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On the Quotient of the Regularized Determinant of Two Elliptic Operators*

R. E. Gamboa Saraví, M. A. Muschietti and J. E. Solomin

Facultad de Ciencias Exactas, Universidad Nacional de La Plata, C.C. No. 67, 1900 La Plata, Argentina

Abstract. We study the quotient of the regularized determinants of two elliptic operators having the same principal symbol. We prove that, under general conditions, a method recently proposed by Tamura coincides with the ζ-function approach.

1. Introduction

In the computation of quadratic path-integrals one is led to the evaluation of determinants of elliptic operators. Nevertheless, these determinants diverge. Hence, it is necessary to adopt some regularization procedure.

Often, it is the quotient of the determinants of two operators D_0 and D_1 that is searched. In this case, one can attempt to connect them by means of a differentiable one-parameter family D_t of operators, and if the determinant regularized according to some prescription results in a differentiable function of the parameter t, the quotient can be computed as the exponential of an integral, i.e.

$$\frac{\det(D_1)}{\det(D_0)}=\exp\int_0^1\frac{d}{dt}\log(\det(D_t))\,dt\,. \tag{1.1}$$

One regularizing prescription that can be used in this approach is the well-known ζ-function method [1], since it has the required differentiability [2, 3].

Recently, Tamura [4] proposed an alternative method to regularize the determinant of the radio of two Dirac operators D_0 and D_1 based on Fujikawa's results [5]. In this approach, it is not the value of the determinant of each operator that is given, indeed it is the change of the logarithm of the determinant that is regularized. In order to do it, he defines the M depending function

$$D_M(D_1;D_0)=\exp\left[\operatorname{Tr}\int_0^1\frac{dD_t}{dt}D_t^{-1}\exp(-D_t^2/M^2)\,dt\right], \tag{1.2}$$

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where M is an arbitrary positive constant. In particular, when the Fredholm determinant of $D_1D_0^{-1}$ exists this function has a limit for $M\to\infty$ and it coincides with $\det(D_1D_0^{-1})$. This limit can exist even in more general cases, and then its value can be taken as the regularized determinant of $D_1D_0^{-1}$.

We shall consider Tamura's function (1.2), for any couple of invertible elliptic pseudodifferential operators D_0 and D_1 sharing the same principal symbol such that the spectrum of its square lies in the right semiplane. We shall establish the relationship between it and the ratio $\mathrm{Det}(D_1)/\mathrm{Det}(D_0)$, where Det is the determinant defined through the ζ-function. More precisely, we are going to show that the ratio of the ζ-function determinants can be written as a simple expression involving only two coefficients of the asymptotic expansion of (1.2), and we shall see that it coincides with $\lim_{M\to\infty} D_M(D_1;D_0)$ when this limit exists.

In the proof of this relation the Mellin transform plays a central role. In order to use it we introduce a complex parameter a to avoid the possibility of having eigenvalues in the left semiplane.

In Sect. 2, we give the definitions and settle the notations we shall use. In Sect. 3, we introduce a generalized Tamura's function depending on the complex parameter a and we analyze its asymptotic expansion for large M. In Sect. 4, we compute the ratio of the determinants regularized by means of the ζ-function method, and finally we establish its relation with Tamura's approach.

2. Definitions and Notations

Given an elliptic invertible pseudodifferential operator D, of order $m>0$ with a cone of Agmon directions for its principal symbol, defined on a compact manifold M without boundary of dimension n, following Seeley [6], we define

$$D^s=\frac{i}{2\pi}\int_\Gamma \lambda^s(D-\lambda)^{-1}\,d\lambda\,, \tag{2.1}$$

where Γ is a curve beginning at ∞, passing along a ray that is an Agmon direction, which eludes the eigenvalues of D, to a small circle about the origin, then clockwise about the circle, and back to ∞ along the ray.

We shall denote by $K(B;x,y)$ the kernel of an operator B.

The kernel $K(D^s;x,y)$ is a continuous function, even at $x=y$, for $\mathrm{Re}\,s<-n/m$. The function $K(D^s;x,x)$ admits a meromorphic extension to the whole s-complex plane, and in particular it is regular at the origin [6].

The generalized ζ-function associated to D is then defined as

$$\zeta(D;s)=\int_M \mathrm{tr}\,K(D^{-s};x,x)\,dx\,, \tag{2.2}$$

and the corresponding regularized determinant is [1]

$$\mathrm{Det}(D)=\exp\left\{-\frac{d}{ds}\zeta(D;s)\right\}_{s=0}. \tag{2.3}$$

We shall denote by $\|B\|_1$ the trace norm of a trace class operator B, by $\|A\|_{r,\ell}$, the norm of an operator A acting from the Sobolev space $H^r(M)$ to $H^\ell(M)$, by $\mathscr{L}(H)$ the space of bounded linear operators from H into itself, and by $\mathscr{S}_1(H)$ its subspace of trace class operators.

We shall consider the class $\mathscr{D}_{\sigma_m}$ of all the invertible pseudodifferential operators on M having $\sigma_m(x,\xi)$ as the principal symbol. Along the work we shall restrict ourselves to elliptic $\sigma_m(x,\xi)$ having no eigenvalues in the cone $\pi/4-\varepsilon<|\arg\lambda|<3\pi/4+\varepsilon$ (i.e. each ray of the cone is an Agmon direction of σ_m).

3. A Generalized Tamura's Function

We shall define a generalization of Tamura's function $D_M(D_1;D_0)$ [4] in order to establish its relation with the ζ-function procedure.

