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Kac-Moody Symmetry of Generalized Non-Linear Schrödinger Equations

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Abstract. The classical non-linear Schrödinger equation associated with a symmetric Lie algebra $\mathscr{g}=\mathscr{k}\oplus\mathscr{m}$ is known to possess a class of conserved quantities which form a realization of the algebra $\mathscr{k}\otimes\mathbb{C}[\lambda]$. The construction is now extended to provide a realization of the Kac-Moody algebra $\mathscr{k}\otimes\mathbb{C}[\lambda,\lambda^{-1}]$ (with central extension). One can then define auxiliary quantities to obtain the full algebra $\mathscr{g}\otimes\mathbb{C}[\lambda,\lambda^{-1}]$. This leads to the formal linearization of the system.

1. Introduction

This is a continuation of the work presented in [1], in which it was shown how to construct conserved quantities for the generalized non-linear Schrödinger (GNLS) equation of Fordy and Kulish [2]:

$$iq_t^\alpha = q_{xx}^\alpha \pm q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta} \tag{1.1}$$

(summation is implied over repeated indices) which is associated with a Lie algebra $\mathscr{g}=\mathscr{k}\oplus\mathscr{m}$. $q(x,t)$ is a matrix field in $1+1$ dimensions whose components lie in $\mathscr{m}$, and $\mathscr{k}$ is the centralizer of a special Cartan subalgebra element E satisfying the property

$$[E, e_\alpha] = -ie_\alpha \tag{1.2}$$

for all $e_\alpha\in\mathscr{m}$ (where α is positive). This means that the algebra $\mathscr{g}$ is "symmetric", i.e.

$$[\mathscr{k},\mathscr{k}]\subset\mathscr{k},\quad [\mathscr{k},\mathscr{m}]\subset\mathscr{m},\quad [\mathscr{m},\mathscr{m}]\subset\mathscr{k}. \tag{1.3}$$

The curvature tensor R has components in $\mathscr{m}$ defined by

$$e_\alpha R^\alpha_{\beta\gamma-\delta} = [e_\beta[e_\gamma, e_{-\delta}]]. \tag{1.4}$$

Equation (1.1) can be written as a zero-curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0, \tag{1.5}$$

where

$$A_x = \lambda E + A_x^0, \tag{1.6a}$$

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]], \tag{1.6b}$$

and

$$A_x^0 = -q^\alpha e_\alpha - q^{-\alpha} e_{-\alpha}. \tag{1.7}$$

The component form of (1.5) is

$$iq_t^\alpha = q_{xx}^\alpha + q^\beta q^\gamma q^{-\delta} R^\alpha_{\beta\gamma-\delta}, \tag{1.8a}$$

$$-iq_t^{-\alpha} = q_{xx}^{-\alpha} + q^{-\beta} q^{-\gamma} q^\delta R^{-\alpha}_{-\beta-\gamma\delta}. \tag{1.8b}$$

The choices $q^{-\alpha} = \pm q^{\alpha *}$ correspond to the restriction to the non-compact (+) or compact (−) real forms of $\mathscr{g}$, and lead to Eq. (1.1) with a plus or minus sign.

One can find other values of A_t as a polynomial in λ such that the new equation of motion (1.5), with A_x given by (1.6a), is still independent of λ. Each such A_t is associated, via (1.5), with an evolution operator ∂_t. It was shown in [1] that when A_t is a polynomial in positive powers only, the collection of evolution operators can be labelled $\partial_{N,k}$, where $k \in \mathscr{k}$ and N is a positive integer, and that they have the commutation relation

$$[\partial_{M,j}, \partial_{N,k}] = \partial_{M+N,[j,k]} \quad \forall M,\ N \geqq 0;\ \forall j,\ k \in \mathscr{k}. \tag{1.9}$$

In this paper, the case will be considered when A_t is a polynomial in negative powers of λ. This will lead to the construction of evolution operators $\partial_{-N,k}$ such that

$$[\partial_{m,j}, \partial_{n,k}] = \partial_{m+n,[j,k]} \quad \forall m,\ n \in \mathbb{Z},\ \forall j,\ k \in \mathscr{k}. \tag{1.10}$$

The complete collection of operators $\partial_{\pm N,k}$ provides a realization of the Kac-Moody algebra $\mathscr{k} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, where $\mathbb{C}[\lambda, \lambda^{-1}]$ is the algebra of Laurent polynomials in the complex variable λ. The parameters $(\pm N, k)$ are thought of as infinitely many independent "time" variables.

