ij} \delta v_j\,,$$

where the transition matrices A, B are defined by:

$$A_{ij} \equiv g_p(\delta t_i, \delta w_j)\,, \qquad B_{ij} \equiv g_p(\delta t_i, \delta v_j)\,.$$

Proceeding to the appropriate dual bases $\{dw^i\}_{i=1}^m$, $\{dv^i\}_{i=1}^m$, $\{dt^i\}_{i=1}^m$ we have:

$$\bigwedge_{i=1}^m dw^i = \det A \bigwedge_{i=1}^m dt^i = \pm \det B \bigwedge_{i=1}^m dt^i = \pm \bigwedge_{i=1}^m dv^i\,, \tag{A.5}$$

and therefore:

$$|\det A| = |\det B|\,. \tag{A.6}$$

The transition matrices have the following form:

$$A=\left[\begin{array}{c|c} a & \dots \\ \hline 0 & \mathbb{1} \end{array}\right], \quad B=\left[\begin{array}{c|c} \mathbb{1} & 0 \\ \hline \dots & b \end{array}\right], \tag{A.7}$$

where:

$$a_{ij}\equiv g_p(\delta v_i, \delta w_j) \qquad i,j=1,\dots,n,$$

$$b_{kl}\equiv g_p(\delta v_k, \delta w_l) \qquad k,l=n+1,\dots,m.$$

From (A.6) and (A.7) it follows that $|\det a|=|\det b|$. Now from an explicit form of the matrix a one can see that $\det a$ can be interpreted as a determinant of the operator $\mathscr{P}_p^{W\perp}$ or $\mathscr{P}_p^V$ evaluated in the orthonormal bases in V_p and $W_p^\perp$. Similarly for the determinant of the matrix b we have:

$$\det b=\det\mathscr{P}_p^{V\perp}=\det\mathscr{P}_p^W.$$

Now we proceed to the second part of the lemma. Let us note that all determinants (A.1) are independent, up to sign, of the choice of an orthonormal bases in the spaces V_p, $V_p^\perp$, W_p, $W_p^\perp$ and their absolute values are well defined functions on P. The G-invariance of these functions immediately follows from the G-invariance of $\det a$ (or $\det b$) which is a consequence of the G-invariance of the metric g. In order to show the relation (A.2), let us consider some base $\{\delta s_i\}_{i=1}^n$ in G' and let $\{\delta v_i\}_{i=1}^n$ and $\{\delta w_i\}_{i=1}^n$ are orthonormal bases in the spaces V_p and respectively $W_p^\perp$. With the convention that $[A]$ denotes the matrix of operator A evaluated in appropriate bases and $[\tilde{h}]$ is a matrix defined by $[\tilde{h}]_{ij}\equiv\tilde{h}(\delta s_i, \delta s_j)$, we have:

$$[\tau_p^+|_{V_p}]=[\tilde{h}]^{-1}\circ[\tau_p]^T,$$

$$[\Delta_p^+|_{W_p^\perp}]=[\tilde{h}]^{-1}\circ[\tau_p]^T\circ[\mathscr{P}_p^{W\perp}]^T$$

and

$$\begin{aligned}\det\tau_p^+\tau_p&=\det\tau_p^+|_{V_p}\circ\tau_p\\&=(\det[\tilde{h}])^{-1}(\det[\tau_p])^2,\\ \det\Delta_p^+\Delta_p&=\det\Delta_p^+|_{W_p^\perp}\circ\Delta_p\\&=(\det[\tilde{h}])^{-1}(\det[\tau_p]\cdot\det[\mathscr{P}_p^{W\perp}])^2.\end{aligned}$$

Therefore the equation (A.2) follows. Let us note that the right-hand side of (A.2) does not depend on the choice of the base in G' and the inner product $\tilde{h}$ in G'.

Proof of Theorem 1. Let us change the variable in the integral I_1 using the global section $\sigma: U\to\Sigma\subset P$:

$$I_1[f]=\int_U f d\omega^{\tilde{g}}=\int_\Sigma f d\tilde{\omega}^\Sigma,$$

where $d\tilde{\omega}^\Sigma$ is the volume element related to the metric $\tilde{g}^\Sigma$ defined at every point $u\in\Sigma$ by:

$$\tilde{g}_u^\Sigma\equiv(\pi^*\tilde{g})_u|_{T_u\Sigma},$$

For every $\delta u, \delta u'\in T_u\Sigma=W_u$ we have:

$$\tilde{g}_u^\Sigma(\delta_u, \delta_{u'})=g_u(\mathscr{P}_u^{V\perp}\delta u, \mathscr{P}_u^{V\perp}\delta u'). \tag{A.8}$$

