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Semi-Infinite Ising Model

I. Thermodynamic Functions and Phase Diagram in Absence of Magnetic Field

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Abstract. For the semi-infinite Ising model in two or more dimensions, we prove analyticity properties of the surface free energy and map out the phase diagram in the absence of an external magnetic field. We prove that this phase diagram contains critical lines where the parallel and/or the transverse correlation lengths diverge. The critical exponent, $\nu_\perp$, of the transverse correlation length is shown to be equal to the exponent ν of the Ising model on an infinite lattice. In a second paper, these results will be used to analyze the wetting transition.

1. Introduction

We consider a binary system in the two-phase region, with phases + and −. We suppose that the system is in the − phase. If we insert a wall, which adsorbs preferentially the + phase, there is formation of a film of the + phase between the wall and the bulk phase. There is a *partial wetting* of the wall when the thickness of the film is microscopic, and *complete wetting* when the thickness is macroscopic. The *wetting transition* is the transition from partial wetting to complete wetting. This phenomenon can be analyzed in the Ising model. Let us consider the Ising model on $\mathbb{Z}^d$, with Hamiltonian

$$-\sum_{\langle ij\rangle} K\sigma(i)\sigma(j)-\sum_i \lambda\sigma(i), \tag{1.1}$$

where $\langle ij\rangle$ indicates a pair of points $\{i,j\}$ such that $|i-j|=1$. We insert a wall by setting $\sigma(i)=1$, for all $i=(i^1,\dots,i^d)\in\mathbb{Z}^d$ with $i^d\leqq 0$. In this way we get a semi-infinite model on the sublattice

$$\mathbb{L}=\{i\in\mathbb{Z}^d;\ i^d>0\}=\mathbb{Z}^{d-1}\times\mathbb{Z}^+ \tag{1.2}$$

with coupling constant K, external field λ and boundary field K. We generalize the model by admitting an arbitrary boundary field h and by choosing a coupling constant J for the interaction of two spins inside the first layer of $\mathbb{L}$,

$$\Sigma=\{i\in\mathbb{L};\ i^d=1\}\cong\mathbb{Z}^{d-1}\,. \tag{1.3}$$

Our final Hamiltonian is

$$H(J,K,h,\lambda)=-\sum_{\langle ij\rangle\subset \mathbb{L}} K(i,j)\sigma(i)\sigma(j)-\sum_{i\in\mathbb{L}}\lambda\sigma(i)-\sum_{i\in\Sigma}h\sigma(i) \tag{1.4}$$

with $K(i,j)=J$ if $\langle ij\rangle\subset\Sigma$, and otherwise $K(i,j)=K$. The model is always taken to be ferromagnetic, i.e. $J>0$ and $K>0$. If $h>0$, the wall adsorbs preferentially the + phase. Notice that both fields h and λ act on the first layer Σ.

In this paper we analyze

1) the surface free energy,
2) the surface equilibrium states,
3) the correlation lengths,
4) the phase diagram at $h=0$ and $\lambda=0$.

The wetting transition is considered in a second paper, where we analyze

5) the wetting transition; phase diagram at $\lambda=0$ and $h>0$,
6) the layering transition; phase diagram at $\lambda<0$ and $h>0$.

The condition $h=0$ means that the wall does *not* adsorb preferentially one of the phases. We have studied this case in [1] for multicomponent spin models. In [1] we also have obtained results on the behaviour of the spin-spin correlation function of the Ising model near the wall, when $J\leqq K\ll 1$, or $K\ll J\ll 1$.

The above model, for $J=K$, $\lambda=0$, and dimension $d=2$, has been solved by McCoy and Wu [2], and for $J\neq K$, by Au Yang [3]. Our approach is different. We use mainly correlation inequalities (or moment inequalities), and therefore our results are valid for $d\geqq 2$. But even in two dimensions some of our results are new.

The results of this paper are formulated in Sect. 2, and the proofs are given in Sects. 4 and 5. In Sect. 3, correlation inequalities and some consequences of these inequalities are summarized. In particular, the duplication trick is explained. Most of the results of this section are known, or are extensions of known results (e.g. Lemma 3.3). This section is basic for understanding the proofs of this and the next paper. The positivity of J and K, as well as the two-body character of the interaction are essential, since F.K.G. inequalities, duplicate variable inequalities and the Lee Yang theorem are used.

Our main results have been described in [4], where a discussion of points 5) and 6) may be found.

2. Results

2.1. Thermodynamic Functions

We start by defining the thermodynamic functions which describe the behaviour of the model near Σ. Let $\Lambda(L,M)$ be the finite box in $\mathbb{L}$, defined by

$$\Lambda(L,M)=\{i\in\mathbb{L},\ |i^k|\leqq L,\ k=1,\dots,d-1,\ 1\leqq i^d\leqq M\}\,. \tag{2.1}$$

Let the spin configuration outside $\Lambda(L,M)$ be fixed, $\sigma(i)=1$ for all $i\in\mathbb{L}\backslash\Lambda(L,M)$. If we restrict the summation in the Hamiltonian (1.4) over all pairs $\langle ij\rangle\subset\mathbb{L}$, such that $\{i,j\}\cap\Lambda(L,M)\neq\emptyset$, then we obtain a Hamiltonian $H^+_{L,M}$ for the model in the box $\Lambda(L,M)$ with + boundary condition (+ b.c.). The corresponding partition function and finite-volume Gibbs state are $Z^+_{L,M}$ and $\langle\,\cdot\,\rangle^+_{L,M}$. Similarly, we define

$-$ b.c. or free b.c. by taking $\sigma(i)=-1$ or $\sigma(i)=0$ outside $\Lambda(L,M)$. The bulk free energy is given by the formula

$$F_B(K,\lambda)=\lim_{L\to\infty}\frac{-1}{|\Lambda(L,L)|}\ln Z^+_{L,L}(J,K,h,\lambda). \tag{2.2}$$

In (2.2) $|\Lambda(L,L)|$ is the number of points inside the box $\Lambda(L,L)$, and the left-hand side does not depend on J, h or the choice of the b.c. Indeed, the boundary terms give a contribution to $\ln Z^+_{L,L}$ of order $O(L^{d-1})$ and $|\Lambda(L,L)|$ is of order $O(L^d)$. The definition of a surface free energy $F(J,K,h,\lambda)$ is more delicate; this problem is partially studied in [5, 6]. It is not true, anymore, that F is independent of the choice of the b.c. when the system is in the two-phase region, i.e. when $\lambda=0$ and $K>K_c(d)$, the critical coupling constant of the d-dimensional Ising model defined by (1.1). One usually considers the surface free energy $F(J,K,h,\lambda)$ for $\lambda\neq 0$, and then one defines $F^+(J,K,h,\lambda)=\lim_{\lambda\downarrow 0}F(J,K,h,\lambda)$ and $F^-(J,K,h,\lambda)=\lim_{\lambda\uparrow 0}F(J,K,h,\lambda)$. F^+ and F^- are the surface coefficients (or surface tensions) of the wall against the $+$ phase, the $-$ phase, respectively. The analysis of the difference $F^- - F^+$ corresponds to the analysis of the wetting phenomenon, and is the subject of our second paper. We proceed in a slightly different way. We define two surface free energies F^+ and F^- using $+$ b.c. and $-$ b.c. For $\lambda\neq 0$ it is likely that $F^+=F^-$, but we cannot prove this equality in full generality.

