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Limit Sets of S-Unimodal Maps with Zero Entropy

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Abstract. One-dimensional mappings "at the limit of period doubling" are studied in this paper without the use of the renormalization theory of Feigenbaum and others. The principal result is that the attracting part of the nonwandering set is a Cantor set of measure zero under the additional assumption that the map has negative Schwarzian derivative.

The topological structure of unimodal maps of the interval with negative Schwarzian derivative has been completely characterized [2]. The measure theoretic properties of these maps are less thoroughly understood. Here we study maps in one particular topological equivalence class, namely those topologically equivalent to the "Feigenbaum fixed point" [1]. These maps lie at the accumulation of period doubling bifurcations in one parameter families. Without appeal to the properties of the fixed point function or renormalization arguments, we give an elementary geometric proof that the limit sets of these mappings are Cantor sets of Hausdorff dimension smaller than one. In particular, the limit sets of all trajectories have Lebesgue measure zero and the mappings do not support absolutely continuous invariant measures.

Theorem. *Let $f:R\to R$ be an even C^3 map with*

(1) *a single critical point* 0 *which is a nondegenerate maximum,*

(2) *negative Schwarzian derivative $Sf=(f'''/f')-(3/2)(f''/f')^2$, and*

(3) *nonwandering set the union of two unstable fixed points, one periodic orbit of period 2^n for each $n>0$, and a Cantor set Λ without periodic points.*

Then Λ has Lebesgue measure zero.

Remark. The theorem is not the most general which can be proved. In particular, one can allow degenerate critical points which are not flat and relax the assumption that f is even. The proof is presented here in only the simplest case for the sake of clarity.

The proof of the theorem proceeds in several steps which we isolate as separate statements. First we recall some of the topological properties of an f satisfying the

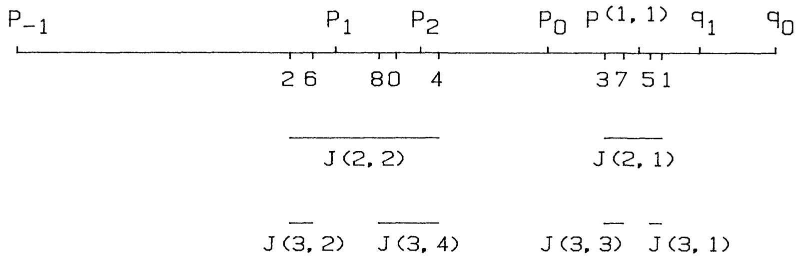

Fig. 1

assumptions of the Theorem. Denote by $c(j)$ the point $f^j(0)$. For each $n>0$, consider the 2^{n-1} intervals $J(n,j)=[c(j), c(2^{n-1}+j)]$. The intervals $J(n,j)$, $0<j<2^{n-1}$, are disjoint. They are permuted by f and each contains exactly one periodic point of period 2^{n-1}. We denote by p_{-1} the fixed point of f with positive slope and by p_n the periodic point of period 2^n closest to the origin. Set $p(n,j) = f^j(p_n)$. As n increases, we have $J(n+1,j) \cup J(n+1, j+2^{n-1}) \subset J(n,j)$ and

$$\Lambda = \bigcap_n \left(\bigcup_j J(n,j)\right).$$

There is a point q_n to the right of $c(1)$ with the property that $f^{2^{n-1}}(q_n) = -p_{n-1}$. Note that $f^{2^n}|(p(n,1), q_n)$ is a diffeomorphism. Figure 1 illustrates these definitions.

Denote the length of an interval J by $\boldsymbol{l}(J)$. To prove the theorem we establish an upper bound $\alpha<1$ on $(\boldsymbol{l}(J(n+1,j)+\boldsymbol{l}(J(n+1,j+2^n))/\boldsymbol{l}(J(n,j))$. We then have $\sum \boldsymbol{l}(J(n,j)) < \alpha^n \boldsymbol{l}(J(1,1))$ which gives an upper bound of $\log(2)/\log(2/\alpha)$ for the Hausdorff dimension of Λ. In proving these estimates we use two basic properties of functions with negative Schwarzian derivative. The general statement of these are formulated as two lemmas that appear in the middle of the proof of the theorem.

Step 1. $|(f^{2^n})'(q_n)|>1$.

