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A Block Spin Construction of Ondelettes[1]. Part I: Lemarié Functions

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Abstract. Using block spin assignments, we construct an L^2-orthonormal basis consisting of dyadic scalings and translates of just a finite number of functions. These functions have exponential localization, and for even values of a construction parameter M one can make them class C^{M-1} with vanishing moments up to order M inclusive. Such a basis has an important application to phase cell cluster expansions in quantum field theory.

1. Introduction

Quite recently Y. Meyer et al. [1, 2, 3] have constructed very useful bases of ondelettes (wavelets) to solve certain problems in functional analysis. These new functions are now expected to have applications to several areas of physics. They have already had an impact on constructive quantum field theory [4, 5, 6].

A basis of ondelettes is defined to be an orthonormal basis—say for $L^2(\mathbb{R}^d)$—whose functions are dyadic scalings (from $2^{-\infty}$ to 2^{∞}) and translates of just a finite number of them. The most familiar example is the standard basis of Haar functions on $\mathbb{R}^d$. Indeed, Battle and Federbush [4, 5] used a polynomial generalization of the Haar basis to develop a phase cell cluster expansion a few years ago. This basis has the following useful properties:

(a) The basis consists of all dyadic scalings and translates of a finite collection $\psi_1, \ldots, \psi_n$ of functions.

(b) ψ_i is a piecewise polynomial supported on the cube associated with it. Thus we have sharp localization but poor regularity.

(c) For all multi-indices α for which $|\alpha|$ is less than or equal to a certain construction parameter,

$$\int \psi_i(x)\, x^\alpha\, dx = 0. \tag{1.1}$$

Equivalently, $\hat{\psi}_i(p)$ vanishes to some finite order at $p=0$.

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Remark. The construction parameter is the maximum degree of the polynomials used, and it determines the size of the collection $\psi_1, \ldots, \psi_n$. For the standard Haar basis the construction parameter is zero and $n = 2^d - 1$.

Although these functions were instrumental in introducing important new ideas in constructive quantum field theory, their lack of regularity created serious technical inconvenience. Battle and Federbush developed a very natural expansion for the hierarchical version [5] of the celebrated ϕ_3^4 model, but it took the more complicated expansion of Williamson [7] to control the real model with these ondelettes.

This past summer Meyer and his co-workers announced the existence of a basis of ondelettes with the following remarkable properties:

(a) The basis consists of all dyadic scalings and translates of functions $\psi_1, \ldots, \psi_{2^d-1}$.

(b) ψ_i is a Schwartz function. (In fact, $\hat{\psi}_i$ is a compactly supported C^∞ function!)

(c) (1.1) holds for *all* multi-indices α. Equivalently, $\hat{\psi}_i(p)$ vanishes to infinite order at $p = 0$.

In addition, a co-worker of Meyer, P. Lemarié, found a basis [3] of ondelettes whose properties complement the preceding properties in the following interesting way:

(a) The basis consists of all dyadic scalings and translates of functions $\psi_1, \ldots, \psi_{2^d-1}$.

(b) ψ_i is class C^N (where N can be made arbitrarily large) and ψ_i has *exponential* localization. (Although the preceding ondelettes fall off faster than any negative power of distance, they cannot have exponential decay, because the vanishing of *all* moments implies that their Fourier transforms cannot be analytic at $p = 0$.)

(c) (1.1) holds for $|\alpha| \leqq N + 1$.

The functions of Lemarié are better suited for constructive field theory because exponential localization is a more useful property than smoothness. In the Battle–Federbush expansion one needs no more regularity for the ondelettes than class C^1. This will be discussed in greater detail in Part II.

In this paper we introduce a machine that actually *finds* the ondelettes of Lemarié. It is a "block spin" construction consisting of two natural stages. The first stage orthogonalizes levels and is carried out in Sects. 2 & 3; the second stage is an orthogonalization (on each scale) preserving translation properties and is carried out in Sect. 4. In Sect. 5 we verify the properties of the functions, and in Sect. 6 we verify completeness.

The orthogonalization of levels is based on a very familiar idea. Let $f_1, \ldots, f_n$ be arbitrary L^2 functions that are not necessarily orthogonal and minimize $\| \varphi \|_2$ with respect to the constraints

$$(\varphi, f_i) = 0, \quad i = 1, \ldots, n. \tag{1.2}$$

Then the solution φ_0 is orthogonal to any f for which some (f, f_i) has a non-zero value. This is why our "block spin assignment" rules schematically look like

$$o\,o\,o\,o + -\, o\,o\,o\,o\,o\,o$$

The solution to this constraint is guaranteed to be orthogonal to the solution of any non-zero block spin assignment we could possibly make at any larger scale, because *this* assignment determines zero block spin values for all larger scales. The essence of this idea goes back to Kupiainen and Gawedzki [8], but we have to introduce a clever modification to obtain class C^N solutions.

The translation invariant orthogonalization is classical. There is a very natural candidate for an orthonormal basis for each subspace, and our task is to show that it makes sense and has all of the desired properties in our case.