In [1] it was shown how to construct a group element of the form,

$$\omega = \exp \sum_{n=1}^{\infty} \lambda^{-n} \omega_n, \tag{1.11}$$

defined by

$$\lambda \omega E \omega^{-1} - \omega_x \omega^{-1} = \lambda E + A_x^0. \tag{1.12}$$

Under the gauge transformation

$$A_x \to \omega^{-1} A_x \omega + \omega^{-1} \omega_x = \lambda E, \tag{1.13a}$$

$$A_t \to \omega^{-1} A_t \omega + \omega^{-1} \omega_t = a_t, \tag{1.13b}$$

where A_t is unknown, the zero curvature condition (1.5) becomes

$$\partial_x a_t + [\lambda E, a_t] = 0. \tag{1.14}$$

This equation can be satisfied by choosing

$$a_t = \lambda^N k, \tag{1.15}$$

where N is a positive integer and $k \in \mathscr{k}$ is constant. The transformation (1.13b) is inverted to obtain

$$A_t = \lambda^N \omega k \omega^{-1} - \omega_t \omega^{-1}. \tag{1.16}$$

If A_t is chosen to have no negative powers of λ, then it is determined uniquely by (1.16) as the positive power part of $\lambda^N \omega k \omega^{-1}$, while the action of ∂_t on ω is determined by the negative power part. A_t and ∂_t defined in this way are denoted $A_N(k)$, $\partial_{N,k}$. Equating coefficients of powers of λ in (1.16) one obtains

$$A_N(k) = \sum_{n=0}^{N} \lambda^n (\omega k \omega^{-1})_{N-n}, \tag{1.17a}$$

$$(\omega_{N,k} \omega^{-1})_n = (\omega k \omega^{-1})_{N+n}, \tag{1.17b}$$

where $(\cdots)_n$ denotes the coefficient of λ^{-n}. The relation (1.9) follows from the definition (1.17b).

Now suppose that one chooses

$$a_t = \lambda^{-N} k \tag{1.18}$$

as a solution to (1.14). Then the inverse gauge transformation (1.13b) gives

$$A_{-N}(k) = \lambda^{-N} \omega k \omega^{-1} - \omega_{-N,k} \omega^{-1} \tag{1.19}$$

which does not determine $A_{-N}(k)$. For example, if $N > 1$, then the coefficient of λ^{-1} in (1.19) is

$$A^1_{-N}(k) = -\partial_{-N,k} \omega_1. \tag{1.20}$$

One can think of (1.19) as defining the action of $\partial_{-N,k}$ on ω in terms of the as yet undetermined $A_{-N}(k)$. To find $A_{-N}(k)$ as a polynomial in negative powers of λ, one would like to have an equation like (1.19) in which ω is replaced by a group element which contains only non-negative powers of λ, i.e. one would like to find $\tilde{\omega}$ of the form

$$\tilde{\omega} = \exp \sum_{n=0}^{\infty} \lambda^n \tilde{\omega}_n \tag{1.21}$$

such that one can perform the transformation

$$A_x \to \tilde{\omega}^{-1} A_x \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_x = \lambda E, \tag{1.22a}$$

$$A_t \to \tilde{\omega}^{-1} A_t \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_t = a_t. \tag{1.22b}$$

Then one can again consider the solutions $a_t = \lambda^{\pm N} k$ for (1.14). For the case $\lambda^{-N} k$, the inverse transformation (1.22b) gives

$$A_{-N}(k) = \lambda^{-N} \tilde{\omega} k \tilde{\omega}^{-1} - \tilde{\omega}_{-N,k} \tilde{\omega}^{-1}, \tag{1.23}$$

which enables one to obtain $A_{-N}(k)$ and the action of $\partial_{-N,k}$ on $\tilde{\omega}$, by equating coefficients of powers of λ. The case $\lambda^N k$ defines the action of $\partial_{N,k}$ on $\tilde{\omega}$ in terms of $A_N(k)$ (1.17a). To construct $\tilde{\omega}$, one writes it in the form

$$\tilde{\omega} = \psi \Omega, \tag{1.24}$$

where ψ is independent of λ, and

$$\Omega = \exp \sum_{n=1}^{\infty} \lambda^n \Omega_n. \tag{1.25}$$

It will be shown in Sect. 2 that Eq. (1.22a) determines Ω to all orders in terms of the auxiliary field ψ. In Sect. 3 the commutation relations of the evolution operators $\partial_{\pm N,k}$ will be investigated, which will show them to form a realization of a Kac-Moody algebra. The class of operators can be extended by allowing the algebra element to be an arbitrary element of $\mathscr{g}$, rather than just of $\mathscr{k}$. In Sect. 4 the Hamiltonians for the operators $\partial_{\pm N,g}$ are considered. Their Poisson bracket algebra provides a realization of the Kac-Moody algebra $\mathscr{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathbb{C}c$. In Sect. 5 it is shown that the formal sum of Hamiltonians for the operators $\partial_{\pm N,e_\alpha}$ can be used to linearize the system. The interpretation of these operators is discussed in Sect. 6.