This is proved inductively. First $q_0 = -p_{-1}$, so the evenness of f implies $|f'(q_0)| = |f'(p_{-1})|>1$. Assume now that $|(f^{2^n})'(q_n)|>1$. Then the chain rule implies $|(f^{2^{n+1}})'(q_n)| = |(f^{2^n})'(q_n)|\,|(f^{2^n})'(p(n,1)|>1$. Observe that q_{n+1} is contained in the interval $(p(n+1,1), q_{n+1})$ and that $f^{2^{n+1}}$ is monotone on this interval. Since the iterates of f all have negative Schwarzian derivative $|(f^{2^{n+1}})'(q_{n+1})|$ is at least as large as the minimum of $|(f^{2^{n+1}})'(q_n)|$ and $|(f^{2^{n+1}})'(p(n+1,1)|$. Now $|(f^{2^{n+1}})'(p(n+1,1)|>1$ since the periodic orbits are unstable, hence $|(f^{2^{n+1}})'(q_{n+1})|>1$.

Step 2. $p_n/c(2^n)<0.71$ for n sufficiently large.

Choose n sufficiently large that f is well approximated in the interval $[0, p_{n-1}]$ by a function of the form $a-bx^2$. This implies that

$$\boldsymbol{l}(p(n,1), c(1))/\boldsymbol{l}(c(2^n+1), c(1)) \approx (p_n/c(2^n))^2.$$

Now $\boldsymbol{l}(c(2^n+1),\ c(1)) = \boldsymbol{l}(c(2^n+1),\ p(n,1)) + \boldsymbol{l}(p(n,1),\ c(1))$ and $\boldsymbol{l}(p(n,1),\ c(1)) < \boldsymbol{l}(c(2^n+1),\ p(n,1))$ because $|(f^{2^n})'|>1$ on the interval $(p(n,1),\ c(1)) \subset (p(n,1),\ q_n)$.

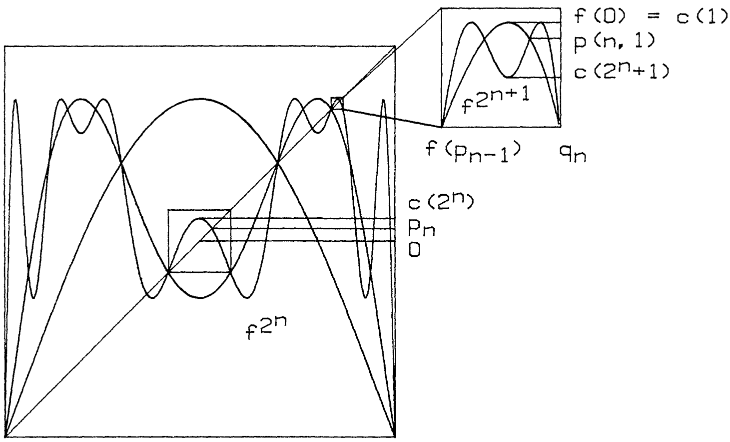

Fig. 2

We conclude that

$$\boldsymbol{l}(p(n,1),c(1))/\boldsymbol{l}(c(2^n+1),c(1))<1/2\,.$$

For $\varepsilon=0.002$, if n is sufficiently large then $|p_n/c(2^n)|<1/\sqrt{2}+\varepsilon<0.71$.

Step 3. $|p_n/c(2^n)|>1/3$.

A lower bound on $|p_n/c(2^n)|$ is also easy to obtain. Consider f^{2^n} on the interval $[p_{n-1},-p_{n-1}]$. The evenness of f, the instability of periodic points and negative Schwarzian derivative imply that f^{2^n} expands on the interval $[p_n,-p_{n-1}]$. Moreover, $c(2^{n+1})\in(0,-p_n)$, implying $\boldsymbol{l}(p_n,c(2^n))<\boldsymbol{l}(c(2^{n+1}),p_n)<2|p_n|$.

Consequently, $|c(2^n)|=|p_n|+\boldsymbol{l}(p_n,c(2^n))<3|p_n|$ and $|p_n/c(2^n)|>1/3$.