Let us consider the precise definition of the surface free energy F^+. It is convenient to consider another copy of the model in a box $\Lambda'(L,M)$, which is obtained by reflection of $\Lambda(L,M)$ with respect to the hyperplane $i^d=1/2$. For both copies we choose $+$ b.c. We may consider these two separate systems as one system contained in the box $\Omega(L,M)=\Lambda(L,M)\cup\Lambda'(L,M)\subset\mathbb{Z}^d$. The corresponding partition function is $(Z^+_{L,M})^2$. In the box $\Omega(L,M)$ we also consider the Ising model with Hamiltonian (1.1) and $+$ b.c. The partition function is $Q^+_{L,M}$. Let

$$F^+_{L,M}(J,K,h,\lambda)=-\frac{1}{2|\Sigma(L)|}\ln\frac{(Z^+_{L,M})^2}{Q^+_{L,M}}, \tag{2.3}$$

where $\Sigma(L)=\Lambda(L,M)\cap\Sigma$. We define

$$F^+(J,K,h,\lambda)=\lim_{L\to\infty}F^+_{L,L}(J,K,h,\lambda). \tag{2.4}$$

In the same way we define $F^-(J,K,h,\lambda)$.

If we replace the $+$ b.c. by the periodic b.c. for $\Omega(L,M)$ we obtain, instead of (2.3), a quantity $F^p_{L,M}(J,K,h,\lambda)$ and $F^p\equiv\lim_{L\to\infty}F^p_{L,L}$. This definition is natural for the following reason: The partition function $Q^p_{L,M}$ in the denominator in (2.3) is the partition function of the Ising model on a torus. One knows, at least for $d=2$, that the leading term for $\ln Q^p_{L,L}$ is of order $O(L^d)$, but that the next term is of order strictly smaller than $O(L^{d-1})$. In contrast, the partition function in the numerator in (2.3) is the partition function of the Ising model on a cylinder with two surfaces of volume $|\Sigma(L)|$. Thus, in (2.3), we extract a boundary contribution which we interpret as the surface free energy coming from the presence of the wall. This definition is used in [2, 3].

In Sect. 4, we prove the following results.

a) *F^+ and F^- are well-defined for any h and any λ.*

b) *F^+ is right-continuous in h and λ. F^- is left-continuous in h and λ.*

In particular, if $F^+ = F^-$ for $\lambda \neq 0$, then the surface tensions of the wall against the + phase and the − phase are given by ($F \equiv F^+ \equiv F^-$, for $\lambda \neq 0$)

$$\lim_{\lambda \downarrow 0} F(h,\lambda) = F^+(h,0), \tag{2.5}$$

and

$$\lim_{\lambda \uparrow 0} F(h,\lambda) = F^-(h,0). \tag{2.6}$$

c) *$F^+ = F^p$ if $\lambda \geqq 0$ and $h \geqq 0$.*

In particular, the free energy computed by McCoy and Wu and by Au Yang is F^+ for $h \geqq 0$, but F^- for $h \leqq 0$, since $F^+(J,K,h,\lambda) = F^-(J,K,-h,-\lambda)$.

d) *F^+ is analytic in h for $\lambda \geqq 0$ and* $\mathrm{Re}(h+\lambda) > 0$.

e) *F^+ is analytic in λ for $h \geqq 0$ and* $\mathrm{Re}\lambda > |\mathrm{Im}\lambda|$.

Once the surface free energy is defined, we can introduce (see [7])

$$M_\Sigma = -\frac{\partial F}{\partial h}, \quad \text{the layer magnetization}, \tag{2.7}$$

$$X_\Sigma = -\frac{\partial M}{\partial h}, \quad \text{the layer susceptibility}, \tag{2.8}$$

$$M_S = -\frac{\partial F}{\partial \lambda}, \quad \text{the excess surface magnetization}, \tag{2.9}$$

$$X_0 = \frac{\partial M_S}{\partial h} = \frac{\partial M_\Sigma}{\partial \lambda}, \tag{2.10}$$

where F is F^+ or F^-.

Remarks. 1) For proving a) and b) we use F.K.G. inequalities.

2) c) is proved in the Appendix.

3) d) is a direct consequence of the Lee Yang theorem.

4) e) is a consequence of correlation inequalities with imaginary angles (see Sect. 3).

5) In (2.3), it is not necessary to choose $M = L$. We may choose $M = L^\alpha$ with $0 < \alpha < 1$. This fact is used in our second paper.

2.2. Surface Equilibrium States

The definition of surface equilibrium states is straightforward. Surface equilibrium states are simply the Gibbs states of the model. Two states are important: one is defined by + b.c.

$$\langle \cdot \rangle^+ = \lim_{L,M \to \infty} \langle \cdot \rangle^+_{L,M}, \tag{2.11}$$

and the other by − b.c.

$$\langle \cdot \rangle^- = \lim_{L,M \to \infty} \langle \cdot \rangle^-_{L,M}. \tag{2.12}$$

Both states are Σ-invariant, i.e. invariant under all translations parallel to Σ (Lemma 3.1), and are extremal Gibbs states. Moreover, there is a unique Gibbs state in the model if and only if $\langle \cdot \rangle^+ = \langle \cdot \rangle^-$ (Lemma 3.2). When $\lambda \geqq 0$ and $h \geqq 0$ $\langle \sigma(i) \rangle^+ = \langle \sigma(i) \rangle^-$, for one i, implies already the uniqueness of the Gibbs state (Lemma 3.3). Just like the free energies F^+ and F^-, the states $\langle \cdot \rangle^+$ and $\langle \cdot \rangle^-$ are right continuous in h and λ, left continuous, respectively. In particular, if for fixed J, K, h there is a unique Gibbs state for $\lambda \neq 0$, $\langle \cdot \rangle(J, K, h, \lambda)$,

$$\langle \cdot \rangle^+(J, K, h, 0) = \lim_{\lambda \downarrow 0} \langle \cdot \rangle(J, K, h, \lambda), \tag{2.13}$$

and the state $\langle \cdot \rangle^-$, at $\lambda = 0$, is obtained by the limit $\lambda \uparrow 0$.