For $D_0, D_1\in\mathscr{D}_{\sigma_m}$, $u>0$ and $a\in\mathbb{C}$, we take

$$D_u(D_t;a)=\exp\left[\operatorname{Tr}\int_0^1 D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]\,dt\right], \tag{3.1}$$

where D_t is a continuous and piecewise differentiable map from $[0,1]$ to $\mathscr{D}_{\sigma_m}$ which connects D_0 and D_1, such that (D_t^2+a) is invertible for every t, and $D_t'=\frac{d}{dt}D_t$. Note that for $u=1/M^2$ and $a=0$ (3.1) becomes Tamura's original function, $D_M(D_1;D_0)$, which is independent of the choice of D_t.

The function defined in (3.1) has the following properties:

Proposition 3.1. *For every $u>0$, $[D_u(D_t;a)]^2$ does not depend on the choice of D_t and it is an analytic function of the variable a in the region $\Omega=\mathbb{C}-\{\mathrm{sp}(-D_0^2)\cup\mathrm{sp}(-D_1^2)\}$.*

Proof. Let us write

$$L_u(D_t;a)=\int_0^1 D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]\,dt\,. \tag{3.2}$$

For any other continuous and piecewise differentiable map $\tilde{D}_t$ connecting D_0 and D_1 we have

$$\operatorname{Tr}L_u(\tilde{D}_t;a)=\operatorname{Tr}L_u(D_t;a)+k\pi i,\qquad k\in\mathbb{Z}, \tag{3.3}$$

so, $[D_u(D_t;a)]^2$ does not depend on the choice of the map D_t. Since the proof of the assertion (3.3) follows with slight modifications the scheme of that of Theorem 1 in [4] we postpone it to the appendix.

Now, given $a_0\in\Omega$, we can take a neighborhood $\mathscr{U}$ of a_0 in Ω and D_t such that

$$\mathscr{U}\cap\bigcup_{t\in[0,1]}\mathrm{sp}(-D_t^2)=\emptyset\,.$$

In fact, arguing again as in [4], we can show that there exists D_t such that

$$a_0\notin\bigcup_{t\in[0,1]}\mathrm{sp}(-D_t^2)\,.$$

Let us call $d(t)$ the distance from a_0 to $\mathrm{Sp}(-D_t^2)$. We can take $C>0$ such that $\|(-D_{t_0}^2-\lambda)^{-1}\|_{0,2m}\leqq C$, for $|\lambda-a_0|<\frac{d(t_0)}{2}$ [6], and $\delta_{t_0}>0$ such that

$$\|-D_{t_0}^2-(-D_t^2)\|_{2m,0}<\frac{1}{C}, \quad \text{for} \quad |t-t_0|<\delta_{t_0}.$$

Now, since

$$(-D_t^2-\lambda)^{-1}=(-D_{t_0}^2-\lambda)^{-1}[I+(D_{t_0}^2-D_t^2)(-D_{t_0}^2-\lambda)^{-1}]^{-1},$$

$\lambda\notin\mathrm{sp}(-D_t^2)$ for $|t-t_0|<\delta_{t_0}$. Then, $d(t)$ is greater than a positive constant for t belonging to the compact set $[0,1]$ and can choose a neighborhood $\mathcal{U}$ in Ω satisfying the required property.

Because of the differentiability of D_t, for every a in $\mathcal{U}$, the trace norm of $D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]$ is uniformly bounded for $t\in[0,1]$. So, (3.1) can be written as

$$D_u(D_t;a)=\exp\left\{\int_0^1 \mathrm{Tr}[D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]]\,dt\right\}. \tag{3.4}$$

On the other hand, as a function of the variable a, $a\in\mathcal{U}$, the map $(D_t^2+a)^{-1}$ is uniformly analytic in the norm $\|\ \|_{0,0}$ for $t\in[0,1]$ (i.e., the remainder of its Taylor expansion tends uniformly to zero for $t\in[0,1]$).

Now, since

$$\begin{aligned}&\mathrm{Tr}[D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]]\\&\quad=e^{-ua}\,\mathrm{Tr}[D_t'D_t(D_t^2+a)^{-1}\exp[-uD_t^2]],\end{aligned}$$

it follows that, for each $u>0$, $\mathrm{Tr}\,L_u(D_t;a)$ is analytic for $a\in\mathcal{U}$, and then $D_u(D_t;a)$ is analytic in $\mathcal{U}$. Hence, so is $[D_u(D_t;a)]^2$ in Ω.

Definition. From Proposition 3.1, we can define the analytic function $D_u^2(D_1;D_0;a)$ as $D_u^2(D_1;D_0;a)=[D_u(D_t;a)]^2$.

In order to analyze the asymptotic behavior of $D_u^2(D_1;D_0;a)$, we need the following expansion:

Proposition 3.2. *For D_t as above, and a in a compact subset of $\mathbb{C}-\bigcup_{t\in[0,1]}\mathrm{Sp}(-D_t^2)$, we have*

a)

$$\begin{aligned}\mathrm{Tr}[D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]]&=C_1(a,t)u^{\frac{-n+1}{2m}}\\&+C_2(a,t)u^{\frac{-n+2}{2m}}+\ldots+C_n(a,t)+\tilde{C}(a,t)\log u+r(a,t,u),\end{aligned} \tag{3.5}$$

with $|r(a,t,u)|\leqq Cu^{\varepsilon}$, $\varepsilon>0$, for u smaller than a positive constant u_0.

b) *If $\mathrm{sp}(D_t^2+a)\subset\{\mathrm{Re}\lambda>\varepsilon_0\}$, then for $0<\varepsilon<\varepsilon_0$,*

$$\begin{gathered}|\mathrm{Tr}(D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)])|\leqq Ce^{-\varepsilon u}\\ \textit{for every } u\geqq 1,\ 0\leqq t\leqq 1.\end{gathered} \tag{3.6}$$