The Lemarié functions themselves are given in Sect. 4 by Eq. (4.7). They are quite explicit, and the fact that our construction reproduces them is an indication of how natural they are. The construction guarantees orthogonality of levels, but since the functions *are* explicitly given, one may wish to see a *direct* verification, so we give here such a calculation. We concentrate on the case $d=1$, where at the unit scale level the functions are even translates of the function ϕ given by

$$\hat{\phi}(p) = c\, w(p)^{-1/2}(1-e^{ip})^{M+1}\frac{\hat{\chi}(p)^{M+1}}{\sum\limits_{n=-\infty}^{\infty} |\hat{\chi}(p+2\pi n)|^{2M+2}}, \tag{1.2}$$

where the function w is given by (4.6) in Sect. 4, χ is the characteristic function of $[0,1]$, and M is an even integer. If we let ϕ_m denote the $2m$-translate of ϕ, then

$$\int \phi_m(p)\overline{\phi_{m'}(2^r p)}dp = c\int \frac{e^{i2mp - i2^{r+1}m'p}}{\sqrt{w(p)}\sqrt{w(2^r p)}}(1-e^{ip})^{M+1}(1-e^{-i2^r p})^{M+1} \cdot \frac{\hat{\chi}(p)^{M+1}\overline{\hat{\chi}(2^r p)^{M+1}}}{[\sum\limits_n |\hat{\chi}(p+2\pi n)|^{2M+2}][\sum\limits_n |\hat{\chi}(2^r p+2\pi n)|^{2M+2}]}dp \tag{1.3}$$

for a given positive integer r. Using the identities

$$\hat{\chi}(2^r p) = (1+e^{i2^{r-1}p})\cdots(1+e^{i2p})(1+e^{ip})\hat{\chi}(p), \tag{1.4}$$

$$(1+e^{i2^{r-1}p})\cdots(1+e^{i2p})(1+e^{ip})(1-e^{ip}) = e^{-i(2^r-1)p}(1-e^{i2^r p}), \tag{1.5}$$

we get

$$\sum_n \int_0^{2\pi} \frac{e^{i2mp - i2^{r+1}m'p}e^{-i(2^r-1)(M+1)p}|1-e^{i2^r p}|^{2M+2}}{\sqrt{w(p)}\sqrt{w(2^r p)}\sum\limits_n |\hat{\chi}(2^r p+2\pi n)|^{2M+2}}dp \tag{1.6}$$

because $\sum\limits_n |\hat{\chi}(p+2\pi n)|^{2M+2}$ cancels out. On the other hand, $w(p)$ *has period* π. So does everything else remaining in the integrand, except

$$e^{-i(2^r-1)(M+1)p}. \tag{1.7}$$

Since M is even, the integral is zero.

We close the Introduction with the claim that we have a basis of exponentially localized ondelettes that are orthogonal with respect to the massless Sobelev norm $\|\vec{\nabla}\varphi\|_2$. This result is non-trivial, because the $|\vec{\nabla}|^{-1}$ potential of the L^2 ondelettes constructed here cannot have exponential decay. We describe the new basis in Part II.

2. Block-Spin Constraints

Consider the set $\mathscr{R}_s$ of mutually disjoint integer-valued translates Γ of the rectangular solid

$$R_s = \{x \in \mathbb{R}^d | 0 \leqq x_\mu \leqq 2 \text{ for } 1 \leqq \mu \leqq s-1 \text{ and } 0 \leqq x_\mu \leqq 1 \text{ for } s \leqq \mu \leqq d\}.$$

obviously this involves even-integer translates in each μ-direction for which $1 \leqq \mu \leqq s-1$. Let R_s' be the unit translate of R_s in the positive s-direction and define the block spin assignment σ_s as follows:

$$\sigma_s(R_s) = 1, \quad \sigma_s(R_s') = -1, \quad \sigma_s(\Gamma) = 0, \quad \Gamma \neq R_s, R_s'.$$

The first step in our construction is to find the function φ_s that minimizes $\|\varphi\|_2^2$ with respect to constraints that are easiest to understand if we initially write them in an ill-posed form:

$$\prod_{\mu=1}^{d} (\textstyle\int dp_\mu)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\varphi}(p)\overline{\hat{\chi}_\Gamma(p)} = \sigma_s(\Gamma), \tag{2.1}$$

where χ_Γ is the characteristic function of Γ. First, it is clear that if $s' > s$, then (2.1) implies

$$\prod_{\mu=1}^{d} (\textstyle\int dp_\mu)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\varphi}(p)\overline{\hat{\chi}_\Gamma(p)} = 0, \quad \Gamma \in \mathscr{R}_{s'}. \tag{2.2}$$

Second, it is equally obvious that *if* $r > 0$, *then (2.1) implies*

$$\prod_{\mu=1}^{d} (\textstyle\int dp_\mu)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\varphi}(p)\overline{\hat{\chi}_\Gamma(p)} = 0, \quad \Gamma \in 2^r\mathscr{R}_{s'}, \quad 1 \leqq s' \leqq d. \tag{2.3}$$

Third, it is easy to see that (2.1) is $(2m_1, \ldots, 2m_s, m_{s+1}, \ldots, m_d)$-translation-covariant—i.e., (2.1) is equivalent to

$$\prod_{\mu=1}^{d} (\textstyle\int dp_\mu)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\varphi}_m(p)\overline{\hat{\chi}_\Gamma(p)} = \sigma_{s,m}(\Gamma), \tag{2.4}$$

where φ_m is the $(2m_1, \ldots, 2m_s, m_{s+1}, \ldots, m_d)$-translate of φ and $\sigma_{s,m}$ is the same translate of the block spin assignment σ_s. Finally, note that the scaling of the block spin assignment can be chosen such that (2.1) is scale-covariant. If

$$\sigma_s^{(r)}(\Gamma) \equiv 2^{r(M+1/2)d}\sigma_s(2^{-r}\Gamma), \quad \Gamma \in 2^r\mathscr{R}_s, \tag{2.5}$$

then (2.1) is equivalent to

$$\prod_{\mu=1}^{d} (\textstyle\int dp_\mu)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\varphi}^{(r)}(p)\overline{\hat{\chi}_\Gamma(p)} = \sigma_s^{(r)}(\Gamma), \quad \Gamma \in 2^r\mathscr{R}_s, \tag{2.6}$$