Let $\lambda > 0$ and $h > 0$. The free energy $F^+(h, \lambda)$ is analytic in λ and in h. Thus the thermodynamic quantities (2.7) through (2.10) are well-defined. We can show that all correlation functions $\langle \sigma_A \rangle^+(h, \lambda) \equiv \left\langle \prod_{i \in A} \sigma(i) \right\rangle^+ (h, \lambda)$ are also analytic functions of h and λ, and we can express (2.7) through (2.10), with $F = F^+$, in terms of correlation functions for the state $\langle \cdot \rangle^+$.

More precisely,

$$M_\Sigma = \langle \sigma(0) \rangle^+, \tag{2.14}$$

$$X_\Sigma = \sum_{i \in \Sigma} \langle \sigma(0); \sigma(i) \rangle^+, \tag{2.15}$$

$$X_0 = \sum_{i \in \mathbb{L}} \langle \sigma(0); \sigma(i) \rangle^+, \tag{2.16}$$

where $\langle \sigma(0); \sigma(i) \rangle^+ \equiv \langle \sigma(0)\sigma(i) \rangle^+ - \langle \sigma(0) \rangle \langle \sigma(i) \rangle^+$.

If $i \equiv (x, z)$, $x \in \mathbb{Z}^{d-1}$ and $z = i^d$,

$$M_S = \sum_{z \geqq 1} (\langle \sigma(0, z) \rangle^+ - \langle \sigma(0, z) \rangle^+_{\text{Is}}), \tag{2.17}$$

where $\langle \sigma(0, z) \rangle^+_{\text{Is}}$ is the magnetization of the Ising model on $\mathbb{Z}^d$ with Hamiltonian (1.1), and is independent of z.

Remarks. 1) Proofs of (2.14) through (2.17) are given in Sect. 4.3. They are based on the Lee Yang theorem and are similar to the proof of (2.15) given in [8]. The proof of (2.14) and (2.15) is also valid for the XY model. Concerning (2.15), see also [9, 10].

2) Formula (2.14) is valid not only for $\lambda > 0$ and $h > 0$, but for any values of λ and h for which F^+ is differentiable in h.

3) If we take the limit $h \downarrow 0$ and $\lambda \downarrow 0$ in (2.14)–(2.16), these formulas still hold in the limit.

Another consequence of the analyticity properties is a proof of normal fluctuations of block spin variables. Let $h > 0$, and let us decompose Σ into blocks of the same size:

$$\Sigma_0 = \{(x, 1) \colon x \in \mathbb{Z}^{d-1},\ -L \leqq x^i \leqq L,\ i = 1, \ldots, d-1\}, \tag{2.18}$$

and Σ_b is the translate of Σ_0 by the translation $a = (2Lb^1, \ldots, 2Lb^{d-1}, 0)$, $b \in \mathbb{Z}^{d-1}$.

We define

$$\sigma(\Sigma_b)=\sum_{i\in\Sigma_b}\sigma(i),\tag{2.19}$$

and the block spin variable

$$\bar{\sigma}(\Sigma_b)=\frac{1}{\sqrt{|\Sigma_b|}}(\sigma(\Sigma_b)-\langle\sigma(\Sigma_b)\rangle^+).\tag{2.20}$$

From [8] we have the following results: The distribution of the block spin $\bar{\sigma}(\Sigma_0)$ in the state $\langle\cdot\rangle^+$ tends to a Gaussian distribution when $L\to\infty$, which is given by the density

$$\frac{1}{(2\pi X_\Sigma)^{1/2}}\exp\left(-\frac{k^2}{2X_\Sigma}\right).\tag{2.21}$$

Moreover, the joint distribution in the state $\langle\cdot\rangle^+$ of p different block spin variables converges to the distribution of a product of p independent Gaussian variables defined by (2.21) when $L\to\infty$.

2.3. Correlation Lengths

Let $h\geqq 0$ and $\lambda\geqq 0$. There are two correlation lengths: the parallel correlation length ξ_Σ, related to the large distance behaviour of the spin-spin correlation function for two spins in or near Σ, and the transverse correlation length, $\xi_\perp$, which indicates how far the presence of the wall influences the behaviour of a spin $\sigma(i)$ inside the system. It is convenient to write $i=(x,z)$, $x\in\mathbb{Z}^{d-1}$, $z=i^d$. By definition

$$\xi_\perp^{-1}=\lim_{z\to\infty}\ -\frac{1}{z}\ln\langle\sigma(0,1);\sigma(0,z)\rangle^+.\tag{2.22}$$

The correlation length of the Ising model on $\mathbb{Z}^d$, with coupling K and field λ, is ξ_{Is}. For this model and for $\lambda=0$ we know, either from exact computation or from [11], that there is a unique Gibbs state with $\xi_{\text{Is}}<\infty$ and $X_{\text{Is}}<\infty$ for $K<K_c(d)$. When $K\uparrow K_c(d)$, ξ_{Is} and X_{Is} diverge. Above $K_c(d)$ there is spontaneous magnetization.

Our results for the semi-infinite model are (Lemmas 5.2, 5.3, and 5.5)

a) $$\xi_\perp(J,K,h,0)=\xi_{\text{Is}}(K,0),\tag{2.23}$$

if $\lambda=0$, $h\geqq 0$, *and* $K<K_c(d)$,

b) $$\langle\sigma(0,1)\sigma(0,L)\rangle^+\sim L^{-(d-2+\eta_\perp)},\qquad L\gg 1,\tag{2.24}$$

if $\lambda=0$, $K=K_c(d)$, $J\geqq K_c(d)$, *and* $h\geqq K_c(d)$, *where in* (2.24) $\eta_\perp=\eta_{\text{Is}}$.

c) *In* (2.24), $\eta_\perp\geqq\eta_{\text{Is}}$ *if* $\lambda=h=0$, $K=K_c(d)$, *and* $J\leqq K_c(d)$.

d) $$\xi_\perp(J,K,h,\lambda)\geqq\xi_{\text{Is}}(K,\lambda),\tag{2.25}$$

if $h\geqq 0$ *and* $\lambda\geqq 0$. *There is equality in* (2.25) *when* $h\geqq K$ *and* $K=J$.

In particular, the critical exponent $\nu_\perp$ of $\xi_\perp$ is equal to the critical exponent ν of ξ_{Is} and is independent of J and h. Therefore, the 2-dimensional plane $\{(J,K,h,\lambda)$: $K=K_c(d)$, $\lambda=0\}$ in the space of parameters of the model is critical.

Remark. In Sect. 5 the behaviour of $\langle\sigma(0,z)\rangle^+$ as $z\to\infty$ is analyzed (Lemma 5.4 and the remark following this lemma).

