The Boltzmann Equation for Weakly Inhomogeneous Data

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Abstract. We solve the initial value problem associated to the nonlinear Boltzmann equation in the case in which the initial distribution has sufficiently small spatial gradients.

1. Introduction and Notation

The nonlinear Boltzmann equation is believed to describe the time evolution of a rarefied gas of particles. It takes the form

$$D_t f := (\partial_t + v \cdot \nabla_x) f = Q(f, f), \quad f(0) = f_0, \tag{1.1}$$

where $D \times \mathbb{R}^3 \ni (x, v) \to f(x, v)$ is the distribution of the particles, x and v denote position and velocity respectively, $D \subset \mathbb{R}^3$ is the domain to which the gas is confined. The bilinear operator Q takes into account the interaction among the particles and will be specified later.

The initial value problem (1.1) has been considered by several authors, and the following three groups of results global in time are available;

a) *Spatially Homogeneous case.* If f_0 depends only on v, f_t has still this property. In this case global existence and uniqueness result can be proved for very general data.

b) *Small Deviations from Equilibrium.* It is well known that the nontrivial equilibria for the problem (1.1) are the Maxwellian distributions. An initial distribution f_0, slightly differing from a Maxwellian, can be proved to evolve globally and uniquely in time, according to Eq. (1.1). Moreover it approaches a Maxwellian asymptotically in time.

c) *Small Deviations from Vacuum.* A global existence and uniqueness theorem for the initial value problem (1.1) can be proved for an initial distribution $f_0(x, v)$, $x \in \mathbb{R}^3$,

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decaying for large values of $|x|$ and $|v|$ and under suitable smallness assumptions on f_0.

The first results in the above directions were obtained by Carleman [1], Ukai [2] (following a previous analysis due to Grad [3]) and Illner and Shinbrot [4] respectively.

Finally it should be noticed that a global existence theorem for the initial value problem (1.1), also for large data, has been proved in [5] in the framework of the Loeb integration. However this result does not clarify whether the solutions to the Boltzmann equation (for more general data than those of the kind a), b), c) above) develop singularities in finite times or not.

In the present paper we construct a unique solution to the Boltzmann problem (1.1) for initial distributions near, in a suitable sense, to an homogeneous datum. Namely, let us consider the gas in a three dimensional flat torus $T = [0, 1]^3$. Let f_0 be the initial distribution and

$$g_0(v) = \int_T dx f_0(x, v), \quad u_0(x, v) = f_0(x, v) - g_0(v). \tag{1.2}$$

One can hope to construct solutions along the following steps.

Step 1 (Local Theorem). The evolution equation for the perturbation u_t is bilinear. So it is natural to expect that, for any fixed arbitrary time t, one should be able to evolve u_0 up to t, provided that u_0 is sufficiently small. Moreover u_t ought to be small together with u_0.

Step 2 (Approach to Equilibrium for the Homogeneous Equation). Let g_t be the solution of the homogeneous equation with initial datum g_0. It is known that g_t approaches a Maxwellian equilibrium when $t \to \infty$. Choose t so large (and accordingly u_0 so small) that g_t is arbitrarily near to a Maxwellian ω.

Step 3 (Perturbation of the Equilibrium). At the time t, f_t is near to g_t, by Step 1. Therefore it is also near to ω by Step 2. Then one can hope to apply the result b) to extend the solution up to time $t = \infty$.

Unfortunately the above program cannot be performed so simply just suitably combining known results. The reason is that the approach to equilibrium for the homogeneous solution (Step 2) is relatively easy to prove in a norm (see $\|\cdot\|_s$ below) in which Steps 1 and 3 become more difficult. In particular $f_t - \omega$, even if small in this norm, does not decay exponentially in the velocity space, and this makes the techniques of Ukai and Grad [2 and 3] not directly applicable since they make essential use of the exponential decay of the perturbations. However, using some of Carleman's techniques [1], we obtain new estimates on the collision operator (see Sect. 3), which allow us to extend the analysis of [2] to the case of perturbations only polynomially decaying (Step 3) as well as to prove the local theorem (Step 1).

We remark that the above results have an intrinsic interest besides the purposes of this paper.

After this preparation the proof of the main theorem follows easily. That is the content of Sect. 4. As for Sect. 2, it is devoted to some known properties of the homogeneous solutions which will be useful in the sequel including a result about the approach to equilibrium.

We conclude this Section by establishing hypotheses, notation and our main theorem.

We define the (unsymmetrized) collision operator as

$$Q(f,g)(v) = \int\limits_{\mathbb{R}^3} dv_1 \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi/2} d\theta \{f(v')g(v_1') - f(v)g(v_1)\}S(\theta, |V|) \tag{1.3}_1$$

$$= J(f,g)(v) - f Rg(v), \tag{1.3}_2$$

where $(1.3)_2$ defines the gain and loss term respectively. Here V denotes the relative velocity $V = v_1 - v$ and v', v_1' are the outgoing velocities after a collision with ingoing velocities v and v_1 and impact parameter $n = n(\theta, \varphi) = -\sin\theta \cos\varphi e_1 - \sin\theta \sin\varphi e_2 + (V/|V|)\cos\theta$, where $(e_1, e_2, V/|V|)$ are an orthonormal system. Explicitly

$$v' = v + (V \cdot n)n, \quad v_1' = v_1 - (V \cdot n)n. \tag{1.4}$$

For S we assume the following form:

$$S(\theta, |V|) = h(\theta)|V|^\beta, \quad C_0 \sin\theta \cos\theta \leq h(\theta) \leq C_0' \sin\theta \cos\theta \tag{1.5}$$

with $0 < \beta \leq 1$, so including the hard sphere case and the k^{-th} power forces with $k > 5$ and angular cut-off [6].

For $f: \mathbb{R}^3 \to \mathbb{R}$ continuous, we introduce the following norm for $s > 1$:

$$\|f\|_s = \sup_{v \in \mathbb{R}^3} (1 + v^2)^s |f(v)| \tag{1.6}$$

and for L^2—functions $f: T \to \mathbb{R}$ we define

$$|f|_l^2 = \sum_{k \in 2\pi\mathbb{Z}^3} (1 + k^2)^l |\hat{f}(k)|^2, \tag{1.7}$$

where $\hat{f}(k)$ denotes the Fourier transform of f.

Finally for measurable functions $f: T \times \mathbb{R}^3 \to \mathbb{R}$, with $|f(\cdot, v)|_l$ continuous in v, we define

$$\|f\|_{l,s} = \sup_{nv \in \mathbb{R}^3} (1 + v^2)^s |f(., v)|_l \tag{1.8}$$

and denote by B_s and $B_{l,s}$ the Banach spaces associated to the norm (1.6) and (1.8).