Proof. a) Since

$$\exp[-u(D_t^2+a)]=e^{-uD_t^2}e^{-ua},$$

we only need to prove part a) for $\mathrm{Tr}[D_t'D_t(D_t^2+a)^{-1}\exp(-uD_t^2)]$. Now this trace is given by

$$\int_M\int_M \mathrm{tr}[K_{t,a}(x,y)\,H_t(u;y,x)]\,dy\,dx; \tag{3.7}$$

with

$$K_{t,a}(x,y)=K(D_t'D_t(D_t^2+a)^{-1};x,y),$$
$$H_t(u;x,y)=K(e^{-uD_t^2};x,y).$$

The principal symbol of D_t does not depend on t, so the order of the operator $D_t'D_t(D_t^2+a)^{-1}$ is not greater than -1; hence its kernel admits the following expansion for x,y in a coordinate neighborhood of M [7],

$$\begin{aligned}K_{t,a}(x,y)=&\sum_{j=-n+1}^{1}\{h_j(x,x-y;a,t)+P_j(x,x-y;a,t)\log|x-y|\}\\&+r(x,x-y;a,t),\end{aligned} \tag{3.8}$$

where $h_j(x,z;a,t)$ is homogeneous in z of the degree j, $P_j(x,z;a,t)$ is a homogeneous polynomial in z of degree j, and, for x in a compact set, $0\leqq t\leqq 1$, a in a compact subset of Ω,

$$|h_j(x,z;a,t)|\leqq C,\quad \text{for}\quad |z|=1$$
$$|P_j(x,z;a,t)|\leqq C,\quad \text{for}\quad |z|=1$$
$$|\partial_x^\alpha\,\partial_z^\beta r(x,z;a,t)|\leqq C_{\alpha,\beta},\quad \text{for}\quad |z|\leqq C_1,\ |\beta|\leqq 1.$$

These inequalities are a consequence of the piecewise continuity of the symbols of D_t^2+a, D_t, D_t' with respect to both parameters t and a.

On the other hand, the kernel $e^{-uD_t^2}$ admits, for x in a coordinate neighborhood V of M, the expansion

$$H_t(u;y,x)=\sum_{k=0}^{n}u^{\frac{-n+k}{2m}}H_k\left(y,\frac{y-x}{u^{1/2m}};t\right)\psi(y)+R(u;y,x;t), \tag{3.9}$$

where the functions $H_k(y,z;t)$ are rapidly decreasing in z, $\operatorname{supp}\psi\subset V$, and $|R(u;y,x;t)|\leqq Cu^\varepsilon$, $\varepsilon>0$, for $u\leqq 1$ and $0\leqq t\leqq 1$, [8].

By means of a suitable choice of a partition of the unity, the integral (3.7) becomes a finite sum of integrals such that in some of them the variable x belongs to the same coordinate neighborhood as the variable y and, in the others, $|x-y|$ is greater than a positive constant. If $|x-y|>C_0$, we have $|H_t(u;y,x)|\leqq Cu$ for $0\leqq t\leqq 1$, $u\leqq 1$ and $|K_{t,a}(x,y)|\leqq C$ for $0\leqq t\leqq 1$ and a in a compact set. Then the corresponding integral is not greater than Cu.

In the other case, for any coordinate neighborhood V, by using expansions (3.8) and (3.9) we can write the restriction of the integral (3.7) to $V\times V$ as the tr of

$$\sum_{\substack{j=-n+1,\dots,1\\ k=0,\dots,n}} u^{\frac{j+k}{2m}} \int h_j(x,z;a,t)\, H_k(x+u^{1/2m}z,x;t)\ \psi(x+u^{1/2m}z)\,dz\,dx$$
$$+\sum_{j=0}^{1}\sum_{k=-n+1}^{1} u^{\frac{j+k}{2m}} \int P_j(x,z;a,t)\log|u^{1/2m}z|\, H_k(x+u^{1/2m}z,x;t)$$
$$\times\psi(x+u^{1/2m}z)\,dz\,dx$$
$$+\sum_{j=-n+1}^{1}\int (h_j(x,x-y;a,t)+P_j(x,x-y;a,t)\log|x-y|)\,R(u;y,x;t)\,dy\,dx$$
$$+u^{\frac{n}{2m}}\int r(x,-u^{1/2m}z;a,t)\,H_t(u;u^{1/2m}z,x)\ \psi(x+u^{1/2m}z)\,dz\,dx\,,$$

[we have set $z=(y-x)/u^{1/2m}$ in some terms].

Expanding the functions H_k and ψ in powers of $u^{1/2m}z$ and rearranging them according to the powers of u, we get a).

b) We have

$$\begin{aligned}&|\mathrm{Tr}\{D_t'D_t(D_t^2+a)^{-1}\exp[-u(D_t^2+a)]\}|\\ &\leqq \|D_t'D_t(D_t^2+a)^{-1}\|_{0,0}\,\|\exp[-(D_t^2+a)]\|_1\\ &\quad\times\|\exp[-(u-1)(D_t^2+a)]\|_{0,0}\,.\end{aligned}$$

The first two factors are independent of u, while the last one satisfies

$$\|\exp[-(u-1)(D_t^2+a)]\|_{0,0}\leqq Ce^{-(u-1)\varepsilon'}$$

for $0<\varepsilon'<\varepsilon_0$, $0\leqq t\leqq 1$, and a in a compact set. This can be shown by writing the exponential of the operator as an integral over a path in the λ-plane, and using the estimate $\|(D_t^2+a-\lambda)^{-1}\|_{0,0}\leqq C(1+|\lambda|)^{-1}$, for $|\arg\lambda|<\pi/2-\varepsilon$ [6].

