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On Charge Conjugation

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Abstract. The group of automorphisms of the conformal algebra su(2, 2) has four components giving the usual four components of symmetries of space time. Only two of these components extend to symmetries of the conformal superalgebra – the identity component and the component which induces the parity transformation, P, on space time. There is no automorphism of the conformal superalgebra which induces T or PT on space time. Automorphisms of su(2, 2) which belong to these last two components induce transformations on the conformal superalgebra which reverse the sign of the odd brackets. In this sense conformal supersymmetry prefers CP to CPT. The operator of charge conjugation acting on spinors, as is found in the standard texts, induces conformal inversion and hence a parity transformation on space time, when considered as acting on the odd generators of the conformal superalgebra. Although it commutes with Lorentz transformations, it does not commute with all of su(2, 2). We propose a different operator for charge conjugation. Geometrically it is induced by the Hodge star operator acting on twistor space. Under the known realization of conformal states from the inclusion SU(2, 2) $\rightarrow$Sp(8) and the metaplectic representations, its action on states is induced by the unique (up to phase) antilinear intertwining operator between the two metaplectic representations. It is consistent with the split orthosymplectic algebras and hence, by the inclusion of the superconformal in the orthosymplectic, with the orthosymplectic algebra.

The conformal superalgebra of Minkowski space-time is a special case of a class of superalgebras defined in [12]. We shall give a definition of a subclass of these superalgebras in Sect. 2, and will be interested in studying their automorphisms. We begin with some notational preliminaries.

Let V be a complex vector space endowed with a (pseudo) Hermitian scalar product (,). That is, (,) assigns a complex number (u, v) to a pair of vectors u and v in V and satisfies

$$(au_1 + bu_2, v) = a(u_1, v) + b(u, v) \qquad \text{linearity in } u\,,$$

$$(v, u) = (\overline{u, v}) \qquad \text{Hermitian property}\,,$$

and

$$(u,v)=0 \quad \text{for all } u \text{ implies} \quad v=0 \qquad \text{non-degeneracy}.$$

This last condition replaces the usual condition $(u,u)>0$ for $u \neq 0$ in the axioms for a Hilbert space which is the reason that some authors use the prefix pseudo. We won't, but simply call (,) a Hermitian form. Just as in the real case any such form on a finite dimensional vector space has a signature p, q. That is, we can find an isomorphism of V with the vector space $\mathbb{C}^n$ (of all column vectors – n tuplets – of complex numbers) with the Hermitian form

$$(\mathbf{z}, \mathbf{w})_{p,q} = z_1\bar{w}_1 + \ldots + z_p\bar{w}_p - z_{p+1}\bar{w}_{p+1} \ldots - \bar{z}_n\bar{w}_n,$$

where $n=p+q$.

If V and W are complex vector spaces, a map $L: V \to W$ is called linear if

$$L(au+bv) = aLu + bLv \qquad \begin{array}{l} a, b \in \mathbb{C} \\ u, v \in V, \end{array}$$

and anti-linear if $L(au+bv)=\bar{a}Lu+\bar{b}Lv$. L is called an isometry if $(Lu, Lu)=(u,u)$ for all $u \in V$. An isometry can be linear or anti-linear. A linear isometry $L: V \to V$ is called unitary while an antilinear isometry is called anti-unitary. A unitary map satisfies $(Lu, Lv)=(u,v)$, while an antiunitary one satisfies $(Lu, Lv)=(\overline{u,v})=(v,u)$. But (especially in the split case, $p=q$) we can also consider anti-isometries which satisfy $(Lu, Lu)=-(u,u)$. They also come in linear and antilinear versions which satisfy $(Lu, Lv)=-(u,v)$, and $(Lu, Lv)=-(\overline{u,v})$, respectively.

1. Weyl Spinors and the Complex Hodge Star Operator

Recall that the group $\mathrm{SL}(2,\mathbb{C})$ (all complex two by two matrices of determinant one) is the universal (in fact double) cover of the proper Lorentz group. Indeed we can identify the vector (x_0, x_1, x_2, x_3) in Minkowski space with the skew adjoint matrix

$$X = i\begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix}$$

so that

$$\det X = -x_0^2 + x_1^2 + x_2^2 + x_3^2 .$$

Then $R \in \mathrm{SL}(2,\mathbb{C})$ acts on X by sending X into RXR^*, where * denotes adjoint (relative to the standard positive Hermitian scalar product on $\mathbb{C}^2$). Since $\det R = \det R^* = 1$, this gives a real representation of $\mathrm{SL}(2,\mathbb{C})$ on Minkowski space as Lorentz transformations. The only elements which act trivially are I and $-I$. Hence it is a double cover.

The group $\mathrm{SL}(2,\mathbb{C})$ has two inequivalent complex representations on $\mathbb{C}^2$. They are given by

$$\begin{Bmatrix} x \\ y \end{Bmatrix} \rightsquigarrow R \begin{Bmatrix} x \\ y \end{Bmatrix}$$

and

$$\begin{Bmatrix} x \\ y \end{Bmatrix} \rightsquigarrow R^{*-1} \begin{Bmatrix} x \\ y \end{Bmatrix},$$

and are usually labeled by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. These are the Weyl spinors (called undotted and dotted Weyl spinors in the physics literature). They are not equivalent as complex representations, in the sense that there does not exist any complex linear isomorphism $L: \mathbb{C}^2 \to \mathbb{C}^2$ such that $LRL^{-1} = R^{*-1}$. (This is clear because the matrix entries on the left are complex linear in the entries of R while those on the right are antilinear.) On the other hand there is an anti-unitary map which does implement the above equation. Indeed, define $\circledast: \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\circledast \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \bar{y} \\ -\bar{x} \end{Bmatrix}.$$

Then $\circledast$ is clearly antilinear and satisfies

$$(\circledast u, \circledast v) = (v, u), \tag{1.1}$$

and

$$\circledast^2 = -\mathrm{id}. \tag{1.2}$$

For any two by two matrix define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^a = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1.3}$$

This is just the "adjoint" operation that appears in Cramer's rule. Thus

$$AA^a = (\det A) \tag{1.4}$$

and

$$(AB)^a = B^a A^a. \tag{1.5}$$

An immediate verification shows that

$$\circledast A = A^{a*} \circledast. \tag{1.6}$$

In particular if $\det R = 1$ so that $R^a = R^{-1}$, we get

$$\circledast R = R^{*-1} \circledast \quad \text{if} \quad \det R = 1. \tag{1.7}$$

Thus, if we regard $\mathbb{C}^2$ as a real four dimensional vector space, then $\circledast$ intertwines the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations as four dimensional representations over the real numbers, even though they are inequivalent as two dimensional representations over the complex numbers. We should also record the infinitesimal version of (1.7) which is

$$\circledast A = -A^* \circledast \quad \text{if} \quad \operatorname{tr} A = 0, \qquad A \in gl(2, \mathbb{C}). \tag{1.8}$$

For any complex vector space we let V^* denote the dual space, the space of all complex linear functions on V. We let $\langle l, u \rangle$ denote the value of a linear function

$l \in V^*$ on the vector $u \in V$. Thus, for example, if $\mathbb{C}^2$ is the vector space of column two vectors, then we may think of $\mathbb{C}^2$ as the space of row vectors with

$$\left\langle \{l_1, l_2\}, \begin{Bmatrix} x \\ x \end{Bmatrix} \right\rangle = l_1 x_1 + l_2 x_2 .$$

A Hermitian form (,) on V induces an antilinear map $J: V \to V^*$ given by

$$\langle Jv, u \rangle = (u, v) . \tag{1.9}$$

That is, Jv is that linear function whose value on any $u \in V$ is given by (u, v). For the case of $\mathbb{C}^2$, for example, $J \begin{Bmatrix} w \\ z \end{Bmatrix} = \{\bar{w}, \bar{z}\}$. Given vectors u and $v \in V$, we can define the linear transformation $H(u, v) = u \otimes Jv$ by

$$H(u, v)w = (w, v)u . \tag{1.10}$$

Thus, for $\mathbb{C}^2$

$$H\begin{pmatrix} x & w \\ y ' & z \end{pmatrix} = \begin{pmatrix} x\bar{w} & x\bar{z} \\ y\bar{w} & y\bar{z} \end{pmatrix}$$

as a matrix. It follows directly from the definitions that

$$H(u, v) = H(v, u)^* , \tag{1.11}$$

and that, for $A: V \to V$ a linear transformation,

$$H(Au, Av) = AH(u, v)A^* . \tag{1.12}$$

In particular the map S given by

$$S(u, v) = i[H(u, v) + H(v, u)] \tag{1.13}$$

is symmetric and real bilinear (i.e., bilinear only over the real numbers) and takes values in the real Lie algebra $u(V)$ of all skew adjoint linear transformations on V. In view of (1.12) it is equivariant for the action of $\mathrm{GL}(V)$. For the case of $V = \mathbb{C}^2$ we thus get a symmetric map from Weyl spinors (of type $(\frac{1}{2}, 0)$) into the translations which is equivariant for the action of $\mathrm{SL}(2, \mathbb{C})$. This is the fundamental building block for space time supersymmetry, cf. [3 or 15].

The space of Dirac spinors is the direct sum $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of Weyl spinors of opposite type. We shall write a typical Dirac spinor as $w = \begin{pmatrix} u \\ v \end{pmatrix}$ and the group $\mathrm{SL}(2, \mathbb{C})$ acts diagonally, i.e.,

$$Rw = \begin{pmatrix} Ru \\ R^{*-1}v \end{pmatrix} .$$

On the space of Dirac spinors we introduce the Hermitian form of signature 2, 2 given by

$$(w, w') = (u, v') + (v, u') . \tag{1.14}$$

This Hermitian form is clearly $\mathrm{GL}(2, \mathbb{C})$ invariant. Up to putting arbitrary non-zero real factors in front of the summands on the right, it is the only $\mathrm{SL}(2, \mathbb{C})$

invariant Hermitian form. We may define

$$\gamma(X)=\begin{pmatrix}0 & X\\ X^a & 0\end{pmatrix} \tag{1.15}$$

for X in Minkowski space (and thought of as a skew adjoint 2×2 matrix).

Then

$$R\gamma(X)R^{-1}=\gamma(RXR^*) \qquad R\in \mathrm{SL}(2,\mathbb{C})$$

and

$$\gamma(X)^2=\det(X)I_4\,, \tag{1.16}$$

so that the $\gamma(X)$ determine a complex representation of the Clifford algebra, $C(3,1)$, of Minkowski space. They are Dirac matrices, to use the language of the physics literature.

In the physics literature one finds an operator, C, acting on Dirac spinors called "charge conjugation." C is to be a map of $\mathbb{C}^4\to\mathbb{C}^4$ which is $\mathrm{SL}(2,\mathbb{C})$ invariant and which interchanges the two Weyl components. It is also to have no effect on space time, and satisfy $C^2=1$. The first two conditions already determine C up to a scalar multiple. So let us take

$$C\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\circledast v\\ -\circledast u\end{pmatrix}. \tag{1.17}$$

Then C is antilinear with

$$C^2=\mathrm{Id} \quad \text{and} \quad (Cw,Cw')=-\overline{(w,w')}\,.$$

Furthermore

$$RC\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}R\circledast v\\ -R^{*-1}\circledast u\end{pmatrix}=\begin{pmatrix}\circledast R^{*-1}\\ -\circledast Ru\end{pmatrix}=CR\begin{pmatrix}u\\ v\end{pmatrix},$$

so

$$RC=CR\,. \tag{1.18}$$

Also

$$\gamma(X)C\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}0 & X\\ X^a & 0\end{pmatrix}\begin{pmatrix}\circledast v\\ -\circledast u\end{pmatrix}=\begin{pmatrix}-X\circledast u\\ X^a\circledast v\end{pmatrix}=\begin{pmatrix}\circledast X^a u\\ -\circledast Xv\end{pmatrix}=C\begin{pmatrix}0 & X\\ X^a & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

Thus

$$C\gamma(X)C^{-1}=\gamma(X)\,, \tag{1.19}$$

which expresses the fact that C has no effect on space time. (We shall modify our view of this interpretation in Sect. 3. We shall introduce a different operator as our choice for charge conjugation in Sect. 5.)