where

$$\varphi^{(r)}(x) = 2^{-rd/2}\varphi(2^{-r}x). \tag{2.7}$$

Now the functions χ_Γ generate a hierarchy of L^2-subspaces that is ordered by containment. In particular, the block-spin constraints at a given level determine the block-spin constraints at all higher levels. It is obvious from (2.2) and (2.3) that the ill-posed constraint (2.1) yields zero constraints for all higher levels. This is a familiar kind of set-up, which has been used by Federbush [9] in the gauge field setting. The idea goes back to Gawedzki and Kupiainen [8], and the point is that the minimization of the Hilbert space norm with respect to such constraints creates a complete set of functions where any two that live on different levels are orthogonal with respect to that norm.

This constrained minimization of the L^2 norm is in ill-posed form because *the linear functionals defining the constraints are unbounded with respect to the L^2 norm.* Our motivation for introducing negative powers of the p_μ is to create orthogonal levels of *smooth* functions for our scale hierarchy, but for our scaling property we also need the homogeneity in the p_μ, so we have an infrared difficulty to cure. The orthogonality of scalings cannot be guaranteed by our constrained minimization unless the constraints can be expressed in terms of *bounded* linear functionals.

However, we can actually form bounded linear functionals from *finite linear combinations* of the unbounded ones. Since multiplication by e^{ip_μ} in momentum space corresponds to translation by a unit in the μ-direction, (2.1) implies

$$\prod_{\mu=1}^{d}\left(\int dp_\mu\right)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\phi}(p)\overline{\prod_{\mu=1}^{s-1}(1-e^{i2p_\mu})^M\prod_{\mu=s}^{d}(1-e^{ip_\mu})^M\hat{\chi}_\Gamma(p)}$$
$$=\Big(\prod_{\mu\neq s}P_M^-(-m_\mu)\Big)\big(P_M^-(-m_s)-P_M^-(1-m_s)\big), \tag{2.8}$$

where Γ is the $(2m_1,\ldots,2m_{s-1},m_s,\ldots,m_d)$-translate of R_s and

$$P_M^{\pm}(n)=(\pm 1)^n\binom{M}{n},\quad 0\leqq n\leqq M,$$

$P_M^{\pm}(n)=0$, otherwise.

The point is that

$$(1-e^{i\nu p_\mu})^M=\sum_n P_M^-(n)e^{in\nu p_\mu}, \tag{2.9}$$

and that our linear functionals are now bounded because $1-e^{i\nu p_\mu}$ cancels the infrared singularity of p_μ^{-1}.

Now (2.8) alone implies that all such integrals are zero for higher levels. The algebra is no longer a triviality, but it is interesting to check. The relevant identities are

$$1-e^{i2p_\mu}=(1+e^{ip_\mu})(1-e^{ip_\mu}),$$
$$2(1-e^{i2p_\mu})^M\hat{\chi}(2p_\mu)=(1+e^{ip_\mu})^{M+1}(1-e^{ip_\mu})^M\hat{\chi}(p_\mu),$$
$$\sum_n P_{M+1}^+(n)\big(P_M^-(-n)-P_M^-(1-n)\big)=0,$$

where χ denotes the characteristic function of $[0,1]$.

Since

$$\hat{\chi}(p_\mu) = i\frac{1 - e^{ip_\mu}}{p_\mu}, \tag{2.10}$$

we will make the replacement

$$p_\mu^{-M}(1 - e^{ip_\mu})^M \hat{\chi}(p_\mu) = i^{-M}\hat{\chi}(p_\mu)^{M+1} \tag{2.11}$$

from this point on.

It is clear from our scaling properties that we may restrict our attention to the s^{th} sub-level of the unit scale level for the remainder of our discussion.

3. Construction of the s^{th} Sub-Level

The next phase in our construction is to derive the formula for φ_s—i.e., to solve the constrained minimization problem posed in the last section. We first minimize

$$\|\varphi\|_2^2 + \alpha^2 \sum_m \Bigg[\prod_{\mu=1}^{d} \left(\int dp_\mu\right)\hat{\varphi}(p) \prod_{\mu=1}^{s-1} \overline{\hat{\chi}(2p_\mu)^{M+1}} \prod_{\mu=s}^{d} \overline{\hat{\chi}(p_\mu)^{M+1}}$$
$$\cdot\exp\left(- i2\sum_{\mu=1}^{s-1} m_\mu p_\mu - i\sum_{\mu=s}^{d} m_\mu p_\mu \right) - \left(\prod_{\mu\neq s} P_M^-(-m_\mu)\right)\left(P_M^-(-m_s) - P_M^-(1-m_s)\right) \Bigg]^2, \tag{3.1}$$

and then take the limit of our α-dependent solution as $\alpha \to \infty$. Since (3.1) is quadratic in φ, we need only to collect terms and complete the square for the inner product to see that (3.1) has the form

$$\|(1 + \alpha^2 \sum_m F_m^s)^{1/2}\varphi - \alpha^2 \sum_l \beta_l^s (1 + \alpha^2 \sum_m F_m^s)^{-1/2} f_l^s\|_2^2, \tag{3.2}$$

plus constant terms, where f_l^s is given by

$$\widehat{f_l^s}(p) = \prod_{\mu=1}^{s-1} \hat{\chi}(2p_\mu)^{M+1} \prod_{\mu=s}^{d} \hat{\chi}(p_\mu)^{M+1} \exp\left(i2\sum_{\mu=1}^{s-1} l_\mu p_\mu + i\sum_{\mu=s}^{d} l_\mu p_\mu \right). \tag{3.3}$$

