Commun. Math. Phys. 110, 427–438 (1987)

Communications in Mathematical Physics

The Spectrum of Operators Elliptic Along the Orbits of $\mathbb{R}^n$ Actions

Jerome Kaminker* and Jingbo Xia*

Department of Mathematical Sciences, IUPUI, 1125 E. 38th St., P.O. Box 647, Indianapolis, IN 46223, USA

Abstract. It is shown that a periodic elliptic operator on $\mathbb{R}^n$ has no eigenvalues off of the set of discontinuities of its spectral density function. The methods involve operator algebras and are based on a "spectral duality" principal first introduced by J. Bellisard and D. Testard. A version of the spectral duality theorem is proved which relates the point spectrum of a certain family of operators to the continuous spectrum of an associated family.

0. Introduction

A general method for studying elliptic operators on open manifolds involves viewing such an operator as a longitudinally elliptic operator on a foliated compact manifold having the original open manifold as a leaf. This is the point of view developed by A. Connes [4]. We shall use this approach to study properties of the spectrum of such an operator. Our work is an extension of the ideas and methods introduced by Bellisard and Testard in [2]. The principal tool is a variant of their "spectral duality theorem." We prove a version of this theorem which holds for operators on the orbits of a locally free action of $\mathbb{R}^n$ on a compact Hausdorff space. The main application states that a periodic elliptic operator on $\mathbb{R}^n$ has no eigenvalues, off of the discontinuities of the spectral density function, when considered as acting on $L^2(\mathbb{R}^n)$.

The paper is organized as follows. Section 1 introduces the necessary operator algebra and discusses two families of representations of it. In Sect. 2 the basic facts on operators elliptic along the orbits of the $\mathbb{R}^n$ action are presented. The spectral duality theorem is proved in Sect. 3 and the application discussed above to the spectrum of periodic elliptic operators on $\mathbb{R}^n$ is covered in Sect. 4.

We remark that it will be quite useful to have a version of the spectral duality theorem which will hold for more general foliations. The case of a foliation transverse to the fibers of a principal G-bundle is particularly appropriate, and we will consider this in a later publication.

* Research sponsored in part by NSF grants

1. The Crossed Product $C(X) \rtimes \mathbb{R}^n$

In this section we will consider some of the facts needed which involve locally free actions of $\mathbb{R}^n$ on topological spaces. Recall that an action is called locally free if each isotropy group is discrete. If X is a smooth manifold then the orbits of a smooth such action yield a foliation of X. However, if no assumption is made about X then the result may not even be a foliated space, (cf. [8]). The analytic facts that we will use are true in the case of general foliations, not merely $\mathbb{R}^n$ actions, but the proofs are often considerably more complicated. For this reason we will develop the necessary analysis in the present generality, guided by the existing theory for foliations of manifolds, and taking advantage of the simplifications which our setting allows.

Let $\mathbb{R}^n$ act on the compact Hausdorff space X. Assume that the action is locally free and minimal, and that there exists an invariant probability measure on X, m, with respect to which the action is ergodic. We will denote the action by $x \to x + t$. Let ds denote Lebesgue measure on $\mathbb{R}^n$, and the product measure, $d\mu = dm \times ds$, will always be used on $X \times \mathbb{R}^n$.

The crossed product, $C(X) \rtimes \mathbb{R}^n$, will be our concern in this section. In the case that X is a manifold it is the same as the C^*-algebra of the associated foliation. Let $C_c(X \times \mathbb{R}^n) \subset C(X) \rtimes \mathbb{R}^n$ be the dense subset consisting of functions of compact support. Recall that the product and involution are defined by

$$(a * b)(x, t) = \int_{\mathbb{R}^n} a(x, s) b(x + s, t - s) ds, \quad a^*(x, t) = \overline{a(x + t, -t)}$$

for a and b in $C_c(X \times \mathbb{R}^n)$.

There is a natural representation $\pi : C(X) \rtimes \mathbb{R}^n \to \mathscr{L}(L^2(X \times \mathbb{R}^n))$ defined by

$$\pi(a)(\xi)(x, t) = \int_{\mathbb{R}^n} a(x, s) \xi(x + s, t + s) ds,$$

where $\xi \in L^2(X \times \mathbb{R}^n)$. This representation is faithful and the weak closure of its image is the von Neumann crossed-product $L^\infty(X) \times \mathbb{R}^n$. Under our assumptions on the action this is a II_∞-factor and has a unique normalized trace which is given on $C_c(X \times \mathbb{R}^n)$ by the formula

$$\tau(a) = \int_X a(x, 0) dm,$$

so that one has

$$\tau(a^* * a) = \int_{X \times \mathbb{R}^n} |a(x, t)|^2 d\mu.$$

The representation π decomposes as a direct integral of faithful representations $\pi_x : C(X) \rtimes \mathbb{R}^n \to \mathscr{L}(L^2(\mathbb{R}^n))$ which are given by

$$\pi_x(a)(f)(t) = \int_{\mathbb{R}^n} a(x + t, s) f(t + s) ds$$

for $a \in C_c(X \times \mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$. To see this, define a unitary operator on $L^2(X \times \mathbb{R}^n)$ by $V\xi(x, t) = \xi(x + t, t)$. Then one easily checks that $[V\pi(a)V^*](\xi)(x, t) =$

$\pi_x(a)(\xi i_x)(t)$, where $i_x:\mathbb{R}^n \to X \rtimes \mathbb{R}^n$ is given by $i_x(t) = (x, t)$. Thus one has

$$L^2(X \rtimes \mathbb{R}^n) \cong \int_X^{\oplus} L^2(\{x\}x\mathbb{R}^n)dm$$

and with respect to this decomposition

$$V\pi(a)V^* = \int_X^{\oplus} \pi_x(a)dm.$$

The representations π_x satisfy $\pi_{x+s} = U_s\pi_x U_{-s}$, where $U_s f(t) = f(t+s)$, for $f \in L^2(\mathbb{R}^n)$. Thus, if x and y are on the same orbit $\sigma(\pi_x(a)) = \sigma(\pi_y(a))$, where σ denotes spectrum. We show that because of the minimality of the action the same is true even if x and y are not in the same orbit. The minimality implies that there is a sequence $t_n \in \mathbb{R}^n$ such that $x + t_n \to y$. A direct computation shows that this implies that $\sigma(\pi_x(a)) \subset \sigma(\pi_y(a))$ for any self-adjoint element $a \in C(X) \rtimes \mathbb{R}^n$. By reversing the roles of x and y we obtain the desired result. We thus obtain