The corresponding expression for the induced metric g^Σ on Σ is as follows:

$$g_u^\Sigma(\delta u, \delta u')=g_u(\delta u, \delta u').$$

Now we compare the volume elements related to the metrics g^Σ and $\tilde{g}^\Sigma$. Let us consider the bases: $\{\delta w_i\}_{i=n+1}^m$ in W_u and $\{\delta v_i\}_{i=n+1}^m$ in $V_u^\perp$ introduced in (A.3). It is clear that $\{\delta w_i\}_{i=n+1}^m$ is the orthonormal base in $T_u\Sigma$ with respect to the metric g^Σ. From the formula (A.8) one can see that the base $\{\delta \tilde{v}_i\}_{i=n+1}^m$ defined by:

$$\delta\tilde{v}_i \equiv (\mathscr{P}_u^{V^\perp})^{-1}\delta v_i \quad \text{for} \quad i=n+1,\dots,m$$

is the orthonormal base with respect to the metric $\tilde{g}$. Without loss of generality one can assume that bases: $\{\delta w_i\}_{i=n+1}^m$, $\{\delta\tilde{v}_i\}_{i=n+1}^m$ are both positively orientated. The transition matrix from $\{\delta w_i\}_{i=n+1}^m$ to $\{\delta\tilde{v}_i\}_{i=n+1}^m$ can be easily calculated:

$$\begin{aligned}\delta\tilde{v}_i &= \sum_{j=n+1}^m g_u^\Sigma(\delta\tilde{v}_i, \delta w_j)\delta w_j \\ &= \sum_{j=n+1}^m g_n((\mathscr{P}_u^{V^\perp})^{-1}\delta v_i, \delta w_j)\delta w_j \\ &= \sum_{j=n+1}^m A_{ij}\delta w_j .\end{aligned}$$

Proceeding to the dual bases $\{dw^i\}_{i=n+1}^m$, $\{d\tilde{v}_i\}_{i=n+1}^m$ we have:

$$d\tilde{\omega}^\Sigma = \bigwedge_{i=n+1}^m d\tilde{v}^i = (\det A)^{-1}\bigwedge_{i=n+1}^m dw^i = (\det A)^{-1}d\omega^\Sigma .$$

Because A is a matrix of the operator $(\mathscr{P}_u^{V^\perp})^{-1}$ evaluated in orthonormal bases of the spaces $V_u^\perp$ and W_u we can apply our lemma to its determinant. From (A.1) it follows that:

$$\det{}^{-1}A = \det\mathscr{P}_u^{V^\perp} = \det\mathscr{P}_u^{W^\perp}$$

and from (A.2) we finally have:

$$d\tilde{\omega}^\Sigma = \frac{(\det\Delta_u^+\Delta_u)^{1/2}}{(\det\tau_u^+\tau_u)^{1/2}}d\omega^\Sigma . \quad \square$$

Proof of Theorem 2. We will prove (2.13) extracting from the integral:

$$\int_P f d\omega^g ,$$

the volume of the "gauge" group G by means of the Fubini theorem on manifolds [26]. In order to apply this theorem in our case we must construct a suitable expression of the volume m-form $d\omega^g$ as an exterior product of two forms on P. At every point $p\in P$ we consider the base $\{\delta t_i\}_{i=1}^m$ in T_pP defined in (A.4) and such that $\{\delta t_i\}_{i=1}^m$ ($\{\delta t_i\}_{i=n+1}^m$) is positively oriented in T_pP (respectively in $T_p\Sigma_a$) with respect to some fixed orientation on P (respectively on Σ_a). An orientation on the fiber $\Pi^{-1}(\Pi(p))$ we determine by demand that the orientation of $\{\delta t_i\}_{i=1}^n$ is positive. Proceeding to the dual base $\{dt^i\}_{i=1}^m$ we define:

$$d\omega_p^\perp \equiv dt^1 \wedge \dots \wedge dt^n ,$$

$$d\omega_p \equiv dt^{n+1} \wedge \dots \wedge dt^m .$$

Because the splitting $T_pP = V_P \oplus W_P$ is continuous and G-invariant $d\omega^\perp$ and $d\omega$ are well defined forms on P. For these forms the following relations are true:

$$\begin{aligned} d\omega^\perp|_{\Pi^{-1}(u)} &= d\omega^g|_{\Pi^{-1}(u)}, \qquad u\in\Sigma , \\ d\omega &= \Pi^* d\omega^\Sigma . \end{aligned} \tag{A.9}$$