Notice that we can consider the real four dimensional subspace M of $\mathbb{C}^4$ consisting of those w which satisfy

$$Cw=w\,, \quad \text{or} \quad w=\begin{pmatrix}u\\ -\circledast u\end{pmatrix}.$$

In view of (1.19), $\gamma(X)M \subset M$. In other words, the γ's act as real transformations of the real four-dimensional space M, and, of course, will satisfy (1.16). The elements of M are called Majorana spinors and a Majorana representation is a matrix representation of the γ relative to a basis of M, so that in a Majorana representation all the matrix entries of the γ's are real. At this point it is essential that we use the signature $+++-$ for our metric on Minkowski space rather than $+---$. (This point was emphasized to me by Mitchell Rothstein and uncovered an error in sign in [13].) Indeed the Clifford algebra $C(3,1)$ is isomorphic to the algebra $\mathbb{R}(4)$ of all four by four real matrices. On the other hand the algebra $C(1,3)$ is isomorphic to $\mathbb{H}(2)$, the algebra of two by two matrices over the quaternions. It has no representation by real matrices in any dimension less than eight. See [6] for a complete discussion of the Clifford algebras of arbitrary signature and their associated spin representations.

The equations (1.1)–(1.8) can all be readily verified by direct calculation. All that is involved is manipulation with complex two vectors and two by two matrices. On the other hand they have a natural extension to higher dimensions once it is recognized that the operator $\circledast$ is just an example of the Hodge star operator. Here are the details: Let V be an n-dimensional complex vector space endowed with a Hermitian form. Then each of the exterior powers $\Lambda^k(V)$ inherits a Hermitian form determined by

$$(v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k) = \det((v_i, w_j)).$$

In particular $\Lambda^n(V)$ is a one dimensional space with Hermitian form. Pick some element $\delta \in \Lambda^n(V)$ with $|(\delta,\delta)|=1$. [Actually $(\delta,\delta)=(-1)^q$, where p,q is the signature.] Define

$$\circledast : \Lambda^k(V) \to \Lambda^{n-k}(V)$$

by

$$(v, \circledast u)\delta = u \wedge v, \quad \text{all} \quad v \in \Lambda^{n-k}(V). \tag{1.20}$$

A direct computation in terms of an "orthonormal" basis shows that the correct generalization of (1.1) is

$$(\circledast u, \circledast v) = (-1)^q \overline{(u, v)}. \tag{1.1$'$}$$

Similarly

$$\circledast\circledast = (-1)^{k(n-k)+q}\mathrm{id}. \tag{1.2$'$}$$

If $A : \Lambda^k(V) \to \Lambda^k(V)$ is any linear map, define $A^a : \Lambda^{n-k}(V) \to \Lambda^{n-k}(V)$ by

$$Au \wedge v = u \wedge A^a v. \tag{1.21}$$

Then (1.5) and (1.6) follow directly from the definitions. Everything else then follows as before except, of course, that $\det X$ is not quadratic unless X is a two by two matrix.

Another identity which we will need later on and which can be verified directly in the $\mathbb{C}^2$ case is

$$H(u,v)^a = (-1)^q H(\circledast v, \circledast u). \tag{1.22}$$

Indeed let u, v, $z \in \Lambda^k V$ and $w \in \Lambda^{n-k} V$. Then

$$H(u,v)z \wedge w = (z,v)u \wedge w = (w, \circledast(z,v)u)\delta = (w, (\overline{z,v})\circledast u)\delta = (z,v)(w, \circledast u)\delta .$$

By (1.1)′ this equals

$$(-1)^q(\circledast v, \circledast z)(w, \circledast u)\delta = (-1)^q z \wedge (w, \circledast u)\circledast v = (-1)^q z \wedge H(\circledast v, \circledast u)w ,$$

proving (1.22).

2. Unitary Lie Algebras and Lie Superalgebras

Let $g = g_{\text{even}} \oplus g_{\text{odd}}$ be a direct sum of vector spaces. Suppose that g_{even} is a Lie algebra and that we are given a representation of g_{even} on g_{odd} and a bilinear map $[\ ,\ ]_{\text{odd}}$ of $g_{\text{odd}} \times g_{\text{odd}} \to g_{\text{even}}$ which satisfies

$$[A, [u,v]_{\text{odd}}] = [Au, v]_{\text{odd}} + [u, Av]_{\text{odd}} , \tag{2.1}$$

and either the $+$ or $-$ version of

$$[u,v]_{\text{odd}} w = \pm [u,w]_{\text{odd}} v - [v,w]_{\text{odd}} u . \tag*{(2.2)$_\pm$}$$

In these equations u, v, w are in g_{odd} and $A \in g_{\text{even}}$. There are now two interesting cases: The map $[\ ,\ ]_{\text{odd}}$ is antisymmetric and $(2.2)_+$ holds. We then define

$$[A, u] = Au = -[u, A] \tag{2.3}$$

and drop the subscript odd. Then we have defined an antisymmetric bracket $[\ ,\ ]$ mapping $g \times g \to g$ and (2.1) and $(2.2)_+$ (together with the fact that $[A,B]u = ABu - BAu$ because we have a representation of g_{even} on g_{odd}) gives the remaining Jacobi identities so we have made g into a Lie algebra. It is an ordinary Lie algebra with a $\mathbb{Z}_2$ grading in the sense that

$$[g_{\text{even}}, g_{\text{even}}] \subset g_{\text{even}} ,$$
$$[g_{\text{even}}, g_{\text{odd}}] \subset g_{\text{odd}} ,$$
$$[g_{\text{odd}}, g_{\text{odd}}] \subset g_{\text{even}} .$$

The other interesting case is where $[\ ,\ ]_{\text{odd}}$ is symmetric and $(2.2)_-$ holds. Then (2.3) makes g into a Lie superalgebra, with (2.1) and $(2.2)_-$ giving the super Jacobi identities, cf. [3]. In [12] we introduced the notion of a Hermitian Lie algebra in which we are given a bilinear map of $g_{\text{odd}} \times g_{\text{odd}} \to g^{\mathbb{C}}_{\text{even}}$ (the complexification of g_{even}) which is Hermitian, and whose real part gives a Lie superalgebra and whose imaginary part gives a Lie algebra. Rather than discuss the general theory, let us illustrate how it works for Hermitian vector spaces. Let W be a Hermitian vector space and let $H: W \times W \to gl(W)$ be defined as in (1.10). There are two ways we can make a skew Hermitian operator from $H(u,v)$: We can antisymmetrize:

$$I(u,v) = H(u,v) - H(v,u) \tag{2.4}$$

is skew Hermitian by (1.11). We can also symmetrize and multiply by i:

$$i[H(u,v) + H(v,u)]$$

is skew Hermitian. Let us first consider the antisymmetrization.

Suppose that the vector space W contains an element e such that $(e,e)=-1$. Let $V=e^{\perp}$ and let u, v, w be elements of V. Then it follows from the definitions (1.10) and (2.4) that

$$I(e,u)I(e,v)e=(v,u)e$$

and

$$I(e,u)I(e,v)w=(w,v)u=H(u,v)w\,.$$

Thus

$$\begin{aligned}[][I(e,u),I(e,v)]e&=-\operatorname{tr}I(u,v)e\,,\\ [I(e,u),I(e,v)]w&=I(u,v)w\,.\end{aligned}\tag{2.5}$$

We can interpret the relations (2.5) as follows: Let g be the Lie algebra $\mathrm{su}(W)$. Let $g_{\mathrm{even}}=u(V)$. We can consider g_{even} as a subalgebra of g by letting $A\in u(V)$ act on e by $Ae=-(\operatorname{tr}A)e$. Then we can take $g_{\mathrm{odd}}=V$, where each $v\in V$ is identified with $I(e,v)$. We have thus identified

$$u(V)\oplus V \quad\text{with}\quad \mathrm{su}(W)\,. \tag{2.5$'$}$$

Of course, if we start with the vector space V we can always adjoin an e with $(e,e)=-1$ to get W. This shows that the map $[u,v]_{\mathrm{odd}}=H(u,v)-H(v,u)$ satisfies the axioms (2.1) and $(2.2)_+$ and that the total algebra obtained is isomorphic to $\mathrm{su}(V\oplus\mathbb{C}^-)$. For example, taking $V=\mathbb{C}^{2,2}$, we see that $u(2,2)\oplus\mathbb{C}^{2,2}$ fit together to give $\mathrm{su}(2,3)$.

Notice, by the way that the map H is already equivariant by (1.10), since $A^*=A^{-1}$ for A unitary.

Now let us consider the symmetrization. Define

$$S(u,v)=i[H(u,v)+H(v,u)]\,. \tag{2.6}$$

As before, adjoin a vector e with $(e,e)=-1$ but now consider W as a super vector space, i.e., $W=W_{\mathrm{even}}+W_{\mathrm{odd}}$, where $W_{\mathrm{even}}=V$ and $W_{\mathrm{odd}}=\mathbb{C}e$. Extend $A\in u(V)$ to be defined on W_{odd} by letting $\hat{A}e=(\operatorname{tr}A)e$ so now the supertrace [7] of $\hat{A}$ vanishes. Let $\hat{S}$ denote the symmetrized H for W considered as a Hermitian vector space, so, for example $\hat{S}(e,u)z=i((z,u)e+(z,e)u)$. Then, as before, $\hat{S}(e,u)\hat{S}(e,v)e=(v,u)e$ and $\hat{S}(e,u)\hat{S}(e,v)w=(w,v)u$, so

$$[\hat{S}(e,u)\hat{S}(e,v)+\hat{S}(e,v)\hat{S}(e,u)]e=[(u,v)+(v,u)]e$$

and

$$[\hat{S}(e,u)\hat{S}(e,v)+\hat{S}(e,v)\hat{S}(e,u)]w=(w,v)u+(w,u)v\,.$$

Thus identify $v\in V$ with $\hat{S}(e,v)$ and define the $[\;,\;]_{\mathrm{odd}}\colon V\times V\to u(V)\oplus u(1)\subsetneqq gl(W)$ by

$$[u,v]_{\mathrm{odd}}=S(u,v)\hat{}=i[\hat{S}(e,u)\hat{S}(e,v)+\hat{S}(e,v)\hat{S}(e,u)]\,.$$

Since, up to a factor of i, $[\;,\;]_{\mathrm{odd}}$ is given by an anticommutator, condition $(2.2)_-$ holds. We thus get a Lie superalgebra. In the terminology of [3 or 12] it is the superalgebra $\mathrm{su}(V/\mathbb{C})$. For the case $V=\mathbb{C}^{2,2}$ this is the superconformal algebra, cf. [3 or 13].