Proposition 1.1. *The representation π is faithful. If the orbit of x is dense, then π_x is faithful. If the action is minimal, then for any self-adjoint element a, the spectrum, $\sigma(\pi_x(a))$, is independent of x.*

There is a second family of representations of $C(X) \rtimes \mathbb{R}^n$ which we must consider. The representations are parametrized by $\hat{\mathbb{R}}^n$, the dual group of $\mathbb{R}^n$. Let $s \in \hat{\mathbb{R}}^n$. Define $\pi^s: C(X) \rtimes \mathbb{R}^n \to \mathscr{L}(L^2(X))$ by

$$[\pi^s(a)(h)](x) = \int_{\mathbb{R}^n} a(x, t)e^{i\langle s,t\rangle}h(x+t)dt$$

for $a \in C_c(X \times \mathbb{R}^n)$ and $h \in C(X)$. One verifies easily that π^s is a $*$-homomorphism, which, since $\mathbb{R}^n$ is amenable, extends to $C(X) \rtimes \mathbb{R}^n$.

We will show that the representation π has a second direct integral decomposition in terms of the π^s. Let $\mathscr{F}: L^2(\mathbb{R}^n, ds) \to L^2(\hat{\mathbb{R}}^n, d\hat{s})$ denote the Fourier transform. Then a direct computation shows that

$$j_s(1 \otimes \mathscr{F})\pi(a)(1 \otimes \mathscr{F})^{-1} = \pi^s(a)j_s,$$

where $j_s: X \to X \times \hat{\mathbb{R}}^n$ is $j_s(x) = (x, s)$. Thus we obtain

$$L^2(X \times \mathbb{R}^n) \cong \int_{\hat{\mathbb{R}}^n}^{\oplus} L^2(X \times \{s\})d\hat{s},$$

and relative to this

$$(1 \otimes \mathscr{F})\pi(a)(1 \otimes \mathscr{F})^* = \int_{\hat{\mathbb{R}}^n}^{\oplus} \pi^s(a)d\hat{s}.$$

Remark. An additional condition is needed to guarantee that the representations π^s are faithful and to relate $\sigma(\pi^s(a))$ to $\sigma(\pi^{s'}(a))$. One must require that the action be effective (i.e. the only element of $\mathbb{R}^n$ fixing everything in X is the identity), [6]. When that is the case one has $\sigma(\pi^0(a)) = \sigma(\pi_x(a))$ for all x. Let $a_s(x, t) = e^{i\langle t,s\rangle}a(x, t)$. Then $\pi_x(a_s)$ is unitarily equivalent to $\pi_x(a)$. Thus, one has the equalities $\sigma(\pi^s(a)) = \sigma(\pi^0(a_s)) = \sigma(\pi_x(a_s)) = \sigma(\pi_x(a))$. This yields the following result.

Proposition 1.2. *If the action of $\mathbb{R}^n$ on X is effective, then the representations π^s are faithful and $\sigma(\pi^s(a))$ is independent of $s\in\hat{\mathbb{R}}^n$.*

2. Pseudo-differential Operators along the Orbits

In this section we will consider the operators to which our main results will apply. They are a class of pseudo-differential operators along the orbits of the $\mathbb{R}^n$ action. Since we wish to work in the generality of a group action on a compact Hausdorff space, we do not assume the existence of local flow boxes. In order to simplify the presentation we consider a restricted set of operators, which are, however, general enough for the applications. Note that it is possible to allow pseudo-differential operators along the orbits which act on sections of vector bundles.

The symbols to be used on $\mathbb{R}^n$ are as follows.

Definition 2.1. A function $\sigma:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is in $\mathcal{S}^m$ if it is smooth and satisfies

$$|\partial_\xi^\alpha \partial_t^\beta \sigma(t,\xi)| \leqq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|},$$

where α and β are multi-indices.

If $\sigma\in\mathcal{S}^m$ then one defines the operator $\mathrm{Op}(\sigma): C_c^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)$ by

$$\mathrm{Op}(\sigma)u(t) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle t-s,\xi\rangle}\sigma(t,\xi)u(s)ds d\xi. \tag{2.2}$$

Definition 2.3. An operator A is pseudo-differential of order m if $A=\mathrm{Op}(\sigma)$ for some $\sigma\in\mathcal{S}^m$. The function σ is the symbol of A. The set of such operators will be denoted by Ψ^m. If $A=\mathrm{Op}(\sigma)$ and $\sigma\in\mathcal{S}^m$ for all m then $A\in\Psi^{-\infty}$. The operator A is elliptic if, further, there is an $R>0$ such that the symbol $\sigma(x,\xi)$ is invertible and satisfies $|\sigma^{-1}(t,\xi)|\leqq C|\xi|^{-m}$ for $|\xi|\geqq R$.

If $A\in\Psi^m$ then it has a kernel

$$K_A(t,s) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle t-s,\xi\rangle}\sigma(t,\xi)d\xi$$

which is continuous if $m\leqq -2n$. Thus, one has

$$Au(t) = \int_{\mathbb{R}^n} K_A(t,s)u(t)dt.$$

It is shown by Shubin and Kozlov, [6], that this class of operators extends continuously to the appropriate Sobelev spaces. We will also have need of the following theorem proved there.

Theorem 2.4. *Let $A\in\Psi^m$, $m>0$, be an elliptic self-adjoint pseudo-differential operator. Then for any $\varphi\in C_c^\infty(\mathbb{R}^n)$, $\varphi(A)\in\Psi^{-\infty}$ and $K_{\varphi(A)}(t,s)$ satisfies*

$$|\partial_t^\alpha\partial_s^\beta K_{\varphi(A)}(t,s)| \leqq C_{\alpha\beta N}(1+|t-s|)^{-N}$$

for any α, β, and N.