From the proof of lemma [relations (A.5–7)] we have:

$$d\omega^g = \det \mathscr{P}_p^{W\perp} d\omega^\perp \wedge \pi^* d\omega^\Sigma$$

(in order to avoid problem with sign one can assume that bases $\{\delta w_i\}_{i=1}^m$ and $\{\delta v_i\}_{i=1}^m$ have positive orientation). Using our lemma (A.2) we have:

$$d\omega^g = \frac{(\det \Delta_p^+ \Delta_p)^{1/2}}{(\det \tau_p^+ \tau_p)^{1/2}} d\omega^\perp \wedge \pi^* d\omega^\Sigma .$$

Now applying the Fubini theorem [26] to the integral:

$$\int_P f d\omega^g = \int_P f \cdot \frac{(\det \Delta_p^+ \Delta_p)^{1/2}}{(\det \tau_p^+ \tau_p)^{1/2}} d\omega^\perp \wedge \pi^* d\omega^\Sigma ,$$

and using (A.9) we have:

$$\int_P f d\omega^g = \int_\Sigma f \cdot \frac{(\det \Delta_u^+ \Delta_u)^{1/2}}{(\det \tau_u^+ \tau_u)^{1/2}} \left(\int_{\pi^{-1}(u)} d\omega^{\pi^{-1}(u)} \right) d\omega^\Sigma \tag{A.10}$$

where $d\omega^{\pi^{-1}(u)}$ denotes the volume element related to the induced metric on $\pi^{-1}(u)$. The volume of the orbit $\pi^{-1}(u)$ can be evaluated by the change of variable $\beta_u : G \to \pi^{-1}(u)$ [see formulae (2.3)]

$$\int_{\pi^{-1}(u)} d\omega^{\pi^{-1}(u)} = \int_G d\omega^{g^u} ,$$

where $d\omega^{g^u}$ is the volume element related to the metric $g^u \equiv \beta_u^* g$ on G. Now we compare the volumes of the group G evaluated in the metric g^u and in the metric h used in the definition of $I_2[f]$. Because both metrices are right-invariant it is sufficient to consider the relation between appropriate volume elements at the neutral element e of G. Suppose $\{\delta s_i\}_{i=1}^n$ is the (suitable oriented) base in $G' \equiv T_e G$ orthogonal with respect to the metric h. The base $\{\delta \tilde{v}_i\}_{i=1}^n$ in G' orthonormal with respect to the metric g^u can be constructed in the following way:

$$\delta \tilde{v}_i \equiv \tau_u^{-1} \delta v_i \quad \text{for} \quad i = 1, \ldots, n ,$$

where τ_u^{-1} is the inverse of the operator τ_u defined in (2.2) and $\{\delta v_i\}_{i=1}^n$ is the base in $T_u \pi^{-1}(u) = V_n$ orthonormal with respect to the metric $g|_{V_n}$. Proceeding to the dual bases $\{d\tilde{v}^i\}_{i=1}^u$, $\{ds^i\}_{i=1}^n$ we obtain:

$$d\omega^{g^u} = \bigwedge_{i=1}^n d\tilde{v}^i = \det \tau_u \cdot \bigwedge_{i=1}^n ds^i = \det \tau_u d\omega^u ,$$

where determinant in the right-hand side is evaluated in the orthonormal bases in V_p and G'. It is clear that:

$$\det \tau_u = (\det \tau_u^+ \tau_u)^{1/2} ,$$

provided that in the definition of τ_u^+ the inner product $h_e \equiv h|_{T_e G}$ in G' is used. Finally we have:

$$\int_{\pi^{-1}(u)} d\omega^{\pi^{-1}(u)} = (\det \tau_u^+ \tau_u)^{1/2} \int_G d\omega^h . \tag{A.11}$$

Inserting (A.11) in (A.10) we have:

$$\int_P f d\omega^g = \int_G d\omega^h \cdot \int_\Sigma f(\det \Delta_u^+ \Delta_u)^{1/2} d\omega^\Sigma ,$$

where operators Δ_u^+ are defined with respect to the inner product $h_e = h|_{T_eG}$ in G'. □

The proof of Theorem 3 is almost identical as in the case of Theorem 2. We must only replace h by h^u, evaluating the volume of the fiber $\pi^{-1}(u)$.