The Lie algebra $\mathrm{su}(V\oplus\mathbb{C}^-)$ and the Lie superalgebra $\mathrm{su}(V/\mathbb{C})$ are both built entirely out of the Hermitian structure of V. Hence any unitary map $A\colon V\to V$

induces an automorphism of these algebras. Indeed, this is a consequence of (1.10) as we remarked above. But let us also consider the case of an antilinear anti-isometry L. Then

$$H(Lu,Lv)w=(w,Lv)Lu=-(\overline{L^{-1}w,v})Lu=-L((L^{-1}w,v)u)=-L[H(u,v)L^{-1}w]$$

In other words

$$H(Lu,Lv)=-LH(u,v)L^{-1}. \tag{2.7}$$

Then $iH(Lu,Lv)=LiH(u,v)L^{-1}$ by the antilinearity of L, so, by (2.6),

$$S(Lu,Lv)=LS(u,v)L^{-1}. \tag{2.8}$$

Thus $u\to Lu$, $A\to LAL^{-1}$ defines an automorphism of the superalgebra su$(V/\mathbb{C})$. In particular, the operator C defines an involutive automorphism of the entire superconformal algebra su$(2,2/1)$. Let us see what the fixed subalgebra is. For the odd piece we have already verified that the fixed vectors are Majorana spinors. Write the most general matrix in $\mathbb{C}^{2,2}$ in block form. Then direct computation shows that

$$C\begin{pmatrix} A & X \\ Y & D\end{pmatrix}C^{-1}=\begin{pmatrix} D^{*a} & -Y^{*a} \\ -X^{*a} & A^{*a}\end{pmatrix}. \tag{2.9}$$

The condition that $\begin{pmatrix} A & X \\ Y & D\end{pmatrix}$ belong to $u(2,2)$ is

$$A=-D^*, \quad X=-X^*, \quad Y=-Y^*.$$

Now for two by two matrices $A=-A^a$ if and only if $\operatorname{tr}A=0$. Thus the subalgebra in question consists of all matrices of the form

$$\begin{pmatrix} A & X \\ X^a & -A^*\end{pmatrix}, \quad A\in \mathrm{sl}(2,\mathbb{C}).$$

As an algebra, it is isomorphic to $o(2,3)$. In particular, C does not commute with all of su$(2,2)$.

We close this section with an observation in the split case where $p=q$. Suppose we choose two complementary totally null subspaces (like the $(\frac{1}{2},0)$ and the $(0,\frac{1}{2})$ type spinors on the $\mathbb{C}^{2,2}$ example). Call one subspace g_1 and the other g_{-1}, so that $V=g_{\mathrm{odd}}=g_{-1}\oplus g_1$. The elements of $g_{\mathrm{even}}=u(V)$ have block form

$$\begin{pmatrix} A & X \\ Y & -A^*\end{pmatrix}, \quad X=-X^*, \quad Y=-Y^*.$$

Thus we can let

$$g_2 \text{ consist of all } \begin{pmatrix} 0 & X \\ 0 & 0\end{pmatrix} X=-X^*,$$

$$g_0 \text{ consist of all } \begin{pmatrix} A & 0 \\ 0 & -A^*\end{pmatrix},$$

$$g_{-2} \text{ consist of all } \begin{pmatrix} 0 & 0 \\ Y & 0\end{pmatrix} Y=-Y^*.$$

Then

$$g = \bigoplus_{-2}^{+2} g_i$$

is a $\mathbb{Z}$ gradation which refines the $\mathbb{Z}_2$ gradation in the sense that $[g_i, g_j] \subset g_{i+j}$ and $g_{\text{odd}} = g_{-1} \oplus g_1$ and $g_{\text{even}} = g_{-2} \oplus g_0 \oplus g_2$. This is true for the Lie algebra and Lie superalgebra cases.

Notice that $g_1 \oplus g_2$ forms a subalgebra. In the case of $\mathbb{C}^{2,2}$ this is the space time supersymmetry algebra provided we identify g_2 with space time translations.

3. Conformally Flat Geometry and Supergeometry

In this section we wish to give a geometric interpretation to the construction of the preceding sections. The basic fact is that su(2, 2) is the Lie algebra of all infinitesimal conformal transformations of Minkowski space. Before explaining this we review the basics of conformal geometry.

Let U be a *real* finite dimensional vector space with a real symmetric nondegenerate bilinear form of type p, q. It is a theorem of Liouville that if $\dim U > 2$ then the group of conformal transformations of U is a finite dimensional Lie group isomorphic to $O(p+1, q+1)$. More precisely, let $U = \mathbb{R}^{p,q}$ and consider the quadratic form of type $p+1$, $q+1$ on $\mathbb{R}^{p+1,q+1} = \mathbb{R}^{n+2}$, where $n = p+q = \dim U$. The set $\bar{U}$ of null lines of this quadratic form is a projective variety which carries an $O(p+1, q+1)$ invariant conformal structure of type p, q. Indeed, a point $x \in \bar{U}$ is a line through the origin in $\mathbb{R}^{p+1,q+1}$. Denote this line by L_x. This defines a line bundle $L \to \bar{U}$. Similarly $L_x^\perp$ is an $n+1$-plane and $L_x^\perp \supset L_x$, since L_x is null. Thus $L_x^\perp / L_x$ is an n-plane canonically associated to x, so we get an n-plane bundle $L^\perp/L$. It is easy to verify that we have an $O(p+1, q+1)$ invariant identification of $T(\bar{U})$, the tangent bundle of $\bar{U}$, with $\mathrm{Hom}(L, L^\perp/L)$. Now $L_x^\perp/L_x$ has a well defined nondegenerate metric of type p, q. Therefore $\mathrm{Hom}(L_x, L_x^\perp/L_x)$ has a metric defined up to a factor. This gives the conformal structure of type p, q on $\bar{U}$. It is easy to check that this structure is conformally flat. Indeed, the space U can be identified as the space of all null lines which are *not* orthogonal to some fixed null line, x_∞, called "the point at infinity." In fact, U is the unique open orbit of the subgroup P^+ fixing x_∞. (In the case of Minkowski space the group P^+ is the Poincaré group together with the scale transformations.) Thus $\bar{U}$ can be regarded as the "conformal completion" of U: any locally defined conformal transformation of U extends to a globally defined transformation of $\bar{U}$ given by an element of $O(p+1, q+1)$. See [10] for a discussion of these facts, particularly the notion of conformal completion from the group theoretical viewpoint.

If $p \geqq 1$ and $q \geqq 1$ then $O(p+1, q+1)$ has four components. This is because in terms of a block decomposition

$$\begin{array}{c c} & \begin{array}{cc} p+1 & q+1 \end{array} \\ \begin{array}{c} p+1 \\ q+1 \end{array} & \left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right) \end{array}$$

(where $\mathbb{R}^{p+1,q+1}=\mathbb{R}^{p+1}\oplus\mathbb{R}^{q+1}$ with $\mathbb{R}^{p+1}$ a positive definite and $\mathbb{R}^{q+1}$ negative definite subspace) an element of $O(p+1,q+1)$ will have both P and S non-singular. Therefore $\det P=\pm 1$ and $\det S=\pm 1$ give the four components. In the case of $O(2,4)$ the components have the familiar geometrical interpretation in that if $C\in O(2,4)$ and if $Cx=x$ so we can think of C as lying in some P^+, then C can have positive or negative determinant (parity conservation or reversal) and preserves the forward light cone or send it into its negative (time reversal). [We should point out that this is so even though the conformally completed space $\bar{U}$ has no global causal structure. That is, despite the fact that $O(p+1,\, q+1)$ has a center with four elements

$$\left(\begin{array}{c|c} \pm I_{p+1} & 0 \\ \hline 0 & \pm I_{q+1} \end{array}\right)$$

the element

$$\left(\begin{array}{c|c} -I_{p+1} & 0 \\ \hline 0 & -I_{q+1} \end{array}\right)$$

acts trivially on projective space. So for the case $p=1,\ q=3$ the space $\bar{U}$ is orientable but without a global sense of future or past. Nevertheless we could pass to the double (or to the universal cf. [9]) cover and regain a global causality. On the covering space it is only the connected component of $O(2,4)$ which preserves both the orientation and the forward light cones. We shall not pass to this covering space since the geometry is more transparent on $\bar{U}$.]

The Lie algebra $o(p+1,q+1)$ can be identified with the space $\Lambda^2(\mathbb{R}^{p+1,q+1})$ according to the usual rule which assigns to $u\wedge v$ the linear transformation given by $(u\wedge v)w=\langle w,v\rangle u-\langle w,u\rangle v$, where $\langle\ ,\ \rangle$ denotes the scalar product on $\mathbb{R}^{p+1,q+1}$. Suppose we pick a "point at infinity" $x_\infty\in\bar{U}$ and an "origin" $x_0\in U$. That is, we pick a pair of non orthogonal null lines x_0 and x_∞ in $\mathbb{R}^{p+1,q+1}$. Together they span a two dimensional subspace, call it W, of signature $1,1$. This choice puts a grading on $o(p+1,q+1)$ as follows: Let e be some non-zero element of x_∞ and f a non-zero element of x_0 with $(e,f)=1$. Set

$$\begin{aligned} g_{+2}&=\{e\wedge v\}\,, \qquad v\in W^\perp\\ g_0&=\Lambda^2W^\perp\oplus\Lambda^2W,\\ g_{-2}&=\{f\wedge v\}\,. \end{aligned}$$

The pieces have the following geometrical interpretation: g_2 consists of all infinitesimal translations of U. The subalgebra $\Lambda^2W^\perp$ is just $o(p,q)$ the algebra of infinitesimal orthogonal transformations, while $\Lambda^2W=\mathbb{R}e\wedge f$ consists of infinitesimal scale transformations and g_{-2} consists of the proper infinitesimal conformal transformations. In terms of local coordinates the vector fields representing elements of g_{-2} vanish to second order at the origin. All of this is explained in [10]. Consider elements of $O(p+1,q+1)$ which act trivially on W and are reflections about a line in $W^\perp$ and interchange the lines x_0 and x_∞. These are the "conformal inversions" which send the origin into the point at infinity.