F_m^s is the un-normalized orthogonal projection $(\cdot, f_m^s) f_m^s$, and

$$\beta_l^s = \left(\prod_{\mu\neq s} P_M^-(-l_\mu)\right)\left(P_M^-(-l_s) - P_M^-(1-l_s)\right). \tag{3.4}$$

The operator calculus makes sense because $\sum_m F_m^s \geqq 0$, but of course the F_m^s are not mutually orthogonal. Thus our problem reduces to solving the linear equation

$$\varphi + \alpha^2 \sum_m F_m^s \varphi = \alpha^2 \sum_l \beta_l^s f_l^s. \tag{3.5}$$

Now for *small* α we have the Neumann series solution

$$\varphi = \alpha^2 \sum_l \beta_l^s \sum_{n=0}^{\infty} (-\alpha^2)^n \left(\sum_m F_m^s\right)^n f_l^s, \tag{3.6}$$

which can be transformed in much the same manner that the corresponding Neumann series was transformed in [9]. The point is that (with $\vec{\lambda}_0 = l$)

$$\begin{aligned}(\sum_m F_m)^n f_l^s &= \sum_{\vec{\lambda}_1,\ldots,\vec{\lambda}_n} \left[\prod_{i=1}^{n} (f^s_{\vec{\lambda}_{i-1}}, f^s_{\vec{\lambda}_i})\right] f^s_{\vec{\lambda}_n} \\ &= \sum_{\vec{\lambda}_1,\ldots,\vec{\lambda}_n} \left[\prod_{i=1}^{n} (f^s_{\vec{\lambda}_{i-1}-\vec{\lambda}_i}, f^s_0)\right] f^s_{\vec{\lambda}_n}, \\ &= \prod_{\mu=1}^{d} \left(\int_0^{2\pi} dk_\mu\right) e^{-il\cdot k} g(k)^n \sum_m e^{im\cdot k} f^s_m,\end{aligned} \tag{3.7}$$

where

$$g(k) = \sum_m (f^s_m, f^s_0) e^{im\cdot k}. \tag{3.8}$$

Remark. $\sum_m e^{im\cdot k} f^s_m$ is a finite sum on any compact set and $g(k)$ is just a trigonometric polynomial, so there are no convergence problems here. Also, $g(k)$ does not depend on s as we see below.

(3.6) becomes

$$\varphi = \alpha^2 \sum_l \beta^s_l \prod_{\mu=1}^{d} \left(\int_0^{2\pi} dk_\mu\right) \frac{e^{-il\cdot k}}{1+\alpha^2 g(k)} \sum_m e^{im\cdot k} f^s_m, \tag{3.9}$$

and by analytic continuation in α, this formula holds for large real α as well. ($g(k) \geqq 0$ because (f^s_m, f^s_0) is of positive type.) Thus our solution φ_s of the constrained minimization problem is given by the limit of this expression as $\alpha \to \infty$. Now the Poisson summation formula enables us to write $g(k)$ in a more useful form:

$$g(k) = \prod_{\mu=1}^{d} \left(\sum_{n=-\infty}^{\infty} |\hat{\chi}(k_\mu + 2\pi n)|^{2M+2}\right). \tag{3.10}$$

Thus $g(k)$ does not vanish anywhere, and so our solution

$$\varphi_s = \sum_l \beta^s_l \prod_{\mu=1}^{d} \left(\int_0^{2\pi} dk_\mu\right) e^{-il\cdot k} g(k)^{-1} \sum_m e^{im\cdot k} f^s_m \tag{3.11}$$

is well-defined and well-behaved. Indeed, φ_s is an L^2 function because $g(k)^{-1}$ is bounded.

The expression for $\hat{\varphi}_s$ is quite explicit. If we set

$$a_m = \prod_{\mu=1}^{d} \left(\int_0^{2\pi} dk_\mu\right) e^{-im\cdot k} g(k)^{-1}, \tag{3.12}$$

then $g(k)^{-1} = \sum_m a_m e^{im\cdot k}$ and

$$\varphi_s = \sum_m \beta^s_l \sum_m a_{l-m} f^s_m. \tag{3.13}$$

Hence

$$\hat{\varphi}_s(p) = \sum_l \beta_l^s \sum_m a_{l-m} \exp\left(i2 \sum_{\mu=1}^{s-1} m_\mu p_\mu + i \sum_{\mu=s}^{d} m_\mu p_\mu \right)$$
$$\cdot \prod_{\mu=1}^{s-1} \hat{\chi}(2p_\mu)^{M+1} \prod_{\mu=s}^{d} \hat{\chi}(p_\mu)^{M+1}$$
$$= \sum_l \beta_l^s \exp\left(i2 \sum_{\mu=1}^{s-1} l_\mu p_\mu + i \sum_{\mu=s}^{d} l_\mu p_\mu \right) g(-2p_1, \dots,$$
$$-2p_{s-1}, -p_s, \dots, -p_d)^{-1} \prod_{\mu=1}^{s-1} \hat{\chi}(2p_\mu)^{M+1} \prod_{\mu=s}^{d} \hat{\chi}(p_\mu)^{M+1}$$
$$= \sum_l \beta_l^s \exp\left(i2 \sum_{\mu=1}^{s-1} l_\mu p_\mu + i \sum_{\mu=s}^{d} l_\mu p_\mu \right)$$
$$\cdot \prod_{\mu=1}^{s-1} \frac{\hat{\chi}(2p_\mu)^{M+1}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(2p_\mu + 2\pi n)|^{2M+2}} \prod_{\mu=s}^{d} \frac{\hat{\chi}(p_\mu)^{M+1}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_\mu + 2\pi n)|^{2M+2}}. \tag{3.14}$$