Now we can state our main result.

Theorem 1. *Given* ρ, $\sigma > 0$, *consider the class of positive continuous functions* f_0 *on* $T \times \mathbb{R}^3$ *with* $\int f_0(v)dvdx = \rho$, $\int v^2 f_0(v)dvdx = \sigma$. *There exist* s_0, l_0 *such that if* $s > s_0, l > l_0$ *and* $g_0 \in B_{s+1}$, *then there is* $a > 0$, *such that for* $\|u_0\|_{l,s} < a$, *there exists a unique, positive, strong solution* f *to the Boltzmann equation (1.1) in* $B_{l-\varepsilon-1,s-(\varepsilon+1)/2}$ *for any* $\varepsilon > 0$. *Moreover* $f_t \in B_{l,s}$, $f \in C^1([0,\infty), B_{l-\varepsilon-1,s-(\varepsilon+1)/2})$, *and* f_t *converges to* ω *in* $B_{l,s}$ *for* $t \to \infty$.

2. Some Results on the Spatially Homogeneous Boltzmann Equation

We need some results for the initial value problem associated to the spatially

homogeneous Boltzmann equation:

$$\partial_t f = J(f,f) - fRf,$$
$$f(0,v) = f_0(v). \tag{2.1}$$

Proposition 2.1. *There is $s_0 > 0$ such that, for all $f_0 > 0$ and $\| f_0 \|_s < \infty$ for some $s > s_0$, there exists a unique positive solution for the problem (2.1) and a constant C_s such that*

$$\sup_{t \geq 0} \| f_t \|_s < C_s. \tag{2.2}$$

Moreover it satisfies:

$$\int f(t,v)dv = \int f_0(v)dv, \quad \int v f(t,v)dv = \int v f_0(v)dv,$$
$$\int v^2 f(v)dv = \int v^2 f_0(v)dv, \tag{2.3}$$
$$\int f(t,v)\log f(t,v)dv \leqq \int f_0(v)\log f_0(v)dv \text{ for all } t > 0.$$

Proposition 2.1 was proved by Carleman [1] for hard spheres and $s_0 = 3$. The proof for the general case is contained in [8].

Proposition 2.2. *Under the conditions of Prop. 2.1 there is a $\nu > 0$ such that*

$$R f_t(v) > \nu(1 + |v|^\beta). \tag{2.4}$$

Proposition 2.2 is proved in [7] Lemma 4. The constant ν depends on the density and the energy of f_0 and on C_s.

Finally, we are interested in the large time behavior of the solution. Since the H-theorem can be proved, it can be shown that the solution approaches the equilibrium for large t. The equilibrium corresponding to the initial datum is given by the Maxwellian with the same mass, energy and momentum as the initial datum. By a suitable choice of coordinates we can reduce it to an initial datum f_0 such that:

$$\int f_0 dv = 1, \quad \int v f_0 dv = 0, \quad \int v^2 f_0 dv = 3. \tag{2.5}$$

In this case

$$\omega(v) = (2\pi)^{-3/2} \exp(-v^2/2). \tag{2.6}$$

For simplicity in this paper we shall always consider initial data such that (2.5) are satisfied.

Proposition 2.3. *Let f_0 be as in Proposition 2.1 and satisfy Eqs. (2.5). Then*

$$\lim_{t \to \infty} \| f_t - \omega \|_r = 0. \tag{2.7}$$

for any $r < s$.

Proposition 2.3 was proved in [1] for hard spheres and then generalized to the present case in [8].

3. Estimates of the Collision Operator

The following useful representation of $J(f,g)$ is due to Carleman [1] p. 32.

It is easy to verify that

$$(v'_1 - v_1) \cdot (v'_2 - v_1) = 0. \tag{3.1}$$

Let us denote by E_{v_1,v'_1} the plane $(v'_1 - v_1) \cdot (\xi - v_1) = 0$. It obviously contains v_1 and is orthogonal to $v'_1 - v_1$. Then

$$J(f,g)(v_1) = \int_{\mathbb{R}^3} dv'_1 f(v'_1) \frac{1}{|v_1 - v'_1|^{2-\beta}} \int_{E_{v_1v'_1}} d\xi g(v'_2) G(\theta), \tag{3.2}$$

where $d\xi$ denotes the Lebesgue measure on the plane $E_{v_1v'_1}$ and

$$G(\theta) = h(\theta)[\sin\theta(\cos\theta)^\beta]^{-1} \leq C'_0(\cos\theta)^{1-\beta} \leq C'_0. \tag{3.3}$$

Denoting by

$$J_0(f,g)(v_1) = \int_{\mathbb{R}^3} dv_2 \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi f(v'_1)g(v'_2)|v_1 - v_2|^\beta \sin\theta \cos\theta \tag{3.4}$$

following Carleman [1] p. 35, one has

$$J_0(f,g) = J_0(g,f). \tag{3.5}$$

Therefore, for positive f and g:

$$J(f,g) + J(g,f) \leq 2C'_0 J_0(f,g). \tag{3.6}$$

Proposition 3.1. *Let $l > 3/2$ and $s > 4$. The following estimate holds:*

$$(1 + v^2)^s |Q(f,g)|_l(v) \leq C(s) \|f\|_{l,s} \|g\|_{l,s} (1 + |v|^\beta). \tag{3.7}$$

Furthermore there exists a function $\varepsilon = \varepsilon(s) \to 0(s \to \infty)$, such that

$$(1 + v^2)^s \{|gRf|_l + |J(g,f)|_l + |J(f,g)|_l\}$$
$$\leq \|f\|_{l,s} \{C(s)\|g\|_{l,s+\beta/2} + \varepsilon(s)\|g\|_{l,3}(1 + |v|^\beta)\}. \tag{3.8}$$

Proof. For any fixed v and w, $f(\cdot, v)$ and $g(\cdot, w) \in H_l(T)$ (Sobolev space), which is a Banach algebra for $l > 3/2$. Then we have

$$|f(\cdot, v)g(\cdot, w)|_l \leq C|f(\cdot, v)|_l|g(\cdot, w)|_l, \tag{3.9}$$

therefore

$$|J(f,g)|_l(v_1) \leq C J_0(|f|_l, |g|_l)(v_1). \tag{3.10}$$

To simplify the notation we denote $|f|_l$ and $|g|_l$ by $\bar{f}$ and $\bar{g}$ respectively. Furthermore, for fixed v_1 we put

$$\chi_i(v) = \begin{cases} 1 & \text{if } |v| < |v_1|/\sqrt{2} \\ 0 & \text{otherwise}, \end{cases} \tag{3.11}$$

