

On the Second Eigenfunctions of the Laplacian in \mathbb{R}^{2*}

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Abstract. A conjecture about the nodal line of a second eigenfunction states that the nodal line of a second eigenfunction divides the domain Ω by intersecting with the boundary of Ω transversely, where Ω is a bounded convex domain of \mathbb{R}^2 . We prove this conjecture provided Ω has a symmetry. Also, we prove the multiplicity of the second eigenvalue is two at most provided Ω is a bounded convex domain of \mathbb{R}^2 .

1. Introduction

An eigenfunction φ is meant to be a solution of Dirichlet's problem:

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{in } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Delta = \sum_{i=1}^n (\partial^2/\partial x_i^2)$ is the Laplacian, Ω is a bounded smooth domain in \mathbb{R}^n , and λ is a constant (i.e. the corresponding eigenvalue). It is well known that the first eigenfunction is positive in Ω , and all higher eigenfunctions must change sign. The nodal set of an eigenfunction φ is defined to be the closure of $\{x \in \Omega \mid \varphi(x) = 0\}$. The Courant nodal domain theorem [2] tells us that the nodal set of a k th eigenfunction divides the domain Ω into at most k subregions. We do not know the topology of the nodal set in general, even for the simplest case $n = 2$. A conjecture about the nodal line (i.e. $n = 2$) of a second eigenfunction states that:

(*) the nodal line of a second eigenfunction divides the domain Ω by intersecting its boundary at exactly two points if Ω is convex. (See [5, 6]).

Throughout the paper, Ω is always assumed a bounded *smooth convex* domain in \mathbb{R}^2 . L. Payne [5] proved the conjecture provided the domain Ω is symmetric with respect to one line. In this paper, we will prove (*) holds true if Ω is symmetric under a rotation with angle $2\pi p/q$, where p, q are positive integers. As a corollary of (*), we

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can prove that if Ω is symmetric with respect to one point, then all the second eigenfunctions are an odd function with respect to this point. Another problem we consider is about the multiplicity of the second eigenvalue. In [1], Cheng actually proved that there are at most three eigenfunctions for the sphere S^2 . His arguments can be carried over to show that the multiplicity of the second eigenvalue of (1.1) is at most 3, provided $n = 2$. We will sharpen his result by proving that the multiplicity is at most two provided Ω is convex, and $n = 2$.

Notation. φ_2 is always denoted as an second eigenfunction and $N = \{x \in \Omega \mid \varphi_2(x) = 0\}$ is the nodal line of φ_2 . A bounded smooth convex domain in Ω in \mathbf{R}^2 is said to have the property S if Ω is symmetric with respect to one line or Ω is invariant under a rotation by an angle θ with respect to one point.

We state some preliminaries about second eigenfunctions. φ is an eigenfunction, and Ω is any bounded domain in \mathbf{R}^2 .

Lemma 1.2. *Suppose $P \in \partial\Omega$. Then $(\partial\varphi/\partial\nu)(P) = 0$ iff $P \in N$, where $\partial\varphi/\partial\nu$ is the outnormal derivative of φ on the boundary.*

Proof. Assume that $P = (0, 0)$ and the tangent at P is x -direction. First, suppose $(\partial\varphi/\partial y)(P) = c \neq 0$, then $\varphi(x, y) = cy + 0(|x|^2 + |y|^2)$, provided $|x|^2 + |y|^2 \leq \varepsilon$, where ε is sufficiently small. Then, given x such that $|x| \leq \varepsilon$, there exists a unique $y(x)$ such that $|y(x)| \leq \varepsilon$ and satisfies $\varphi(x, y) = cy + 0(|x|^2 + |y|^2) = 0$. Because $\varphi|_{\partial\Omega} = 0$, $(x, y(x)) \in \partial\Omega$. Hence $\varphi(x, y) \neq 0$, for $(x, y) \in \Omega \cap B_\varepsilon$.

Secondly, suppose $P \notin N$, then, by a generalized Hopf boundary point lemma [3], we have $(\partial\varphi/\partial\nu)(P) \neq 0$. #

Considering the simple topology of the two dimensional plane, and using the Courant nodal line theorem and Lemma 1.2, we have the following.

Lemma 1.3. *$\partial\varphi_2/\partial\nu$ may have at most two zeros at the boundary of Ω .*

Remark. By Lemma 1.3, (*) is equivalent to the statement that $\partial\varphi_2/\partial\nu$ has two zeros at the boundary of Ω .