We now show how to identify $\mathrm{su}(2,2)$ with $o(2,4)$. Consider the space $\Lambda^2(\mathbb{C}^{2,2})$. It inherits from $\mathbb{C}^{2,2}$ a Hermitian form which is easily computed to have signature

2, 4. Also $\circledast : \Lambda^2(\mathbb{C}^{2,2}) \to \Lambda^2(\mathbb{C}^{2,2})$ with $\circledast^2 = \mathrm{id}$ and $(\circledast\omega, \circledast\sigma) = (\overline{\omega, \sigma})$ for $\omega, \sigma \in \Lambda^2(\mathbb{C}^{2,2})$. Hence the real six dimensional space consisting of all ω satisfying $\circledast\omega = \omega$ has a quadratic form of type 2, 4. We claim that the group SU(2, 2) leaves this space invariant. Indeed, for any $A \in \mathrm{GL}(4, \mathbb{C})$ let $B = \Lambda^2 A$ denote the induced transformation on $\Lambda^2(\mathbb{C}^{2,2})$. Then

$$B^a = (\det A) B^{-1} = (\det A) \Lambda^2 A^{-1},$$

while

$$B^* = \Lambda^2 A^*.$$

If $\det A = 1$ and $A^* = A^{-1}$ [which are the conditions for A to lie in SU(2, 2)], then $\circledast B = B^{a*} \circledast = B \circledast$, so B preserves the equation $\circledast\omega = \omega$. The restriction of SU(2, 2) to the set of solutions of this equation thus gives a homomorphism from SU(2, 2) onto SO(2, 4). (It is a double cover.) Notice that for ω satisfying $\omega = \circledast\omega$, the equations $\omega \wedge \omega = 0$ and $(\omega, \omega) = 0$ are the same. But the equation $\omega \wedge \omega = 0$ is precisely the condition that ω be decomposable, hence determines a plane, with ω and $\lambda\omega$ $(\lambda \neq 0)$ determining the same plane. This will be a null plane if and only if $\circledast\omega = \omega$. Thus the conformal completion of Minkowski space can be identified with the set of all

$$\begin{matrix} \text{null planes} & \text{or of} & \text{null lines} \\ \text{in } \mathbb{C}^{2,2} & \text{all} & \text{in } \mathbb{R}^{2,4} \end{matrix}.$$

Two null planes (corresponding to σ and ω say) are non-singularly paired if and only if $\sigma \wedge \omega \neq 0$, which means that they have no line in common. The group SU(2, 2) acts transitively on the set of all such pairs. Thus we may take x_0 to be the space $\left\{\begin{pmatrix} u \\ 0 \end{pmatrix}\right\}$ of undotted Weyl spinors and $x_\infty = \left\{\begin{pmatrix} 0 \\ v \end{pmatrix}\right\}$ to be the space of dotted spinors. With this choice, the gradation of this section coincides with the gradation of Sect. 2.

The group of automorphisms of SU(2, 2) has four components. [This is because the maximal compact subgroup is $\mathrm{S}(U(2) \times \mathrm{U}(2))$. We may interchange the two U(2) components and also conjugate the center, U(1).] We have already seen what they are: They are given by conjugation by linear isometries, linear anti-isometries, anti-linear isometries, and antilinear anti-isometries. Each of these elements acts as conformal transformations, giving the four components of $O(2, 4)$. To see what they are in terms of the discrete symmetries of space time it is enough to choose representatives, conjugate with the translation

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \qquad X = i \begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix},$$

and see what we get. The linear isometries, of course, map onto the connected component of $O(2, 4)$. Here is a linear anti-isometry:

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Conjugation by it gives

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}=\begin{pmatrix} 0 & -X \\ 0 & 0 \end{pmatrix}.$$

So it induces the discrete symmetry PT (parity and time reversal) on space time.

It follows from (2.9) that for the antilinear anti-isometry C we have

$$C\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}C=\begin{pmatrix} 0 & 0 \\ X^a & 0 \end{pmatrix}.$$

Thus C sends x_0 into x_∞ and a computation shows that it is conformal inversion. We can apply the linear isometry $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ to move x_∞ black to x_0. Thus further conjugation by $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ gives

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}C\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}C\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}=\begin{pmatrix} 0 & X^a \\ 0 & 0 \end{pmatrix}.$$

Now

$$X^a=i\begin{pmatrix} x_0+x_3 & -x_1-ix_2 \\ -x_1+ix_2 & x_0-x_3 \end{pmatrix}.$$

Thus the operation $X \rightsquigarrow X^a$ is just the parity transformation P. Thus in terms of the discrete space time symmetries we have the table

linear isometries → parity and time conservation
linear anti-isometries → parity and time reversal (PT)
antilinear isometries → time reversal (T)
antilinear anti-isometries → parity reversal (P).

We have already seen that linear isometries and antilinear anti-isometries induce automorphisms of the full superalgebra $g=u(V)\oplus V$. It is easy to see that the elements of the other two components do not. Indeed, since the automorphisms form a group, it is enough to prove the following:

There does not exist any automorphism of g which restricts to conjugation by $L=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ *on* $\mathrm{su}(V)\subset g_{\mathrm{even}}$.

Indeed an automorphism preserves the center and so is given by a real linear transformation $D: V\to V$ satisfying $Dc=cD$ or $Dc=\bar{c}D$ for $c\in C$ and

(i) $LAL^{-1}D=DA$ all $A\in \mathrm{su}(V)$

and

(ii) $S(Du, Dv)=LS(u,v)L^{-1}$.

Conjugation by linear and antilinear maps belong to different components of $\mathrm{Aut}(\mathrm{su}(V))$ so (i) implies that D is linear. Since $U(V)$ acts irreducibly on V, Schur's lemma implies that $L^{-1}D=cI$ so $D=cL$. But now (2.7) and the linearity of D implies that

$$S(Du, Dv)=-LS(u,v)L^{-1}$$

contradicting (ii).

We have now reached one of the main points of this paper. Whereas the group of automorphisms of su(2, 2) has four components, only the identity and the parity components extend to be automorphisms of the conformal superalgebra. The remaining two components reverse the signs of the odd brackets. As charge conjugation is intimately connected with P, it is CP which is an automorphism. (We shall see the geometrical connection between C and P more clearly in the next two sections where we consider the action on states.) In particular CPT can never be implemented so as to be an automorphism of the conformal superalgebra. It must change the sign of the odd brackets. To the extent that one is troubled by this change of sign in the odd brackets, conformal supersymmetry prefers CP to CPT.

Now there are many conformally supersymmetric field theories in the literature. To the extent that CPT is a formal consequence of the axioms of relativistic quantum field theory, cf. [14], this would appear to contradict what we have written above. To see what is involved in these models, we should observe that (since Wigner) time reversal is implemented by an antiunitary operator. Now the even elements of any superalgebra must be represented by skew-Hermitian operators in order that the corresponding group be represented by unitaries. Since conjugation by an antiunitary carries unitaries into unitaries, it also carries skew Hermitian operators into skew Hermitian operators and induces an automorphism of the commutator bracket. For the odd elements we have two choices:

a) Represent all the odd generators by self-adjoint operators. Then odd brackets will go over, not into the anticommutator but into i times the anticommutator. That is, if ξ and η are odd elements and ϱ_ξ and ϱ_η their images as operators, then

$$\varrho_{[\xi,\eta]} = i[\varrho_\xi \varrho_\eta + \varrho_\eta \varrho_\xi]\,.$$

With this choice, conjugation by a unitary can preserve the odd brackets but conjugation by an antiunitary changes the sign of the odd brackets since

$$i[\bar{u}\varrho_\xi \bar{u}^{-1}\bar{u}\varrho_\eta \bar{u}^{-1} + \bar{u}\varrho_\eta \bar{u}^{-1}\bar{u}\varrho_\xi \bar{u}^{-1}] = -\bar{u}i[\varrho_\xi\varrho_\eta + \varrho_\eta\varrho_\xi]\bar{u}\,,$$

where $\bar{u}$ is antiunitary, due to the fact that $\bar{u}i = -i\bar{u}$.

b) Represent the odd generators by operators ϱ_ξ whose spectrum lies on the lines through $e^{\pm \pi i/4}$ (cf. [12]) and so that odd bracket goes into anticommutator: $\varrho_{[\xi,\eta]} = \varrho_\xi\varrho_\eta + \varrho_\eta\varrho_\xi$, and so that the $\varrho_{[\xi,\eta]}$ are all skew-Hermitian.

Without having examined all the models, I suspect that the proponents of these models have chosen a) in which case they are prepared to accept the change in sign of the odd bracket as a necessary concommitant of the existance of antiunitaries in *any* supersymmetric theory. It seems to me that the choice b) is more natural. A consequence of the theorem proved above is that for conformal supersymmetry there is no way of choosing b) (or a)) so as to induce an automorphism of the conformal superalgebra which gives PT on spacetime (and does not reverse the sign of the odd brackets) even if antiunitaries are avoided.

Perhaps some more comments on the choice between CP and CPT are in order, this time from the point of view of group theory rather than supersymmetry: If a group H is a subgroup of a group G, disconnected components of H might become connected in G. For example, if we embed Sl(2, C) in its complexification, we find that this complexified group contains a subgroup containing PT together with $\mathrm{Sl}(2, C)$. In other words, PT becomes connected to the identity in the complexification of $\mathrm{Sl}(2, C)$. Now $\mathrm{Sl}(2, C)$ is the natural group to consider in a local Poincaré field theory, and the passage from $\mathrm{Sl}(2, C)$ to its complexification is a consequence of the locality and causality of the quantum fields. This is the essential line of reasoning of the CPT theorem, [14].

In a conformal theory, a natural group to consider is the global symmetry group $\mathrm{SU}(2,2)$. We could embed this group into its complexification, giving the embedding $\mathrm{SU}(2,2) \to \mathrm{Sl}(4, C)$. Since the linear isometries belong to $\mathrm{Sl}(4, C)$, PT becomes connected to the identity in $\mathrm{Sl}(4, C)$. Thus the above embedding prefers CPT. However, we also have the inclusion $\mathrm{SU}(2,2) \hookrightarrow \mathrm{Sp}(8, \mathbb{R})$, where $\mathrm{Sp}(8, \mathbb{R})$ is realized as the group of real linear transformation of $\mathbb{R}^8 = \mathbb{C}^{2,2}$ which preserve the imaginary part of the Hermitian form (,). [The imaginary part, im(,), is an anti-symmetric non-degenerate real bilinear form.] From many points of view, particularly representation theory and symplectic geometry, this inclusion is the more natural one. Notice that the anti-linear antiisometries belong to $\mathrm{Sp}(8)$, so that this inclusion prefers CP to CPT.

There are thus two arguments appealing to "mathematical naturality" which would indicate that the conformal group "prefers" CP to CPT. Now a superficial reading of the Fitch Cronin experiment [2] shows that both CP and CPT are not conserved. It requires a more detailed analysis using the language of local Lagrangians to conclude that CPT is conserved and CP violated. But CPT is a formal consequence of the axioms of quantum field theory according to the CPT theorem [14]. So perhaps the framework used to interpret the experiment determines the interpretation. It is, of course, difficult to interpret any experiment without placing it in some theoretical context. But the above appeals to mathematical "naturality" suggest that we reexamine the experimental evidence of CP violation, preferably from a viewpoint independent of the axioms of quantum field theory and of local Lagrangians. This requires, as a first step, a description of the one particle states and their discrete symmetries from the point of view of conformal geometry. This we do in the next two sections. (Of course CP violation does not contradict conformal supergeometry since the symmetry group of the dynamics might only be the connected component.)

Let us turn to a less controversial topic. Let us examine what is involved in the choice of a charge conjugation.

The group $U(\mathbb{C}^{2,2})$ acts transitively, by conjugation, on the set of all C which are antilinear anti-isometries satisfying $C^2 = \mathrm{id}$. Indeed, any such C is determined by the real four dimensional space $M = M_C$ of solutions $Cw = w$ – the space of associated Majorana spinors. Indeed

$$\mathbb{C}^{2,2} = M + iM \tag{3.1}$$

and

$$C(w + iw') = Cw - iCw', \qquad w, w' \in M. \tag{3.2}$$

The space M is also real isotropic, i.e.,

$$\mathrm{Re}(w, w') = 0, \qquad w, w' \in M. \tag{3.3}$$

Conditions (3.1) and (3.3) imply that M is symplectic for the imaginary part of (,), i.e., that

$$\mathrm{Im}(\ ,\ ) \text{ is non-degenerate on } M. \tag{3.4}$$

From (3.4) it follows that we can find a basis w_1, w_2, w'_1, w'_2 of M and hence of $\mathbb{C}^{2,2}$ such that

$$(w_i, w_j) = (w'_i, w'_j) = 0$$

and

$$(w_k, w'_l) = \delta_{kl} i .$$

The group $U(2,2)$ clearly acts transitively on the set of all such bases, hence on the set of all M's satisfying (3.1) and (3.3), hence on the set of all C's.