Corollary 3.3. *For an open set* $\mathscr{U}\subset\Omega$ *and* D_t *such that* $\mathscr{U}\cap\bigcup_{t\in[0,1]}\mathrm{sp}(-D_t^2)=\emptyset$, *and* L_u *defined in* (3.2),

$$\mathrm{Tr}\,L_u(D_t;a)=\mathfrak{C}_1(a)u^{\frac{-n+1}{2m}}+\ldots+\mathfrak{C}_n(a)+\tilde{\mathfrak{C}}(a)\log u+R(a,u)\,, \qquad (3.10)$$

with $|R(a,u)|\leqq Cu^{\varepsilon}$, $\varepsilon>0$, *for a in a compact subset of* $\mathscr{U}$ *and u smaller than a positive constant* u_0.

Proof. It follows immediately integrating both terms of (3.5) with respect to t from 0 to 1.

Corollary 3.4. *Under the same conditions of Corollary 3.3 the coefficients* $\mathfrak{C}_j(a)$, $j=1,\ldots,n$, $\tilde{\mathfrak{C}}(a)$ *are analytic in* $\mathscr{U}$.

Proof. It is a direct consequence of Corollary 3.3 and of the analyticity of $\mathrm{Tr}\,L_u(D_t;a)$ shown in the proof of Proposition 3.1.

In fact, since $\mathrm{Tr}\,L_u(D_t;a)-R(a,u)$ is bounded in a for each $u<u_0$ and a in a compact subset of $\mathscr{U}$, and the functions $u^{(-n+1)/2m},\ldots,1$, $\ln u$ are linearly independent, we can express each coefficient $\mathfrak{C}_j(a)$ and $\tilde{\mathfrak{C}}(a)$ as a linear combination of the

functions $\operatorname{Tr} L_{u_k}(D_t; a) - R(a, u_k)$, with u_k $n+2$ different points, so the coefficients are bounded for a in a compact subset of $\mathscr{U}$.

Now, multiplying both sides of (3.10) by suitable positive powers of u [$(\ln u)^{-1}$ in the last step] and taking $u \to 0$, we see that each coefficient is the limit of analytic functions, uniformly on compact subsets.

Note that this corollary could also be obtained by means of a detailed analysis of the asymptotic expansion in the proof of Proposition 3.1.

Remark. From the above results it follows that we can define, for each positive u, and a in Ω,

$$\operatorname{Tr} L_u(a) = \operatorname{Tr} L_u(D_t; a)$$

for some D_t such that $a \notin \bigcup_{t \in [0,1]} \operatorname{sp}(-D_t^2)$, and it is an analytic function from Ω to $\mathbb{C}/\pi i\mathbb{Z}$. Analogously, the multivalued function $\mathbb{C}_n(a)$ gives rise to an analytic function from Ω to $\mathbb{C}/\pi i\mathbb{Z}$. Note that the other coefficients of the expansion (3.10) are analytic from Ω to $\mathbb{C}$.

Now, from the definition of $L_u(D_t; a)$, (3.2) and the expansion (3.10) we can write

$$D_u^2(D_1; D_0; a) = \exp 2\left\{ \sum_{j=1}^{n} \mathbb{C}_j(a) u^{\frac{-n+j}{2m}} + \tilde{\mathbb{C}}(a) \log u + O(u^{\varepsilon}) \right\}. \tag{3.11}$$

Note that $D_u^2(D_1; D_0; a)$ is monovalued in spite of the multivaluation of $\mathbb{C}_n(a)$.

Taking into account that $D_{1/M^2}(D_t; 0) = D_M(D_1; D_0)$ we see that if Tamura's function has a limit when M goes to infinite, and it is non-trivial, all coefficients but $\mathbb{C}_n$ vanish at $a=0$. In this case, one has

$$\lim_{M \to \infty} D_M(D_1; D_0) = \exp\{\mathbb{C}_n(0)\}. \tag{3.12}$$

4. The Relation with the ζ-Function Regularization Method

In this section, we shall establish the relation between Tamura's prescription and the ζ-function regularization method [1].

We recall that given an elliptic invertible operator D of order $m>0$ on a compact manifold M without boundary of dimension n, we can take as its regularized determinant

$$\operatorname{Det}(D) = \exp\left\{ -\frac{d}{ds}\zeta(D; s) \right\}_{s=0}, \tag{4.1}$$

where $\zeta(D; s)$ is the analytic extension of the trace of the complex power D^{-s} [6]. This approach can be also extended to non-invertible operators [9].

Let us consider two operators $D_0, D_1 \in \mathscr{D}_{\sigma_m}$. For $a \in \Omega = \mathbb{C} - \{\operatorname{sp}(-D_0^2) \cup \operatorname{sp}(-D_1^2)\}$, we can construct the determinants $\operatorname{Det}(D_0^2 + a)$ and $\operatorname{Det}(D_1^2 + a)$. We have the following property:

Lemma 4.1. *The ratio*

$$\frac{\operatorname{Det}(D_1^2 + a)}{\operatorname{Det}(D_0^2 + a)} \tag{4.2}$$

is an analytic function of a in the region Ω.

Proof. It is a direct consequence of the analyticity of Det( ) established in [3].

Note that, as a changes, one could need different curves Γ's in order to define the powers $(D^2+a)^s$ according to (2.1). This could give rise to an ambiguity in the definition of the associated ζ-functions, since the determination of the argument of a finite number of eigenvalues could change. In spite of this fact, the determinant is unambiguously defined since the derivates at $s=0$ of the corresponding ζ-functions differ in a multiple of $2\pi i$.