Section 2

In this section, a second eigenfunction φ_2 is meant to be a solution of Dirichlet's problem:

$$\begin{cases} \Delta\varphi_2 + \lambda_2\varphi_2 = 0 & \text{in } \Omega \\ \varphi_2|_{\partial\Omega} = 0 \end{cases}, \tag{2.1}$$

where $\Delta = \sum_{i=1}^2 (\partial^2/\partial x_i^2)$, Ω is a bounded convex domain in \mathbf{R}^2 and λ_2 is the second eigenvalue. Throughout this section, Ω is always convex. The Courant nodal line theorem tells us that the nodal line N of φ_2 must divide the domain Ω into exactly two components. A conjecture about the second eigenfunction is the following:

(*) *the nodal line N of a second eigenfunction φ_2 must intersect the boundary $\partial\Omega$ at exactly two points*

Our main results are the following:

Theorem 2.2. *Suppose Ω has the property S. then (*) holds true.*

Theorem 2.3. *The multiplicity of the second eigenvalue is at most two.*

To prove Theorem 2.2 and Theorem 2.3, we need a lemma.

Lemma 2.4. *Let $\varphi \neq 0$ be a solution of (2.1). If $\partial\varphi/\partial\nu \geq 0$ on $\partial\Omega$, then φ is the only second eigenfunction of (2.1).*

Proof. Suppose that there exists another second eigenfunction ψ , then we can always choose ψ to be orthogonal to φ , i.e.

$$\int_{\Omega} \psi \varphi dx = 0. \tag{2.5}$$

Fix a point (x_0, y_0) (which will be chosen later). Set

$$T = (x - x_0) \frac{\partial}{\partial x} + (y - y_0) \frac{\partial}{\partial y}.$$

A straightforward computation shows that

$$\Delta(T\psi) + \lambda_2(T\psi) = 2\Delta\psi = -2\lambda_2\psi.$$

Multiplying φ on both sides and integrating over Ω , we have

$$\int_{\Omega} [\varphi \Delta(T\psi) - (T\psi \Delta \varphi)] dx = -2\lambda_2 \int \psi \varphi = 0$$

by (2.5). By Green's theorem,

$$0 = \int_{\partial\Omega} T\psi \frac{\partial\varphi}{\partial\nu} ds = \int_{\partial\Omega} [\langle x - x_0, y - y_0 \rangle \cdot \nu] \frac{\partial\psi}{\partial\nu} \frac{\partial\varphi}{\partial\nu} ds. \tag{2.6}$$

Suppose that $\partial\psi/\partial\nu$ has only one sign on $\partial\Omega$, say $\partial\psi/\partial\nu \geq 0$ on $\partial\Omega$. Then, choosing $(x_0, y_0) \in \Omega$ in (2.6), we have

$$\frac{\partial\psi}{\partial\nu} \cdot \frac{\partial\varphi}{\partial\nu} \equiv 0 \quad \text{on} \quad \partial\Omega,$$

which is a contradiction to the Hopf boundary point lemma. Therefore $\partial\psi/\partial\nu$ must change sign on $\partial\Omega$. By Lemma 1.3, it then implies the nodal line N intersects the boundary $\partial\Omega$ at exactly two points, say P and Q . If the tangents of $\partial\Omega$ at P, Q are not parallel, then choosing (x_0, y_0) to be the intersection of the tangent lines of $\partial\Omega$ at P and at Q , we have $T\psi$ has only one sign on $\partial\Omega$, say, $T\psi \geq 0$. And (2.6) implies that either $\partial\psi/\partial\nu \equiv 0$ on $\partial\Omega$ or $\partial\varphi/\partial\nu \equiv 0$ on $\partial\Omega$, which again is a contradiction to the Hopf boundary point lemma. If the tangents at P and at Q are parallel, say, to the x -axis, then we define $T' = \partial/\partial x$. Repeating the above arguments for $T'\psi$, we get the same contradiction. Hence the proof of Lemma (2.4) is complete. #