Our next goal is to extend the preceding theory to operators along the orbits. Assume $\mathbb{R}^n$ acts locally freely and minimally on X with invariant measure m.

Definition 2.5. Let $\mathcal{S}^m(X)$ denote the set of symbols, $\sigma: X\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ such that $\sigma_x=\sigma(x,\cdot,\cdot)\in\mathcal{S}^m$ for each $x\in X$. A symbol σ is called invariant if there is a function

$\sigma^{\#}: X \times \mathbb{R}^n \to \mathbb{C}$ such that

$$\sigma_x(t,\xi) = \sigma^{\#}(x+t,\xi).$$

Given a symbol, σ, define a family of operators, A_x on $L^2(\mathbb{R}^n)$ by $A_x = \mathrm{Op}(\sigma_x)$. If the symbol is invariant then $U_s A_x U_{-s} = A_{x+s}$. In this case one is able to define an operator $A^{\#}$ on $L^2(X \times \mathbb{R}^n)$ by

$$A^{\#} f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle t-s,\xi\rangle} \sigma^{\#}(x,\xi) f(x+s-t,s)\, ds\, d\xi.$$

As before, using the operator $Vf(x,s) = f(x+s,s)$, one sees that

$$VA^{\#}V^* = \int_X^{\oplus} A_x\, dm(x).$$

Suppose now that $\sigma \in \mathcal{S}^m(X)$ satisfies the condition that $\sigma_x \in \mathcal{S}^m$, $m > 0$, and $x \to \sigma^{\#}(x+t,s)$ is a continuous function from X to $C^\infty(\mathbb{R}^{2n})$. Assume also that the following estimates hold:

$$|\partial_t^\alpha \partial_s^\beta \sigma^{\#}(x+t,s)| \leqq C_{\alpha\beta}(1+|s|)^{-m} \tag{2.6}$$

with $C_{\alpha\beta}$ independent of x, and

$$|(\sigma^{\#})^{-1}(x,s)| \leqq C|s|^{-m} \tag{2.7}$$

for $|s| \geqq R > 0$. Thus, each operator $A_x = \mathrm{Op}(\sigma_x)$ will be elliptic. If $A^{\#}$ is self-adjoint, then each A_x will be as well.

It was noted in Theorem 2.4 that if $\varphi \in C_c^\infty(\mathbb{R}^n)$ then $\varphi(A_x) \in \Psi^{-\infty}$. We now consider the analogous situation for $A^{\#}$.

Theorem 2.8. *Let $A^{\#}$ be a self-adjoint elliptic pseudo-differential operator with $\sigma^{\#}$ satisfying* (2.6) *and* (2.7). *Then for any $\varphi \in C_0(\mathbb{R})$, $\varphi(A^{\#}) \in C(X) \rtimes \mathbb{R}^n$. In particular, for $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(A^{\#} - z)^{-1}$ belongs to $C(X) \rtimes \mathbb{R}^n$. If $\varphi \in C_c^\infty(\mathbb{R}^n)$, then $\varphi(A^{\#})$ is in the trace ideal.*

Proof. The kernel $K_{A_x}(s,t)$ of $\varphi(A_x)$ satisfies $K_{A_{x+r}}(s,t) = K_{A_x}(s+r,t+r)$. This implies that one has the inequality, for $\varphi \in C_c^\infty(\mathbb{R})$,

$$|\partial_t^\alpha \partial_s^\beta K_{\varphi(A_x)}(t,s)| \leqq C_{\alpha\beta N}(1+|t-s|)^{-N}$$

with $C_{\alpha\beta N}$ independent of x. Thus the kernel for $\varphi(A^{\#})$ satisfies

$$|\partial_t^\alpha \partial_s^\beta K_{\varphi(A^{\#})}(x+t,s)| \leqq C_{\alpha\beta N}(1+|t-s|)^{-N} \tag{2.9}$$

again with $C_{\alpha\beta N}$ independent of x. This implies that if $\psi \in C_c^\infty(\mathbb{R})$ then $\psi(A^{\#})$ is in the trace ideal. By approximation we see that if $\varphi \in C_0(\mathbb{R})$, then $\varphi(A^{\#}) \in C(X) \rtimes \mathbb{R}^n$. □

Remark. This result is the analog of a theorem of Alain Connes and John Roe to the effect that applying a function in $C_0^\infty(\mathbb{R})$ to a longitudinally elliptic operator yields a result in the foliation C^*-algebra. Unfortunately, our theorem does not follow from theirs since we do not assume that we are working with a foliation. This generality is needed for future applications to almost-periodic Schrödinger operators. On the other hand, we work with a rather restricted class of operators and foliations obtained from locally free $\mathbb{R}^n$ actions are relatively well behaved.

Proposition 2.10. *Let $A^\#$ be a self-adjoint elliptic pseudo-differential operator with $\sigma^\#$ satisfying* (2.6) *and* (2.7). *Let*

$$A^\# = \int_{\mathbb{R}} \lambda dE^\#(\lambda)$$

be the spectral decomposition of $A^\#$. Then, for any bounded Borel set $\Delta \subset \mathbb{R}, E^\#(\Delta)$ belongs to the ideal of elements of finite trace in $L^\infty(X) \rtimes \mathbb{R}^n$.

Proof. Since $\varphi(A^\#) \in C(X) \rtimes \mathbb{R}^n \subset L^\infty(X) \rtimes \mathbb{R}^n$ for $\varphi \in C_0(\mathbb{R})$, each $E^\#(\Delta)$ is in $L^\infty(X) \rtimes \mathbb{R}^n$. For a given bounded Borel set $\Delta \subset \mathbb{R}$, let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\varphi \geqq \chi_\Delta$, where χ_Δ is the characteristic function of Δ. Then $\varphi(A^\#) \geqq E^\#(\Delta)$. According to (2.8), $\varphi(A^\#)$ has finite trace. Therefore $\tau(E^\#(\Delta)) < \infty$. □

One further concept must be introduced before proving the spectral duality theorem. If A is a self-adjoint operator affiliated to a semi-finite von Neumann algebra $\mathscr{W}$ with trace τ, then the spectral density function of A is defined to be $N_A(\lambda) = \tau(E_A(\lambda))$. It is also called the "integrated density of states" in the mathematical physics literature [9], or the "state density" [10]. It is closely related to the notion of rotation number for Schrödinger operators on $\mathbb{R}$ with continuous potential [5]. The main fact we need is that if λ_0 is a point of discontinuity of $N_A(\lambda)$ then λ_0 is an eigenvalue of A. We will show in Sect. 4 that certain operators have purely continuous spectrum off of the set of discontinuities of their spectral density functions. It is indeed possible that $N_A(\lambda)$ has points of discontinuity. On the other hand, to give a feeling for the possibilities we consider an example where $N_A(\lambda)$ is continuous.