2. Construction of $\tilde{\omega}$

Let $\tilde{\omega}$ be an element of the Lie group G, of the form

$$\tilde{\omega} = \psi \Omega, \tag{2.1}$$

where ψ is independent of λ, and

$$\Omega = \exp \sum_{n=1}^{\infty} \lambda^n \Omega_n. \tag{2.2}$$

Now fix $\tilde{\omega}$ by choosing

$$\lambda \tilde{\omega} E \tilde{\omega}^{-1} - \tilde{\omega}_x \tilde{\omega}^{-1} = A_x, \tag{2.3}$$

where A_x is given by (1.6a); i.e.

$$\lambda E + A_x^0 = \lambda \psi \Omega E \Omega^{-1} \psi^{-1} - \psi \Omega_x \Omega^{-1} \psi^{-1} - \psi_x \psi^{-1}. \tag{2.4}$$

Equating coefficients of powers of λ^0, one can see that

$$A_x^0 = -\psi_x \psi^{-1}. \tag{2.5}$$

Notice that ψ is the group element which arises in the transformation between the GNLS system and the generalized Heisenberg ferromagnet [2]. (This will be explained more fully in Sect. 6.) The λ-dependent part of (2.4) becomes

$$\lambda \Omega E \Omega^{-1} - \Omega_x \Omega^{-1} = \lambda \psi^{-1} E \psi. \tag{2.6}$$

Now, by expanding (2.2) as a power series in λ, one can obtain the identities

$$(\Omega E \Omega^{-1})_n = \sum_{r=1}^{n} (r!)^{-1} \Sigma_{k_i : \Sigma k_i = n} [\Omega_{k_1} [\Omega_{k_2} [\cdots [\Omega_{k_r}, E] \cdots]]], \tag{2.7a}$$

$$(\Omega_x \Omega^{-1})_n = \sum_{r=1}^{n} (r!)^{-1} \Sigma_{k_i : \Sigma k_i = n} [\Omega_{k_1} [\Omega_{k_2} [\cdots [\Omega_{k_{r-1}}, \partial_x \Omega_{k_r}] \cdots]]], \tag{2.7b}$$

where $(\cdots)_n$ denotes the coefficient of λ^n. Use these to equate coefficients of λ^n in (2.6):

$$\lambda^1: \partial_x \Omega_1 = E - \psi^{-1} E \psi,$$

i.e

$$\Omega_1 = xE - \partial^{-1}(\psi^{-1} E \psi), \tag{2.8a}$$

$$\lambda^2: \partial_x \Omega_2 + 1/2[\Omega_1, \partial_x \Omega_1] = [\Omega_1, E],$$

i.e.

$$\Omega_2 = 1/2\partial^{-1}([E, \partial^{-1}(\psi^{-1}E\psi)] + x[E, \psi^{-1}E\psi] + [\psi^{-1}E\psi, \partial^{-1}(\psi^{-1}E\psi)]), \tag{2.8b}$$

and so on. In general one has

$$(\Omega_x \Omega^{-1})_n = (\Omega E \Omega^{-1})_{n-1} \tag{2.9}$$

for all $n > 1$, and so $\partial_x \Omega_n$ is determined in terms of $\Omega_{m<n}$. In this way, Ω is determined to all orders non-locally in terms of ψ.

3. The Evolution Operators

Recall the zero curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0, \tag{3.1}$$

where A_x is given by (1.6a) and A_t is unknown. Consider the gauge transformation

$$A_x \to \omega^{-1} A_x \omega + \omega^{-1} \omega_x = \lambda E, \tag{3.2a}$$

$$A_t \to \omega^{-1} A_t \omega + \omega^{-1} \omega_t = a_t, \tag{3.2b}$$

where ω is the group element defined by (1.12), of the form $\omega = \exp \sum_{n=1}^{\infty} \lambda^{-n} \omega_n$. Under the transformation (3.2), the zero curvature condition (3.1) becomes

$$\partial_x a_t + [\lambda E, a_t] = 0. \tag{3.3}$$

One can choose the solutions

$$a_t = \lambda^{\pm N} k \tag{3.4}$$

for (3.3) (where N is a positive integer). Then (3.2b) can be inverted to obtain

$$A_N(k) = \lambda^N \omega k \omega^{-1} - \omega_{N,k} \omega^{-1}, \tag{3.5a}$$

$$\omega_{-N,k} \omega^{-1} = \lambda^{-N} \omega k \omega^{-1} - A_{-N}(k), \tag{3.5b}$$

where $A_N(k)$ is chosen to be a polynomial in non-negative powers of λ, while $A_{-N}(k)$ is a polynomial in negative powers. Equation (3.5a) defines $A_N(k)$ and the action of $\partial_{N,k}$ on ω, while (3.5b) is regarded as defining the action of $\partial_{-N,k}$ on ω in terms of $A_{-N}(k)$, which is still undetermined.