$\chi_0 = 1 - \chi_i$ and f_i, f_0 will stand for $f\chi_i$ and $f\chi_0$ respectively. By the energy conservation law $J_0(f_i, \bar{g}_i) = 0$ and hence

$$J_0(\bar{f}, \bar{g}) = J_0(\bar{f}_i, \bar{g}_0) + J_0(\bar{f}_0, \bar{g}_i) + J_0(\bar{f}_0, \bar{g}_0). \tag{3.12}$$

We estimate separately the above three terms.

$$(1 + v_1^2)^s J_0(\bar{f}_0, \bar{g}_0) \leqq C \| \bar{f} \|_s \| \bar{g} \|_s (1 + v_1^2)^s$$

$$\cdot \int_{\mathbb{R}^3} dv'_1 \chi_0(v'_1) \frac{1}{|v_1 - v'_1|^{2-\beta}(1 + v_1'^2)^s} \int_{E_{v_1 v'_1}} d\xi \chi_0(v'_2) \frac{1}{(1 + v_2'^2)^s}. \qquad (3.13)$$

The last integral equals

$$\int_{\mathbb{R}^2 \cap \{P^2 + \lambda^2 > v_1^2/2\}} \frac{d\lambda}{(1 + P^2 + \lambda^2)^s} \leqq \frac{C}{(1 + v_1^2)^{s-1}}, \qquad (3.14)$$

where P is the distance of the origin from the plane $E_{v_1 v'_1}$ and λ is the distance of the point v'_2 (on the plane) from the orthogonal projection of the origin on the same plane. If $s > 3/2$ the first integral appearing in the right-handside of (3.13) is bounded by $C(1 + |v_1|)^{-2+\beta}$ so that,

$$(1 + v_1^2)^s J_0(\bar{f}_0, \bar{g}_0) \leqq C \| \bar{f} \|_s \| \bar{g} \|_s (1 + |v_1|)^\beta. \qquad (3.15)$$

Furthermore, making use of Eq. (3.14) and proceeding as in (3.13),

$$(1 + v_1^2)^s J_0(\bar{f}_i, \bar{g}_0) \leqq C(1 + v_1^2)^s \| \bar{f} \|_s \| \bar{g} \|_s (1 + v_1^2)^{-(s-1)}$$

$$\cdot \int_{\mathbb{R}^3} dv'_1 \chi_i(v'_1) \frac{1}{(1 + v_1'^2)^s |v_1 - v'_1|^{2-\beta}}. \qquad (3.16)$$

If $v_1'^2 \leqq v_1^2/2$ we have

$$|v_1 - v'_1| \geqq |v_1| - |v'_1| \geqq (1 - \sqrt{2}^{-1})|v_1|, \qquad (3.17)$$

so that the last integral is bounded by $C(1 + |v_1|)^{-2+\beta}$ provided that $s > 3/2$. Finally

$$(1 + v_1^2)^s J_0(\bar{f}_i, \bar{g}_0) \leqq C(1 + |v_1|^\beta) \| \bar{f} \|_s \| \bar{g} \|_s. \qquad (3.18)$$

The same estimate is obtained by means of the relation (3.5) for the term $J_0(\bar{f}_0, \bar{g}_i)$.

We now estimate the loss term. Since $\overline{Rf} \leqq R\bar{f}$, we have:

$$|\bar{g}\overline{Rf}|(1 + v_1^2)^s \leqq C \| \bar{g} \|_s \int dv_2 \bar{f}(v_2)|v_1 - v_2|^\beta$$

$$\leqq C \| \bar{f} \|_s \| \bar{g} \|_s \int dv_2 |v_1 - v_2|^\beta \frac{1}{(1 + v_2^2)^s}$$

$$\leqq C \| \bar{f} \|_s \| \bar{g} \|_s (1 + |v_1|^\beta), \qquad (3.19)$$

provided that $s > (3 + \beta)/2$. This achieves the proof of (3.8).

To obtain the proof of the estimate (3.8) we first remark that proceeding as above

$$|\bar{g}\overline{Rf}|(1 + v_1^2)^s \leqq C \| \bar{f} \|_s \| \bar{g} \|_{s+\beta/2}. \qquad (3.20)$$

Furthermore, using (3.6)

$$| J(g, f)|_l + | J(f, g)|_l \leqq C J_0(\bar{g}, \bar{f}) \leqq C\{ J_0(\bar{g}_0, \bar{f}_i) + J_0(\bar{g}_0, \bar{f}_0) + J_0(\bar{g}_i, \bar{f}_0)\}. \qquad (3.21)$$

For the term $J_0(\bar{g}_0, \bar{f}_0)$ and $J_0(\bar{g}_0, \bar{f}_i)$ we slightly modify (3.15) and (3.16) respectively

to get

$$(1 + v_1^2)^s (J_0(\bar{g}_0, \bar{f}_0) + J_0(\bar{g}_0, \bar{f}_i)) \leq C(s) \| \bar{f} \|_s \| \bar{g} \|_{s + \beta/2}. \tag{3.22}$$

The last term in the right-handside of (3.21) requires more care. We split $J_0(\bar{g}_i, \bar{f}_0)$ in three terms:

$$J_0^1(\bar{g}_i, \bar{f}_0) = J_0(\bar{g}_i \chi_\delta^c, \bar{f}_0), \tag{3.23}$$

$$J_0^2(\bar{g}_i, \bar{f}_0) = J_0(\bar{g}_i \chi_\delta, \chi_\eta \bar{f}_0), \tag{3.24}$$

$$J_0^3(\bar{g}_i, \bar{f}_0) = J_0(\bar{g}_i \chi_\delta, \bar{f}_0 \chi_\eta^c), \tag{3.25}$$

and χ_δ, χ_η are the characteristic functions of the sets $\{ |v_1'| < \delta |v_1| \}$ and $\{ |v_2| < \eta |v_1| \}$ for $\delta < \eta < 1$ and $\chi^c = 1 - \chi$. We notice that, fixed v_1, v_1' and a point ξ in the plane $E_{v_1 v_1'}$ (and therefore fixed v_2'), v_2 is determined by the collision laws so that the above definitions make sense.