Lemma 1. *Let h be a C^3 diffeomorphism of* $[0,1]$ *with $S(h)<0$, $h(0)=0$ and $h(1)=1$. On the interval $h^{-1}(\delta,1-\delta)$ we have $|h''(x)/(h'(x))^2|<2/\delta$. If x and y are in the interval $h^{-1}(\delta,1-\delta)$, then $|h'(x)/h'(y)|<e^{2/\delta}$.*

Assume x satisfies $\delta<h(x)<1-\delta$ and $|h''(x)/(h'(x))^2|>2/\delta$. We fit a hyperbola $\xi(y)$ to h by matching its value and first two derivatives at x: if $h''(x)\neq 0$,

$$\xi(y)=h(x)+(y-x)h'(x)/(1-(y-x)h''(x)/2h'(x))\,.$$

The property $S(h)<0$ is equivalent to the convexity of $(h')^{-1/2}$. Denoting $g=(h')^{-1/2}-(\xi')^{-1/2}$, we have $g''>0$, $g(x)=0$, and $g'(x)=0$. Therefore, $g(y)>0$ and $h'(y)<\xi'(y)$ for all $y\neq x$. Then $\xi(y)>h(y)$ if $y>x$ and $\xi(y)<h(y)$ if $y<x$. If $h''(x)<0$, we estimate

$$h(1)<\xi(1)<h(x)-2(h'(x))^2/h''(x)<h(x)+\delta<1\,,$$

contradicting the properties of h. Similarly, if $h''(x)>0$,

$$h(0)>\xi(0)>h(x)-2(h'(x))^2/h''(x)>h(x)-\delta>0\,,$$

again contradicting properties of h. This proves the first part of the lemma. The second part follows from the usual Denjoy estimate together with a variable change. We have

$$\log|h'(x)/h'(y)| = \int_y^x |h''(\zeta)/h'(\zeta)|\,d\zeta = \int_{h(y)}^{h(x)} |h''(h^{-1}(\eta))/(h'(h^{-1}(\eta)))^2|\,d\eta\,,$$

where $\eta = h(\zeta)$ in the passage between the two integrals.

Lemma 2. *Let h be a C^3 diffeomorphism of $[0,1]$ with $h'(0)>0$ and $S(h)<0$. If g is a fractional linear transformation continuous on $[0,1]$ with $g'(0)=h'(0)$ and $g'(1)=h'(1)$, then $h'(x)>g'(x)$ for all $x\in(0,1)$.*

Again we use the property that $S(h)<0$ is equivalent to the convexity of $(h')^{-1/2}$. Let g be a fractional linear transformation with $g'(0)=h'(0)$ and $g'(1)=h'(1)$. Since $(g')^{-1/2}$ is linear for a fractional linear transformation g, $(h')^{-1/2}(x)<(g')^{-1/2}(x)$ for all $x\in(0,1)$ and the lemma follows.

Step 4. There exists an $\varepsilon>0$ such that $|p_n/p_{n-1}|>\varepsilon$ independently of n.

We know that $|(f^{2^n})'(p_n)|>1$ and $|(f^{2^n})'(-p_{n-1})|>1$. Lemma 1 implies that f^{2^n-1} restricted to $[f(p_n), f(p_{n-1})]$ has uniformly bounded distortion on the inverse image of the interval $L=[p_{n-1}/2, -p_{n-1}/2]$ because $[f(p_n), f(-p_{n-1})]$ is contained in an interval mapped monotonically onto $[p_{n-1}, -p_{n-1}]$ by f^{2^n-1}. If $|p_n/p_{n-1}|<1/4$, then $[-p_{n-1}/4, -p_{n-1}/2]$ is contained in $[p_n, -p_{n-1}/2]$ and we assert that $f^{2^n}([p_n, -p_{n-1}/2])\subset L$. Observe that f^{2^n} is monotone on $[p_n, -p_{n-1}/2]$, so the assertion can be proved by checking that $f^{2^n}(p_n)$, $f^{2^n}(-p_{n-1}/2)\in L$. First, $f^{2^n}(p_n)=p_n$ and $|p_n|<|p_{n-1}/2|$. Second,

$$f^{2^n}(-p_{n-1}/2)=f^{2^n}(p_{n-1}/2)\in(p_{n-1},p_n)\subset(p_{n-1}, -p_{n-1}/2)$$