Now consider the gauge transformation (3.2) with ω replaced by $\tilde{\omega}$ as constructed in Sect. 2. Then, from the definition (2.3),

$$A_x \to \tilde{\omega}^{-1} A_x \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_x = \lambda E, \tag{3.6a}$$

$$A_t \to \tilde{\omega}^{-1} A_t \tilde{\omega} + \tilde{\omega}^{-1} \tilde{\omega}_t = a_t, \tag{3.6b}$$

where A_t is considered unknown. The zero curvature condition again takes the form (3.3), and the solutions (3.4) can be used to invert (3.6b) to give

$$A_{-N}(k) = \lambda^{-N} \tilde{\omega} k \tilde{\omega}^{-1} - \tilde{\omega}_{-N,k} \tilde{\omega}^{-1}, \tag{3.7a}$$

$$\tilde{\omega}_{N,k} \tilde{\omega}^{-1} = \lambda^{N} \tilde{\omega} k \tilde{\omega}^{-1} - A_N(k). \tag{3.7b}$$

Equation (3.7a) defines $A_{-N}(k)$ as the negative power part of $\lambda^{-N} \tilde{\omega} k \tilde{\omega}^{-1}$, and defines $\tilde{\omega}_{-N,k} \tilde{\omega}^{-1}$ as the positive power part. Equation (3.7b) defines the action of $\partial_{N,k}$ on $\tilde{\omega}$ in terms of $A_N(k)$, which was defined by the positive power part of (3.5a). Explicitly, one has

$$A_N(k) = \sum_{n=0}^{N} \lambda^n (\omega k \omega^{-1})_{N-n} = \sum_{n=0}^{N} \lambda^n A_N^n(k), \tag{3.8a}$$

$$(\omega_{N,k} \omega^{-1})_n = (\omega k \omega^{-1})_{N+n}, \tag{3.8b}$$

(from (3.5a)), where $(f(\omega))_n$ denotes the coefficient of λ^{-n} in $f(\omega)$,

$$A_{-N}(k) = \sum_{n=1}^{N} \lambda^{-n} \psi (\Omega k \Omega^{-1})_{N-n} \psi^{-1} = \sum_{n=1}^{N} \lambda^{-n} A_{-N}^n(k), \tag{3.9a}$$

$$(\Omega_{-N,k} \Omega^{-1})_n = (\Omega k \Omega^{-1})_{N+n}, \tag{3.9b}$$

for $N > 0$ (from (3.7a), using (2.1)), where $(f(\Omega))_n$ denotes the coefficient of λ^n in $f(\Omega)$,

$$\psi_{N,k} \psi^{-1} = -A_N^0(k) = -(\omega k \omega^{-1})_N \quad (\forall N > 0), \tag{3.10a}$$

$$\psi_{0,k} \psi^{-1} = \psi k \psi^{-1} - k \tag{3.10b}$$

(from the coefficient of λ^0 in (3.7b), using (2.1) and (3.8a)),

$$\psi_{-N,k} \psi^{-1} = \psi (\Omega k \Omega^{-1})_N \psi^{-1} \quad (\forall N > 0) \tag{3.11}$$

(from the coefficient of λ^0 in (3.7a)). Lastly, (3.5b) and (3.7b) become

$$\omega_{-N,k} \omega^{-1} = \lambda^{-N} \omega k \omega^{-1} - \sum_{n=1}^{N} \lambda^{-n} \psi (\Omega k \Omega^{-1})_{N-n} \psi^{-1} \quad (\forall N > 0), \tag{3.12a}$$

$$\Omega_{N,k} \Omega^{-1} = \lambda^N \Omega k \Omega^{-1} - \sum_{n=1}^{N} \lambda^n \psi^{-1} (\omega k \omega^{-1})_{N-n} \psi \quad (\forall N > 0), \tag{3.12b}$$

$$\Omega_{0,k} \Omega^{-1} = \Omega k \Omega^{-1} - k \tag{3.12c}$$

(using (3.9a), (3.8a) and (3.10)).