We now consider ξ_Σ for $h=\lambda=0$ and in the one-phase region, which is characterized by $\langle\sigma(i)\rangle^+(J,K)=0$. We define

$$\xi_\Sigma^{-1}=\lim_{L\to\infty} -\frac{1}{L}\ln\langle\sigma(0,1)\sigma(x_L,1)\rangle^f \tag{2.26}$$

with $x_L=(L,0,\ldots,0)\in\mathbb{Z}^{d-1}$. Here we choose free b.c., for technical reasons, and define the layer susceptibility X_Σ by

$$X_\Sigma=\sum_{j\in\Sigma}\langle\sigma(i)\sigma(j)\rangle^f\,,\qquad i\in\Sigma\,. \tag{2.27}$$

Remark. In the Appendix we prove that $F^+=F^f$ when $\lambda=0$ and $h\geqq 0$. By repeating an argument of Aizenman [11] we can prove that $X_\Sigma<\infty$, as defined in (2.27), implies $\langle\sigma(i)\rangle^+=0$. In other words, as long as (2.27) is finite, we are in the one-phase region and (2.27) is equal to (2.15).

The function $\xi_\Sigma^{-1}(J,K)$ is a continuous function of J and K. The proof is the same as for the Ising model (see e.g. [12]). By adapting an argument of Simon [13] we also show that

$$K<K_c(d)\quad\text{and}\quad X_\Sigma<\infty\quad\text{imply}\quad \xi_\Sigma<\infty\,. \tag{2.28}$$

Moreover, X_Σ^{-1} is a continuous function of J,

$$0\leqq X_\Sigma^{-1}(J_1)-X_\Sigma^{-1}(J_2)\leqq 2(d-1)(J_2-J_1)\,,\qquad J_2>J_1\,. \tag{2.29}$$

These results are proved in Sect. 5.2. Summary: *let the high-temperature region be defined by*

$$\{(J,K)\colon X_\Sigma \text{ defined by (2.27) is finite}\}\,. \tag{2.30}$$

From the above results we see that this region is inside the one-phase region, where there is uniqueness of the Gibbs state. Moreover, ξ_Σ is finite in the interior of the high-temperature region, and the boundary of this region is critical, since ξ_Σ diverges when one reaches the boundary.

Remark. If $K>K_c(d)$, then $\langle\sigma(i)\rangle^+(J,K)>0$, and by the preceding remark (2.27) is infinite.

2.4. *Phase Diagram at $h=0$ and $\lambda=0$*

For $d=2$, the phase diagram is known exactly. In the plane (J,K) there is a critical line $K=K_c(2)$. If $K<K_c(d)$ there is a unique Gibbs state, and therefore, $\langle\sigma(i)\rangle^+(J,K)=0$ for any $J\geqq 0$. For $K>K_c(d)$ the surface Σ and the bulk are ordered. The critical exponents of the layer susceptibility X_Σ and of the correlation length ξ_Σ are $\gamma_\Sigma=0$ and $\nu_\Sigma=1$. The exponent $\eta_\Sigma=1$; the definition of η_Σ is

$$\langle\sigma(0,1)\sigma(x_L,1)^+\rangle\sim L^{-(d-2+\eta_\Sigma)}\,,\qquad K=K_c(2)\,,\qquad L\gg 1\,. \tag{2.31}$$

From Sect. 2.3 we know that $\nu_\perp=1$. The exponent β of the spontaneous magnetization is $\frac{1}{2}$.

For $d\geqq 3$, the phase diagram is different, because the surface can be ordered before the bulk is ordered [14]. When $J=K$ we know that there is a unique Gibbs

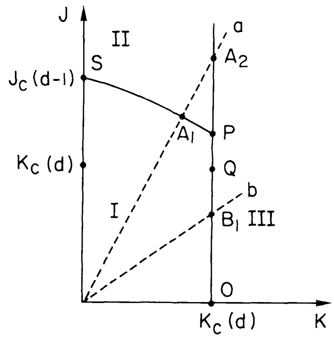

Fig. 1

state if and only if $K \leqq K_c(d)$ [15]. By correlation inequalities, this also holds for $J \leqq K$. Let $K \leqq K_c(d)$. We define

$$\hat{J}(K) = \inf\{J: \langle\sigma(0)\rangle^+(J,K) > 0\}. \tag{2.32}$$

[We have used the shorthand $\langle\sigma(0)\rangle^+$ for $\langle\sigma(0,\ldots,0,1)\rangle^+$.] If $J < \hat{J}(K)$ and $K \leqq K_c(d)$, there is a unique Gibbs state. $\hat{J}(K)$ is a monotone decreasing function of K, $\hat{J}(0) = J_c(d-1)$, $\hat{J}(K_c(d)) \geqq K_c(d)$. The phase diagram looks like Fig. 1.

From Sect. 2.3, $K = K_c(d)$ is a critical line, since $\xi_\perp$ diverges when one crosses this line, moreover, $v_\perp = v_{\mathrm{Is}}$. The line OP is the ordinary transition line and the critical exponents are called surface exponents. The vertical line from P to infinity is the extraordinary transition line. The line from P to S is the surface transition line. Here the exponents are those of the $(d-1)$-dimensional Ising model. Region I corresponds to the region where there is a unique Gibbs state. It contains the high-temperature region defined by (2.30). Assuming that it coincides with the high-temperature region and assuming that the function $K \to \hat{J}(K)$ is differentiable, it follows that X_Σ and ξ_Σ diverge when one crosses this line, e.g. along the line a at A_1 in Fig. 1. Moreover, $\gamma_\Sigma \geqq 1$ (Lemma 5.13). At A_2 one has a second transition. Here the bulk becomes ordered, and $\xi_\perp$ diverges. If one moves along line b, there is one transition at B_1, where ξ_Σ and $\xi_\perp$ diverge. [The divergence of X_Σ is not proved, because in (2.28) we need $K < K_c(d)$, and we have proved only continuity in J for X_Σ^{-1}.]

3. Correlation Inequalities

Let Λ be a finite subset. In the applications $\Lambda \subset \mathbb{Z}^d$ or $\Lambda \subset \mathbb{L}$. The interaction is given by a finite range two-body interaction $K(i,j)$ which is always ferromagnetic: $K(i,j) \geqq 0$. The assumption of finite range is not important. We also add an external magnetic field h, which is a real-valued function on the lattice. The Hamiltonian is

$$H_\Lambda(h) = -\sum_{\{i,j\}\subset\Lambda} K(i,j)\sigma(i)\sigma(j) - \sum_{i\in\Lambda} h(i)\sigma(i). \tag{3.1}$$

The corresponding Gibbs state is $\langle\cdot\rangle_\Lambda(h)$. If $\Lambda \subset \mathbb{Z}^d$ or $\Lambda \subset \mathbb{L}$, a boundary condition for the system in Λ is specified by fixing the values of the spins outside Λ and taking into account in (3.1) their interaction with the spins inside Λ.