Suppose that we have an action of $\mathbb{R}$ on X. Let d/dt denote differentiation along the flow. Let $q_0, \ldots, q_k \in C(X)$ be real valued functions which are smooth along the flow and satisfy $|q_k(x)| \geqq \delta > 0$ on X. Consider the operator

$$A = q_0 + \frac{d}{dt} q_1 \frac{d}{dt} + \cdots + \frac{d^k}{dt^k} q_k \frac{d^k}{dt^k}$$

on $L^2(X \times \mathbb{R})$. Then we have

Proposition 2.11. *The spectral density function of A, $N_A(\lambda)$, is continuous.*

Proof. Since $|q_k| \geqq \delta > 0$, A is elliptic and $\varphi(A) \in C(X) \rtimes \mathbb{R}$ whenever $\varphi \in C_0(\mathbb{R})$. We must show that $E(\{\lambda_0\}) = 0$ for any fixed $\lambda_0 \in \mathbb{R}$, where E denotes the spectral resolution of A. Let $1 \geqq \varphi_m \geqq 0$ be a sequence of functions in $C_c^\infty(\mathbb{R})$ such that $\varphi_m(\lambda_0) = 1$ and $\lim\limits_{m \to \infty} \varphi_m = \delta_{\lambda_0}$. Let A_x denote A restricted to an orbit. For each m and $x \in X$,

$$(\varphi_m(A_x)f)(t) = \int_{\mathbb{R}} a_m(x+t, s-t) f(s) ds = \int_{\mathbb{R}} a_m(x+t, s) f(s+t) ds,$$

where a_m is the kernel of $\varphi_m(A^\#)$ which, according to Theorem 2.7, represents an element of $C(X) \rtimes \mathbb{R}^n$.

Let Tr denote the type I_∞ trace on $\mathscr{L}(L^2(\mathbb{R}))$ and let χ_r be the characteristic function of $(-r, r)$. Then the operator $\chi_r \varphi_m(A_x) \chi_r$ is trace class and

$$\operatorname{Tr}(\chi_r \varphi_m(A_x) \chi_r) = \operatorname{Tr}(\varphi_m(A_x) \chi_r) = \int_{-r}^{r} a_m(x+s, 0) ds.$$

Hence,

$$\int_X \mathrm{Tr}\,(\chi_r \varphi_m(A_x)\chi_r)dm(x) = \int_X \int_{-r}^{r} a_m(x+s,0)ds\,dm(x)$$
$$= 2r\tau(\varphi_m(A^\#)) \geqq 2r\tau(E^\#(\{\lambda_0\})) = 2r(N(\lambda_0+0) - N(\lambda_0-0)).$$

On the other hand, if $A_x = \int \lambda dE_\lambda^x$, then $E^x(\{\lambda_0\})$ is an operator of rank at most $2k$. This is because the eigenvalue problem $(A_x - \lambda_0)\varphi = 0$ has only $2k$ solutions, (which may or may not be square integrable). Since $\varphi_m(A_x) \to E^x(\{\lambda_0\})$ strongly, and Tr is a normal trace, we have

$$2k \geqq \int_X \mathrm{Tr}\,(\chi_r E^x(\{\lambda_0\})\chi_r)dm(x) = \lim_{m\to\infty} \int_X \mathrm{Tr}\,(\chi_r \varphi_m(A_x)\chi_r)dm(x) \geqq 2rN(\lambda_0).$$

Since r is arbitrary, we have $N(\lambda_0) = 0$.

3. The Spectral Duality Theorem

The idea behind the spectral duality theorem is the use of the two direct integral decompositions of $L^2(X \times \mathbb{R}^n)$. If A is an operator on $L^2(X \times \mathbb{R}^n)$ then spectral properties of the family A^s are "dual" to those of A_x. Roughly, if the A^s have pure point spectrum, then the A_x have purely continuous spectrum. These notions were first developed by Bellisard and Testard in [2]. In their setting, $\mathbb{R}^n$ was allowed to be a general locally compact abelian group H, X was required to be a compact abelian group and the action was given by a homomorphism of H into X. The present version does not require X to be a topological group. Although we work with $\mathbb{R}^n$, the proof goes through for a locally free action of a general locally compact abelian group.

Throughout this section we shall view $C(X) \rtimes \mathbb{R}^n$ as contained in the von Neumann algebra $L^\infty(X) \rtimes \mathbb{R}^n$. Note that the direct integral decompositions $VAV^* = \int_X^\oplus \pi_x(A)dm(x)$ and $(1 \otimes \mathscr{F})A(1 \otimes \mathscr{F}^*) = \int_{\mathbb{R}^n}^\oplus \pi^s(A)ds$ for $A \in C(X) \rtimes \mathbb{R}^n$ can be extended to elements of $L^\infty(X) \rtimes \mathbb{R}^n$. This is because $C(X) \rtimes \mathbb{R}^n$ is strongly dense in $L^\infty(X) \rtimes \mathbb{R}^n$. Thus, for each $B \in L^\infty(X) \rtimes \mathbb{R}^n$, VBV^* commutes with multiplication by $\varphi \in C(X)$, if the Hilbert space $L^2(X \times \mathbb{R}^n)$ is decomposed as $\int_X^\oplus L^2(\{x\} \times \mathbb{R}^n)dm(x)$. Therefore VBV^* has a direct integral decomposition $\int_X^\oplus B_x dm(x)$, where, for a.e. $x \in X$, B_x is a bounded operator on $L^2(\mathbb{R}^n)$. For the same reason, for any $B \in L^\infty(X) \rtimes \mathbb{R}^n$, $(1 \otimes \mathscr{F})B(1 \otimes \mathscr{F}^*)$ has a direct integral decomposition $\int_{\mathbb{R}^n}^\oplus B^s ds$ corresponding to $\int_{\mathbb{R}^n}^\oplus L^2(X \times \{s\})ds = L^2(X \times \mathbb{R}^n)$. Clearly, $x \mapsto B_x$ and $s \mapsto B^s$ are strongly measurable.