We have:

$$J_0^1(\bar{g}_i, \bar{f}_0)(v_1)(1 + v_1^2)^s \leq \| \bar{g} \|_3 \| \bar{f} \|_s (1 + v_1^2)^s \int\limits_{\mathbb{R}^3} dv_1' \chi_i(v_1') \chi_\delta^c(v_1')$$

$$\cdot \frac{1}{(1 + v_1'^2)^3 |v_1 - v_1'|^{2 - \beta}} \int\limits_{E_{v_1 v_1'}} d\xi \chi_0(v_2') \frac{1}{(1 + v_2'^2)^s}$$

$$\leq (\text{by virtue of Eq. (3.14))}.$$

$$C \| \bar{g} \|_3 \| \bar{f} \|_s (1 + v_1^2) \int\limits_{\mathbb{R}^3} dv_1' \chi_i(v_1') \chi_\delta^c(v_1') \frac{1}{(1 + v_1'^2)^3 |v_1 - v_1'|^{2 - \beta}}$$

$$\leq C \| \bar{g} \|_3 \| \bar{f} \|_s \sup_{v_1} \frac{(1 + v_1^2)}{(1 + \delta^2 v_1^2)} \int\limits_{\mathbb{R}^3} dv_1' \frac{1}{(1 + v_1'^2)^2 |v_1 - v_1'|^{2 - \beta}}$$

$$\leq C \| \bar{g} \|_3 \| \bar{f} \|_s \delta^{-2}. \tag{3.25}$$

On the other hand, if $|v_2| < \eta |v_1|$ and $|v_1'| < \delta |v_1|$, by the energy and momentum conservation we have

$$v_2'^2 = v_1^2 + v_2^2 - v_1'^2 > (1 - \delta^2) v_1^2, \tag{3.26}$$

and

$$|v_2' - v_1| < |v_2| + |v_1'| < (\eta + \delta)|v_1|. \tag{3.27}$$

so that v_2' is at most in a sphere centered in v_1 and with radius $(\eta + \delta)|v_1|$.

Its intersection with $E_{v_1 v_1'}$ has measure at most $\pi(\eta + \delta)^2 |v_1|^2 < 4\pi\eta^2 |v_1|^2$. Therefore

$$J_0^2(\bar{g}_i, \bar{f}_0)(v_1)(1 + v_1^2)^s \leq C \| \bar{g} \|_3 \| \bar{f} \|_s (1 + v_1^2)^s \frac{\eta^2 v_1^2}{[1 + (1 - \delta^2) v_1^2]^s}$$

$$\cdot \int\limits_{\mathbb{R}^3} dv_1' \chi_i(v_1') \chi_\delta(v_1') \frac{1}{(1 + v_1'^2)^3 |v_1 - v_1'|^{2 - \beta}}$$

$$\leq C\eta^2 (1 - \delta^2)^{-s} (1 + |v_1|^\beta) \| \bar{g} \|_3 \| \bar{f} \|_s. \tag{3.28}$$

Finally, since

$$\chi_\delta(v_1') \int_{E_{v_1 v_1'}} d\xi \chi_\eta^c(v_2') \frac{1}{(1 + v_2'^2)^s} \leq \frac{C}{(s-1)(1 + \bar{v}_2'^2)^{s-1}}, \tag{3.29}$$

where $\bar{v}_2'^2$ is the minimum value of $v_2'^2$ compatible with the constraints, we have

$$v_2'^2 = v_1^2 + v_2^2 - v_1'^2 > (1 + \eta^2 - \delta^2)v_1^2, \tag{3.30}$$

and hence

$$J_0^3(\bar{g}_i, \bar{f}_0)(v_1)(1 + v_1^2)^s \leq C \|\bar{g}\|_3 \|\bar{f}\|_s (1 + |v_1|^\beta)(s-1)^{-1}(1 + \eta^2 - \delta^2)^{1-s}, \tag{3.31}$$

by the inequality (3.17).

In conclusion, collecting all the above estimates, we have:

$$J_0(\bar{g}_i, \bar{f}_0)(v_1)(1 + v_1^2)^s \leq \|\bar{g}\|_3 \|\bar{f}\|_s \{C\delta^{-2} + A(1 + |v_1|^\beta)$$
$$\cdot (\eta^2(1 - \delta^2)^{-s} + (s-1)^{-1}(1 + \eta^2 - \delta^2)^{1-s})\}, \tag{3.32}$$

for some $A > 0$ not depending on s.

We choose $\eta = 1/\sqrt{s}, \delta = 1/s$ for which $\delta^2 \leq \eta^2/2 = 1/2s$. Therefore

$$(1 + \eta^2 - \delta^2)^{1-s} \leq (1 + \eta^2/2)^{1-s} = (1 + 1/2s)^{1-s} \leq C. \tag{3.33}$$

On the other hand

$$\eta^2(1 - \delta^2)^{-s} \leq \frac{1}{s(1 - s^{-2})^s} \leq Cs^{-1}, \tag{3.34}$$

so that (3.8) is proved with $\varepsilon(s) = Cs^{-1}$.

4. Proof of the Main Theorem

Let $f_0(x, v), (x, v) \in T \times \mathbb{R}^3$ be positive and in $B_{l,s}$. Denote by g_0 its space average

$$g_0(v) = \int_T dx f_0(x, v) \tag{4.1}$$

and by $u_0 = f_0 - g_0$ its deviation. Let g_t and u_t be solutions of the initial value problems

$$\partial_t g = Q(g, g), \quad g(0) = g_0 \tag{4.2}$$

and

$$D_t u + uRg = \Phi_t := Q(g, u) + J(u, g) + Q(u, u), \quad u(0) = u_0. \tag{4.3}$$

Denote by

$$f_t(x, v) = g_t(v) + u_t(x, v), \tag{4.4}$$

then f_t is a solution of the initial value problem for the Boltzmann equation (1.1). The initial value problem (4.2) can be uniquely solved for all t if s is large enough since $g_0 \in B_s$ and the solution has the properties established in Sect. 2. Now we prove a local existence and uniqueness theorem for the problem (4.3). Let $\hat{u}_t(k, v), k \in 2\pi\mathbb{Z}^3$

be the Fourier transform of $u_t(x, v)$. Consider the following integral equation in $B_{l,s}$:

$$\hat{u}_t(k, v) = \hat{u}_0(k, v) \exp - \left\{ ik \cdot vt + \int_0^t d\tau (Rg_\tau)(v) \right\}$$

$$+ \int_0^t d\tau \exp \left\{ - \int_\tau^t d\sigma (ik \cdot v\sigma + Rg_\sigma(v)) \right\} \hat{\Phi}_\tau(k, v), \qquad (4.5)$$

where $\hat{\Phi}_\tau(k, v)$ denotes the Fourier transform of $\Phi_\tau(x, v)$.