and

$$|f^{2^n}(p_{n-1}/2)-p_{n-1}|>|p_{n-1}/2-p_{n-1}|=|p_{n-1}/2|\,,$$

implying that $|f^{2^n}(p_{n-1}/2)|<|p_{n-1}/2|$. Now $|f'(x)\cdot y/x\cdot f'(y)|$ is bounded away from zero, so there is a constant $\delta>0$ such that

$$|(f^{2^n})'(x)/(f^{2^n})'(y)| = |(f^{2^n-1})'(f(x))/(f^{2^n-1})'(f(y))|\,|f'(x)/f'(y)|>\delta|x/y|$$

for $x,y\in[p_n, -p_{n-1}/2]$. In particular, $|(f^{2^n})'(x)|>\delta|x/p_n|$, since $|(f^{2^n})'(p_n)|>1$. Assume now $|p_n/p_{n-1}|<\delta/32$. Then $|p_{n-1}/4p_n|>8/\delta$ and $|(f^{2^n})'(x)|>8$ on the interval $K=[-p_{n-1}/4, -p_{n-1}/2]$. But then $\boldsymbol{l}((f^{2^n})(K))>2|p_{n-1}|$, contradicting that the image of K is a subinterval of $[p_{n-1}, -p_{n-1}]$. We conclude that $|p_n/p_{n-1}|>\delta/32$ as was to be proved.

Step 5. There is a constant $\beta>0$ such that

$$\boldsymbol{l}(p_{n-1}, c(2^{n+1}))/\boldsymbol{l}(c(2^{n+1}), c(2^{n+2}))>\beta$$

and

$$\boldsymbol{l}(c(2^{n+2}), p_n)/\boldsymbol{l}(c(2^{n+1}), c(2^{n+2}))>\beta\,.$$

If $p_{n-1}<0$, there is the following ordering of points:

$$p_{n-1}<-p_n<c(2^{n+1})<0<c(2^{n+2})<p_n<c(2^n)<-p_{n-1}.$$

From the ratio estimate of Step 2, we have $|c(2^{n+1})/p_{n-1}|<|p_n/c(2^n)|<0.71$ and similarly $|c(2^{n+2})/p_n|<0.71$. Since the ratios $|p_n/p_{n-1}|$ are bounded away from zero, Steps 2 and 3 imply that $|c(2^{n+1})/c(2^n)|$ is bounded away from 0. This establishes the existence of β. There are gaps of fixed minimum ratio on either side of $J(n+1,2^{n+1})$ in (p_{n-1},p_n).

All of the estimates for the proof of the theorem are now in place. At the nth step of the Cantor construction, we remove from the intervals $[c(k),c(k+2^n)]$, $0<k<2^n$, the intervals $[c(k+2^{n+1}),c(k+3\cdot 2^n)]$. Now $[c(k),c(k+2^n)]$ is contained in an interval which is mapped diffeomorphically by f^{2^n-k} onto $[p_{n-1},p_{n-2}]$ $\supset[c(2^{n+1}),c(2^n)]$. Step 5 and Lemma 1 imply that the f^{2^n-k} have uniformly bounded distortion on the intervals $[c(k),c(k+2^n)]$ independent of n and k. We assert that $\boldsymbol{l}([c(3\cdot 2^n),c(2^{n+2})])/\boldsymbol{l}([c(2^n),c(2^{n+1})])$ is bounded away from zero. This follows from the following observations:

(1) $[c(3\cdot 2^n),c(2^{n+2})]\supset[p_n,c(2^{n+2})]$,

(2) $\boldsymbol{l}([c(2^{n+1}),c(2^{n+2})])/\boldsymbol{l}([c(2^n),c(2^{n+1})])$ is bounded away from zero (Step 4), and

(3) $\boldsymbol{l}([p_n,c(2^{n+2})])/\boldsymbol{l}([c(2^{n+1}),c(2^{n+2})])$ is bounded away from zero (Step 5). There is a $\gamma>0$ such that each step of the Cantor construction removes a minimum proportion γ from each subinterval. These facts together give a minimal ratio of the total length of the intervals in the nth stage of the Cantor set construction that is removed in proceeding to the $(n+1)$st stage. More precisely, $\sum\boldsymbol{l}(J(n+1,j))<\alpha$ $\sum\boldsymbol{l}(J(n,j))$ with $\alpha=1-\gamma$. The theorem is proved.

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