In order to compute the ratio (4.2) we shall take a map D_t connecting D_0 and D_1 as in Sect. 3. One of us, M.A.M., has proved that if the map D_t is analytic, so is $\mathrm{Det}(D_t^2+a)$ [3]. Similar arguments allow us to show that $\mathrm{Det}(D_t^2+a)$ is continuous and piecewise differentiable when D_t is. Thus, we can move in a continuous and piecewise differentiable way from $\mathrm{Det}(D_0^2+a)$ to $\mathrm{Det}(D_1^2+a)$ by means of $\mathrm{Det}(D_t^2+a)$.

Now, we are going to give a suitable expression of the ratio (4.2) in order to establish its relation with Tamura's prescription.

Proposition 4.2. *For* $D_0, D_1 \in \mathscr{D}_{\sigma_m}$, $a \in \Omega$ *with* Rea *large enough,*

$$\frac{\mathrm{Det}(D_1^2+a)}{\mathrm{Det}(D_0^2+a)} = \exp 2\{\mathfrak{C}_n(a) + \Gamma'(1)\tilde{\mathfrak{C}}(a)\}, \tag{4.3}$$

where $\mathfrak{C}_n(a)$ *and* $\tilde{\mathfrak{C}}(a)$ *are the coefficients appearing in* (3.10).

Proof. Let D_t be a map connecting D_0 and D_1 as in Sect. 3, and a such that $\mathrm{Re}(\lambda^2+a) > \varepsilon_0 > 0$ for every eigenvalue of D_t, $0 \leqq t \leqq 1$. Firstly, we shall prove

$$\frac{\mathrm{Det}(D_1^2+a)}{\mathrm{Det}(D_0^2+a)} = \exp 2 \int_0^1 \frac{d}{ds} \left\{ -\frac{s}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[D_t' D_t (D_t^2+a)^{-1} \times \exp[-u(D_t^2+a)]] u^{s-1}\, du \right\}_{s=0} dt. \tag{4.4}$$

We have

$$\frac{\mathrm{Det}(D_1^2+a)}{\mathrm{Det}(D_0^2+a)} = \prod_{j=1}^{p} \frac{\mathrm{Det}(D_{t_j}^2+a)}{\mathrm{Det}(D_{t_{j-1}}^2+a)} = \exp\left\{ -\sum_{j=1}^{p} \int_{t_{j-1}}^{t_j} \frac{d}{dt} \zeta'(D_t^2+a;0)\, dt \right\}, \tag{4.5}$$

for $0 = t_0 < t_1 < \ldots < t_p = 1$ such that the restriction of D_t to (t_{j-1}, t_j) is differentiable.

But, according to [2] we have

$$\frac{d}{dt}\zeta'(D_t^2+a;0) = \frac{d}{ds}\left\{ -s\,\mathrm{Tr}\left[(D_t^2+a)^{-s-1} \frac{d}{dt}(D_t^2+a) \right] \right\}_{s=0}. \tag{4.6}$$

By using Mellin's transformation,

$$(D_t^2+a)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-u(D_t^2+a)} u^{s-1}\, du.$$

We can write (4.6) as

$$\frac{d}{dt}\zeta'(D_t^2+a;0)=\frac{d}{ds}\left[-\frac{s}{\Gamma(s)}\operatorname{Tr}\int_0^\infty e^{-u(D_t^2+a)}\times(D_t^2+a)^{-1}(D_t'D_t+D_tD_t')u^{s-1}\,du\right]_{s=0}. \tag{4.7}$$

Now, since $T(t,a)=(D_t^2+a)^{-1}(D_tD_t'+D_t'D_t)$ is bounded in $H^{-N}(M)$ and

$$\int_0^\infty \|e^{-u(D_t^2+a)}\|_{-N,N}\,u^{\mathrm{Re}(s)-1}\,du<\infty,$$

for $N=\left[\frac{n}{2}\right]+1$, $\mathrm{Re}(s)>0$, we have

$$K\left(\int_0^\infty e^{-u(D_t^2+a)}\,T(t,a)u^{s-1}\,du;x,y\right)=\int_0^\infty K(e^{-u(D_t^2+a)}\,T(t,a)u^{s-1};x,y)\,du.$$

So, we can commute Tr and the integration in (4.7) getting (4.4).

Now, defining

$$g(t,u,a)=\operatorname{Tr}(D_t'D_t(D_t^2+a)^{-1}e^{-u(D_t^2+a)}),$$

from Proposition 3.2a) we have

$$\int_0^1 g(t,u,a)u^{s-1}\,du=\sum_{j=1}^n C_j(t,a)\frac{1}{s-\frac{n-j}{2m}}-\tilde{C}(t,a)\frac{1}{s^2}+\varphi(t,a,s)$$

with $\varphi(t,a,s)$ analytic at $s=0$. Analogously, from Proposition 3.2b), we have that $\int_1^\infty g(t,u,a)u^{s-1}\,du$ is an entire function of s.

Then,

$$\frac{s}{\Gamma(s)}\int_0^\infty g(t,u,a)u^{s-1}\,du=-\tilde{C}(t,a)\frac{1}{\Gamma(s+1)}+C_n(t,a)\frac{s}{\Gamma(s+1)}+\frac{s^2}{\Gamma(s+1)}\psi(t,a,s),$$

with $\psi(t,a,s)$ analytic in a neighborhood of $s=0$.

Hence,

$$\frac{d}{ds}\left[-\frac{s}{\Gamma(s)}\int_0^\infty g(t,u,a)u^{s-1}\,du\right]_{s=0}=C_n(t,a)+\Gamma'(1)\,\tilde{C}(t,a).$$

Finally, carrying out the integration with respect to t from 0 to 1, we get the proposition.