Proof of Theorem 2.3. Suppose that there are three second eigenfunctions φ_1, φ_2 and φ_3 . We want to construct a second eigenfunction φ such that $\partial\varphi/\partial\nu$ has only one sign on $\partial\Omega$. Then, by Lemma 2.4, we obtain a contradiction. Fix two points P and P_i , on $\partial\Omega$; we can always choose three constants C_i^1, C_i^2, C_i^3 such that

$$(C_i^1)^2 + (C_i^2)^2 + (C_i^3)^2 = 1, \tag{2.7}$$

and the linear combination

$$\varphi_i = C_i^1 \varphi_1 + C_i^2 \varphi_2 + C_i^3 \varphi_3$$

satisfies

$$0 = \nabla \varphi_i(P) = \nabla \varphi_i(P_i).$$

By Lemma 1.3, P and P_i are exactly two zeros of $\partial \varphi_i / \partial v$ on $\partial \Omega$. Taking $P_i \rightarrow P$, and by (2.7), there is a subsequence of φ_i which converges to φ , and obviously $\varphi \neq 0$ is such a second eigenfunction that $\partial \varphi / \partial v$ has only one sign on $\partial \Omega$. Hence the proof of Theorem 2.3 is complete. #

Proof of Theorem 2.2. First, we prove the case when Ω is symmetric with respect to the y -axis. This was proved by L. Payne [5]. We include the proof here for completion. If φ is odd in x (i.e. $\varphi(-x, y) \equiv \varphi(x, y)$), the nodal line N is just the y -axis. And (*) is obviously true. Suppose φ is even in x (i.e. $\varphi(-x, y) = \varphi(x, y)$). Assume that (*) is not true. Then for $P \in \partial \Omega$, $(\partial \varphi / \partial x)(P) \neq 0$, except the tangent of $\partial \Omega$ at P is the x -direction. Without loss of generality, we may assume that $(\partial \varphi / \partial x)(P) \geq 0$ for $P \in \partial \Omega \cap \{(x, y) | y \leq 0\}$. Set $\Omega^- = \bar{\Omega} \cap \{(x, y) | y \leq 0\}$. $\partial \varphi / \partial x$ must change sign in Ω^- . Otherwise $\varphi(x, y) \geq 0$ in Ω^- , by evenness, $\varphi(x, y) \geq 0$ in $\bar{\Omega}$, which leads to a contradiction. Hence the nodal line $\{(x, y) \in \Omega^- | (\partial \varphi / \partial x)(x, y) = 0\}$ encloses a subregion Ω_*^- of $\{(x, y) \in \Omega^- | (\partial \varphi / \partial x)(x, y) < 0\}$. Let $\Omega_* = \{(x, y) \in \Omega | \text{either } (x, y) \in \Omega_*^- \text{ or } (-x, y) \in \Omega_*^-\}$. Then $\partial \varphi / \partial x$ satisfies

$$\begin{cases} \Delta \left(\frac{\partial \varphi}{\partial x} \right) + \lambda_2 \left(\frac{\partial \varphi}{\partial x} \right) = 0 & \text{in } \Omega_* \\ \frac{\partial \varphi}{\partial x} = 0 & \text{on } \partial \Omega_* \end{cases}.$$

Since $(\partial \varphi / \partial x)(-x, y) = (\partial \varphi / \partial x)(x, y)$, $\partial \varphi / \partial x$ must change sign. Hence $\lambda_2 \geq \lambda_2(\Omega_*)$, where $\lambda_2(\Omega_*)$ is the second eigenvalue of the Laplacian for the domain Ω_* . But $\Omega_* \subset \Omega$ implies

$$\lambda_2 \geq \lambda_2(\Omega_*) > \lambda_2,$$

which is a contradiction.

In general a second eigenfunction φ can be written as $\varphi = \varphi_1 + \varphi_2$, where φ_1 is odd in x , φ_2 is even in x and both are second eigenfunctions. By the above proof, we know that there exist two points $P = (x_0, y_0)$ and $Q = (-x_0, y_0) \in \partial \Omega$, where $x_0 \neq 0$ such that $(\partial \varphi_2 / \partial v)(P) = (\partial \varphi_2 / \partial v)(Q) = 0$. Since $\partial \varphi_1 / \partial v$ is odd on $\partial \Omega$, we assume $(\partial \varphi_1 / \partial v)(P) > 0$ and $(\partial \varphi_1 / \partial v)(Q) < 0$. Hence $(\partial \varphi / \partial v)(P) > 0$ and $(\partial \varphi / \partial v)(Q) < 0$. It implies that there exist two points \tilde{P} and \tilde{Q} on $\partial \Omega$ such that

$$\frac{\partial \varphi}{\partial v}(\tilde{P}) = \frac{\partial \varphi}{\partial v}(\tilde{Q}) = 0,$$

and the theorem for this case follows.

Now suppose Ω is invariant under a rotation of angle θ , i.e. $\theta = \pi q$, q is a rational number. Let Ω_t be continuously deformed from $\Omega_0 = \Omega$ to a ball Ω_1 . $\lambda_2(t)$ is denoted as the second eigenvalue for the domain Ω_t . We need the following lemma.