Notice that (3.8b) implies

$$(\omega_{1,E} \omega^{-1})_n = (\omega E \omega^{-1})_{n+1} = (\omega_x \omega^{-1})_n \tag{3.13}$$

(by (1.12)), i.e.

$$\partial_{1,E} = \partial_x \tag{3.14}$$

and so, by (3.10a),

$$\psi_x\psi^{-1} = -(\omega E\omega^{-1})_1 = -A_x^0 \tag{3.15}$$

(using (1.12) again). This is consistent with (2.5). Now, ω satisfies identities like (2.7), where $(\cdots)_n$ is taken to denote the coefficient of λ^{-n}, so that (from (3.15))

$$\begin{aligned}\partial_{N,k}A_x^0 &= [\partial_{N,k}\omega_1, E] = [(\omega_{N,k}\omega^{-1})_1, E] \quad \text{(using (2.7b))}\\ &= [(\omega k\omega^{-1})_{N+1}, E] \qquad \text{(from (3.8b))},\end{aligned} \tag{3.16a}$$

and

$$\partial_{-N,k}A_x^0 = [\partial_{-N,k}\omega_1, E] = [E, A_{-N}^1(k)] \quad \text{(from the coefficient of } \lambda^{-1} \text{ in (3.5b))}$$

i.e.

$$-\partial_{-N,k}(\psi_x\psi^{-1}) = [E, \psi(\Omega k\Omega^{-1})_{N-1}\psi^{-1}] \tag{3.16b}$$

(by (3.9a) and (2.5)). Equations (3.16) are the equations of motion corresponding to $A_{\pm N}(k)$. (One could also obtain them by substitution of (3.8a), (3.9a) in (3.1).)

The commutation properties of the evolution operators will now be investigated. Recall Eq. (3.5):

$$\omega_{n,k}\omega^{-1} = \lambda^n\omega k\omega^{-1} - A_n(k) \quad \forall n\in\mathbb{Z},\ k\in\mathscr{k} \tag{3.17}$$

(or alternatively use $\tilde{\omega}$, i.e. Eq. (3.7)) and consider the identity

$$([\partial_{n,k}, \partial_{m,j}]\omega)\omega^{-1} = \partial_{n,k}(\omega_{m,j}\omega^{-1}) - \partial_{m,j}(\omega_{n,k}\omega^{-1}) + [\omega_{m,j}\omega^{-1}, \omega_{n,k}\omega^{-1}] \tag{3.18}$$

for all m, $n\in\mathbb{Z}$, k, $j\in\mathscr{k}$. Using (3.17), one has

$$\partial_{n,k}(\omega_{m,j}\omega^{-1}) = \lambda^m[\lambda^n\omega k\omega^{-1} - A_n(k), \omega j\omega^{-1}] - \partial_{n,k}A_m(j). \tag{3.19}$$

One finds that (3.18) becomes

$$\begin{aligned}&([\partial_{n,k}, \partial_{m,j}]\omega)\omega^{-1}\\ &\quad = \partial_{m,j}A_n(k) - \partial_{n,k}A_m(j) - [A_n(k), A_m(j)] + \lambda^{n+m}\omega[k, j]\omega^{-1},\end{aligned} \tag{3.20}$$

and so (using (3.17) to rewrite the last term in (3.20)) one can deduce that the relation

$$[\partial_{n,k}, \partial_{m,j}] = \partial_{n+m,[k,j]} \tag{3.21}$$

is equivalent to the condition

$$\partial_{n,k}A_m(j) - \partial_{m,j}A_n(k) + [A_n(k), A_m(j)] = A_{n+m}([k, j]), \tag{3.22}$$

which must now be verified using (3.8a) and (3.9a). One needs to consider separately the cases where n, m are of the same or different sign. Suppose they are of different sign, i.e. $n = N(\geqq 0)$ and $m = -M(\leqq 0)$, and suppose $(N-M) \geqq 0$. Split (3.22) into coefficients of λ^{-p} and λ^q (where $p, q \geqq 0$):

$$\partial_{N,k}A_{-M}^p(j) + [A_N(k), A_{-M}(j)]_{-p} = 0, \tag{3.23a}$$

$$-\partial_{-M,j}A_N^q(k) + [A_N(k), A_{-M}(j)]_q = A_{N-M}^q([k, j]). \tag{3.23b}$$

Using (3.8a) and (3.9a), the left side of Eq. (3.23a) becomes

$$[\psi_{N,k}\psi^{-1}, \psi(\Omega j\Omega^{-1})_{M-p}\psi^{-1}] + \psi([\Omega_{N,k}\Omega^{-1}, \Omega j\Omega^{-1}])_{M-p}\psi^{-1}$$
$$+ \sum_{r=0}^{M-p} [(\omega k\omega^{-1})_{N-r}, \psi(\Omega j\Omega^{-1})_{M-p-r}\psi^{-1}]$$
$$= \sum_{r=1}^{M-p} [(\omega k\omega^{-1})_{N-r}, \psi(\Omega j\Omega^{-1})_{M-p-r}\psi^{-1}]$$
$$+ \sum_{r=1}^{M-p} \psi[(\Omega_{N,k}\Omega^{-1})_r, (\Omega j\Omega^{-1})_{M-p-r}]\psi^{-1} \quad \text{(using (3.10))}$$
$$= 0$$

as required (using (3.12b) and noting that $r < N$). The left side of (3.23b) becomes

$$-([\omega_{-M,j}\omega^{-1}, \omega k\omega^{-1}])_{N-q} + \sum_{r=q+1}^{M+q} [(\omega k\omega^{-1})_{N-r}, A^{r-q}_{-M}(j)]$$
$$= (\omega[k, j]\omega^{-1})_{N-M-q} \quad \text{(using (3.9a), (3.12a))}$$
$$= A^q_{N-M}([k, j]),$$

as required (using (3.8a)).