Proposition 4.1. *There exists $s_0 > 0$ such that, for $s > s_0$ and $l > 3/2$ if $u_0 \in B_{l,s}$ and $g_0 \in B_{s+\beta/2}$, then any solution in $B_{l,s}$ of Eq. (4.5) satisfies the estimate:*

$$\| u_t \|_{l,s} \le \| u_0 \|_{l,s} \exp [(2C \ln 4)t] \qquad (4.6)$$

for some constant C depending only on s and g_0, provided that

$$\| u_0 \|_{l,s} \le (24C)^{-1} \exp [- (2C \ln 4)t]. \qquad (4.7)$$

Proof. Denote $Y(t) = \| u_t \|_{l,s}$ and $\bar{Y}(\tau) = \sup_{0 \le \sigma \le \tau} Y(\sigma)$.

By the estimates on the collision term in Proposition 3.1 and by Eq. (2.2), fixed $\varepsilon > 0$, we can find s_0 such that

$$|\Phi_t(\cdot, v)|_l (1 + v^2)^s \le (A_s + \varepsilon|v|^\beta) \| u_t \|_{l,s} + A_s(1 + |v|^\beta)(\| u_t \|_{l,s})^2 \qquad (4.8)$$

for some constant A_s and $s > s_0$. Therefore, by Eq. (4.5), using (2.4), we have:

$$Y(\sigma) \le Y(0) + \tau A_s \bar{Y}(\tau) + (\varepsilon/v)\bar{Y}(\tau) + (A_s/v)\bar{Y}(\tau)^2 \quad \text{for} \quad 0 \le \sigma \le \tau. \qquad (4.9)$$

Putting $C = \max(A_s + 1, A_s/v)$, if $\varepsilon < v\tau$,

$$0 \le Y(0) - (1 - \tau C)\bar{Y}(\tau) + C\bar{Y}(\tau)^2. \qquad (4.10)$$

Thus, for τ such that $\tau C = 1/2$ we have, by continuity Y as function of σ

$$Y(\sigma) \le \bar{Y}(\tau) \le (1/4 - \sqrt{1/16 - Y(0)C})/C \le 4Y(0) \quad \text{for} \quad 0 \le \sigma \le \tau, \qquad (4.11)$$

if $Y(0)C < 1/24$. In the last step we used the obvious inequality $\sqrt{1 - 16Y(0)C} > (1 - 8Y(0)C/\sqrt{1 - 16Y(0)C})$. By iteration of the estimate (4.11) n times, with $n = [t/\tau] - 1$ ([x] = integer part of x), we get (4.6) from (4.7).

Proposition 4.2. *Given $s > s_0$, if u_0 satisfies (4.7) with C large enough, and $g_0 \in B_{s+\beta/2}$, there exists a unique solution up to the time t, of Eq. (4.5), satisfying (4.6).*

Proof. We define, by iteration

$$(D_t + Rg)u^{(n)} = \Phi^{(n-1)} n \ge 1, \quad u^{(n)}(0) = u_0 = u^{(0)}, \qquad (4.12)$$

where

$$\Phi^{(n)} = Q(g, u^{(n)}) + J(u^{(n)}, g) + Q(u^{(n)}, u^{(n)}). \qquad (4.13)$$

For $n \ge 2 \varphi^{(n)} := u^{(n)} - u^{(n-1)}$, $\varphi^{(1)} := u^{(1)}$ satisfies

$$(D_t + Rg)\varphi^{(n)} = F^{(n-1)}, \qquad (4.14)$$

where

$$F^{(n-1)} = \Phi^{(n-1)} - \Phi^{(n-2)} = Q(g, \varphi^{(n-1)}) + J(\varphi^{(n-1)}, g)$$
$$+ Q(u^{(n-1)}, \varphi^{(n-1)}) + Q(\varphi^{(n-1)}, u^{(n-2)}), \quad n \geq 2, \tag{4.15}$$
$$F^{(0)} = \Phi^{(0)}. \tag{4.16}$$

By Proposition 3.1 we have

$$|F^{(n-1)}(\cdot, v)|_l(1 + v^2)^s \leq (A_s + \varepsilon|v|^\beta) \| \varphi^{(n-1)} \|_{l,s}$$
$$+ A_s(1 + |v|^\beta)\{ \| u^{(n-1)} \|_{l,s} + \| u^{(n-2)} \|_{l,s}\} \| \varphi^{(n-1)} \|_{l,s}. \tag{4.17}$$

Obviously the sequence $u^{(n)}$ satisfies the estimate (4.6) provided that (4.7) is satisfied. Therefore, proceeding as in Proposition 4.1, we get for $s > s_0$:

$$\bar{\varphi}_\tau^{(n)} \leq \alpha \bar{\varphi}_\tau^{(n-1)}, \quad \alpha < 1, \tag{4.18}$$

where

$$\bar{\varphi}_\tau = \sup_{0 \leq \sigma \leq \tau} \| \varphi_\sigma \|_{l,s}, \tag{4.19}$$

provided that τ and $1/C$ are small enough. Therefore, if ε and τ are small enough the iteration converges for $\sigma \leq \tau$. This procedure can be iterated to get the convergence up to time t if $\| u_0 \|_{l,s}$ is small. The uniqueness follows trivially.

Remark. Further regularity properties of the solution constructed above will be discussed later.

Now we consider the evolution of a perturbation of the equilibrium. We assume f_0 such that

$$\int f_0(x, v)dxdv = 1, \quad \int v f_0(x, v)dxdv = 0, \quad \int v^2 f_0(x, v)dxdv = 3. \tag{4.20}$$

Then f_t is expected to converge to ω defined in (2.6).