Combining this with (3.4), we obtain

$$\hat{\varphi}_s(p) = \prod_{\mu=1}^{s-1} \frac{(1 - e^{-i2p_\mu})^M \hat{\chi}(2p_\mu)^{M+1}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(2p_\mu + 2\pi n)|^{2M+2}} \cdot \frac{(1 - e^{-ip_s})^{M+1} \hat{\chi}(p_s)^{M+1}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_s + 2\pi n)|^{2M+2}}$$
$$\cdot \prod_{u=s+1}^{d} \frac{(1 - e^{-ip_\mu})^M \hat{\chi}(p_\mu)^{M+1}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_\mu + 2\pi n)|^{2M+2}}. \tag{3.15}$$

Remark. The manipulation above is justified by the fact that $g(k)^{-1}$ is smooth—i.e., that $\{a_m\}$ has rapid fall-off.

4. Translation-Invariant Orthogonalization

Having decomposed $L^2(\mathbb{R}^d)$ into subspaces corresponding to sublevels of scales, we now seek a function in each subspace with the property that all translates associated with the subspace yield an orthonormal basis for the subspace. This is the final step in our construction. As before, we consider the s^{th} sublevel for the unit scale without loss. Let $\mathscr{H}_s$ be the subspace that we have constructed for this sublevel. $\mathscr{H}_s$ is spanned by the set of functions $\varphi_{s,m}$ given by

$$\hat{\varphi}_{s,m}(p) = \exp\left(i2 \sum_{\mu=1}^{s} m_\mu p_\mu + i \sum_{\mu=s+1}^{d} m_\mu p_\mu \right) \hat{\varphi}_s(p). \tag{4.1}$$

We should first point out that there is a canonical solution of such a problem—under the right conditions. The natural candidate ϕ_s is given by

$$\hat{\phi}_s(p) = h_s(p)^{-1/2} \hat{\varphi}_s(p), \tag{4.2}$$

$$h_s(p) = \sum_l |\hat{\varphi}_s(p_1 + l_1\pi, \dots, p_s + l_s\pi, p_{s+1} + 2\pi l_{s+1}, \dots, p_d + 2\pi l_d)|^2 \tag{4.3}$$

and the translates $\phi_{s,m}$ are given by

$$\hat{\phi}_{s,m}(p) = \exp\left(i2 \sum_{\mu=1}^{s} m_\mu p_\mu + i \sum_{\mu=s+1}^{d} m_\mu p_\mu \right) \hat{\phi}_s(p)$$

$$= h_s(p)^{-1/2} \hat{\varphi}_{s,m}(p). \tag{4.4}$$

There are a number of things to be verified, however.

First we insert (3.15) in (4.3) to obtain a more explicit expression for h_s. We have

$$h_s(p) = \prod_{\mu=1}^{s-1} \frac{|1 - e^{i2p_\mu}|^{2M}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(2p_\mu + 2\pi n)|^{2M+2}} w(p_s)$$

$$\cdot \prod_{\mu=s+1}^{d} \frac{|1 - e^{ip_\mu}|^{2M}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_\mu + 2\pi n)|^{2M+2}}, \tag{4.5}$$

$$w(t) = \frac{|1 - e^{it}|^{2M+2}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(t + 2\pi n)|^{2M+2}} + \frac{|1 + e^{it}|^{M+2}}{\sum_{n=-\infty}^{\infty} |\hat{\chi}(t + 2\pi n + \pi)|^{2M+2}}. \tag{4.6}$$

Thus (since we are assuming M to be even)

$$\hat{\phi}_s(p) = \prod_{\mu=1}^{s-1} \frac{(-1)^{(1/2)M} e^{iMp_\mu} \hat{\chi}(2p_\mu)^{M+1}}{\left(\sum_{n=-\infty}^{\infty} |\hat{\chi}(2p_\mu + 2\pi n)|^{2M+2} \right)^{1/2}}$$

$$\cdot \frac{(1 - e^{ip_s})^{M+1} \hat{\chi}(p_s)^{M+1}}{w(p_s)^{1/2} \sum_{n=-\infty}^{\infty} |\hat{\chi}(p_s + 2\pi n)|^{2M+2}} \prod_{\mu=s+1}^{d} \frac{(-1)^{(1/2)M} e^{i(1/2)Mp_\mu} \hat{\chi}(p_\mu)^{M+1}}{\left(\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_\mu + 2\pi n)|^{2M+2} \right)^{1/2}}. \tag{4.7}$$

Now the denominators in (4.6) are periodic functions that do not vanish anywhere, so $w(t)$ is a periodic C^∞ function. Furthermore $w(t)$ is a positive function that cannot vanish anywhere, so all of the denominators in (4.7) are well-defined and do not vanish anywhere. Indeed, by continuity and periodicity they are bounded below by positive constants, and so we have established:

Theorem 4.1. *ϕ_s is an L^2 function.*

Now consider the inner product $(\phi_{s,m}, \phi_{s,m'})$ with $m' \neq m$. Returning to (4.4) we see that

$$(\phi_{s,m}, \phi_{s,m'}) = \prod_{\mu=1}^{d} (\textstyle\int dp_\mu) h_s(p)^{-1} \hat{\varphi}_{s,m}(p) \overline{\hat{\varphi}_{s,m'}(p)}$$

$$= \prod_{\mu=1}^{d} (\textstyle\int dp_\mu) h_s(p)^{-1} |\hat{\varphi}_s(p)|^2$$

$$\cdot \exp\left(i2 \sum_{\mu=1}^{s} (m_\mu - m'_\mu) p_\mu + i \sum_{\mu=s+1}^{d} (m_\mu - m'_\mu) p_\mu \right). \tag{4.8}$$