The calculation for $(N-M) < 0$ proceeds along similar lines, noting that for this case $A_{N-M}([k, j])$ has the form (3.9a). One can also check the cases where n, m in (3.22) are of the same sign. (The case $n, m \geqq 0$ was considered in [1]). Having verified (3.22), one can conclude, from (3.21), that the evolution operators $\partial_{\pm N,k}$ provide a realization of the Kac-Moody algebra $\mathscr{k} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$.

Notice that (3.22) is a generalization of the zero curvature condition (3.1). Putting $(n, k) = (1, E)$ and recalling (3.14), Eq. (3.22) becomes

$$\partial_x A_m(j) - \partial_{m,j} A_x + [A_x, A_m(j)] = 0 \quad \forall m \in \mathbb{Z}, j \in \mathscr{k}. \tag{3.24}$$

A less illuminating (but quicker) method of obtaining (3.21) is to use ψ in place of ω in (3.18). For example, using (3.10), (3.11) and Eq. (3.5), (3.7), one obtains

$$([\partial_{N,k}, \partial_{-M,j}]\psi)\psi^{-1}$$
$$= \sum_{r=1}^{N} [(\omega j\omega^{-1})_{r-M} - \psi(\Omega j\Omega^{-1})_{M-r}\psi^{-1}, (\omega k\omega^{-1})_{N-r}]$$
$$+ \sum_{r=1}^{M} \psi[(\Omega k\Omega^{-1})_{r-N} - \psi^{-1}(\omega k\omega^{-1})_{N-r}\psi, (\Omega j\Omega^{-1})_{M-r}]\psi^{-1}. \tag{3.25}$$

If $(N-M) \geqq 0$ then (3.25) becomes

$$([\partial_{N,k}, \partial_{-M,j}]\psi)\psi^{-1} = \sum_{s=0}^{N-M} [(\omega j\omega^{-1})_{N-M-s}, (\omega k\omega^{-1})_s]$$
$$= -(\omega[k, j]\omega^{-1})_{N-M} = \psi_{N-M,[k,j]}\psi^{-1} \tag{3.26}$$

with a similar result for $(N-M) < 0$. One can also check the equal sign cases in the same fashion.

The evolution operators $\partial_{\pm N,k}$ can be extended in the obvious way, simply by replacing k by a general constant element $g \in \mathscr{g}$, i.e.

$$A_n(g) = \lambda^n \omega g \omega^{-1} - \omega_{n,g}\omega^{-1} = \lambda^n \tilde{\omega} g \tilde{\omega}^{-1} - \tilde{\omega}_{n,g}\tilde{\omega}^{-1} \quad \forall n \in \mathbb{Z},\ g \in \mathscr{g}. \tag{3.27}$$

One can then verify in the same way as earlier the equation

$$\partial_{n,g} A_m(h) - \partial_{m,h} A_n(g) + [A_n(g), A_m(h)] = A_{n+m}([g,h]) \quad \forall n, m \in \mathbb{Z},\ g, h \in \mathscr{g}, \tag{3.28}$$

which is equivalent to the condition

$$[\partial_{n,g}, \partial_{m,h}] = \partial_{n+m,[g,h]}, \tag{3.29}$$

and so one obtains a realization of the Kac-Moody algebra $\mathscr{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. Notice that for $m \in \mathscr{m}$ the quantities $A_{\pm N}(m)$ do not satisfy the zero curvature condition, which is why they were not obtained in the original construction (where one sought solutions of (3.1)), and so the parameters $(\pm N, m)$ cannot be regarded as true "times." Their interpretation will be discussed later.

4. Hamiltonians

The Hamiltonians $H_n(g)$ for the operators $\partial_{n,g}$ are defined by

$$\partial_{n,g} f = \{f, H_n(g)\}, \tag{4.1}$$

where f is any function, and the Poisson bracket is given by

$$\{f_1, f_2\} = \sum_\alpha \int dz (\partial f_1/\partial q^{-\alpha}(z) \cdot \partial f_2/\partial q^\alpha(z) - \partial f_1/\partial q^\alpha(z) \cdot \partial f_2/\partial q^{-\alpha}(z)). \tag{4.2}$$