It can be shown that if $\mathscr{A}$ is a norm separable C^*-subalgebra of $L^\infty(X) \rtimes \mathbb{R}^n$, then there is a set $S \subset X$ of measure 0 such that for each $x \in X \backslash S$, $B \mapsto B_x$ defines a C^*-algebra homomorphism. The same holds true for $B \mapsto B^s$. If it happens that $B \in$

$C(X) \rtimes \mathbb{R}^n$, then $B_x = \pi_x(B)$ and $B^s = \pi^s(B)$. Since these decompositions are spatial, $\underset{m\to\infty}{s\text{-lim}}\, B_m = B$ in $L^\infty(X) \rtimes \mathbb{R}^n$ implies that $\underset{m\to\infty}{s\text{-lim}}\, B_{m,x} = B_x$ almost everywhere and $\underset{m\to\infty}{s\text{-lim}}\, B^s_m = B^s$ almost everywhere.

We must extend this to self-adjoint operators affiliated to $L^\infty(X) \rtimes \mathbb{R}^n$. First, note that if A is a self-adjoint operator in $L^\infty(X) \rtimes \mathbb{R}^n$ with spectral decomposition $A = \int \lambda dE_\lambda$, then for almost every $x \in X$, (respectively, almost every $s \in \mathbb{R}^n$), $\{E_{\lambda,x}\} = \{(E_\lambda)_x\}$ (respectively, $\{E^s_\lambda\} = \{(E_\lambda)^s\}$) is a resolution of the identity. In the decompositions $VAV^* = \int_X^\oplus A_x dm(x)$ and $(1 \otimes \mathscr{F})A(1 \otimes \mathscr{F}^*) = \int_{\mathbb{R}^n}^\oplus A^s ds$, one has $A_x = \int \lambda dE_{\lambda,x}$ almost everywhere, and $A^s = \int \lambda dE^s_\lambda$ almost everywhere. If one only assumes that A is affiliated with $L^\infty(X) \rtimes \mathbb{R}^n$, then by the usual approximation, it is still true that $A_x = \int \lambda dE_{\lambda,x}$ and $A^s = \int \lambda dE^s_\lambda$.

Before proving the theorem, we will describe A_x and A^s if A is a self-adjoint differential operator along the orbits. In what follows α will denote a multi-index, $\alpha = (\alpha_1, \ldots, \alpha_n)$. Let b_α be real valued functions in $C(X)$ which are smooth in the direction of the flow. Consider the operator on $L^2(X \times \mathbb{R}^n)$ given by

$$A = \sum b_\alpha(x) \hat{D}^\alpha,$$

where $\hat{D}^\alpha = \hat{D}_1^{\alpha_1} \cdots \hat{D}_n^{\alpha_n}$, and $\hat{D}_j f(x,r) = -i(\partial f/\partial t_j(x+t, r+t)|_{t=0})$. If $|\sum_{|\alpha|=m} b_\alpha(x)\xi^\alpha| \geqq \varepsilon|\xi|$, then A is affiliated to $L^\infty(X) \rtimes \mathbb{R}^n$ by Theorem 2.8. The operators A_x associated to A are given by

$$A_x = \sum b_\alpha(x + \cdot) D^\alpha,$$

where $D^\alpha = (-i)^{|\alpha|}(\partial/\partial t_1)^{\alpha_1} \cdots (\partial/\partial t_n)^{\alpha_n}$. Finally, $A^s = \sum b_\alpha(x)(\tilde{D} + s)^\alpha$, where $(\tilde{D} + s)^\alpha = (\tilde{D}_1 + s_1)^{\alpha_1} \cdots (\tilde{D}_n + s_n)^{\alpha_n}$ and $\tilde{D}_j f(x) = -i(\partial f(x+t)/\partial t_j)|_{t=0}$.

The following is the main result of the paper.

Theorem 3.1. *Let $A = \int \lambda dE_\lambda$ be a self-adjoint operator affiliated to $L^\infty(X) \rtimes \mathbb{R}^n$ with the property that for each bounded Borel set $S \subset \mathbb{R}$, $E(S)$ belongs to the trace ideal of $L^\infty(X) \rtimes \mathbb{R}^n$. Let F be the set of discontinuities of the increasing function $\tau(E_\lambda - E_0)$. If $\Delta \subset \mathbb{R}\backslash F$ is a Borel set such that for almost every $s \in \mathbb{R}^n$, A^s has pure point spectrum on Δ, then for almost every $x \in X$, A_x has purely continuous spectrum on Δ.*

For the operator A described explicitly above, the theorem states that, if for almost every $s \in \mathbb{R}^n$, $\sum b_\alpha(x)(\tilde{D}^\alpha + s)$ (as an operator on $L^2(X)$) has pure point spectrum on a Borel set $\Delta \subset \mathbb{R}\backslash F$, then $\sum b_\alpha(x + \cdot)D^\alpha$ (as an operator on $L^2(\mathbb{R}^n)$) has purely continuous spectrum, on Δ, for almost all $x \in X$.

If A is bounded from below, (e.g. if A is a Schrödinger operator), then F is the set of discontinuities of the spectral density function, $N_A(\lambda)$ of A.