We decompose the integration as follows:

$$\int dp_\mu = \sum_{n=-\infty}^{\infty} \int_{\pi n}^{\pi n+\pi} dp_\mu, \quad 1 \leqq \mu \leqq s,$$

$$\int dp_\mu = \sum_{n=-\infty}^{\infty} \int_{2\pi n}^{2\pi n+2\pi} dp_\mu, \quad s+1 \leqq \mu \leqq d.$$

Then for the n^{th} term we make the change of variable $p'_\mu = p_\mu - \pi n$ or $p'_\mu = p_\mu - 2\pi n$ as the case may be. Both h_s and the exponential are unaffected by these changes, so by (4.3) our inner product reduces to

$$\prod_{\mu=1}^{s} \left(\int_0^{\pi} dp'_\mu \right) \prod_{\mu=s+1}^{d} \left(\int_0^{2\pi} dp'_\mu \right) \exp\left(i2 \sum_{\mu=1}^{s} (m_\mu - m'_\mu) p'_\mu + i \sum_{\mu=s+1}^{d} (m_\mu - m'_\mu) p'_\mu \right)$$

which is obviously zero for $m' \neq m$. Thus we have proven:

Theorem 4.2. *$\{\phi_{s,m}\}$ is an orthogonal set.*

We shall not bother to normalize $\phi_{s,m}$. We observe that the normalization constant is independent of m, so we need say no more about it.

Our next concern is to verify:

Theorem 4.3. $\phi_{s,m} \in \mathscr{H}_s$.

Proof. Let $\varphi_{s,m}^{(N)}$ be defined inductively by

$$\varphi_{s,m'}^{(N)} = \sum_m \left[\prod_{\mu=1}^{s} \left(\int_0^{\pi} dp'_\mu \right) \prod_{\mu=s+1}^{d} \left(\int_0^{2\pi} dp'_\mu \right) \exp\left(i2 \sum_{\mu=1}^{s} (m_\mu - m'_\mu) p'_\mu \right.\right.$$
$$\left.\left. + i \sum_{\mu=s+1}^{d} (m_\mu - m'_\mu) p'_\mu \right) h_s(p')^{(-1/6)M} \right] \varphi_{s,m}^{(N-1)} \tag{4.9}$$

with $\varphi_{s,m}^{(0)} = \varphi_{s,m}$. The point of this definition is that we have to worry about the zeros of h_s. Let $b_{m'-m}$ be the coefficient defined by this negative fractional power of h_s. Then

$$\varphi_{s,m'}^{(N)} = \sum_m b_{m'-m} \varphi_{s,m}^{(N-1)}, \tag{4.10}$$

$$\widehat{\varphi_{s,m}^{(N-1)}}(p) = \exp\left(i2 \sum_{\mu=1}^{s} m_\mu p_\mu + i \sum_{\mu=s+1}^{d} m_\mu p_\mu \right) \widehat{\varphi_s^{(N-1)}}(p), \tag{4.11}$$

$$h_s(p)^{(-1/6)M} = \sum_m b_m \exp\left(i2 \sum_{\mu=1}^{s} m_\mu p_\mu + i \sum_{\mu=s+1}^{d} m_\mu p_\mu \right), \tag{4.12}$$

so we have

$$\widehat{\varphi_{s,m'}^{(N)}}(p) = h_s(p)^{-(1/6)M} \widehat{\varphi_{s,m'}^{(N-1)}}(p). \tag{4.13}$$

Thus (with $\vec{\lambda}_0 = m$)

$$\phi_{s,m} = \varphi_{s,m}^{(3M)} = \sum_{\vec{\lambda}_1, \ldots, \vec{\lambda}_{3M}} \prod_{i=1}^{3M} b_{\vec{\lambda}_{i-1} - \vec{\lambda}_i} \varphi_{s, \vec{\lambda}_{3M}}, \tag{4.14}$$

so $\phi_{s,m} \in \mathscr{H}_s$ provided this multiple sum converges in L^2. Now, in particular, $\{\varphi_{s,m}^{(N)}(x)\}$ is uniformly summable on compact sets for $N \leqq 3M$. On the other hand

$\{b_m\}$ *is square summable because* $h(p)^{-(1/6)M}$ *is square integrable on* $[0,\pi]^s \times [0,2\pi]^{d-s}$. Hence

$$\sum_m b_{m'-m} \varphi_{s,m}^{(N-1)}(x) \tag{4.15}$$

converges uniformly on compact sets in the m'-square-summable sense, and so by translation decomposition of integrals it converges in the L^2 sense. Thus $\varphi_{s,m}^{(N)} \in \mathscr{H}_s$ for all m provided $\varphi_{s,m}^{(N-1)} \in \mathscr{H}_s$ for all m. The induction carries us up to $N = 3M$, so $\phi_{s,m} \in \mathscr{H}_s$ for all m. $\square$

Remark. The uniform summability of $\{\varphi_{s,m}^{(N)}(x)\}$ on compact sets follows from

$$\widehat{\varphi_{s,m}^{(N)}}(p) = h_s(p)^{-(N/6)M} \widehat{\varphi_{s,m}}(p) \tag{4.16}$$

together with the observation that $h_s(p)^{-(N/6)M} \hat{\varphi}_s(p)$ is an integrable C^∞ function whose derivatives are integrable, provided $N \leqq 3M$.