We are now in condition to establish the relation between Tamura's method and the ζ-function one.

Theorem 4.3. *If there exists a non-trivial limit for Tamura's function* (1.2) *then*

$$\lim_{M\to\infty} D_M(D_1;D_0)^2 = \mathrm{Det}(D_1^2)/\mathrm{Det}(D_0^2)\,. \tag{4.8}$$

Proof. By Lemma 4.1 the left-hand side of (4.3) is analytic for a in the domain $\Omega=\mathbb{C}-\{\mathrm{sp}(-D_0^2)\cup\mathrm{sp}(-D_1^2)\}$, and by Corollary 3.4, so is the right-hand side. Then, the equality (4.3) holds in the whole Ω. In particular, at $a=0$ it reads

$$\frac{\mathrm{Det}(D_1^2)}{\mathrm{Det}(D_0^2)} = \exp 2\{\mathbb{C}_n(0)+\Gamma'(1)\,\tilde{\mathbb{C}}(0)\}\,. \tag{4.9}$$

On the other hand, when the limit exists when M goes to ∞, and it is non-trivial, considering that $\tilde{\mathbb{C}}(0)$ must vanish, from (4.9) and (3.12) we get (4.8).

5. Concluding Remarks

We have discussed the relation between the ζ-function and Tamura's methods. We have seen that for elliptic invertible pseudodifferential operators defined on compact manifolds both approaches coincide provided Tamura's function limit exists. Since the former is always meaningful for this class of operators, it has a wider range of application. Another difference is that the ζ-function prescription provides a value not only to the quotient of determinants as Tamura's does, but also to the determinant of each operator itself.

On the other hand, the ζ-function method is so far restricted to pseudodifferential operators, while Tamura's could be used in other cases. In fact, it can, in principle, be applied to the sum of a pseudodifferential operator and a bounded one, even acting on non-compact manifolds.

It is worthwhile mentioning that when facing explicit computations by using the ζ-function method from (1.1) and [2] one has

$$\frac{\mathrm{Det}(D_1)}{\mathrm{Det}(D_0)} = \exp\int_0^1 \frac{d}{ds}[s\,\mathrm{Tr}(D_t^{-s-1}D_t')]_{s=0}\,dt\,.$$

If D_t' is a multiplicative operator, it becomes

$$\frac{\mathrm{Det}(D_1)}{\mathrm{Det}(D_0)} = \exp\int_0^1\int_M \frac{d}{ds}\{s\,\mathrm{tr}[K_{-s-1}(D_t;x,x)D_t']\}_{s=0}\,dx\,dt\,, \tag{5.1}$$

where $K_s(D;x,x)=K(D^s;x,x)$. Thus, one is, a priori, led to the evaluation of the kernel K_{-1}, which cannot, in general, be written as a function of Seeley's coefficients exclusively [6]. Nevertheless, sometimes [2, 10], owing to the particular D_t' involved, one can write the trace in (5.1) in terms of the kernel $K_0(D_t;x,x)$ which is always obtained from only one Seeley's coefficient. In this case the coefficient $\tilde{\mathbb{C}}(a)$ in the expansion (3.11) of Tamura's function should vanish, as it occurs in the example of [4].

On the other hand, when the Green function of D_t is available the kernel $K_{-1}(D_t;x,x)$ can be explicitly written in terms of it and some Seeley's coefficient [11].

Appendix

In this appendix, we present the proof of the assertion (3.3). We assume throughout it the same hypothesis of Sect. 3.

Lemma A.1. *Let $E_t=[I+A(t)]E$ with $A(t)$ continuous from $[0,1]$ to $\mathscr{L}(H^0)$, $A'(t)$ piecewise continuous from $[0,1]$ to $\mathscr{S}_1(H)$ and E^{-1} belonging to $\mathscr{L}(H^0)$.*

Then, for $\lambda \notin \bigcup_{t\in[0,1]} \mathrm{sp}(E_t)$, $\int_0^1 E_t'(\lambda-E_t)^{-1}\,dt$ is a trace class operator, its trace is continuous in λ and can differ only by $2k\pi i$, with $k\in\mathbb{Z}$, for different $A(t)$ with fixed $A(0)$ and $A(1)$.

Proof. In order to compute the trace we can write

$$E_t'(\lambda-E_t)^{-1}=A'(t)\,E[\lambda-(I+A(t))E]^{-1}=K_\lambda'(t)\,[I+K_\lambda(t)]^{-1}$$

with $K_\lambda(t)=A(t)-\lambda E^{-1}$.

Then, arguing as in [4] the lemma follows from the equality,

$$\begin{aligned}\mathrm{Tr}\int_0^1 E_t'(\lambda-E_t)^{-1}\,dt&=\mathrm{Tr}\int_0^1 K_\lambda'(t)\,(I+K_\lambda(t))^{-1}\,dt\\&=\ln\det[(I+K_\lambda(1))\,(I+K_\lambda(0))^{-1}],\end{aligned}$$

that is obtained from the properties of Fredholm's determinants.

Lemma A.2. *Let D_t and $\tilde{D}_t$ be as in Proposition 3.1, and $a\notin \bigcup_{t\in[0,1]}\{\mathrm{sp}(-D_t^2)\cup\mathrm{sp}(-\tilde{D}_t^2)\}$. If $D_t^2+a=[I+A(t)]D_0^2$ and $\tilde{D}_t^2+a=[I+\tilde{A}(t)]D_0^2$ with $A'(t)$ and $\tilde{A}'(t)$ trace class operators, thus, for $u>0$,*

$$\mathrm{Tr}(L_u(D_t;a))-\mathrm{Tr}(L_u(\tilde{D}_t;a))\quad \textit{is independent of } u\,.$$

Proof. Let Γ be a contour enclosing $\bigcup_{t\in[0,1]}(\mathrm{sp}(D_t^2+a)\cup\mathrm{sp}(D_t^2+a))$. Note that such a contour exists because of the properties assumed for D_t and $\tilde{D}_t$ [6].