We write, for $t \geq 0$,

$$f_t = \omega + u_t. \tag{4.21}$$

We prove the following:

Proposition 4.3. *Fixed s_0 and l_0 sufficiently large, for any $s > s_0$ and $l > l_0$ we can find $b, b', \gamma > 0$ such that, if $\| u_0 \|_{l,s} < b$, then there exists a unique solution of the Boltzmann equation in the strong sense of $B_{l-\varepsilon-1, s-(\varepsilon+1)/2}$ for $\varepsilon > 0$ and u_t satisfies the bound*

$$\| u_t \|_{l,s} \leq b' \exp - \gamma t. \tag{4.22}$$

Proof. Let $h_\alpha(v)\alpha = 1, \ldots, 5$ be defined as follows:

$$h_1(v) = 1, \quad h_{1+i}(v) = v_i \, i = 1, \ldots, 3, \quad h_5(v) = (\sqrt{6})^{-1}(v^2 - 3), \tag{4.23}$$

and let

$$\psi_\alpha = h_\alpha \sqrt{\omega}, \quad \alpha = 1, \ldots, 5. \tag{4.24}$$

$\{\psi_\alpha\}_{\alpha=1,\ldots,5}$ are in $L_2(\mathbb{R}^3, H_l(T))$ and are orthonormal with respect to the scalar product

$$(f, g) = \int_{\mathbb{R}^3} dv \sum_{k \in 2\pi\mathbb{Z}^3} \hat{f}^*(k, v)\hat{g}(k, v)(1 + k^2)^l. \tag{4.25}$$

Let P be the projector on the subspace spanned by $\{\psi_\alpha\}$:

$$Pf = \sum_{\alpha=1,5} (f, \psi_\alpha)\psi_\alpha. \qquad (4.26)$$

Now we decompose u_0 as follows: $u_0 = u_0^{(1)} + u_0^{(2)} + u_0^{(3)}$ with

$$u_0^{(1)}(x,v) = \sqrt{\omega}(1-P)[\sqrt{\omega}^{-1}\chi_M(v)u_0(x,v)],$$
$$u_0^{(2)}(x,v) = \chi_M^c(v)u_0(x,v),$$
$$u_0^{(3)}(x,v) = \sqrt{\omega}P[\sqrt{\omega}^{-1}\chi_M(v)u_0(x,v)], \qquad (4.27)$$

where, for $M > 0$, $\chi_M(v) = \begin{cases} 1 \text{ if } |v| < M \\ 0 \text{ otherwise} \end{cases}$, and $\chi_M^c = 1 - \chi_M$.

Furthermore we put

$$f_0^{(i)} = u_0^{(i)}\sqrt{\omega}^{-1}, \quad i = 1, 2, 3. \qquad (4.28)$$

Since $u_0 \in B_{l,s}$, $f_0^{(1)}$ and $f_0^{(3)}$ are in $L_2(\mathbb{R}^3, H_l(T))$.

Putting

$$Lf = -(Q(\omega, f) + Q(f, \omega)),$$
$$\tilde{L}f = \sqrt{\omega}^{-1}L(\sqrt{\omega}f),$$
$$vf = fR\omega,$$
$$Kf = vf - Lf, \qquad (4.29)$$

consider the linear initial value problems, for $z_i \in L_\infty([0, +\infty), B_{l,s})$, $i = 1, 2, 3$:

$$(D_t + v)u_2 = \chi_M^c Ku_2 + [Q(z_1, z_2) + Q(z_2, z_3)], \qquad (4.30)_1$$

$$(D_t + \tilde{L})f_1 = (1-P)\sqrt{\omega}^{-1}\chi_M Ku_2, \qquad (4.30)_2$$

$$(D_t + \tilde{L})f_3 = P\sqrt{\omega}^{-1}\chi_M Ku_2, \qquad (4.30)_3$$

$$u_2(0) = u_0^{(2)}, \quad f_1(0) = f_0^{(1)}, \quad f_3(0) = f_0^{(3)}. \qquad (4.30)_4$$

We put $u^{(i)}(t) = \sqrt{\omega}f^{(i)}(t)$ and $u(t) = u_1(t) + u_2(t) + u_3(t)$. For $z_1 = z_2 = u$ and $z_3 = 0$, $\omega + u(t)$ is a solution of the Boltzmann equation.

We begin by studying Eq. $(4.30)_1$ in integral form (the same as (4.5) with g replaced by ω and ϕ replaced by right-handside of $(4.30)_1$.

Lemma 1. *There is $\bar{v} > 0$ such that $v(v) \geq \bar{v}(1 + |v|^\beta)$.*

Proof. See Grad [3] or alternatively Proposition 2.2.

We put:

$$\|f\|_{l,s,\gamma} = \sup_{t \geq 0} e^{\gamma t}\|f(t)\|_{l,s}. \qquad (4.31)$$

Lemma 2: *For s and l sufficiently large, there are $C, \bar{C}$ such that any solution of $(4.30)_1$ satisfies*

$$\|u_2\|_{l,s\gamma} \leq \|u_0\|_{l,s} + C\|z_2\|_{l,s,\gamma}(\|z_1\|_{l,s,\gamma} + \|z_3\|_{l,s,\gamma})$$
$$+ 2\bar{v}^{-1}[\varepsilon(s) + C(1 + M^\beta)^{-1}]\|u_2\|_{l,s,\gamma} \qquad (4.32)$$

for $0 < \gamma < v_0/2 = 1/2 \inf_{v \in \mathbb{R}^3} v(v)$.

Proof. We have

$$\hat{u}_2(k, v, t) = e^{-\{(ik \cdot v + v(v))t\}} \hat{u}_0^{(2)}(k, v) + \int_0^t d\tau e^{-(ik \cdot v + v(v))(t-\tau)}$$

$$\cdot [\chi_M^c K u_2(\tau) + Q(z_1(\tau), z_2(\tau)) + Q(z_2(\tau), z_3(\tau))]\hat{}(k, v). \tag{4.33}$$

By (3.8) and Lemma 1

$$(1 + v^2)^s \left(\sum_{k \in 2\pi \mathbb{Z}^3} (1 + k^2)^l \int_0^t |d\tau e^{-(ik \cdot v + v(v))(t-\tau)} \chi_M^c (K u_2(\tau))\hat{}(k, v)|^2 \right)^{1/2}$$

$$\leqq 2\bar{v}^{-1}(C(s)(1 + M^\beta)^{-1} + \varepsilon(s)) \| u_2 \|_{l,s,\gamma} e^{-\gamma t}. \tag{4.34}$$

On the other hand by (3.7),

$$(1 + v^2)^s |Q(z_1(\tau), z_2(\tau))|_l \leqq C(1 + |v|^\beta) \| z_1 \|_{l,s,\gamma} \| z_2 \|_{l,s,\gamma} e^{-2\gamma\tau}, \tag{4.35}$$

and therefore by Lemma 1

$$\left\| \int_0^t ds e^{-(ik \cdot v + v(v))(t-\tau)} Q(z_1(\tau), z_2(\tau)) \right\|_{l,s}$$

$$\leqq C \| z_1 \|_{l,s,\gamma} \| z_2 \|_{l,s,\gamma} \sup_{v \in \mathbb{R}^3} \int_0^t d\tau (1 + |v|^\beta) e^{-2\gamma\tau} e^{-v(v)(t-\tau)}$$

$$\leqq 2\bar{v}^{-1} C \| z_1 \|_{l,s,\gamma} \| z_2 \|_{l,s,\gamma} e^{-\gamma t}. \tag{4.36}$$