Hamilton's equations take the form

$$\partial_{n,g} q^\alpha = -\partial H_n(g)/\partial q^{-\alpha}, \tag{4.3a}$$

$$\partial_{n,g} q^{-\alpha} = \partial H_n(g)/\partial q^\alpha. \tag{4.3b}$$

Now, the Jacobi identity, together with (4.1) and (3.29), implies that (for any function f)

$$\{f\{H_n(g), H_m(h)\}\} = [\partial_{n,g}, \partial_{m,h}] f = \partial_{n+m,[g,h]} f = \{f, H_{n+m}([g,h])\} \quad \forall n, m \in \mathbb{Z},\ g, h \in \mathscr{g}, \tag{4.4}$$

and so

$$\{H_n(g), H_m(h)\} = H_{n+m}([g,h]) + C^{g,h}_{n,m}, \tag{4.5}$$

where $C^{g,h}_{n,m}$ is a constant. Since the Hamiltonians are only defined up to addition of constants, it is possible to arrange for the Poisson bracket to take the form [3]

$$\{H_n(g), H_m(h)\} = H_{n+m}([g,h]) + n\delta_{n,-m}\delta_{g,h} c, \tag{4.6}$$

where c is a constant; i.e. the Hamiltonians provide a realization of the centrally extended algebra $\mathscr{g}\otimes\mathbb{C}[\lambda,\lambda^{-1}]\oplus\mathbb{C}c$.

The Hamiltonian of the GNLS equation is $H_2(E)$, so Eq. (4.6) implies that $H_{\pm N}(k)$ are conserved quantities for the GNLS equation, except for $H_{-2}(E)$ if $c\neq 0$. Notice also that (4.6) and (4.1) imply

$$\partial_{\pm N,k}H_0(E)=0 \quad \forall N\geqq 0,\, k\in\mathscr{k}, \tag{4.7}$$

and one can use (3.16a) (with $N=0$, $k=E$) to deduce that

$$H_0(E)=-i\int q^\alpha q^{-\alpha} \tag{4.8}$$

(summation implied). Next, observe that

$$\begin{aligned}\int \mathrm{Tr}\,(E[A_x^0,\partial_{n,g}A_x^0]) &= i\int (q^\alpha_{n,g}q^{-\alpha}-q^\alpha q^{-\alpha}_{n,g})\\ &= 2i\int q^\alpha_{n,g}q^{-\alpha} = -2i\int q^{-\alpha}_{n,g}q^\alpha,\end{aligned} \tag{4.9}$$

if $g\in\mathscr{k}$ (using (4.7), (4.8)). Differentiation of (4.9) with respect to $q^{\pm\alpha}$ gives Hamilton's equations (4.3), so that one can use (3.16) to write

$$H_N(g)=-ia\int \mathrm{Tr}\,(A_x^0\omega g\omega^{-1})_{N+1}, \tag{4.10a}$$

$$H_{-N}(g)=ia\int \mathrm{Tr}\,(A_x^0\psi(\Omega g\Omega^{-1})_{N-1}\psi^{-1}), \tag{4.10b}$$

where $a=1/2$ if $g\in\mathscr{k}$, and $a=1$ if $g\in\mathscr{m}$.

5. Linearization

For step operators $e_{\pm\alpha}\in\mathscr{m}$, define the formal power series

$$\Gamma_{\pm\alpha}(\lambda)=\sum_{n=-\infty}^{\infty}\lambda^{-(n+1)}H_n(e_{\pm\alpha})=i\int \mathrm{Tr}\,(A_x^0(\psi\Omega e_{\pm\alpha}\Omega^{-1}\psi^{-1}-\omega e_{\pm\alpha}\omega^{-1})) \tag{5.1}$$

(by (4.10)). Then (4.6) and (1.2) imply

$$\partial_{N,E}\Gamma_{\pm\alpha}(\lambda)=\{\Gamma_{\pm\alpha}(\lambda),H_N(E)\}=\pm i\lambda^N\Gamma_{\pm\alpha}(\lambda), \tag{5.2}$$

and so $\Gamma_{\pm\alpha}(\lambda)$ linearizes the equations of motion of the GNLS hierarchy.

Using the cyclic property of the trace, (5.1) can be written as

$$\Gamma_{\pm\alpha}(\lambda)=i\int((\Omega^{-1}\psi^{-1}A_x^0\psi\Omega)^{\mp\alpha}-(\omega^{-1}A_x^0\omega)^{\mp\alpha}), \tag{5.3}$$

where $(X)^{\mp\alpha}$ denotes the $e_{\mp\alpha}$ component of X. Now, the restriction to the compact or non-compact form corresponds to

$$(X)^{\alpha *}=\mp(X)^{-\alpha} \tag{5.4}$$

so that

$$\{\Gamma_\alpha(\lambda),\Gamma^*_\beta(\mu)\}=\mp\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\lambda^{-n-1}\mu^{-m-1}H_{n+m}([e_\alpha,e_{-\beta}]) \tag{5.5}$$

when $q^{-\alpha}=\mp q^{\alpha *}$. Also

$$\{\Gamma_\alpha(\lambda),\Gamma_\beta(\mu)\}=0 \tag{5.6}$$

(since (1.2) implies that $[e_\alpha, e_\beta] = 0$). Equation (5.5) shows that the transformation

$$q^\alpha \to \Gamma_\alpha \tag{5.7}$$

is not canonical.