Lemma 2.7. *Suppose Ω_t is a continuous family of bounded domains in R^n and $\lambda(t)$ is the k^{th} eigenvalue, then $\lambda(t)$ is continuous in t .*

The proof of Lemma 2.7, e.g., see Courant and Hilbert [2].

Going back to the proof of the theorem, we let $t_0 = \inf \{t | (*) \text{ is not true for } \Omega_t\}$. By the above proofs, $(*)$ holds true for Ω_1 . Hence $0 \leq t_0 < 1$. There exists a sequence $t_i \geq t_0$ and $t_i \rightarrow t_0$ as $i \rightarrow +\infty$ such that there exists a normalized second eigenfunction φ_i :

$$\begin{cases} \Delta \varphi_i + \lambda_i \varphi_i = 0 & \text{in } \Omega_{t_i} \\ \varphi_i = 0 & \text{on } \partial \Omega_{t_i} \end{cases}$$

and $\partial \varphi_i / \partial \nu$ has two zeros on $\partial \Omega_{t_i}$.

By standard estimates for elliptic equations, a subsequence of $\{\varphi_i\}$ converges to a function φ_0 in C^2 . By Lemma 2.7, φ_0 is a second eigenfunction in Ω_{t_0} and $\partial \varphi_0 / \partial \nu$ has at least one zero on $\partial \Omega$. Considering a sequence $t_i \leq t_0$, and $t_i \rightarrow t_0$ as $i \rightarrow \infty$, and repeating the same argument as the above, we obtain a second eigenfunction $\tilde{\varphi}_0$ in Ω_{t_0} such that $\partial \tilde{\varphi}_0 / \partial \nu \geq 0$ on $\partial \Omega$. By Lemma 2.4, we may assume $\tilde{\varphi}_0 = \varphi_0$. Hence $\partial \varphi_0 / \partial \nu \geq 0$ on $\partial \Omega$ has at least one zero on $\partial \Omega$. Since Ω_{t_0} is invariant under a rotation, by Lemma 2.4 again, $\partial \varphi_0 / \partial \nu$ has two zeros on $\partial \Omega$. But $\partial \varphi_0 / \partial \nu$ could not have two zeros on $\partial \Omega$, otherwise, $\partial \varphi_0 / \partial \nu$ must change sign on $\partial \Omega$ (because the nodal line of φ_0 intersects with $\partial \Omega$ at points wherever $\partial \varphi_0 / \partial \nu$ vanish). Therefore we have reached a contradiction. And Theorem 2.2 follows. #

Corollary 2.8. *Suppose Ω is symmetric with respect to the origin, then all the second eigenfunctions are odd (i.e. $\varphi(-x, -y) = -\varphi(x, y)$).*

Proof. Suppose not. Then there exists a second eigenfunction φ which is even. The normal derivative $\partial \varphi / \partial \nu$ is also even when restricted on $\partial \Omega$. By Theorem 2.2, there are exactly two points P, Q on $\partial \Omega$ such that

$$\frac{\partial \varphi}{\partial \nu}(P) = \frac{\partial \varphi}{\partial \nu}(Q) = 0,$$

and $\partial \varphi / \partial \nu$ must change sign on $\partial \Omega$. Since $\partial \varphi / \partial \nu$ is even, $\partial \varphi / \partial \nu$ must change sign on either of the boundary arcs connecting P and Q . $\partial \varphi / \partial \nu$ has at least four zeros on $\partial \Omega$ which is a contradiction. Corollary 2.9 follows. #

Corollary 2.9. *Suppose Ω is symmetric with respect to the y -axis, then there is at most one second eigenfunction which is even (or odd) in x -variable.*

Proof. Suppose that the intersection of $\partial \Omega$ with the y -axis are $(0, y_0)$ and $(0, -y_0)$. From the proof of Theorem 2.2, the nodal line of a second eigenfunction φ which is even in x does not intersect the boundary $\partial \Omega$ at $(0, -y_0)$ nor at $(0, y_0)$. Namely, $(\partial \varphi / \partial \nu)(0, y_0) \neq 0$ and $(\partial \varphi / \partial \nu)(0, -y_0) \neq 0$. The uniqueness of the even second eigenfunction follows. The uniqueness of the odd second eigenfunction is obvious. #

Remark. When the Laplacian Δ is replaced by $\Delta + V$, $(*)$ does not hold true. In [4], Lin and Ni have found a counterexample.

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