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References

1. Fradkin, E.S., Vilkoviski, G.A.: Phys. Lett. **55B**, 224 (1975)
   Batalin, I.A., Vilkoviski, G.A.: Phys. Lett. **69B**, 309 (1977)
   Fradkin, E.S., Fradkina, T.E.: Phys. Lett. **72B**, 343 (1978)
2. Henneaux, M.: Phys. Rep. **126**, 1–66 (1985)
3. Faddeev, L.D.: Theor. Math. Phys. **1**, 3 (1969)
4. Faddeev, L.D., Popov, V.N.: Phys. Lett. **25B**, 30 (1967)
5. Faddeev, L.D., Slavnov, A.A.: Gauge fields: introduction to quantum theory. Reading MA: Benjamin/Cummings 1980
6. Gribov, V.N.: Nucl. Phys. B **139**, 1 (1978)
7. Singer, I.M.: Commun. Math. Phys. **60**, 7 (1978)
8. Cotta Ramusino, P., Reina, C.: JGP **1**, 121 (1984)
9. Kondracki, W., Rogulski, J.: Preprint No. 293, Inst. of Math. Polish Academy of Science (1984)
10. Babelon, O., Viallet, C.M.: Phys. Lett. **85B**, 264 (1979); Commun. Math. Phys. **81**, 515 (1981)
11. Fujikawa, K.: Phys. Rev. Lett. **42**, 1195 (1979), Phys. Rev. Lett. **44**, 1733 (1980); Phys. Rev. D **23**, 2262 (1981); Phys. Rev. D **25**, 2584 (1982); Nucl. Phys. B **226**, 437 (1983)
12. Polyakov, A.M.: Phys. Lett. **103B**, 207 (1981)
13. Alvarez, O.: Nucl. Phys. B **216**, 125 (1983)
14. Gilbert, G.: Nucl. Phys. B **277**, 102 (1986)
15. D'Hoker, E., Phong, D.H.: Nucl. Phys. B **269**, 205 (1986)
16. Namazie, M.A., Rajeev, S.: Nucl. Phys. B **277**, 332 (1986)
17. Gava, E., Iengo, R., Jayaraman, T., Ramachandran, R.: Phys. Lett. **168** B, 207 (1986)
18. Jaskólski, Z.: J. Math. Phys. **27**, 2570 (1986)
19. Elworthy, K.D.: In: Functional integration and its applications, p. 60. Arthurs, A.M. (ed.). Oxford: Clarendon 1975
20. Polchinski, J.: Commun. Math. Phys. **104**, 37 (1986)
21. Cohen, A., Moore, G., Nelson, P., Polchinski, J.: Nucl. Phys. B **267**, 143 (1986)
22. Moore, G., Nelson, P.: Nucl. Phys. B **226**, 58 (1986)
23. Durhuus, B.: Lectures given at NBI/NORDITA, Autumn 1982, NORDITA preprint 82/36 (1982) (unpublished)
24. Schwarz, A.S.: Commun. Math. Phys. **64**, 233 (1979)
25. Killingback, T.P.: Commun. Math. Phys. **100**, 267 (1985)
26. Sulanke, R., Wintgen, P.: Differential geometrie und Faserbundel. Berlin: VEB Deutscher Verlag der Wissenschaften 1972
27. Brink, L., Di Vecchia, P., Howe, P.: Nucl. Phys. B **118**, 76 (1977)
28. Ebin, D.: In: Proc. Symp. Pure Math., Vol. 15, pp. 11–40. Providence, RI: American Mathematical Society 1970
29. Hawking, S.W.: Commun. Math. Phys. **55**, 133 (1977)

30. Alvarez, O.: Lectures presented at the Workshop on Unified String Theories, Santa Barbara, Aug. 1985, preprint LBL-20483 (1985)
31. Abikoff, W.: The real analytic theory of Teichmüller space. Lecture Notes in Mathematics, Vol. 820. Berlin, Heidelberg, New York: Springer 1980
32. Niemi, A.J., Semenoff, G.W.: Phys. Lett. B **176**, 108 (1986)
33. Marnelius, R., Nilsson, B.: ITP-Goteborg preprint (Dec. 1979) unpublished
34. Fradkin, E.S., Tseytlin, A.A.: Ann. Phys. (N.Y.) **143**, 413 (1982)
35. Marnelius, R.: Nucl. Phys. B **211**, 14 (1983) see also Hwang, S., Marnelius, R.: ITP-Goteborg preprint 85–43 (1985)
36. Catenacci, R., Martinellini, M.: Phys. Lett. **138B**, 263 (1984)

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