Finally, we need to establish:

Theorem 4.4. $\{\phi_{s,m}\}$ *spans* $\mathscr{H}_s$.

Proof. Let

$$c_m = \prod_{\mu=1}^{s} \left(\int_0^{\pi} dp'_\mu \right) \prod_{\mu=s+1}^{d} \left(\int_0^{2\pi} dp'_\mu \right) \exp\left(-i2 \sum_{\mu=1}^{s} m_\mu p'_\mu - i \sum_{\mu=s+1}^{d} m_\mu p'_\mu \right) \sqrt{h_s(p)}. \tag{4.17}$$

Thus

$$\sqrt{h_s(p)} = \sum_m c_m \exp\left(i2 \sum_{\mu=1}^{s} m_\mu p_\mu + i \sum_{\mu=s+1}^{d} m_\mu p_\mu \right), \tag{4.18}$$

and since

$$\hat{\varphi}_{s,m'}(p) = \sqrt{h_s(p)}\, \hat{\phi}_{s,m'}(p), \tag{4.19}$$

it follows that

$$\hat{\varphi}_{s,m'}(p) = \sum_m c_{m'-m} \hat{\phi}_{s,m}(p). \tag{4.20}$$

Since $\{c_m\}$ is square summable and $\{\phi_{s,m}(x)\}$ is summable uniformly on compact sets, we know that

$$\sum_m c_{m'-m} \phi_{s,m}(x) \tag{4.21}$$

converges in the L^2 sense. Having expressed the $\varphi_{s,m}$ as linear combinations of the $\phi_{s,m}$ we have shown that the latter span $\mathscr{H}_s$. $\square$

5. Moment Properties and Exponential Decay

In this section we examine the properties of ϕ_s. By (4.7) we have

$$\hat{\phi}_s(p) = O\left(\prod_\mu |p_\mu|^{-M-1}\right) \tag{5.1}$$

for large p, and

$$\hat{\phi}_s(p) = O(|p_s|^{M+1}) \tag{5.2}$$

for small p_s. The former property implies that ϕ_s is class C^{M-1} smooth, while the latter property means that

$$\int_{-\infty}^{\infty} \phi_s(x) x_s^N \, dx_s = 0 \tag{5.3}$$

for arbitrary $\hat{x}_s \equiv (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_d)$ and $N \leqq M$. Equation (5.3) is a strong condition; it implies that

$$\int \phi_s(x) x^\alpha dx = 0 \tag{5.4}$$

for all multi-indices α for which $\alpha_s \leqq M$. There are more than enough vanishing moments to give us the desired power law for the long-distance fall-off of a massless potential of ϕ_s—provided that ϕ_s itself has enough decay. The fact that $\hat{\phi}_s$ is an integrable C^∞ function with integrable derivatives gives us as much as we want for that purpose.

Indeed, our final task is to show that ϕ_s has *exponential* fall-off. This property is a consequence of the following theorem.

Theorem 5.1. *$\hat{\phi}_s$ extends to an analytic function bounded by $c \prod_\mu (1 + |\mathrm{Re}\, z_\mu|)^{-M-1}$ on the product of strips $|\mathrm{Im}\, z_\mu| < \delta$ for some $\delta > 0$ ($p_\mu = \mathrm{Re}\, z_\mu$).*

Proof. Returning to (4.7), we first note that the numerators extend to entire functions. Moreover, they are bounded on arbitrary strips centered about the real axis because

$$|e^{i\xi}| \leqq e^{|\mathrm{Im}\,\xi|},$$

$$|\hat{\chi}(\xi)| = \frac{|1 - e^{i\xi}|}{|\xi|} = \begin{cases} O(1), & |\xi| \leqq 1, \\ O\left(\dfrac{e^{|\mathrm{Im}\,\xi|}}{(1 + |\mathrm{Re}\,\xi|)}\right), & |\xi| \geqq 1. \end{cases}$$

To control the denominators we must appeal to:

Lemma 5.2. $\sum_{n=-\infty}^{\infty} |\hat{\chi}(t + 2\pi n)|^{2M+2}$ *extends to an entire function whose real part is positive and bounded away from zero on the strip $|\mathrm{Im}\,\xi| < \delta$ for some $\delta > 0$.*

Proof. The entire function is

$$4^{M+1} \sum_{n=-\infty}^{\infty} \frac{\sin^{2M+2}(\frac{1}{2}\xi)}{(\xi + 2\pi n)^{2M+2}}, \tag{5.5}$$

and it is periodic. Hence, its derivative is periodic and therefore bounded on any strip centered about the real axis. On the other hand, the function is strictly positive on the real axis, so the periodicity also tells us that it is greater than some positive constant on the whole real axis. The boundedness of the derivative certainly implies boundedness of the partial derivative of the real part of (5.5) with respect to Im ξ, so we have the desired conclusion. $\square$

Proof of Theorem 5.1. (continued) An immediate consequence of the lemma is that

$$\left(\sum_{n=-\infty}^{\infty} |\hat{\chi}(t+2\pi n)|^{2M+2}\right)^{-\alpha}, \quad \alpha > 0$$

extends to a function that is both bounded *and analytic* on the strip $|\mathrm{Im}\,\xi| < \delta$. Our argument is complete if we can show that $w(t)^{-\alpha}$ can also be extended in this way. Now Lemma 5.2 applies to the function

$$\sum_{n=-\infty}^{\infty} |\hat{\chi}(t+2\pi n+\pi)|^{2M+2}$$

as well, so $w(t)$ itself extends to a bounded analytic function on some strip centered about the real axis. But $w(t)$ is also periodic and strictly positive on the real axis, so we may use the same reasoning as before. $\square$