3.1. F.K.G. Inequalities

A spin configuration is a function σ defined on the lattice with values ± 1. By definition $\sigma \leqq \sigma'$, if and only if $\sigma(i) \leqq \sigma'(i)\ \forall i$. A real-valued function f, defined on the configurations, is (monotone) increasing if $\sigma \leqq \sigma'$ implies $f(\sigma) \leqq f(\sigma')$. Examples of increasing functions are

$$f_A^{\pm}(\sigma) = \sum_{i \in A} \sigma(i) \pm \prod_{i \in A} \sigma(i), \qquad n_A(\sigma) = \prod_{i \in A} \tfrac{1}{2}(1 + \sigma(i)). \tag{3.2}$$

Let f and g be increasing functions. The F.K.G. inequality for f and g says that [16]

$$\langle f \cdot g \rangle_\Lambda(h) \geqq \langle f \rangle_\Lambda(h) \langle g \rangle_\Lambda(h). \tag{3.3}$$

There is a related inequality for an increasing function f (Holley [17]),

$$\langle f \rangle_\Lambda(h) \leqq \langle f \rangle_\Lambda(h') \quad \text{if} \quad h \leqq h'. \tag{3.4}$$

Applications. Let us consider some standard consequences. (See e.g. the review [18].) Let $\Lambda \subset \mathbb{Z}^d$ and for each $a = (a^1, \ldots, a^{d-1}, 0) \in \mathbb{Z}^d$ we suppose that $K(i+a, j+a) = K(i,j)$ and $h(i+a) = h(i)$. Let $\langle \cdot \rangle_\Lambda^+$ be the Gibbs state in Λ with $+$ b.c., i.e. $\sigma(i) = 1$ if $i \notin \Lambda$. By the F.K.G. inequality, we have for all $A \subset \Lambda_1 \cap \Lambda_2$,

$$\langle n_A \rangle_{\Lambda_1}^+ \leqq \langle n_A \rangle_{\Lambda_2}^+, \qquad \Lambda_1 \supseteqq \Lambda_2. \tag{3.5}$$

Indeed, for the Gibbs state $\langle \cdot \rangle_{\Lambda_2}^+$ the boundary condition means $\sigma(i) = +1$ when $i \notin \Lambda_2$. In particular, $\sigma(i) = +1$ for $i \in \Lambda_1 \backslash \Lambda_2$. If we add a term $\sum_{i \in \Lambda_1 \backslash \Lambda_2} \mu \cdot \sigma(i)$ in the Hamiltonian for the system in Λ_1, and let $\mu \to \infty$, then $\sigma(i) = 1$ for all $i \in \Lambda_1 \backslash \Lambda_2$. Thus

$$\lim_{\mu \to \infty} \langle n_A \rangle_{\Lambda_1}^+(\mu) = \langle n_A \rangle_{\Lambda_2}^+, \tag{3.6}$$

and the inequality (3.5) follows. Therefore, for any sequence $\Lambda_n \subset \Lambda_{n+1}$, with $\Lambda_n \uparrow \mathbb{Z}^d$, we have a limit

$$\lim_n \langle n_A \rangle_{\Lambda_n}^+ \equiv \langle n_A \rangle^+, \tag{3.7}$$

which is independent of the sequence. This defines a Gibbs state $\langle \cdot \rangle^+$ of the infinite system on $\mathbb{Z}^d$.

Lemma 3.1. *The Gibbs state $\langle \cdot \rangle^+$ is an extremal Gibbs state and has the properties*

$$\langle n_A \rangle^+ = \langle n_{A+a} \rangle^+, \tag{3.8}$$

where $A + a = \{i + a;\ i \in A\}$ and $a = (a^1, \ldots, a^{d-1}, 0)$.

For any $\varepsilon > 0$, there is a finite subset of $\mathbb{Z}^d$, $\Delta(A, \varepsilon)$, such that for all $\Lambda \supset \Delta(A, \varepsilon)$

$$|\langle n_A \rangle_\Lambda^+ - \langle n_A \rangle^+| \leqq \varepsilon. \tag{3.9}$$

Moreover, for $A + a$, with a as above, we can choose $\Delta(A+a, \varepsilon) = \Delta(A, \varepsilon) + a$. The state $\langle \cdot \rangle^+$ has the clustering property

$$\lim_{|a| \to \infty} \langle n_{A+a} n_B \rangle^+ = \langle n_A \rangle^+ \langle n_B \rangle^+. \tag{3.10}$$

Remarks. 1) We can define, in a similar way, a Gibbs state $\langle\cdot\rangle^-$ with $-$ b.c. In this case (3.5) is replaced by

$$\langle n_A\rangle^-_{\Lambda_1} \geqq \langle n_A\rangle^-_{\Lambda_2}, \qquad \Lambda_1 \supseteqq \Lambda_2. \tag{3.11}$$

Lemma 3.1 is valid for $\langle\cdot\rangle^-$.

2) We have chosen the lattice $\mathbb{Z}^d$, but it is obvious that we could take $\mathbb{L}$ and get the same results.

Lemma 3.2 [19]. *If $\langle\sigma(i)\rangle^+ = \langle\sigma(i)\rangle^-$ for all i, the two Gibbs states $\langle\cdot\rangle^+$ and $\langle\cdot\rangle^-$ coincide, and this implies the uniqueness of the Gibbs state.*

The proof follows from the fact that the functions f_A^+, defined in (3.2) are increasing, and therefore,

$$|\langle\sigma_A\rangle^+ - \langle\sigma_A\rangle^-| \leqq \sum_{i\in A} (\langle\sigma(i)\rangle^+ - \langle\sigma(i)\rangle^-). \tag{3.12}$$

Moreover, for any Gibbs state $\langle\cdot\rangle$, we have

$$\langle n_A\rangle^- \leqq \langle n_A\rangle \leqq \langle n_A\rangle^+. \tag{3.13}$$

The extremality of $\langle\cdot\rangle^+$ and $\langle\cdot\rangle^-$ follows from (3.13). We can deduce from (3.5) and (3.11) continuity properties. Let $h(i)=h$ for all i. Then

$$\lim_{h_1\uparrow h_2} \langle n_A\rangle^-(h_1) = \langle n_A\rangle^-(h_2), \tag{3.14}$$

and

$$\lim_{h_1\downarrow h_2} \langle n_A\rangle^+(h_1) = \langle n_A\rangle^+(h_2). \tag{3.15}$$

Indeed, $\langle n_A\rangle^+_\Lambda(h_1) \geqq \langle n_A\rangle^+(h_1) \geqq \langle n_A\rangle^+(h_2)$, so that for all Λ

$$\langle n_A\rangle^+_\Lambda(h_2) \geqq \lim_{h_1\downarrow h_2} \langle n_A\rangle^+(h_1) \geqq \langle n_A\rangle^+(h_2). \tag{3.16}$$