Lemma 3. *Suppose M, l, s large enough and $\| z_i \|_{l,s,\gamma} < +\infty$, $i = 1, 2, 3$. There exists a unique solution of Eq. (4.33), $u_2 \in L_\infty([0, +\infty), B_{l,s}) \cap C_0([0, +\infty), B_{l-\varepsilon,s-\varepsilon/2})$ for any $\varepsilon > 0$. Suppose in addition that $z_i \in C_0([0, +\infty), B_{l-\varepsilon,s-\varepsilon/2})$, $i = 1, 2, 3$ for some $\varepsilon > 0$. Then $u_2 \in C_1([0, +\infty), B_{l-\varepsilon-1,s-(\varepsilon+1)/2})$ and satisfies Eq. (4.30)$_1$ strongly in $B_{l-\varepsilon-1,s-(\varepsilon+1)/2}$.*

Proof. Existence and uniqueness follow easily by iteration and estimate (4.32). Since $Q(z_1, z_2)$, $Q(z_2, z_3)$ and $K u_2 \in L_\infty([0, +\infty), B_{l,s-\beta/2})$, then, by Eq. (4.33) $u_2 \in C_0([0, +\infty), B_{l-\varepsilon,s-\varepsilon/2})$.
Moreover, putting

$$\Psi = (-v \cdot \nabla_x - v + \chi_M^c K) u_2 + [Q(z_1, z_2) + Q(z_2, z_3)], \tag{4.37}$$

we get $\Psi \in C_0([0, +\infty), B_{l-\varepsilon-1,s-(\varepsilon+1)/2})$ provided that $z_i \in C_0([0, +\infty), B_{l-\varepsilon,s-\varepsilon/2})$, $i = 1, 2, 3$. The convergence of $h^{-1}(u_2(t + h) - u_2(t))$ to $\Psi(t)$ in $B_{l-\varepsilon-1,s-(\varepsilon+1)/2}$, for $h \to 0$, is standard.

The following Proposition summarizes the properties of the operator $\tilde{L}$ we use in the proof:

Proposition 4.4. *Consider the initial value problem*

$$(D_t + \tilde{L}) w(t) = \phi(t), \quad w(0) = w_0. \tag{4.38}$$

Suppose $w_0 \in B_{l,s}$ and $\phi \in L_\infty([0, +\infty), B_{l,s}) \cap C_0([0, +\infty), B_{l-\varepsilon-1,s-\varepsilon/2-1/2})$ for $\varepsilon > 0$ and some l and s sufficiently large. Then Eq. (4.38) has a unique solution $w(t) \in B_{l,s}$ for $t \geqq 0$ which is strongly differentiable in $B_{l-\varepsilon-1,s-(\varepsilon+1)/2}$. Moreover if

$\phi(t) \in \operatorname{Ker} P$ for $t \in [0, +\infty)$, and $w_0 \in \operatorname{Ker} P$, then $w(t) \in \operatorname{Ker} P$ and there are positive constants μ, d, d' such that for $0 < \gamma < \mu$,

$$\| w \|_{l,s,\gamma} \leq d \| w_0 \|_{l,s} + d' \| \phi \|_{l,s,\gamma}. \tag{4.39}$$

Proof. Proposition 4.4 follows easily by ref. [2], Lemma 4.4.

Given u_2 as in Lemma 3 with $z_1 = z_2$ and $z_3 = 0$, we next consider the solution of $(4.30)_2$ and $(4.30)_3$ as given by Proposition 4.4.

Lemma 4. *Suppose* $u(t) = u_1(t) + u_2(t) + u_3(t) \in B_{l,s} \forall t > 0$, *for* $s > 3$. *Then* $(h_\alpha, u(t)) = 0$, $\alpha = 1, \ldots, 5$.

Proof. The scalar products make sense since $s > 3$. Furthermore the scalar products $(h_\alpha, Q(\omega, u(t)))$ and $(h_\alpha, Q(u(t), u(t)))$ make sense by the estimates on the collision operators and vanish by direct computation. Therefore $(h_\alpha, u(t)) = 0$ since $(h_\alpha, u_0) = 0$.

We have, for any $f \in B_{l,s}, l > 1, s > 1/2$,

$$(v \cdot \nabla_x + \tilde{L}) P f = 0 = P(v \cdot \nabla_x + \tilde{L}) f \tag{4.40}$$

by the same arguments as in the proof of Lemma 4.

Therefore, by Proposition 4.4, $P f_1(t) = 0, \forall t > 0$, and, by Lemma 4

$$P f_2(t) = -P f_3(t) \forall t > 0. \tag{4.41}$$

Lemma 5. *There is a constant* C *such that*

$$\| u_3(t) \|_{l,s} \leq C \| u_2(t) \|_{l,s}. \tag{4.42}$$

Proof. $f_3(t) = P f_3(t)$ by Eq. (4.40) and Proposition 4.4, since $(1 - P) f_3(0) = 0$. Therefore

$$\begin{aligned}
\| u_3(t) \|_{l,s} &= \| \sqrt{\omega} f_3 \|_{l,s} = \| \sqrt{\omega} P f_2 \|_{l,s} \leq \| P f_2 \|_{l,s} \\
&\leq \sum_{\alpha = 1, 5} (\sqrt{\omega}^{-1} u_2, \psi_\alpha) \sup_{v \in \mathbb{R}^3} [(1 + v^2)^s | \psi_\alpha(v) |] \\
&\leq C_s \sum_{\alpha = 1, 5} \int dx dv | u_2(x, v) | h_\alpha(v) \\
&\leq C_s \| u_2(t) \|_{l,s} \int dv h_\alpha(v)(1 + v^2)^{-s} \leq C \| u_2(t) \|_{l,s} \text{ for } s > 3.
\end{aligned} \tag{4.43}$$

Lemma 6. *For* l, s *as in Proposition 4.4 there is a constant* C_M *such that, for* $0 < \gamma < \mu$,

$$\| u_1 \|_{l,s,\gamma} < C_M (\| u_2 \|_{l,s,\gamma} + \| u_0 \|_{l,s}). \tag{4.44}$$