Let us make a remark on the algebra $L^\infty(X) \rtimes \mathbb{R}^n$ before we start the proof of the theorem. If an operator $B \in L^\infty(X) \rtimes \mathbb{R}^n$ is of Hilbert–Schmidt class in the sense that $\tau(B^*B) < \infty$, then B is represented by a square-integrable kernel on $X \times \mathbb{R}^n$, i.e. $(B\xi)(x,t) = \int_{\mathbb{R}^n} b(x,s)\xi(x+s, t+s)ds$ with $b \in L^2(X \times \mathbb{R}^n)$. To see this, note that $\mathscr{C}_2 =$

$\{K \in L^\infty(X) \times \mathbb{R}^n : K$ is represented by a square-integrable kernel$\}$ is a strongly dense ideal in $L^\infty(X) \rtimes \mathbb{R}^n$. Indeed, if $K \in L^\infty(X) \rtimes \mathbb{R}^n$ is represented by $k \in L^2(X \times \mathbb{R}^n)$, then for $D \in L^\infty(X) \rtimes \mathbb{R}^n$, DK is represented by $V^{-1}DVk \in L^2(X \times \mathbb{R}^n)$, where $(Vf)(x,t) = f(x,-t)$. Let $\{\iota_m\}$ be a countable approximate identity in $\mathscr{C}_2$. For an operator B of Hilbert–Schmidt class, the strong convergence of $B\iota_m$ to B implies that there is a sequence of convex combinations of the $B\iota_m$'s, which are operators in $\mathscr{C}_2$, that converge to B in the Hilbert–Schmidt norm. Therefore B is also represented by a square-integrable kernel.

We start the proof with two lemmas.

Lemma 3.2. *Let $B \in C(X) \rtimes \mathbb{R}^n$ and let $\eta \in L^2(\mathbb{R}^n)$ be such that $\hat{\eta} \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where $\hat{\eta}(s) = \int_{\mathbb{R}^n} \eta(t) e^{i\langle s,t\rangle} dt$. Then, for almost every $t \in \mathbb{R}^n$*

$$\int_X |(B_x \eta)(t)|^2 dm(x) = \iint e^{i\langle t, s'-s\rangle} \hat{\eta}(s) \overline{\hat{\eta}(s')} \langle B^s 1, B^{s'} 1\rangle ds ds'.$$

Proof. Since $B_m \to B$ strongly implies $B_{m,x} \to B_x$ strongly for almost all x and $B_m^s \to B^s$ strongly for almost all s, it suffices to prove the identity for $B \in C_c(X \times \mathbb{R}^n)$. Suppose that B is represented by $b(x,t)$. Then we have $B_x = \pi_x(B)$ and $B^s = \pi(B^s)$, and

$$\begin{aligned}
&\int_X |(\pi_x(B)\eta)(t)|^2 dm(x) \\
&= \int_X \left(\int b(x+t,s)\eta(t+s)ds\right)\overline{\left(\int b(x+t,s')\eta(t+s')ds'\right)} dm(x) \\
&= \int_X \left[\int \left(\int b(x+t,k)e^{-i\langle s,k\rangle}dk\right)\left(\int \eta(t+k')e^{i\langle s,k'\rangle}dk'\right)ds\right] \\
&\quad \cdot \overline{\left[\int \left(\int b(x+t,p)e^{-i\langle s',p\rangle}dp\right)\left(\int \eta(t+p')e^{i\langle s',p'\rangle}dp'\right)ds'\right]} dm(x) \\
&= \int_X \left[\int (\pi^s(B)1)(x+t)\hat{\eta}(s)e^{-i\langle s,t\rangle}ds\right] \\
&\quad \cdot \overline{\left[\int (\pi^{s'}(B)1)(x+t)\hat{\eta}(s')e^{-i\langle s',t\rangle}ds'\right]} dm(x) \\
&= \iint e^{i\langle t,s'-s\rangle}\hat{\eta}(s)\overline{\hat{\eta}(s')}\langle \pi^s(B)1, \pi^{s'}(B)1\rangle ds ds'.
\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Let $B \in L^\infty(X) \rtimes \mathbb{R}^n$ be an element of Hilbert–Schmidt class. Then $\|B^s 1\|^2$, considered to be a function of s, is integrable on $\mathbb{R}$ and*

$$\int_{\mathbb{R}^n} \|B^s 1\|^2 ds = \tau(B^* B).$$

Proof. Let $f \in L^2(\mathbb{R}^n)$ and let $F(x,t) = f(t)$. From the definition of B^s, one obtains $[(1 \otimes \mathscr{F})BF](x,s) = (B^s 1)(x)\hat{f}(s)$, where $\hat{f}$ is the Fourier transform of f. Therefore $\int \|B^s 1\|^2 |\hat{f}(s)|^2 ds = \|BF\|^2$. Since B is Hilbert–Schmidt it is represented by a square-integrable kernel $b(x,t)$. Thus, $\|BF\|^2 = \iint |\int b(x,s) f(t-s) ds|^2 dt dm(x)$ and $\tau(B^*B) = \iint |b(x,s)|^2 ds dm(x)$. Let $\varphi_m \in L^2(\mathbb{R}^n)$ be such that

$$\hat{\varphi}_m(s) = \begin{cases} 1 & \text{if } |s| \le m \\ 0 & \text{otherwise} \end{cases}.$$

The operators defined by convolution with the φ_m act as an approximate identity on $L^2(\mathbb{R}^n)$. Hence $\lim_{m\to\infty} \| B\Phi_m \|^2 = \lim_{m\to\infty} \int\int|\int b(x,s)\varphi_m(t-s)ds|^2 dtdm(x) = \int\int|b(x,s)|^2 dsdm(x) = \tau(B^*B)$, where $\Phi_m(x,t)=\varphi_m(t)$. On the other hand, $\lim_{m\to\infty} \| B\Phi_m \|^2 = \lim_{m\to\infty} \int \| B^s 1 \|^2 |\hat{\varphi}_m(s)|^2 ds = \int \| B^s 1 \|^2 ds$. □