Any C carries a null complex two dimensional subspace of $\mathbb{C}^{2,2}$ into another such, hence it acts on $\bar{U}$, the conformal completion of Minkowski space. Furthermore, if x and y are two points of $\bar{U}$ which have a complex line in common, so will Cx and Cy. Thus C carries the "null cone" through x into the null cone through Cx. Hence C acts as a conformal transformation.

We have seen that for the particular C we chose in Sect. 2 we have that

$$C \text{ acts as a conformal inversion.} \tag{3.5}$$

Of course the C interchanging x_0 and x_∞ is not uniquely determined by this property. Indeed, we can multiply C by any non-zero complex number, z. Multiplication by a real number r changes the conformal inversion which involves a choice of element $e \in x_\infty$, once we choose the plane W. Equally, it involves the choice of a negative definite line in this plane. Since the stabilizer of C is $o(2, 3)$, we see that this is the choice of a negative line in $\mathbb{R}^{2,4}$. Multiplication by $e^{i\varphi}$ has no effect on the conformal geometry.

Up until this point we have been treating the superalgebra and the conformal geometry separately. The natural setting is in the framework of supermanifolds [5, 7]. We briefly indicate how the transcription of the preceding results goes over into superconformal geometry. Let g be a superalgebra and h a subsuperalgebra. Given any h module N, we can form the "produced" g module [1, 8], $\mathrm{Hom}_{U(h)}(U(g), N)$, where $U(g)$ and $U(h)$ are the associated universal enveloping algebras [3], and $U(g)$ is regarded as a right $U(h)$ module. In particular, we can take $h = g_{\mathrm{even}}$. If H is a Lie group whose Lie algebra is h, and we are given an H action on some manifold, Q, then the sheaf of C^∞ functions on Q is a sheaf of h modules. Applying the above construction gives a sheaf over Q which then defines the structure of a supermanifold $\tilde{Q}$ with an action of g as sheaf derivations. Applied to our setting gives the super conformal geometry. Then the two components of our automorphism group can be realized as the (two component) group of motions of conformal superspace.

4. The Action on States

In the next few sections we wish to study the action of $SU(2,2)$ and its group of automorphisms on the physical states. In particular this will allow us to determine, from purely geometrical considerations, the action of the discrete symmetries. In contrast to the usual theory based on the Poincaré group, there will be no arbitrary choices involved in the specification of this action – it will be determined by the geometry of the Grassmann variety. As the mathematics that we will need to use is somewhat involved, it may be useful to give a brief summary of what we will present in these next few sections. The mass zero irreducible representations of the Poincaré group all extend to representations of SU(2, 2). In fact, the following is true: SU(2, 2) is a subgroup of Sp(8), the group of real linear transformations of $C^{2,2}$ which preserve the antisymmetric real bilinear form given by the imaginary part of the Hermitian form (,). The double cover, $Mp(8)$, of Sp(8) has two distinguished representations, called the metaplectic representations. Each of these can be slightly modified so as to yield a representation of SU(2, 2) itself and not only its double cover, and decomposes into a direct sum of irreducible representations, each occurring with multiplicity one. These component representations remain irreducible upon further restriction to the Poincaré group, regarded as a subgroup of SU(2, 2), and all such restrictions remain inequivalent as Poincaré group representations. Each of the positive energy mass zero (arbitrary spin) representations of the Poincaré group occurs by this process of restriction from one of these metaplectic representations, and each of the negative energy mass zero representations occurs from the other. These mass zero representations of SU(2, 2) can be realized as acting on spaces of sections of vector bundles defined over the Grassmann variety of all complex two-dimensional subspaces of complex four-dimensional space. The positive energy representations will be realized on spaces of sections which are holomorphic in the domain, D^-, of all negative definite planes, while the negative energy representations will be realized on spaces of sections which are holomorphic on D^+, the space of positive definite planes. Our space $\bar{U}$, the conformal completion of Minkowski space, is the Shilov boundary of both D^- and D^+. These facts are true with the necessary minor modifications for arbitrary $U(k, l)$, see [12 or 17] for a detailed discussion.

The situation for the positive mass representation is similar, but somewhat more complicated. Since the scale transformation does not preserve mass, we must, for a given spin, s, combine all the $(m>0, s)$ representations of the Poincaré group, P_{10}, into a single representation of the Poincaré plus scale, P_{11} [which is the connected component of a maximal parabolic subgroup of SU(2, 2)]. Indeed, these representations (for varying s) are irreducibles of P_{11} and are precisely the ones determined by the Wigner-Mackey little group method applied to the orbit consisting of the interior of the forward light cone. Let us call an irreducible representation of SU(2, 2) a "positive energy" representation if the spectrum of the space-time translations $x \in P_{11}$ are of the form $e^{i \operatorname{tr} k \cdot x}$, where k lies in the forward light cone. Mack [18] has determined all the positive energy representations of SU(2, 2), see also [17]. In fact, [18], it is easy to see that any positive energy representation must contain a lowest weight vector (or highest weight vector depending on notation) and the set of all unitarizable highest weight modules for SU(2, 2) has been determined by Williams [19]. They all arise from the decomposition of the metaplectic representations of $MP(8)$, $MP(16)$, or $MP(24)$. The metaplectic representation of $Mp(8)$ gives rise to the mass zero particles as we have already mentioned. Let us consider the irreducibles coming from $Mp(16)$ (where the Howe pair is $U(2, 2) \times U(2)$, cf. [20 and 21]).

They are clearly described [indeed the general case $U(p, q) \times U(k)$ is completely treated] in the paper by Kashiwara and Vergne [17]. They each remain irreducible when restricted to P_{11} (cf. [21]). Each $(m>0, s)$ representation of P_{11}

occurs but now an infinite number of distinct representations (parametrized by the integers) restrict to the same $(m>0, s)$ representation of P_{11}. We hope to discuss the physical significance of this quantum number, which is present for SU(2,2) but disappears when we break the conformal symmetry down to P_{11}, in a future publication. But, as is shown in [17], all of these representations can be realized on spaces of holomorphic sections of vector bundles quite similar to the ones that arise in the mass zero case, and our discussion carries over with little change. We shall therefore concentrate on the $m=0$, and return to the positive mass case later.

We will follow the paper [16] by Jakobsen and Vergne in our summary. This paper gives a crystal clear description of all the mass zero representations of SU(2,2) together with complete proofs of all the relevant facts, including the relation of these representations to the zero mass equations. We strongly urge the reader to turn to this beautiful paper for a complete discussion of many of the facts summarized here.

We must begin, however, with a short digression on notation. In the first three sections of this paper we have chosen a basis of $\mathbb{C}^{2,2}$ so that the Hermitian scalar product was represented by the matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

i.e., so that the scalar product (w_1, w_2) is given by

$$(w_2)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} w_1,$$

where w_2^* denotes the row vector which is the conjugate transpose of w_2. If we make some change of basis then the scalar product will be represented by

$$N \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} N^*,$$

where N is the change of basis matrix. Then all the matrix representations of elements of $U(2,2)$, etc., will have to be conjugated by N. There are various conventional choices for the matrix form of the scalar product in the literature each with its own advantages. In [12] we used the matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ which is convenient for describing the mass zero representations in the Fock-Bargmann representation (in terms of creation and annihilation operators). In much of the physics literature (for example [14]) the matrix representation that is used is

$$\begin{pmatrix} 0 & \varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \varepsilon^* \end{pmatrix}, \quad \text{where} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Jakobsen and Vergne use the matrix

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} e^{-\pi i/4} & 0 \\ 0 & e^{+\pi i/4} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} e^{+\pi i/4} & 0 \\ 0 & e^{-\pi i/4} \end{pmatrix},$$

which has the advantage of making the action of SU(2,2) on the space of negative definite planes look like the action of SL(2, $\mathbb{R}$) on the upper half plane. For the rest of this paper we will switch to this new system of coordinates. (There will still be a slight discrepancy between our notation in that our space coordinates differ from theirs by an interchange of the x_1 and x_3 axes. But this difference will not be noticeable, and our notation is more consistent with most of the physics literature.) With this notation, a matrix

$$U=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{array}{l} a,b,c,d \\ \text{two by two complex matrices} \end{array}$$

belong to $U(2,2)$ if and only if it satisfies

$$U\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}U^*=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

which amounts to the conditions

$$ab^*-ba^*=0\,, \qquad dc^*-cd^*=0\,, \qquad ad^*-bc^*=I\,. \tag{4.1}$$

The group $U(2,2)$ acts transitively on the space, D, of negative definite two planes (as does the set of antilinear isometries; an anti-isometry carries a negative definite plane into a positive definite plane). The space D can be parametrized as follows: Fix the null plane $x_\infty=\left\{\begin{pmatrix} u \\ 0 \end{pmatrix}\right\}$. If $\lambda\in D$ is a negative definite plane, then $\lambda\cap x_\infty=\{0\}$. So λ can be regarded as the graph of a linear map $z:x_0\to x_\infty$, where $x_0=\left\{\begin{pmatrix} 0 \\ v \end{pmatrix}\right\}$. Thus

$$\lambda=\operatorname{graph} z=\left\{\begin{pmatrix} zv \\ v \end{pmatrix}\right\} \; v\in\mathbb{C}^2\,. \tag{4.2}$$

The condition that λ be negative definite is that

$$0>\left(\begin{pmatrix} zv \\ v \end{pmatrix},\begin{pmatrix} zv \\ v \end{pmatrix}\right)=i[(zv,v)(v,zv)]=i((z-z^*)v,v)\,.$$

Writing $z=x+iy$ where x and y are self-adjoint, we get no condition on x and the condition that y be positive definite. Thus

$$z=x+iy\,, \qquad y\gg 0\,. \tag{4.3}$$

We will use z to parametrize D, and frequently, by abuse of language, talk about "the point z," meaning the $\lambda\in D$ given by (4.2). (In this sense, and in what immediately follows, D looks like the upper half plane.) Actually, the condition $y\gg 0$ is the same as saying that $\|y\|^2=\det y>0$ and $y_0=\frac{1}{2}\operatorname{tr} y>0$ in the sense of Minkowski space. Thus condition (4.3) says that y belongs to the interior of the forward light cone and that D is a "tube domain" associated with this cone over Minkowski space, cf. [14].) This system of coordinates makes D into a complex manifold, so it makes sense to talk of holomorphic functions on D. The action of $U(2,2)$ on D is easy to describe:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} zv \\ v \end{pmatrix}=\begin{pmatrix} (az+b)v \\ (cz+d)v \end{pmatrix}.$$

Since, if $v\neq 0$, the vector on the right must have negative length squared, we conclude that $(cz+d)v\neq 0$. Thus $cz+d$ is an invertible matrix, and we can write the right-hand side of the preceding equation as

$$\begin{pmatrix} (az+b)(cz+d)^{-1}w \\ w \end{pmatrix}, \quad \text{where} \quad w=(cz+d)v. \tag{4.4}$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z=(az+b)(cz+d)^{-1}. \tag{4.5}$$

Notice that the positive definite planes can be described as the set of all $x+iy$, $y \leqslant 0$, with the same action, (4.5) of $U(2,2)$. The map $z \to z^*$ sends a negative definite plane into a positive definite plane and commutes with the action of $U(2,2)$:

$$(Uz)^* = Uz^*, \qquad U \in U(2,2). \tag{4.6}$$

The space M of null planes is the Shilov boundary of D in the sense that a function holomorphic in D is determined by its boundary values (in the distributional or hyperfunction sense) on M. In fact, we shall see (as was pointed out in [16]) that the various mass zero representations can be realized on spaces of holomorphic sections of certain vector bundles that we shall soon describe. Notice that if we replace (4.3) by $y=0$ we get ordinary Minkowski space, $\mathbb{R}^{1,3}$. As we have seen, this is just M with the light cone at x_∞ removed. Thus x gives a local coordinate system on this open subset of M. The formula (4.5) for the action of $U(2,2)$ is no longer globally valid on M (with $y=0$) since $cx+d$ need not be invertible for all x.