Proof. Using Proposition 4.4, with $w_0 = f_0^{(1)}$ and $\phi(t) = (1 - P) \sqrt{\omega}^{-1} \chi_M K u_2$, we have, since $f_0^{(1)}$ and $(1 - P) \sqrt{\omega}^{-1} \chi_M K u_2$ are in $\operatorname{Ker} P$,

$$\| f_1 \|_{l,s,\gamma} \leq d \| f_0^{(1)} \|_{l,s} + d' \| (1 - P) \sqrt{\omega}^{-1} \chi_M K u_2 \|_{l,s,\gamma}. \tag{4.45}$$

By (3.7) and the arguments used in the proof of Lemma 5 we have

$$\| (1 - P) \sqrt{\omega}^{-1} \chi_M K u_2 \|_{l,s,\gamma} \leq C_M \| u_2 \|_{l,s,\gamma}. \tag{4.46}$$

Moreover

$$\| f_0^{(1)} \|_{l,s} = \| (1 - P) \sqrt{\omega}^{-1} \chi_M u_0 \|_{l,s} \leq C_M \| u_0 \|_{l,s}. \tag{4.47}$$

Now we can prove Proposition 4.3. From now on we fix s and M such that $2\bar{v}^{-1}(\bar{C}(1+M^\beta)^{-1}+\varepsilon(s))<1/2$. By Lemma 2:

$$\|u_2\|_{l,s,\gamma} \leq 2\|u_0\|_{l,s} + 2C\|z_2\|_{l,s,\gamma}(\|z_1\|_{l,s,\gamma} + \|z_3\|_{l,s,\gamma}). \qquad (4.48)$$

By Lemma 5, Lemma 6 and inequality (4.48) we see that for $l > l_0, s > s_0$ and $0 < \gamma < \min(\mu, \bar{v}_0/2)$, $\|u\|_{l,s,\gamma}$ must satisfy the inequality:

$$\|u\|_{l,s,\gamma} \leq C_1\|u_0\|_{l,s} + C_2\|z_2\|_{l,s,\gamma}(\|z_1\|_{l,s,\gamma} + \|z_3\|_{l,s,\gamma}) \qquad (4.49)$$

(here C_1 and C_2 depend only on the choice of M and s).

Now we consider the following iterative procedure:

$$(D_t + L)u^{(n)} = Q(u^{(n-1)}, u^{(n-1)}), \quad n \geq 1,$$
$$u^{(0)} = 0, \quad u^{(n)}(0) = u_0, \quad n \geq 1. \qquad (4.50)$$

By virtue of the estimate (4.49) (with $z_3 = 0, z_1 = z_2 = u^{(n-1)}$) we easily obtain

$$\|u^{(n)}\|_{l,s,\gamma} < (1 - \sqrt{1 - 4C_1C_2\|u_0\|_{l,s}})(2C_2)^{-1} := C' \qquad (4.51)$$

provided that $\|u_0\|_{l,s} < (4C_1C_2)^{-1}$. Moreover, again by (4.49) (with $z_1 = u^{(n-1)}$, $z_2 = u^{(n-1)} - u^{(n-2)}$ and $z_3 = u^{(n-2)}$),

$$\|u^{(n)} - u^{(n-1)}\|_{l,s,\gamma} \leq \alpha \|u^{(n-1)} - u^{(n-2)}\|_{l,s,\gamma} \qquad (4.52)$$

with $\alpha < 1$ if $\|u_0\|_{l,s}$ is sufficiently small (by (4.51)).

Denoting by u limit of $u^{(n)}$, it is not hard to prove that it satisfies:

$$\hat{u}(k,v,t) = e^{-(ik \cdot v + v(v))t}\hat{u}_0(k,v) + \int_0^t d\tau\, e^{-(ik \cdot v + v(v))(t-\tau)}[Ku(\tau) + (Q(u(\tau), u(\tau)))]\hat{}(k,v).$$

$$(4.53)$$

The strong differentiability of u in $B_{l-\varepsilon-1,s-\varepsilon-1/2}$ follows by the same arguments as in Lemma 3.

Proof of Theorem 1. We fix s and l according to the previous lemmas and assume $f_0 \in B_{l,s+1}, g_0$ to be the space average of f_0 and u_0 its deviation. By Prop. 2.3 $g_t \in B_{s+1}$ and approaches the equilibrium in B_s. Therefore there is $t^* > 0$ such that

$$\|g_t - \omega\|_s < b/2 \quad \forall t \geq t^*, \qquad (4.54)$$

where b is the one introduced in Proposition 4.3. By Proposition 4.1 $\|u(t^*)\|_{l,s} \leq b/2$ if $\|u_0\|_{l,s} < a$, where a is fixed by (4.6) and (4.7). Therefore at time t^* we have

$$\|f_{t^*} - \omega\|_{l,s} \leq \|f_{t^*} - g_{t^*}\|_{l,s} + \|g_{t^*} - \omega\|_s < b. \qquad (4.55)$$

Using Prop. 4.3 with initial time t^* and initial datum $f_t - \omega$, it follows that f_t exists uniquely also for $t > t^*$ and converges to ω for $t \to \infty$. By the same arguments as in Lemma 3 $f \in C^1((0,\infty), B_{l-\varepsilon-1,s-(\varepsilon+1)/2})$. Therefore Eq. (1.1) can be understood in $B_{l-\varepsilon-1,s-(\varepsilon+1)/2}$ for all $t > 0$.

Remark 1. Since (4.54) follows by the H-theorem, t^* is not explicitly known and then we have no control on the rate of convergence of the solution to equilibrium.

Remark 2. By the same arguments of Sect. 4, it also follows that the solution constructed in Theorem 1 is stable in $B_{l,s}$. Namely, the constant μ of Proposition 4.4 is locally uniformly (with respect to ω) bounded away from zero, as can be seen by a slight elaboration of the argument in [2]. It then follows from the proof of Proposition 4.3, that its constant s_0, l_0, b, b', γ can be chosen locally independent of ω, and that we can take $b' \to 0$, when $b \to 0$. Therefore, analogously to the proof of Theorem 1, we can conclude that the solutions constructed in Theorem 1 are stable in $B_{l,s}$, i.e. given $\varepsilon > 0$, and f_0 as in Theorem 1, there is a $\delta > 0$ such that

$$\| f_t - \tilde{f}_t \|_{l,s} < \varepsilon, \quad t > 0, \text{if } \tilde{f}_0 \in B_{l,s} \text{ and } \| f_0 - \tilde{f}_0 \|_{l,s} < \delta. \tag{4.56}$$

In particular the space homogeneous Boltzmann Equation is stable in this sense also when the initial value is far from equilibrium.

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