Now, using Cauchy's formula, we can write

$$\mathrm{Tr}(L_u(D_t;a))=\frac{1}{2}\mathrm{Tr}\left[\int_0^1\frac{1}{2\pi i}\int_\Gamma e^{-u\lambda}\lambda^{-1}\frac{d}{dt}(D_t^2+a)\,[\lambda-(D_t^2+a)]^{-1}\,d\lambda\,dt\right],$$

where the integrals in the right-hand side are norm convergent in $\mathscr{L}(H^0)$. Moreover,

$$\begin{aligned}&\left\|\frac{d}{dt}(D_t^2+a)\,[\lambda-(D_t^2+a)]^{-1}\right\|_1\\&\quad\leqq\|A'(t)\|_1\,\|D_0^2\|_{2m,0}\,\|[\lambda-(D_t^2+a)]^{-1}\|_{0,2m}\leqq c\,.\end{aligned}$$

So, we have

$$\begin{aligned}\mathrm{Tr}(L_u(D_t;a))-\mathrm{Tr}(L_u(\tilde{D}_t;a))=\frac{1}{4\pi i}\int_\Gamma e^{-u\lambda}\lambda^{-1}\,\mathrm{Tr}\left\{\int_0^1\left[\frac{d}{dt}(D_t^2+a)\,(\lambda-(D_t^2+a))^{-1}\right.\right.\\\left.\left.-\frac{d}{dt}(\tilde{D}_t^2+a)\,(\lambda-(\tilde{D}_t^2+a))^{-1}\right]dt\right\}d\lambda\,.\end{aligned}$$

Then, by Lemma A.1

$$\mathrm{Tr}(L_u(D_t;a))-\mathrm{Tr}(L_u(\tilde{D}_t;a))=\frac{1}{4\pi i}\int_\Gamma e^{-u\lambda}\lambda^{-1}2\pi i\ell(\lambda)\,d\lambda,$$

with $\ell(\lambda)$ locally constant, for it is continuous and takes integer values.

Lemma A.3. *For D_t as in Lemma A.2,*

$$\lim_{u\to 0^+}\left\{\mathrm{Tr}\left[\int_0^1\frac{d}{dt}(D_t^2+a)(D_t^2+a)^{-1}e^{-u(D_t^2+a)}\,dt\right]\right\}$$
$$=\mathrm{Tr}\left[\int_0^1\frac{d}{dt}(D_t^2+a)(D_t^2+a)^{-1}\,dt\right]. \qquad \text{(A.1)}$$

Proof. By Grümm's lemma (see for instance [4]),

$$\lim_{u\to 0^+}\mathrm{Tr}\left[\frac{d}{dt}(D_t^2+a)(D_t^2+a)^{-1}e^{-u(D_t^2+a)}\right]=\mathrm{Tr}\left[\frac{d}{dt}(D_t^2+a)(D_t^2+a)^{-1}\right]$$

and, since $\left\|\frac{d}{dt}(D_t^2+a)(D_t^2+a)^{-1}e^{-u(D_t^2+a)}\right\|_1$, is uniformly bounded for $t\in[0,1]$, by Lebesgue's dominated convergence theorem, we get (A.1) after interchanging Tr and the integration.

Theorem A.4. *Under the same hypothesis of Proposition 3.1,*

$$\mathrm{Tr}\,L_u(\tilde{D}_t;a)=\mathrm{Tr}\,L_u(D_t;a)+k\pi i, \quad \textit{with} \quad k\in\mathbb{Z}. \qquad \text{(A.2)}$$

Proof. First, let us suppose D_t and $\tilde{D}_t$ as in Lemma A.2. By Lemma A.3,

$$\lim_{u\to 0^+}\{\mathrm{Tr}[2L_u(D_t;a)]-\mathrm{Tr}[2L_u(\tilde{D}_t;a)]\}=\mathrm{Tr}\left[\int_0^1\frac{d}{dt}(D_t^2+a)(D_t^2+a)^{-1}\,dt\right]$$
$$-\mathrm{Tr}\left[\int_0^1\frac{d}{dt}(\tilde{D}_t^2+a)(\tilde{D}_t^2+a)^{-1}\,dt\right].$$

Then, we get (A.2) from Lemmas A.1 and A.2.

In the general case, if we write

$$D_t^2+a=(I+A(t))D_0^2,$$
$$\tilde{D}_t^2+a=(I+\tilde{A}(t))D_0^2,$$

$A'(x)$ and $\tilde{A}'(x)$ are pseudodifferential operators (of negative order). Then we can choose a pseudodifferential elliptic operator D of a large enough positive order such that, for $k\in(0,k_0]$, $I+kD$ is invertible and $A'(t)[I+kD]^{-1}$ and $\tilde{A}'(t)[I+kD]^{-1}$ are trace class operators.

Let us take

$$E_k(t)=[I+A(t)(I+kD)^{-1}]D_0^2,$$
$$\tilde{E}_k(t)=[I+\tilde{A}(t)(I+kD)^{-1}]D_0^2.$$

Since $E_k(t) \to D_t^2 + a$ and $\tilde{E}_k(t) \to \tilde{D}_t^2 + a$ in the $\| \ \|_{2m,0}$ norm, uniformly in t when $k \to 0^+$, there exists $k_1 > 0$ such that $E_k(t)$ and $\tilde{E}_k(t)$ are invertible for $0 < k \leqq k_1$, and $t \in [0,1]$. Arguing as in the first case, we see that, for $0 < k \leqq k_1$,

$$\operatorname{Tr}\left[\int_0^1 E_k'(t) E_k^{-1}(t) e^{-uE_k(t)}\, dt\right] - \operatorname{Tr}\left[\int_0^1 \tilde{E}_k'(t) \tilde{E}_k^{-1}(t) e^{-u\tilde{E}_k(t)}\, dt\right] = 2\pi i \ell\,,$$

with $\ell \in \mathbb{Z}$.