The various spaces we have been considering, D, M, the space D^+ of positive definite planes, are all subsets of the complex Grassmann variety $G(2,4)$ consisting of all complex two-dimensional subspaces of the complex four-dimensional space $\mathbb{C}^{2,2}$. The group $\mathrm{GL}(4,C)$ acts transitively on $G(2,4)$. It also acts as automorphisms of certain natural vector bundles over $G(2,4)$ which we now describe:

E – the canonical or tautological bundle, which assigns to each $\lambda \in G(2,4)$ the two dimensional subspace, E_λ, of $\mathbb{C}^{2,2}$ given by λ. If $\lambda \in D$ then E_λ consists of all vectors appearing on the right-hand side of (4.2). A similar description is valid over $\mathbb{R}^{1,3} \subset M$ (with $y=0$) but not globally over M. For example, $E_{x_\infty} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \right\}$.

F – the polar bundle. F_λ is the subspace of the dual space, $(\mathbb{C}^{2,2})^*$ consisting of those linear functions which vanish on E. In other words, $F_\lambda = (E_\lambda)^0$. Under $U(2,2)$, we have an antilinear identification (given by the scalar product), of F_λ with $(E_\lambda)^\perp$.

$\Lambda^2 E$ – a line bundle. Over M [and under $U(2,2)$] it can be identified with the complexification of the canonical line bundle of the conformal structure, i.e., the bundle of metrics in the conformal class.

From each of the above vector bundles we can form symmetric or exterior powers. We can reformulate one of the important observations of [16] as saying that the states of the $m=0$, $s=0, \frac{1}{2}, 1, \ldots$ representations can be regarded as certain holomorphic sections in D of

$$S^n(E) \otimes \Lambda^2(E), \qquad n=2s, \tag{4.7}$$

while the states in the $m=0$, $s=-\frac{1}{2}$, -1, $-3/2, \ldots$ can be regarded as holomorphic sections in D of the vector bundles

$$S^n(F) \otimes \Lambda^2 E, \qquad n=-2s. \tag{4.8}$$

Thus, if we wish to regard the states as geometrical objects over M, then we must regard them as generalized (or hyperfunction) sections of these vector bundles over M, which are the boundary values of sections holomorphic in the interior.

In order to formulate this result more precisely and explain the relation to the metaplectic representation we review some general facts about homogeneous vector bundles, cf. [22]. Let $D=G/K$ be a homogeneous space, and let $E\to D$ be a homogeneous vector bundle. Such a vector bundle is determined by a representation, τ, of K on a vector space, V, which can be identified with the fiber E_{λ_0}, where λ_0 is the point of D fixed by K. A section, s, of the vector bundle E can be identified with a function $\psi: G\to V$ which satisfies

$$\psi(gk)=\tau(k)^{-1}\psi(g) \quad \text{all} \quad k\in K. \tag{4.9}$$

The relation between s and ψ is given by

$$s(g\lambda_0)=[(g,\psi(g))], \tag{4.10}$$

where [ ] denotes the equivalence class of $G\times V$ under the diagonal action of K:

$$(g,v)\sim(gk^{-1},\tau(k)v).$$

Suppose we are given a function $J=J_\tau: G\times D\to \mathrm{GL}(V)$ which satisfies

$$J(g_1g_2,\lambda)=J(g_1,g_2\lambda)J(g_2,\lambda), \qquad J(k,\lambda_0)=\tau(k), \qquad J(e,\lambda)=I, \tag{4.11}$$

where e denotes the identity element of G and I the identity element of $\mathrm{Gl}(V)$. Then it follows from (4.9) and (4.11) that

$$J(gk,\lambda_0)\psi(gk)=J(g,\lambda_0)\psi(g),$$

and so we may define the function $f: D\to V$ by

$$f(\lambda)=J(g,\lambda_0)\psi(g). \tag{4.12}$$

We thus have three descriptions of the space, $\Gamma(E)$, of all sections of E given by s, ψ, or f, related to one another by (4.10) and (4.12). The group G acts on $\Gamma(E)$. In terms of the three descriptions, an element $h\in G$ acts by sending

$$s\rightsquigarrow r_h s, \quad \text{where} \quad r_h(\lambda)=hs(h^{-1}\lambda), \tag{4.13}$$

$$\psi\rightsquigarrow r_h\psi, \quad \text{where} \quad (r_h\psi)(g)=\psi(h^{-1}g), \tag{4.14}$$

$$f\rightsquigarrow r_h f, \quad \text{where} \quad (r_h f)(\lambda)=[J(h^{-1},\lambda)]^{-1}f(h^{-1}\lambda). \tag{4.15}$$

It is a useful exercise to check that these three definitions are consistent under the identifications given by (4.10) and (4.12).

Let us illustrate these identifications in the case where we take D to be the negative definite planes under $U(2,2)$. Let us choose $\lambda_0=\left\{\begin{pmatrix} iv \\ v\end{pmatrix}\right\}$. The condition that a matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ stabilizes λ_0 is that $(ai+b)(ci+d)^{-1}=i$ or $ai+b=-c+di$. Clearly all matrices with $b=-c$, $a=d$ have this property. Multiplying on the right by the inverse of $\begin{pmatrix} a & b \\ -b & a\end{pmatrix}$ we may assume that $a=I$ and $b=0$. Then

$ad^* - bc^* = ad^* = I$ by one of the defining conditions of $U(2,2)$ so $d = I$ and we get the condition $i = -c + i$ or $c = 0$. Thus the stabilizer group of λ_0 is given by

$$K = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\}, \tag{4.16}$$

and the conditions (4.1) give

$$(a+ib)(a+ib)^* = I, \qquad (a-ib)(a-ib)^* = I.$$

Thus

$$\text{the map } \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \rightsquigarrow ((a+ib), (a-ib))$$
$$\text{gives an isomorphism of } K \text{ with } U(2) \times U(2), \tag{4.17}$$

which shows that K is a maximal compact subgroup of $U(2,2)$. Let us consider the tautological bundle $E \to D$. The action of K on $E_{\lambda_0} = V = \mathbb{C}^2$ is given by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} iv \\ v \end{pmatrix} = \begin{pmatrix} i(a+ib)v \\ (a+ib)v \end{pmatrix} \quad \text{or} \quad \tau \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = (a+ib).$$

We can now define

$$J(g, \lambda) = (cz + d) \quad \text{if} \quad \lambda = \left\{ \begin{pmatrix} zv \\ v \end{pmatrix} \right\}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and check that (4.11) holds. Thus we can think of a section of E as being given by a function $f: D \to \mathbb{C}^2$ where the corresponding function $\psi: G \to \mathbb{C}^2$, and section s are given by

$$\psi(g) = (ci+d)^{-1} f(z) \quad \text{if} \quad z = gi = (ai+b)(ci+d)^{-1}$$

and

$$s(\lambda) = s(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i\psi(g) \\ \psi(g) \end{pmatrix} = \begin{pmatrix} (ai+b)(ci+d)^{-1} f(z) \\ f(z) \end{pmatrix} = \begin{pmatrix} zf(z) \\ f(z) \end{pmatrix}.$$

The action of $h \in G$ on sections is given as follows: Suppose that $h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then, by (4.15),

$$(r_h f)(z) = (c'z + d')^{-1} f((a'z + b')(c'z + d')^{-1}).$$

The corresponding action on the section $s(z) = \begin{pmatrix} zf(z) \\ f(z) \end{pmatrix}$ should be

$$\begin{aligned} (r_h s)(z) &= hs(h^{-1}z) = h \begin{pmatrix} (a'z + b')(c'z + d')^{-1} f(h^{-1}z) \\ f(h^{-1}z) \end{pmatrix} \\ &= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} z(c'z + d')^{-1} f(h^{-1}z) \\ (c'z + d')^{-1} f(h^{-1}z) \end{pmatrix} = \begin{pmatrix} z(r_h f)(z) \\ (r_h f)(z) \end{pmatrix} \end{aligned}$$

as required.

Let τ_n denote the representation of $\mathrm{GL}(2,\mathbb{C})$ on the space $S^n(\mathbb{C}^2)$ – the n^{th} symmetric power of $\mathbb{C}^2$. $\left[\text{Restricted to } \mathrm{SU}(2)\subset \mathrm{GL}(2,\mathbb{C}) \text{ this is exactly the spin } \frac{n}{2} \text{ representation.}\right]$ Then the "automorphic factor" J corresponding to the vector bundle $S^n(E)\otimes \Lambda^2 E$ is

$$J(g,z)=\tau_n(cz+d)\det(cz+d).$$

The corresponding action on sections is given by

$$(r_h f)(z)=\tau_n(c'z+d')^{-1}\det(c'z+d')^{-1}f((a'z+b')(c'z+d')^{-1}) \quad \text{if} \quad h^{-1}=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{4.18}$$

Next let us consider the bundle $F\to M$. Over a point of D we can identify the space $F_\lambda=E_\lambda^0\subset(\mathbb{C}^{2,2})^*$ with the orthogonal complement

$$E_\lambda^\perp=\left\{\begin{pmatrix} z^*v \\ v \end{pmatrix}\right\} \quad \text{if} \quad E_\lambda=\left\{\begin{pmatrix} zv \\ v \end{pmatrix}\right\}.$$

We are interested in holomorphic sections of F_λ and its tensor powers. As the identification of F_λ with $E_\lambda^\perp$ is antilinear, let us write a section of $E_\lambda^\perp$ as

$$s(z)=\begin{pmatrix} z^*\circledast f \\ \circledast f \end{pmatrix} \qquad \circledast \text{ is the star operator of } \mathbb{C}^2,$$

with f holomorphic. Then, if $h^{-1}=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$,

$$\begin{aligned}(r_h s)(z)=hs(h^{-1}z)&=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}\begin{pmatrix} (a'z^*+b')(c'z^*+d')^{-1}\circledast f(h^{-1}z) \\ \circledast f(h^{-1}z) \end{pmatrix}\\ &=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} z^*(c'z^*+d')^{-1}\circledast f(h^{-1}z) \\ (c'z^*+d')^{-1}\circledast f(h^{-1}z) \end{pmatrix}=\begin{pmatrix} z^*\circledast(r_h f)(z) \\ \circledast(r_h f)(z) \end{pmatrix},\end{aligned}$$

if we define

$$(r_h f)(z)=[(c'z^*+d')^{-1}]^{a*}f(h^{-1}z),$$

so

$$(r_h f)(z)=(zc'^*+d'^*)\det(zc'^*+d'^*)^{-1}f((a'z+b)(c'z+d')^{-1}),$$

$$\text{where } \circledast A=A^{a*}\circledast \text{ and } A^{a-1}=A(\det A)^{-1} \text{ for any } A\in \mathrm{GL}(2,\mathbb{C}).$$

Thus, the automorphic factor corresponding to the vector bundle $S^n(F)\otimes\Lambda^2 E$ is

$$J_{S^nF\otimes\Lambda^2E}(g,z)=\tau_n(zc^*+d^*)\det(zc^*+d^*)^{-n}\det(cz+d) \quad \text{if} \quad g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.19}$$

[This explains the mysterious $-(n+1)^{\text{th}}$ power of the determinant occurring in the formulas of Jakobsen-Vergne. We are also using a slightly different factor from theirs which fits together better with the metaplectic representations.]