6. Completeness

It is easy to convince oneself that our construction guarantees completeness of the orthogonal set of Lemarié functions, provided that our original set of block-spin constraints is complete. Now recall that $\varphi_{s,m}$ was constructed with the set of constraints

$$\prod_{\mu=1}^{d} \left(\int dp_\mu\right)\hat{\varphi}(p) \prod_{\mu=1}^{s-1} \overline{\hat{\chi}(2p_\mu)^{M+1}} \prod_{\mu=s}^{d} \overline{\hat{\chi}(p_\mu)^{M+1}} \exp\left(-i2\sum_{\mu=1}^{s-1} l_\mu p_\mu - i\sum_{\mu=s}^{d} l_\mu p_\mu\right)$$
$$= \left(\prod_{\mu\neq s} P_M^-(m_\mu - l_\mu)\right)\left(P_M^-(m_s - l_s) - P_M^-(1+m_s-l_s)\right) \tag{6.1}$$

with l running over all d-tuples of integers. These constraints follow from (2.8), (2.10), and (4.1), and the first observation to make is that the functions on $\mathbb{Z}^d$ given by

$$l \mapsto \left(\prod_{\mu\neq s} P_M^-(m_\mu - l_\mu)\right)\left(P_M^-(m_s - l_s) - P_M^-(1+m_s-l_s)\right) \tag{6.2}$$

span the Hilbert space $l^2(\mathbb{Z}^d)$. This is an obvious consequence of the fact that the functions

$$(1-e^{ik_s})^{M+1} \prod_{\mu\neq s} (1-e^{ik_\mu})^M \prod_{\mu} e^{im_\mu k_\mu} \tag{6.3}$$

span $L^2(T^d)$. (Linear combinations form an algebra that separates points.)

On the other hand, we know that $\hat{\varphi}_{s,m}$ is a square-summable combination of the functions

$$\widehat{f_l^s}(p) = \prod_{\mu=1}^{s-1} \hat{\chi}(2p_\mu)^{M+1} \prod_{\mu=s}^{d} \hat{\chi}(p_\mu)^{M+1} \exp\left(i2\sum_{\mu=1}^{s-1} l_\mu p_\mu + i\sum_{\mu=s}^{d} l_\mu p_\mu\right) \tag{6.4}$$

because $\varphi_{s,m}$ minimizes $\|\varphi\|_2$ with respect to (6.1). The implication of our remark above is that the $\hat{\varphi}_{s,m}$ span the same subspace of $L^2(\mathbb{R}^d)$ as the $\widehat{f_l^s}$. Our proof of completeness therefore reduces to establishing the following theorem:

Theorem 6.1. *Let* $\zeta \in L^2(\mathbb{R}^d)$ *such that*

$$\int \hat{\zeta}(p)\widehat{f_l^s}(2^r p)dp = 0 \tag{6.5}$$

for all integers r, *all* $l \in \mathbb{Z}^d$, *and* $1 \leqq s \leqq d$. *Then* $\zeta = 0$.

Proof. Pick an arbitrary *negative* value for r. We know that

$$\begin{aligned} \hat{\chi}(2^v p_\mu) = (1 + e^{i2^{v-1}p_\mu})(1 + e^{i2^{v-2}p_\mu})\cdots \\ (1 + e^{i2^{v+r+1}p_\mu})(1 + e^{i2^{v+r}p_\mu})\hat{\chi}(2^{v+r}p_\mu), \quad v = 0, 1, \end{aligned} \tag{6.6}$$

so by taking appropriate linear combinations of (6.5) we can infer that

$$\int \hat{\xi}(p) \prod_{\mu=1}^{s-1} \hat{\chi}(2^{r+1}p_\mu)^M \prod_{\mu=s}^{d} \hat{\chi}(2^r p_\mu)^M \exp\left(i2^{r+1} \sum_{\mu=1}^{s-1} l_\mu p_\mu + i2^r \sum_{\mu=s}^{d} l_\mu p_\mu \right) dp = 0 \tag{6.7}$$

for all $l \in \mathbb{Z}^d$, where $\xi \in L^2(\mathbb{R}^d)$ is given by

$$\hat{\xi}(p) = \hat{\zeta}(p) \prod_{\mu=1}^{s-1} \hat{\chi}(2p_\mu) \prod_{\mu=s}^{d} \hat{\chi}(p_\mu). \tag{6.8}$$

Since $\hat{\zeta} \in L^2(\mathbb{R}^d)$, we see that $\hat{\xi} \in L^1(\mathbb{R}^d)$, and so ξ is continuous. Now

$$\lim_{r \to -\infty} \hat{\chi}(2^{v+r}p_\mu) = 1, \tag{6.9}$$

so if we consider $l_\mu = 2^{r'+1}l'_\mu$, $r' > r$ and then take the limit of (6.7) for fixed l'_μ as $r \to -\infty$, it follows from dominated convergence that

$$\int \hat{\xi}(p) \exp\left(i2^{r'+1} \sum_{\mu=1}^{s-1} l'_\mu p_\mu + i2^{r'} \sum_{\mu=s}^{d} l'_\mu p_\mu \right) = 0, \tag{6.10}$$

i.e.,

$$\xi(2^{r'+1}l'_1, \ldots, 2^{r'+1}l'_{s-1}, 2^{r'}l'_s, \ldots, 2^{r'}l'_d) = 0. \tag{6.11}$$

Since this equation holds for arbitrarily negative r', ξ vanishes at all dyadic points, and so by continuity ξ vanishes everywhere. Hence $\zeta = 0$. □

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