Proof of Theorem 3.1. For the operator $A=\int \lambda dE_\lambda$, let $\{E_{\lambda,x}:\lambda\in\mathbb{R}\}$ and $\{E^s_\lambda:\lambda\in\mathbb{R}\}$ be the spectral resolutions of A_x and A^s, respectively. Recall that $(E_\lambda)_x = E_{\lambda,x}$ for almost every x and $(E_\lambda)^s = E^s_\lambda$ for almost every s. Given a Borel set Ω, denote $E_x(\Omega) = \int_\Omega dE_{\lambda,x} (= (E(\Omega))_x$ almost everywhere) and $E^s(\Omega) = \int_\Omega dE^s_\lambda (= (E(\Omega))^s$ almost everywhere). Let Δ be the Borel set in the statement of the theorem. To prove the theorem, it suffices to show that there is a set $N\subset X$ of measure 0 such that the Borel measure $\mu_{\eta,x}(\Omega) = \langle E_x(\Omega\cap\Delta)\eta, \eta\rangle$ has no atoms whenever $x\in X\backslash N$ and $\eta\in C_c^\infty(\mathbb{R}^n)$. This is equivalent to showing that for the same x and η, $\mu_{\eta,x}\times\mu_{\eta,x}(\mathscr{D})=0$ [12], where $\mathscr{D}$ is the diagonal of $\Delta\times\Delta$. This will follow if one proves that $\int \mu_{\eta,x}\times\mu_{\eta,x}(\mathscr{D})dm(x)=0$ whenever $\eta\in C_c^\infty(\mathbb{R}^n)$.

There is a sequence of finite Borel partitions $\{\mathscr{P}_m\}$, of Δ such that $\mathscr{P}_{m+1}$ is finer than $\mathscr{P}_m$ and one has $\mathscr{D} = \bigcap_{m\geqq 1} [\bigcup_{S\in\mathscr{P}_m} S\times S]$. Since $\mu_{\eta,x}\times\mu_{\eta,x}(S) = \langle E_x(S)\eta,\eta\rangle^2$ for $S\subset\Delta$, it suffices to show that $\lim_{n\to\infty} I_n = 0$, where

$$I_n = \int_X \sum_{S\in\mathscr{P}_n} \langle E_x(S)\eta,\eta\rangle^2 dm(x).$$

By changing the order of integration and using Cauchy's inequality, one obtains

$$\begin{aligned}
I_n &= \int\int\overline{\eta(t)\eta(s)}\,[\sum_{S\in\mathscr{P}_n}\int_X (E_x(S)\eta)(t)(E_x(S)\eta)(s)dm(x)]dtds \\
&\leqq \int\int|\eta(t)\eta(s)|[\sum_{S\in\mathscr{P}_m}(\int_X |(E_x(S)\eta)(t)|^2 dm(x)\int_X |(E_x(S)\eta)(s)|^2 dm(x))^{1/2}]dtds \\
&\leqq \int\int|\eta(t)||\eta(s)|[\sum_{S\in\mathscr{P}_m}\int_X |(E_x(S)\eta)(t)|^2 dm(x)]^{1/2} \\
&\quad\cdot[\sum_{S\in\mathscr{P}_m}\int_X |(E_x(S)\eta)(s)|^2 dm(x)]^{1/2} dtds \\
&= [\int|\eta(t)|(\sum_{S\in\mathscr{P}_m}\int_X |(E_x(S)\eta)(t)|^2 dm(x))^{1/2} dt]^2 \\
&\leqq \|\eta\|_1^2 \operatorname{ess\,sup}_t(\sum_{S\in\mathscr{P}_m}\int_X |(E_x(S)\eta)(t)|^2 dm(x)).
\end{aligned}$$

Applying Lemma 3.2, one has

$$I_m \leqq \|\eta\|_1^2 \int\int|\hat{\eta}(s)||\hat{\eta}(s')||\sum_{S\in\mathscr{P}_m}\langle E^s(S)1, E^{s'}(S)1\rangle|dsds'.$$

The factor $|\sum_{S\in\mathscr{P}_m}\langle\cdot,\cdot\rangle|$ in the integral is bounded by 1. Hence, the dominated convergence theorem implies

$$\overline{\lim_{m\to\infty}} I_m \leqq \|\eta\|_1^2 \int\int|\hat{\eta}(s)\hat{\eta}(s')||\mu^{s,s'}(\mathscr{D})|dsds',$$

where $\mu^{s,s'}$ is the Borel measure on $\Delta\times\Delta$ whose value on a rectangle $U\times V$ is

$\langle E^s(U)1, E^{s'}(V)1\rangle$. We now wish to show that for each fixed s', $\int |\hat{\eta}(s)||\mu^{s,s'}(\mathcal{D})|ds = 0$. Let $\mathcal{E} = \{\lambda \in \Delta : E^{s'}(\{\lambda\})1 \neq 0\}$. Then it is easy to see that

$$\mu^{s,s'}(\mathcal{D}) = \sum_{\lambda \in \mathcal{E}} \langle E^s(\{\lambda\})1, E^{s'}(\{\lambda\})1\rangle.$$

Hence,

$$|\mu^{s,s'}(\mathcal{D})| \leqq \sum_{\lambda \in \mathcal{E}} \| E^s(\{\lambda\})1 \| \, \| E^{s'}(\{\lambda\})1 \|.$$

Since $\mathcal{E} \subset \Delta \subset \mathbb{R} \backslash F$, we have, according to Lemma 3.3, $\int \| E^s(\{\lambda\})1 \|^2 ds = \tau(E(\{\lambda\})) = 0$. Therefore, for a fixed s', $|\mu^{s,s'}(\mathcal{D})| = 0$, almost everywhere. This completes the proof. □

4. Applications

In this section we apply the previous theory to some geometric examples. As a first step we obtain a criterion for an operator to have pure point spectrum.

Proposition 4.1. *Let S and T be commuting essentially self-adjoint operators with the same domain. If S has pure point spectrum, with each eigenvalue of finite multiplicity, then T has pure point spectrum.*

Proof. Assume that there is a basis consisting of eigenvectors for S. Since S and T maps each eigenspace into itself. By diagonalizing T on each eigenspace one obtains a basis consisting of eigenvectors for T. □

Now, let $\mathbb{R}^n$ act locally freely and minimally on the closed manifold V. This is determined by a Lie algebra homomorphism of Lie $(\mathbb{R}^n)$ into $\mathcal{X}(V)$, the Lie algebra of vector fields on V. Assume that a Riemannian metric has been fixed on V and that the action is via orientation preserving isometries. Then the normal boundle to the orbits, $N\mathcal{F}$, has a holonomy invariant Riemannian metric and a unique torsion free connection, ∇, which preserves that metric. This data allows one to define a transverse signature operator

$$D_{\mathcal{S}} : \Omega^+ \to \Omega^-$$

as in [7], where $\Omega^{\pm}$ are the eigenspaces of the Hodge operator on $C^\infty(\Lambda^*(N\mathcal{F}))$. It is a 1st order differential operator on V which is invariant under the $\mathbb{R}^n$ action and elliptic transverse to the orbits.