Let us now recall some facts about the symplectic group $\mathrm{Sp}(8,\mathbb{R})$ [or, more generally, $\mathrm{Sp}(2n,\mathbb{R})$]. It is generated by matrices of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A^{t-1}\end{array}\right)\ A\in \mathrm{Gl}(n,\mathbb{R}),\qquad \left(\begin{array}{c|c} I & B \\ \hline 0 & I\end{array}\right)\ B=B^t,$$

and the single matrix

$$\begin{pmatrix} 0 & I_4 \\ -I_4 & 0\end{pmatrix}.$$

Actually, a smaller collection will do, as is described in detail in [23, pp. 27–30]. The group $U(2,2)$ [or more generally $U(n,n)$] is generated by matrices of the form

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & a^{*-1}\end{array}\right),\ \left(\begin{array}{c|c} I & b \\ \hline 0 & I\end{array}\right)\ b=b^* \quad\text{and}\quad \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0\end{array}\right),$$

where $a\in \mathrm{GL}(2,\mathbb{C})$ [more generally, $\mathrm{GL}(n,\mathbb{C})$, etc.]. The embedding of $U(n,n)\to \mathrm{Sp}(2n,\mathbb{R})$ is then given by foregetting the underlying complex structure, that is, by considering a as a real linear transformation, A, and b as a symmetric real bilinear form, B, instead of as a Hermitian form. Thus, in establishing relations between the metaplectic representation and representations of $U(2,2)$, it is enough to check them on these generators. This is the method of Jakobsen and Vergne.

One realization of the metaplectic representation is the following: The Hilbert space on which the representation takes place is $L^2(\mathbb{R}^n)$. (In our case $\mathbb{R}^n=\mathbb{R}^4=\mathbb{C}^2$ with the standard Lebesgue measure $d^4u=-\frac{1}{4}du_1du_2d\bar{u}_1d\bar{u}_2$.) The generating elements are represented by

$$\varrho_M\begin{pmatrix} A & 0 \\ 0 & A^{t-1}\end{pmatrix}\varphi=(\det A)^{1/2}\varphi(A^t) \tag{4.20}$$

(it is the ambiguity in the definition of the square root of the determinant which requires passage to the double cover)

$$\left[\varrho_M\begin{pmatrix} I & B \\ 0 & I\end{pmatrix}\varphi\right](u)=e^{-iB(u,u)}\varphi(u) \tag{4.21}$$

and

$$\varrho_M\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}=(-1)^{n/2}\mathscr{F}, \tag{4.22}$$

where $\mathscr{F}$ denotes the Fourier transform. In (4.21) we can write $B(u,u)=\mathrm{tr}\, b\cdot H(u,u)$ if b is a self-adjoint two by two matrix and B the associated real quadratic form. (Here H is the map of $\mathbb{C}^2\times\mathbb{C}^2$ into linear operators on $\mathbb{C}^2$ described in Sect. 1.)

In (4.20) we would have to pass to the double cover of $U(2,2)$. However, as was pointed out in [23, 16, 17], by multiplying by $(\det_{\mathbb{C}} g)^{1/2}$, we can modify the formula (4.20) so as to get a representation of $U(2,2)$ itself.

This representation of $U(2,2)$ is then given by (4.22) and

$$i\varrho_M\begin{pmatrix} a & 0 \\ 0 & a^{*-1}\end{pmatrix}\varphi(\cdot)=(\det a)\,\varphi(a^*\cdot(\cdot)) \tag{4.23}$$

and

$$\left[\varrho_M\begin{pmatrix} I & b \\ 0 & I\end{pmatrix}\varphi\right](u)=e^{-i\operatorname{tr} b\cdot H(u,u)}\varphi(u)\,. \tag{4.24}$$

We can write

$$L^2(\mathbb{C}^2)=\overline{\oplus H_n}\quad \text{(Hilbert space direct sum)}, \tag{4.25}$$

where

$$H_n=\{\varphi\in L^2(\mathbb{C}^2)|\varphi(e^{i\theta}u)=e^{-in\theta}\varphi(u)\}\,. \tag{4.26}$$

In fact, (4.25) is the eigenspace decomposition of the center, $U(1)=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix}\right\}$ of $U(2,2)$, and the general theory of Howe pairs [20, 21] (or see [16] for a direct proof for $U(2,2)$ and [12, 17] for a direct proof for $U(p,q)$) guarantees that (4.25) is a decomposition of $L^2(\mathbb{C}^2)$ into irreducibles under $U(2,2)$, each occurring with multiplicity one.

For each $n\geqq 0$, consider the map $T_n: H_n\to$ Holomorphic functions on D with values in $S^n(\mathbb{C}^2)$, given by

$$(T_n\varphi)(z)=\int_{\mathbb{C}^2} e^{i\operatorname{tr} zH(u,u)}\varphi(u)u^n d^4u\,. \tag{4.27}$$

For $z=x+iy$ and $y\gg 0$, the exponential factor $e^{-\operatorname{tr} yH(u,u)}=e^{-u^*yu}$ is more than enough to counteract the polynomial growth of u^n so the integral in (4.27) converges and defines a holomorphic function of z. For fixed n and $\varphi\in H_n$ we shall write $f=T_n\varphi$. Notice that

$$\begin{aligned}\left[T_n\varrho_M\begin{pmatrix} a & 0 \\ 0 & a^{*-1}\end{pmatrix}\varphi\right](z)&=\det a\int e^{i\operatorname{tr} zH(u,u)}\varphi(a^*v)u^n d^4u\\ &=(\det a^*)^{-1}\int e^{i\operatorname{tr} zH(a^*v,a^*v)}\varphi(v)(a^{*-1}v)^n d^4v\\ &=\tau_n(a^*)^{-1}(\det a^*)^{-1}f(a^{-1}za^{*-1}),\\ &\quad v=a^*u\,,\quad d^4u=|\det a|^{-1}d^4v\,.\end{aligned}$$

Thus, for $n\geqq 0$ and g of the form $\begin{pmatrix} a & 0 \\ 0 & a^{*-1}\end{pmatrix}$ we have

$$T_n\varrho_M(g)\varphi=r_g f, \tag{4.28}$$

where r_g is given by (4.18). For $g=\begin{pmatrix} I & b \\ 0 & I\end{pmatrix}$ we have

$$\begin{aligned}(T_n\varrho_M(g)\varphi)(z)&=\int e^{i[\operatorname{tr} zH(u,u)\,b(u,u)]}\varphi(u)u^n du\\ &=\int e^{i\operatorname{tr}(z-b)H(u,u)}\varphi(u)u^n du=f(z-b)\,,\end{aligned}$$

so (4.28) again holds. Thus to check that (4.28) holds for all $g\in U(2,2)$ it is enough to check it for $g=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. For this operator it is enough to check (4.28) on a dense set of elements in H_n. This idea is also due to Jakobsen and Vergne.

Functions of the form

$$e^{-\langle u,u\rangle} u_1^{\alpha_1} u_2^{\alpha_2} \bar{u}_1^{\beta_1} \bar{u}_2^{\beta_2} \tag{4.29}$$

span a dense subspace of $L^2(\mathbb{C}^n)$. (This is just the expansion into Hermite polynomials.) Thus

functions of the form $e^{-\langle u,u\rangle} e^{i\langle wu,u\rangle} \bar{u}_1^{\beta_1} \bar{u}_2^{\beta_2} (\beta_1+\beta_2=n)$ span $H_n \cdot (w \gg 0)$.

Now the element $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts as

$$[(\varrho_M(\sigma))\varphi](u) = \frac{-1}{\pi^2} \int e^{-2i\mathrm{Re}\langle u,v\rangle} \varphi(v)\, d^4v$$

(which is the convenient normalization for the Fourier transform). A straightforward Gaussian integral shows that

$$\varrho_M(\sigma) e^{-\langle\cdot,\cdot\rangle} = -e^{-\langle\cdot,\cdot\rangle}, \tag{4.31}$$

i.e., that

$$\int e^{-2i\mathrm{Re}\langle u,v\rangle} e^{-\langle v,v\rangle} d^n v = \pi^2 e^{-\langle u,u\rangle}, \tag{4.31$'$}$$

or, more generally, that

$$\varrho_M(\sigma) e^{i\langle w\cdot,\cdot\rangle} = -(\det w)^{-1} e^{i\langle w^{-1}\cdot,\cdot\rangle}. \tag{4.32}$$

The same Gaussian integral shows that

$$(T_0 e^{i\langle w\cdot,\cdot\rangle})(z) = \det\left(\frac{z-w^*}{2i}\right)^{-1}, \tag{4.33}$$

and the case $n=0$, $g=\sigma$ of (4.28) is then a straightforward verification.

Applying the operator $\left(\frac{\partial}{\partial u_1}\right)^{\beta_1} \left(\frac{\partial}{\partial k_2}\right)^{\beta_2}$ to both sides of (4.32) and applying T_n to both sides gives (4.28) in general.

So far we have treated $n \geqq 0$. For $n<0$ define

$$(T_n\varphi)(z) = \int e^{i\,\mathrm{tr}\, zH(u,u)} \varphi(u) (\circledast u)^{|n|} d^4u. \tag{4.34}$$

The same argument as above then gives

$$\begin{aligned} &\det a \int e^{i\,\mathrm{tr}\, z\cdot H(u,u)} \varphi(a^* u) (\circledast u)^{|n|} d^4u \\ &= (\det a^*)^{-1} \int e^{i\,\mathrm{tr}\, z\cdot H(a^{*-1}v,\, a^{*-1}v)} \varphi(v) \bigcirc (a^{*-1}v)^{a^{*-1}u=v} \\ &= (\det a^*)^{-1} (\det a)^{-n} \tau_n(a) f(a^{-1} z a^{*-1}), \end{aligned}$$

giving (4.28) for $n<0$.

We have thus described a class of irreducible representations of $\mathrm{SU}(2,2)$ as spaces of sections of canonical vector bundles on $G(2,4)$ which are holomorphic over D. To see that these restrict to the $m=0$, $s=\frac{n}{2}$ representations of the Poincaré group it is enough to make the following observations: The map $u \rightsquigarrow H(u,u)=k$ sends $\mathbb{C}^2 - \{0\}$ onto the set of all k with $\det k = 0$, $k_0 > 0$, i.e., the forward light cone, and $H(u',u')=H(u,u)$ if and only if $u'=e^{i\theta}u$. In other words, we can consider $\mathbb{C}^2-\{0\}$ as a circle bundle over the forward light cone. The isotropy group, in $SL(2,\mathbb{C})$ of the point $\begin{pmatrix}1\\0\end{pmatrix}$ is $\left\{\begin{pmatrix}1 & b\\0 & 1\end{pmatrix} b\in\mathbb{C}\right\}$ and so the vector spaces H_n can be regarded as sections of the line bundle over the forward light cone associated to the representation $\begin{pmatrix}e^{i\theta} & b\\0 & e^{-i\theta}\end{pmatrix}\rightsquigarrow e^{in\theta}$ of the "little group" $\left\{\begin{pmatrix}e^{i\theta} & b\\0 & e^{-i\theta}\end{pmatrix}\right\}$ – the stabilizer of the point $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}=H\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)$. But the action of the Poincaré group [given by (4.23) and (4.24)] on this space of vectors is precisely the mass zero spin $n/2$ representation of the Poincaré group.