3.2. G.K.S. Inequalities

We take $h \geqq 0$ in (3.1), i.e. $h(i) \geqq 0$ for all i. G.K.S. inequalities are [20]

$$\langle\sigma_A\rangle_\Lambda(h) \geqq 0, \tag{3.17}$$

$$\langle\sigma_A\sigma_B\rangle_\Lambda(h) \geqq \langle\sigma_A\rangle_\Lambda(h)\langle\sigma_B\rangle_\Lambda(h). \tag{3.18}$$

We can apply these inequalities in the situation described in 3.1. For example, we have

$$\langle\sigma_A\rangle^+ \geqq \langle\sigma_A\rangle^-. \tag{3.19}$$

Moreover, we can define the Gibbs state $\langle\cdot\rangle^f$ with free boundary condition exactly as the states $\langle\cdot\rangle^+$ or $\langle\cdot\rangle^-$. Lemma 3.1 is also valid for $\langle\cdot\rangle^f$, with the exception of the clustering property and the extremality property. The following continuity property is valid,

$$\lim_{J_1\uparrow J_2} \langle\sigma_A\rangle^f(J_1) = \langle\sigma_A\rangle^f(J_2). \tag{3.20}$$

3.3. Duplicate Variable Inequalities

Let us consider two independent copies of the same system in Λ. The spins in the second copy are $\sigma(i)'$, $i \in \Lambda$, and the Hamiltonian is $H_\Lambda(h')$. [We suppose here that $K'(i,j)=K(i,j)$.] We can consider the two systems as one system, called the duplicate system, which is defined in Λ, and for each $i \in \Lambda$ we have two independent spin variables $\sigma(i)$ and $\sigma(i)'$. Let

$$s(i)=\sigma(i)+\sigma(i)' , \qquad t(i)=\sigma(i)-\sigma(i)' . \tag{3.21}$$

The Hamiltonian for the duplicate system can be written

$$\begin{aligned} H_\Lambda(h)+H_\Lambda(h') = &-\tfrac{1}{2} \sum_{\{i,j\}\subset \Lambda} K(i,j)(s(i)s(j)+t(i)t(j)) \\ &-\tfrac{1}{2} \sum_{i \in \Lambda} (h(i)+h(i)')s(i) - \sum_{i\in\Lambda} (h(i)-h(i)')t(i) . \end{aligned} \tag{3.22}$$

Let $t_A = \prod_{i \in A} t(i)$ and $s_A = \prod_{i \in A} s(i)$. We also write $\langle s_A \rangle_\Lambda$ for the expectation value of s_A in the duplicate system. Let $h(i) \pm h(i)' \geqq 0$. Duplicate variable inequalities are [21]

$$0 \leqq \langle t_A s_B \rangle_\Lambda \leqq \langle t_A \rangle_\Lambda \langle s_B \rangle_\Lambda , \tag{3.23}$$

$$\langle t_A t_B \rangle_\Lambda \geqq \langle t_A \rangle_\Lambda \langle t_B \rangle_\Lambda , \tag{3.24}$$

$$\langle s_A s_B \rangle_\Lambda \geqq \langle s_A \rangle_\Lambda \langle s_B \rangle_\Lambda . \tag{3.25}$$

Applications. We consider again the model defined in 3.1. In the first application the invariance by translation a is not required. It is, however, important that $K(i,j)>0$ if $|i-j|=1$. Let $\langle \cdot \rangle$ be another Gibbs state of the infinite system. We suppose that $\langle \cdot \rangle = \lim_n \langle \cdot \rangle_{\Lambda_n}$, where $\langle \cdot \rangle_{\Lambda_n}$ is a Gibbs state in the finite volume Λ_n with some boundary condition.

Lemma 3.3. *Under the above conditions, and if* $h(i) \geqq 0$ $\forall i$, *we have*

1) $\langle \sigma(i) \rangle^+ > 0$, $\forall i$, *or* $\langle \sigma(i) \rangle^+ = 0$, $\forall i$.
2) *If* $\langle \sigma(i) \rangle^+ = \langle \sigma(i) \rangle$ *for some* i, *then* $\langle \cdot \rangle^+ = \langle \cdot \rangle$.

Proof. Since $K(i,j)>0$ if $|i-j|=1$, we have by G.K.S. inequalities,

$$\langle \sigma(i)\sigma(j) \rangle^+ \geqq \langle \sigma(i)\sigma(j) \rangle^f_{\{i,j\}} > 0 . \tag{3.26}$$

Therefore, 1) follows from

$$\langle \sigma(i) \rangle^+ = \langle \sigma(i)\sigma(j)^2 \rangle^+ \geqq \langle \sigma(i)\sigma(j) \rangle^+ \langle \sigma(j) \rangle^+ . \tag{3.27}$$

We consider the duplicate system in Λ_n constructed with the Hamiltonians giving the Gibbs states $\langle \cdot \rangle^+_{\Lambda_n}$ and $\langle \cdot \rangle_{\Lambda_n}$. For this duplicate system we can apply inequalities (3.23)–(3.25). We first prove that

$$\langle t(i)t(j) \rangle_{\Lambda_n} > 0 \quad \text{if} \quad |i-j|=1 . \tag{3.28}$$

To prove this we proceed as for (3.5). We add an external field $\mu \cdot s(k)$ for all $k \in \Lambda_n \backslash \{i,j\}$, and let $\mu \to \infty$. In this limit, $s(k)=2$ and $t(k)=0$, for all $k \in \Lambda_n \backslash \{i,j\}$. Therefore,

$$\langle t(i)t(j) \rangle_{\Lambda_n} \geqq \langle t(i)t(j) \rangle'_{\{i,j\}} > 0 , \tag{3.29}$$

where on the right-hand side the expectation value is that given by the Gibbs state defined on $\{i,j\}$ with Hamiltonian

$$-\tfrac{1}{2}K(i,j)(s(i)s(j)+t(i)t(j))-\sum_{l\neq j}s(i)(h(i)+\tfrac{1}{2}K(i,l))-\sum_{l\neq i}s(j)(h(j)+\tfrac{1}{2}K(j,l)).\tag{3.30}$$

Therefore, in the duplicate system

$$\lim_n\langle t(i)t(j)\rangle_{\Lambda_n}\equiv\langle t(i)t(j)\rangle>0.\tag{3.31}$$

As for 1), we have $\langle t(i)\rangle>0$, $\forall i$, or $\langle t(i)\rangle=0$, $\forall i$, since

$$\begin{aligned}\langle t(i)\rangle&=1/4\langle t(i)(t(j)^2+s(j)^2)\rangle\\&\geqq 1/4\langle t(i)t(j)^2\rangle\geqq 1/4\langle t(i)t(j)\rangle\langle t(j)\rangle.\end{aligned}\tag{3.32}$$

If $\langle\sigma(i)\rangle^+=\langle\sigma(i)\rangle$ for some i, $\langle t(i)\rangle=0$, and therefore, $\langle\sigma(j)\rangle^+=\langle\sigma(j)\rangle$ for all j.