Now, the proof will be accomplished if we can show that

$$\lim_{k \to 0^+} \operatorname{Tr}\left[\int_0^1 E_k'(t) E_k^{-1}(t) e^{-uE_k(t)}\, dt\right] = \operatorname{Tr}[2L_u(D_t; a)] \tag{A.3}$$

and analogously for $\tilde{E}_k(t)$ and $\tilde{D}_t$.

Let us take $r \in \mathbb{N}$ such that $(D_t^2 + a)^{-r}$ and $(\tilde{D}_t^2 + a)^{-r}$ are trace class operators and Γ a contour enclosing $\bigcup_{t \in [0,1]} \operatorname{sp}(E_k(t))$. Then, we can write

$$\operatorname{Tr}\left[\int_0^1 E_k'(t) E_k^{-1}(t) e^{-uE_k(t)}\, dt\right] = \int_0^1 \frac{1}{2\pi i} \int_\Gamma \lambda^{r-1} e^{-u\lambda} E_k'(t) E_k(t)^{-r} [\lambda - E_k(t)]^{-1}\, d\lambda\, dt\,,$$

and a similar expression for $\operatorname{Tr}[2L_u(D_t; a)]$. Hence,

$$\begin{aligned}&\left|\operatorname{Tr}\left[\int_0^1 E_k'(t) E_k^{-1}(t) e^{-uE_k(t)}\, dt\right] - \operatorname{Tr}[2L_u(D_t; a)]\right| \\ &\quad \leqq \frac{1}{2\pi} \int_0^1 \int_\Gamma \left\| E_k'(t) E_k^{-r}(t) [\lambda - E_k(t)]^{-1} - \frac{d}{dt}(D_t^2 + a)(D_t^2 + a)^{-r} [\lambda - (D_t^2 + a)]^{-1} \right\|_1 \\ &\qquad \times |\lambda^{r-1} e^{-u\lambda}|\, |d\lambda|\, dt\,.\end{aligned}$$

In order to complete the proof of the equality (A.3), we are going to show that

$$\lim_{k \to 0^+} \left\| E_k'(t) E_k^{-r}(t) [\lambda - E_k(t)]^{-1} - \frac{d}{dt}(D_t^2 + a)(D_t^2 + a)^{-r} [\lambda - (D_t^2 + a)]^{-1} \right\|_1 = 0\,, \tag{A.4}$$

uniformly in t and λ.

In fact, we have

$$\begin{aligned}&E_k'(t) E_k^{-r}(t) [\lambda - E_k(t)]^{-1} - \frac{d}{dt}(D_t^2 + a)(D_t^2 + a)^{-r} [\lambda - (D_t^2 + a)]^{-1} \\ &\quad = \left\{ E_k'(t) [\lambda - E_k(t)]^{-1} E_k^{-r}(t) (D_t^2 + a)^r - \frac{d}{dt}(D_t^2 + a) [\lambda - (D_t^2 + a)]^{-1} \right\} (D_t^2 + a)^{-r}.\end{aligned}$$

Since $\|(D_t^2 + a)^{-r}\|_1$ is bounded, uniformly in t, and the other factor in the right-hand side tends to zero in $\mathscr{L}(H^0)$ when $k \to 0^+$, uniformly in λ and t, we get (A.4).

References

1. Hawking, S.W.: Zeta function regularization of path integrals in curved spacetime. Commun. Math. Phys. **55**, 133–148 (1977)
2. Gamboa Saraví, R.E., Muschietti, M.A., Solomin, J.E.: On perturbation theory for regularized determinants of differential operators. Commun. Math. Phys. **89**, 363–373 (1983)
3. Muschietti, M.A.: Sobre las potencias complejas de operadores elípticos. Tesis, Universidad Nacional de La Plata (1984)
4. Tamura, H.: Regularized determinants for quantum field theories with fermions. Commun. Math. Phys. **98**, 355–367 (1985)
5. Fujikawa, K.: Path-integral measure for gauge-invariant fermion fields. Phys. Rev. Lett. **42**, 1195–1198 (1979)
6. Seeley, R.T.: Complex powers of an elliptic operator. Am. Math. Soc. Sym. Pure Math. **10**, 288–307 (1967)
7. Calderón, A.P.: Lecture notes on pseudodifferential operators and elliptic boundary value problems, I. Instituto Argentino de Matemática (1976)
8. Greiner, P.: An asymptotic expansion for the heat equation. Arch. Rat. Mech. Anal. **41**, 163–218 (1971)
9. Gamboa Saraví, R.E., Muschietti, M.A., Solomin, J.E.: On the regularized determinant for non-invertible elliptic operators. Commun. Math. Phys. **93**, 407–415 (1984)
10. Gamboa Saraví, R.E., Muschietti, M.A., Schaposnik, F.A., Solomin, J.E.: Chiral symmetry and functional integral. Ann. Phys. **157**, 360–393 (1984)
11. Gamboa Saraví, R.E., Muschietti, M.A., Schaposnik, F.A., Solomin, J.E.: ζ-function method and the evaluation of fermion currents. J. Math. Phys. **26**, 2045–2049 (1985)

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