5. The Action of the Discrete Components

The various components of Aut SU(2, 2) act as geometrical transformations on the Grassmann variety and on the associated vector bundles. Hence we may determine the action on states using the formulas of the preceding section. For example, the anti-linear anti-isometries will map negative definite planes into positive definite planes, and hence map sections of various vector bundles defined over D^- into sections defined over D^+. Let us examine the antilinear anti-isometry

$$P=\begin{pmatrix}\circledast & 0\\0 & \circledast\end{pmatrix}.$$

Then

$$P\begin{pmatrix}zv\\v\end{pmatrix}=\begin{pmatrix}z^{a*}\circledast v\\\circledast v\end{pmatrix}, \tag{5.1}$$

so

$$Pz=z^{a*}. \tag{5.2}$$

Furthermore

$$P: E_z\to E_{z^{a*}}, \tag{5.3}$$

this map being antilinear by (5.1). Thus $P:\Lambda^2E_z\to\Lambda^2E_{z^{a*}}$. If s is a section of $E\circledast\Lambda^2E$ defined over D^-, the geometrical action of P on s is given by

$$(R_ps)(w)=Ps(P^{-1}w).$$

If

$$s(z)=\begin{pmatrix}zf(z)\\f(z)\end{pmatrix}\otimes\begin{pmatrix}z\delta_1\\\delta_1\end{pmatrix}\wedge\begin{pmatrix}z\delta_2\\\delta_2\end{pmatrix}\qquad \begin{array}{l}\delta_1=\begin{pmatrix}1\\0\end{pmatrix},\\ \delta_2=\begin{pmatrix}0\\1\end{pmatrix}\end{array}$$

then, by (5.1) and (5.2)

$$(R_Ps)(w)=\begin{pmatrix}w\circledast f(w^{a*})\\\circledast f(w^{a*})\end{pmatrix}\otimes\begin{pmatrix}w\delta_1\\\delta_1\end{pmatrix}\wedge\begin{pmatrix}w\delta_2\\\delta_2\end{pmatrix}.$$

Thus the action of P in terms of the function f is given by

$$(R_P f)(w) = \circledast f(w^{a*}). \tag{5.4}$$

Notice that if f is a holomorphic function of $z \in D^-$, then $f(w^{a*})$ is an antiholomorphic function of w and $\circledast f(w^{a*})$ is a holomorphic function on D^+. Furthermore, if

$$f(z) = \int e^{i \operatorname{tr} zH(u,u)} \varphi(u) u d^4 u, \qquad \varphi \in H_1,$$

then

$$\circledast f(w^{a*}) = \int e^{-i \operatorname{tr} w^a H(u,u)} \overline{\varphi(u)} \circledast u d^4 u.$$

But $\operatorname{tr} A^a B^a = \operatorname{tr} AB$ and $H(u,u)^a = H(\circledast u, \circledast u)$, so we can rewrite this last integral as

$$\int e^{-iwH(\circledast u, \circledast u)} \overline{\varphi(u)} \circledast u d^4 u = \int e^{-iwH(v,v)} \overline{\varphi(-\circledast v)} v dv.$$

We are to interpret this result as follows: Let

$H^+ = L^2(\mathbb{C}^2)$ giving the positive energy metaplectic representation

and

$H^- = L^2(\mathbb{C}^2)$ giving the negative energy metaplectic representation.

Then

$$R_P : H^+ \to H^-, \tag{5.5}$$

$$(R_P \varphi)(u) = \overline{\varphi(\circledast u)}. \tag{5.6}$$

We should emphasize once again that (5.4)–(5.6) shows that the geometrical action on states associated with P has "charge conjugation" built into it, in that it carries positive energy states into negative energy states. We can isolate the geometrical character of this "charge conjugation" as follows: Recall from Sect. 4 that the map which assigns to each two plane in $\mathbb{C}^{2,2}$ its orthocomplement commutes with the action of $U(2,2)$. For positive or negative definite planes this expresses itself by the assertion $(Mz)^* = Mz^*$, $M \in U(2,2)$. In particular we have an antilinear identification

$$l : E_z \xrightarrow{\sim} F_{z^*}.$$

Recall from Sect. 3 that the four dimensional star operator $\circledast_4 : \Lambda^2(\mathbb{C}^{2,2}) \to \Lambda^2(\mathbb{C}^{2,2})$ commutes with the action of $\mathrm{SU}(2,2)$. Furthermore $\circledast_4 : \Lambda^2 E_z \to \Lambda^2 E_{z^*}$. In fact, if we take

$$\delta = \begin{pmatrix} \delta_1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \delta_2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix},$$

then, by definition

$$\left(\begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}, \circledast_4 \left(\begin{pmatrix} z\delta_1 \\ \delta_1 \end{pmatrix} \wedge \begin{pmatrix} z\delta_2 \\ \delta_2 \end{pmatrix} \right) \right) \delta = \begin{pmatrix} z\delta_1 \\ \delta_1 \end{pmatrix} \wedge \begin{pmatrix} z\delta_2 \\ \delta_2 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}$$
$$= (\det z)\delta,$$

and

$$\left(\begin{pmatrix}0\\ \delta_1\end{pmatrix}\wedge\begin{pmatrix}0\\ \delta_2\end{pmatrix},\begin{pmatrix}z^*\delta_1\\ \delta_1\end{pmatrix}\wedge\begin{pmatrix}z^*\delta_2\\ \delta_2\end{pmatrix}\right)=\det\left[-i\begin{pmatrix}(\delta_1,z^*\delta_1)\,(\delta_1,z^*\delta_2)\\ (\delta_2,z^*\delta_1)\,(\delta_2,z^*\delta_2)\end{pmatrix}\right]=-\det z\,.$$

Thus, up to an overall scalar factor, (independent of z), $\circledast_4$ carries the element $\begin{pmatrix}z\delta_1\\ \delta_1\end{pmatrix}\wedge\begin{pmatrix}z\delta_2\\ \delta_2\end{pmatrix}$ into $\begin{pmatrix}z^*\delta_1\\ \delta_1\end{pmatrix}\wedge\begin{pmatrix}z^*\delta_2\\ \delta_2\end{pmatrix}$. Thus

$$l\otimes\circledast_4 : E_z\otimes\Lambda^2E_z\to F_{z^*}\otimes\Lambda^2E_{z^*}$$

in an antilinear fashion (and similarly $S^n(l)\otimes\circledast_4$ maps $S^nE_z\otimes\Lambda^2E_z\to S^nF_{z^*}\otimes\Lambda^2E_{z^*}$ and $S^n(l^{-1})\otimes\circledast_4$ maps $S^nF_z\otimes\Lambda^2E_z\to S^nE_{z^*}\otimes\Lambda^2E_{z^*}$. We thus get a map, $\mathscr{C}$ taking sections of $E\otimes\Lambda^2E$ over $D^\mp$ into sections of $F\otimes\Lambda^2E$ over $D^\pm$ by

$$(\mathscr{C}s)(z^*)=(l\otimes\circledast_4)s(z) \tag{5.7}$$

(with similar definitions for each of our other bundles). If we represent our section as

$$s(z)=\begin{pmatrix}zf(z)\\ f(z)\end{pmatrix}\otimes\begin{pmatrix}z\delta_1\\ \delta_1\end{pmatrix}\wedge\begin{pmatrix}z\delta_2\\ \delta_2\end{pmatrix}$$

then, up to an overall constant factor

$$(\mathscr{C}S)(z^*)=\begin{pmatrix}z\circledast\circledast f(z)\\ \circledast\circledast f(z)\end{pmatrix}\otimes\begin{pmatrix}z^*\delta_1\\ \delta_1\end{pmatrix}\wedge\begin{pmatrix}z^*\delta_2\\ \delta_2\end{pmatrix},$$

so

$$(\mathscr{C}f)(w)=\circledast f(w^*)\quad\text{if}\quad w=z^*\,. \tag{5.8}$$

Notice that if f is a holomorphic function of z then $\circledast f(w^*)$ is a holomorphic function of w. If

$$f(z)=\int e^{i\,\mathrm{tr}\,zH(U,U)}\varphi(u)u\,du\,,$$

then

$$\circledast f(w^*)=\int e^{-i\,\mathrm{tr}\,wH(u,u)}\bar{\varphi}(u)\circledast u\,du\,,$$

so

$$\mathscr{C}: H^\pm\to H^\mp \tag{5.9}$$

is given by

$$(\mathscr{C}\varphi)=\bar{\varphi}\,. \tag{5.10}$$

In particular, if we compose (5.8) with (5.9) we obtain the map sending

$$\varphi(u)\rightsquigarrow\varphi(\circledast u)$$

of H^+ into itself. Now the element P preserves $\mathrm{Im}(\ ,\ )$ (as do all antilinear anti-isometries), i.e., $P\in\mathrm{Sp}(8)$. Then, we see from (·) that up to a phase factor we have

$$\mathscr{C}\cdot R_P=\varrho_P\,. \tag{5.11}$$

In order to understand (5.9) and (5.10) a little better, let us make the following observations: The spaces H^+ and H^- are not irreducible under $Mp(n)$ (each decomposing into two components – the even and the odd polynomials, say in the Fock representation). However, they are irreducible as representations of the semi-direct product of $Mp(n)$ with the Heisenberg group – in fact, they are irreducible under the Heisenberg group above. [Here we have fixed some definite value, h, of Planck's constant, and H^+ is then identified with the unique irreducible representation of the Heisenberg algebra with $\varrho_+(\mathbb{1})=hi$, where $\mathbb{1}$ generates the center and H^- is the representation with $\varrho_-(\mathbb{1})=-hi$.] The two representations are inequivalent over the complex numbers, but are real equivalent by the anti-unitary operator **c**. By irreducibility, **c** is uniquely determined, up to a phase factor, and can be given, in the Schrödinger representation by (5.10). *It is this operator*, **c**, uniquely determined (up to a phase) by the underlying symplectic structure, *that we propose as the charge conjugation operator*. Notice that it commutes with all of Sp(8), and hence, a fortiori, with all of SU(2, 2). It is implemented geometrically by the Hodge star operator acting on sections as described above. The "charge conjugation" used in the standard literature, as exemplified in Sect. 3, only commutes with an $O(2,3)$ subgroup of $O(2,4)$.

One can check that this operation can be extended so as to work for the orthosymplectic algebras $\mathrm{osp}(2n/k, k)$. The superconformal algebra can be embedded in $\mathrm{osp}(8/1, 1)$, for example, and hence the entire picture works in the superconformal setting. We shall discuss this point in a future publication.

Let us close this section by noting that the transformation PT is implemented in holomorphic sections by the rule $f(z)\to f(-z)$, as is to be expected.

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