The result follows from a variant of Lemma 3.2 [see (3.12)]. We also could use the results of [22], since either $\langle\sigma(i)\rangle^+>0$, for all i, or $\langle\sigma(i)\sigma(j)\rangle^+=\langle\sigma(i)\sigma(j)\rangle>0$, for all i and j when $\langle\sigma(i)\rangle\equiv 0$:

$$\begin{aligned}0&\leqq\langle\sigma(i)\sigma(j)\rangle^+-\langle\sigma(i)\sigma(j)\rangle=\tfrac{1}{2}(\langle s(i)t(j)\rangle+\langle s(j)t(i)\rangle)\\&\leqq\tfrac{1}{2}\langle s(i)\rangle\langle t(j)\rangle+\tfrac{1}{2}\langle s(j)\rangle\langle t(i)\rangle.\quad\square\end{aligned}\tag{3.33}$$

Remarks. 1) Every Gibbs state is a convex combination of extremal Gibbs states and for those Gibbs states the hypothesis which we made on $\langle\cdot\rangle$ is verified. Moreover, if $\langle\cdot\rangle$ is any Gibbs state, $\langle\sigma(i)\rangle^+\geqq\langle\sigma(i)\rangle$. From this it follows that Lemma 3.2 is in fact true for any Gibbs state $\langle\cdot\rangle$.

2) In our application the lattice is $\mathbb{Z}^d$. The only property which is important is that any two points i and j of the lattice can be joined by a path of points $i_0=i$, $i_1,\dots,i_n=j$ such that $K(i_l,i_{l+1})>0$.

3) In the duplication trick it is not necessary to consider two independent systems. We can make a self-duplication if we can divide the spins in the box into two sets, Λ_1 and Λ_2, such that for each $i\in\Lambda_1$ there is one and only one $\bar{i}\in\Lambda_2$. Then we introduce the variables

$$s(i)=\sigma(i)+\sigma(\bar{i}),\qquad t(i)=\sigma(i)-\sigma(\bar{i}).\tag{3.34}$$

Several well-known inequalities follow from inequalities (3.23)–(3.25) [21]. Let us consider, for example, the model defined in 3.1 on the lattice $\mathbb{L}$ or $\mathbb{Z}^d$. The G.H.S. inequality [23] is

$$\begin{aligned}&\langle\sigma(i)\sigma(j)\sigma(k)\rangle^+-\langle\sigma(i)\rangle^+\langle\sigma(j)\sigma(k)\rangle^+-\langle\sigma(j)\rangle^+\langle\sigma(i)\sigma(k)\rangle^+\\&\quad-\langle\sigma(k)\rangle^+\langle\sigma(i)\sigma(j)\rangle^++2\langle\sigma(i)\rangle^+\langle\sigma(j)\rangle^+\langle\sigma(k)\rangle^+\leqq 0,\end{aligned}\tag{3.35}$$

provided $h(i)\geqq 0\ \forall i$. We can also replace in (3.35) the state $\langle\cdot\rangle^+$ by the state $\langle\cdot\rangle^f$. The second very important inequality is the u_4-inequality [21], which is valid if $h(i)=0$, $\forall i$, and $\langle\sigma(i)\rangle^+=0$, $\forall i$:

$$\begin{aligned}&\langle\sigma(i)\sigma(j)\sigma(k)\sigma(l)\rangle^+-\langle\sigma(i)\sigma(j)\rangle^+\langle\sigma(k)\sigma(l)\rangle^+-\langle\sigma(i)\sigma(k)\rangle^+\langle\sigma(j)\sigma(l)\rangle^+\\&\quad-\langle\sigma(i)\sigma(l)\rangle^+\langle\sigma(j)\sigma(k)\rangle^+\leqq 0.\end{aligned}\tag{3.36}$$

Using the duplicate system we can conveniently express the two-point connected function $\langle\sigma(i);\sigma(j)\rangle\equiv\langle\sigma(i)\sigma(j)\rangle^{+}-\langle\sigma(i)\rangle^{+}\langle\sigma(j)\rangle^{+}$. In the duplicate system, where this time both Hamiltonians are identical, we find

$$2\langle\sigma(i);\sigma(j)\rangle^{+}=\langle t(i)t(j)\rangle. \tag{3.37}$$

From (3.37) it follows that

$$2\langle\sigma(i);\sigma(j)\rangle^{+}\geqq\langle\sigma(i);\sigma(k)\rangle^{+}\langle\sigma(k);\sigma(j)\rangle^{+}. \tag{3.38}$$

This inequality has been improved by Graham [24],

$$\langle\sigma(i);\sigma(j)\rangle^{+}\geqq\langle\sigma(i);\sigma(k)\rangle^{+}\langle\sigma(k);\sigma(j)\rangle^{+}. \tag{3.39}$$

3.4. Ellis-Monroe Inequalities

We consider a duplicate system in Λ whose Hamiltonian is

$$-\sum_{\{i,j\}\subset\Lambda}K(i,j)(s(i)s(j)+t(i)t(j))-\sum_{i\in\Lambda}\lambda(i)s(i)-\sum_{i\in\Lambda}\mu(i)t(i). \tag{3.40}$$

The Gibbs state corresponding to (3.40) is $\langle\cdot\rangle_{\Lambda}$. The t-b.c. is, by definition, $t(i)=2$, $\forall i\notin\Lambda$, and the s-b.c. is $s(i)=2$, $\forall i\notin\Lambda$. Notice that $t(i)\neq 0 \Leftrightarrow s(i)=0$. The corresponding states are $\langle\cdot\rangle_{\Lambda}^{t}$, respectively, $\langle\cdot\rangle_{\Lambda}^{s}$. We make a duplication of the systems in Λ and consider in the second system a Hamiltonian (3.40) with $\lambda'(i)$ and $\mu'(i)$ instead of $\lambda(i)$ and $\mu(i)$. The corresponding Gibbs state is $\langle\cdot\rangle'_{\Lambda}$ and $\langle\!\langle\cdot\rangle\!\rangle_{\Lambda}$ is the expectation value with respect to $\langle\cdot\rangle_{\Lambda}\otimes\langle\cdot\rangle'_{\Lambda}$. If $K(i,j)\geqq 0$, $\lambda(i)\geqq|\lambda'(i)|$ and $\mu'(i)\geqq|\mu(i)|$ $\forall i\in\Lambda$, then the Ellis-Monroe inequalities are [25]

$$\left\langle\!\!\left\langle\prod_{i}(t(i)+t(i)')^{n_1(i)}\cdot(t(i)-t(i)')^{n_2(i)}\cdot(s(i)'+s(i))^{n_3(i)}\cdot(s(i)'-s(i))^{n_4(i)}\right\rangle\!\!\right\rangle_{\Lambda}\geqq 0, \tag{3.41}$$

where $n_k(i)\in\mathbb{N}$, $k=1,2,3,4$.