6. Discussion

Recalling Eq. (3.14), one notes the following special cases of (3.29):

$$[\partial_x, \partial_{n,k}] = 0 \quad \forall n \in \mathbb{Z},\ k \in \mathscr{k}, \tag{6.1a}$$

$$[\partial_x, \partial_{n,e_{\pm\alpha}}] = \mp i\partial_{1+n,e_{\pm\alpha}} \quad \forall n \in \mathbb{Z},\ e_{\pm\alpha} \in \mathscr{m}. \tag{6.1b}$$

The parameters (n, k) could be regarded as "time" variables, but $(n, e_{\pm\alpha})$ cannot. Transformations which do not commute with translations are regarded, in the context of gauge theories, as "internal symmetries" [4]. The mutually commuting class $\{\partial_{n,e_\alpha}\}$ can be thought of as the basis of an "internal space." It is the generator of translations in this space (Γ_α) which linearizes the GNLS system. It should be noted, however, that what one is really considering is the phase space of the system. The construction seems, in fact, to be a generalization of the conventional approach to the $SU(2)$ non-linear Schrödinger equation [5], where one considers the so-called "monodromy matrix" whose diagonal elements give rise to conserved quantities, while the off-diagonal elements lead to the linearization of the system.

The use of the gauge transformation $\tilde{\omega}$ is similar to the method used by Olive and Turok for deriving the conserved quantities of the Toda equation [6]. In that case, a λ-independent local gauge transformation was composed with a local transformation of the form (1.25) so that the transformed gauge potential was a series belonging to $\mathscr{k}$. In the present case, as was mentioned earlier, the λ-independent element ψ is associated with the generalized Heisenberg ferromagnet (GHF) [2]. Consider the transformation

$$A_x \to \tilde{A}_x = \psi^{-1}A_x\psi + \psi^{-1}\psi_x = \lambda\psi^{-1}E\psi, \tag{6.2a}$$

$$A_2(E) \to \tilde{A}_t = \psi^{-1}A_2(E)\psi + \psi^{-1}\psi_{2,E} = \lambda^2\psi^{-1}E\psi + \lambda\psi^{-1}A_x^0\psi \tag{6.2b}$$

(by (3.10)) and define

$$S = \psi^{-1}E\psi. \tag{6.3}$$

Then

$$\partial_x S = \psi^{-1}[E, \psi_x\psi^{-1}]\psi = \psi^{-1}[A_x^0, E]\psi, \tag{6.4}$$

and so

$$[S, S_x] = \psi^{-1}[E[A_x^0, E]]\psi = \psi^{-1}A_x^0\psi, \tag{6.5}$$

i.e. the transformed gauge potentials are

$$\tilde{A}_x = \lambda S, \tag{6.6a}$$

$$\tilde{A}_t = \lambda^2 S + \lambda[S, S_x], \tag{6.6b}$$

and the zero curvature condition becomes the GHF equation

$$\partial_t S = [S, S_{xx}]. \tag{6.7}$$

The conserved quantities which have been constructed for the GNLS system are non-local, because of the non-locality of the gauge transformations used to construct them. Non-local conserved quantities were constructed for the non-linear σ-model in [7], and it was shown in [8] that these are associated with infinitesimal transformations which form a centre-free Kac-Moody algebra. However, the charges themselves do not form an algebra [9], and the Kac-Moody symmetry is interpreted as a property of the solution space, rather than of the phase space. The infinitesimal symmetries of the $SU(2)$ non-linear Schrödinger equation were investigated in [10]. In that construction, only the "positive" subalgebra was realized non-trivially.

The linearization of the GNLS system using step operators of a Kac-Moody algebra (Eq. (5.2)) seems to be related to the work of the Kyoto group [11], who use vertex operators to construct soliton solutions for a large class of equations. It would be interesting to establish the connection of these ideas with the approach of Adler and van Moerbeke [12]. Other topics worth pursuing include the investigation of the central term in (4.6) (e.g. the conditions under which it vanishes), and the quantization of the system. For the $SU(2)$ case, the quantization of the action-angle variables (i.e. the canonical linearizing variables) gives the "Bethe ansatz" creation operators [13]. In the general case, quantization should lead to vertex operators of some sort.

The methods which have been presented here can be generalized to cover a wide range of integrable systems. This will be discussed in a subsequent paper [14].

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