Next, consider a self-adjoint invariant elliptic operator on $\mathbb{R}^n$. If one views it as an element of the universal enveloping algebra $\mathfrak{U}(\text{Lie}\,\mathbb{R}^n)$), then its image in $\mathfrak{U}(\mathcal{X}(V))$ defines an operator $D: C^\infty(V) \to C^\infty(V)$, elliptic along the orbits. Let $\mathcal{D} = D \otimes_\nabla I$ denote the extension of D to Ω^+. Since $D_{\mathcal{S}}$ is invariant under the action of $\mathbb{R}^n$, it commutes with $\mathcal{D}$.

We shall apply the previous theory to $\mathcal{D}$. Since $\mathcal{D}$ acts along the orbits, it yields $\tilde{\mathcal{D}}$ on $L^2(V \times \mathbb{R}^n)$, and it is given, as in Sect. 2, by an invariant symbol. Thus, we may consider $\tilde{\mathcal{D}}_s$ and $\tilde{\mathcal{D}}_x$. Each $\tilde{\mathcal{D}}^s$ has the property that it commutes with $D_{\mathcal{S}}$ and that $\tilde{\mathcal{D}}^s + D^*_{\mathcal{S}} D_{\mathcal{S}}$ is self-adjoint and elliptic. By Proposition 4.1 $(\tilde{\mathcal{D}}^s)^2$, and hence $\tilde{\mathcal{D}}^s$, has pure point spectrum for each s. Thus, by the spectral duality theorem we obtain

Theorem 4.2. *The operator $D \otimes_{\nabla} I$ has purely continuous spectrum off the discontinuities of the spectral density function $N_{\mathscr{D}}(\lambda)$.*

A variant of the above example having some independent interest goes as follows. Let $D: C^{\infty}(\mathbb{R}^n) \to C^{\infty}(\mathbb{R}^n)$ be a self-adjoint periodic elliptic differential operator of order k. Thus, there is a lattice $\Gamma \subset \mathbb{R}^n$ such that $D^{\gamma} = D$ for any $\gamma \in \Gamma$. Let $\alpha: \Gamma \to T^k$ be an embedding of Γ as a dense subset of a torus obtained by choosing independent irrational numbers $\alpha_2, \ldots, \alpha_k$ and setting $\alpha(x_1, \ldots, x_k) = (e^{i\pi x_1}, e^{i\pi\alpha_2 x_2}, \ldots, e^{i\pi\alpha_k x_k})$. Then $V = \mathbb{R}^n x_{\Gamma} T^n$ is a compact manifold acted upon locally freely by $\mathbb{R}^n$, and D induces an operator $\hat{D}$ on V which is elliptic along the orbits. This operator yields $\tilde{D}$ on $L^2(V \times \mathbb{R}^n)$ such that the symbol of $\tilde{D}$ is invariant in the sense described in Sect. 2. Thus the theory applies. The operators $\tilde{D}^s$ are all invariant under the action of T^n on V, hence they commute with the Laplacian $\Delta_T n$, (cf. [1]). One then has $\tilde{D}^s + (\Delta_T n)^{k/2}$ self-adjoint elliptic and commuting with $\tilde{D}^s$. As before, we deduce that $\tilde{D}^s$ has pure point spectrum. Thus, $\tilde{D}_x$ has purely continuous spectrum off the discontinuities of $N(\lambda)$, for almost all x. However, in the present case, each of the operators $\tilde{D}_x$ is the same as the original operator D, so one obtains

Theorem 4.3. *Let D be a periodic elliptic operator on $\mathbb{R}^n$. Then, off of the discontinuities of the spectral density function, D has no eigenvalues with eigenfunctions in $L^2(\mathbb{R}^n)$.*

References

1. Atiyah, M. F.: *Elliptic operators and compact groups.* Lecture Notes in Mathematics Vol. **41**, Berlin, Heidelberg, New York: Springer 1970
2. Bellisard, J., Testard, D.: Almost periodic hamiltonians: An algebraic approach, pre-print, 1981, Centre de Physique Theorique, CNRS Marseille
3. Connes, A.: Sur la theorie non commutative de l'interation. Lecture Notes in Mathematics, Vol. **725**. Berlin, Heidelberg, New York: Springer 1979
4. ———, A survey of foliations and operator algebras. Proc. Symposia Pure Math. **38**, Part I, 521–628 (1982)
5. Johnson, R., Moser, J.: The rotation number for almost periodic potentials. Commun. Math. Phys. **84**, 403–438 (1982)
6. Kozlov, S. M., Shubin, M. A.: On the coincidence of the spectra of random elliptic operators. Math. USSR Sbornik **51**, 455–471 (1985)
7. Lazarov, C.: Spectral invariants of foliations. Mich. Math. J. **33**, 231–243 (1986)
8. Moore, C., Schochet, C.: Global analysis on foliated spaces. Berlin, Heidelberg, New York: Springer (to appear)
9. Simon, B.: Almost periodic Schrödinger operators: A review. Adv. Appl. Math. **3**, 463–490 (1982)
10. Shubin, M. A.: The spectral theory and the index of elliptic operators with almost periodic coefficients. Russ. Math. Surveys **34**, 109–157 (1979)
11. Shubin, M. A.: Spectrum and its distribution function for a transversally elliptic operator. Funct. Anal. Appl. **15**, 74–76 (1981)
12. Wiener, N.: The fourier integral and certain of its applications. Cambridge: The University Press 1933

Communicated by C. H. Taubes

Received June 19, 1986