Remark. If $K(i,j)\geqq 0$, $\lambda(i)'\geqq|\lambda(i)|$ and $\mu(i)\geqq|\mu'(i)|$, then we have similar inequalities with $t(i)'\pm t(i)$ and $s(i)\pm s'(i)$.

Application. We consider a duplicate model in Λ with Hamiltonian (3.40) and $\Lambda\subset\mathbb{Z}^d$. We suppose that $\lambda(i)\geqq 0$, $\mu(i)\geqq 0$, and $K(i,j)\geqq 0$. By duplicate variable inequalities we have for $\Lambda_1\subseteqq\Lambda_2$,

$$\langle s_A\rangle_{\Lambda_1}^{t}\leqq\langle s_A\rangle_{\Lambda_2}^{t},\qquad\langle t_A\rangle_{\Lambda_1}^{t}\geqq\langle t_A\rangle_{\Lambda_2}^{t}, \tag{3.42}$$

and

$$\langle s_A\rangle_{\Lambda_1}^{s}\geqq\langle s_A\rangle_{\Lambda_2}^{s},\qquad\langle t_A\rangle_{\Lambda_1}^{s}\leqq\langle t_A\rangle_{\Lambda_2}^{s}. \tag{3.43}$$

Using Ellis-Monroe inequalities we have

$$|\langle s_At_B\rangle_{\Lambda_1}^{t}-\langle s_At_B\rangle_{\Lambda_2}^{t}|\leqq\langle t_B\rangle_{\Lambda_1}^{t}\langle s_A\rangle_{\Lambda_2}^{t}-\langle t_B\rangle_{\Lambda_2}^{t}\langle s_A\rangle_{\Lambda_1}^{t}, \tag{3.44}$$

and

$$|\langle s_At_B\rangle_{\Lambda_1}^{s}-\langle s_At_B\rangle_{\Lambda_2}^{s}|\leqq\langle s_A\rangle_{\Lambda_1}^{s}\langle t_B\rangle_{\Lambda_2}^{s}-\langle s_A\rangle_{\Lambda_2}^{s}\langle t_B\rangle_{\Lambda_1}^{s}. \tag{3.45}$$

In the first case we use (3.41) for $(t_B\pm t'_B)(s'_A\mp s_A)$, and in the second case we use the remark and apply it to $(t'_B\pm t_B)(s_A\mp s'_A)$.

Using these inequalities we can define an infinite-volume Gibbs state $\langle \cdot \rangle^t$, and a Gibbs state $\langle \cdot \rangle^s$ as in 3.1. We have a lemma similar to Lemma 3.1.

We state it for the state $\langle \cdot \rangle^t$, and we suppose that $K(i,j)$, $\lambda(i)$ and $\mu(i)$ satisfy the invariance properties as in Sect. 3.1.

Lemma 3.4. a) *The Gibbs state $\langle \cdot \rangle^t$ is invariant by translation $a=(a^1,\ldots,a^{d-1},0)$:*

$$\langle s_A t_B \rangle^t = \langle s_{A+a} t_{B+a} \rangle . \tag{3.46}$$

b) *For any $\varepsilon>0$ there is a finite subset of $\mathbb{Z}^d$, $\Delta(A,\varepsilon)$ such that for all $\Lambda \supset \Delta(A,\varepsilon)$*

$$|\langle s_A \rangle_\Lambda^t - \langle s_A \rangle^t| \leqq \varepsilon , \quad |\langle t_A \rangle_\Lambda^t - \langle t_A \rangle^t| \leqq \varepsilon . \tag{3.47}$$

Moreover, we can choose Δ so that $\Delta(A+a,\varepsilon)=\Delta(A,\varepsilon)+a$.

c) *The state $\langle \cdot \rangle^t$ has clustering property in t_A:*

$$\lim_{|a|\to\infty} \langle t_{A+a} t_B \rangle^t = \langle t_A \rangle^t \langle t_B \rangle^t , \tag{3.48}$$

where a is as above.

Remarks. 1) The proofs of a) and b) follow directly from the monotonicity properties (3.42). To prove (3.46) we use (3.44).

2) The lemma is still true if we add interactions of the kind $-\tilde{\lambda}(i)s(i)^2$ or $-\tilde{\mu}(i)t(i)^2$ with $\tilde{\lambda}(i)\geqq 0$, $\tilde{\mu}(i)\geqq 0$ which satisfy the invariance property stated in the lemma. This is useful in a self-duplication of the model, for example in the Ising model with $\pm$ b.c. It is not true in general that the state $\langle \cdot \rangle^t$ has clustering properties in s_B.

3) If we take $\langle \cdot \rangle^s$ in Lemma 3.4 the clustering property in c) is now valid for s_A, and not for t_A.

3.5. Ginibre Inequalities

There is another way to express the duplicate system. Let $\theta(i)$ be defined by

$$\tfrac{1}{2}(\sigma(i)+\sigma(i)')=\cos\theta(i), \quad \tfrac{1}{2}(\sigma(i)-\sigma(i)')=\sin\theta(i). \tag{3.49}$$

Thus $\theta(i)=k\cdot\frac{\pi}{2}$, $k=0,1,2,3$. The duplicate Hamiltonian becomes

$$\begin{aligned} H_\Lambda(h)+H_\Lambda(h') = &- \sum_{\{i,j\}\subset\Lambda} 2K(i,j)\cos(\theta(i)-\theta(j)) - \sum_{i\in\Lambda}(h(i)+h(i)')\cos\theta(i) \\ &- \sum_{i\in\Lambda}(h(i)-h(i)')\sin\theta(i). \end{aligned} \tag{3.50}$$

If $h(i)=h(i)'$, $\forall i$, (3.50) is a Hamiltonian of the type considered by Ginibre,

$$-\sum_m K(m)\cos m\theta , \tag{3.51}$$

where m is a $\mathbb{Z}$-valued function defined on the lattice (with finite support) and

$$m\theta = \sum_i m(i)\cdot\theta(i).$$

If $K(m)\geqq 0$, we have the Ginibre inequalities [26]

$$\langle \cos m\theta \rangle_\Lambda \geqq 0 , \tag{3.52}$$

$$\langle \cos m\theta \cdot \cos n\theta \rangle_\Lambda \geqq \langle \cos m\theta \rangle_\Lambda \cdot \langle \cos n\theta \rangle_\Lambda . \tag{3.53}$$

In [27], a more general Hamiltonian than (3.51) is introduced,

$$-\sum_m K(m)\cos(m\theta-\psi(m)), \tag{3.54}$$

where $\psi(m)$ is a real number. If $\langle\cdot\rangle_\Lambda$ is the Gibbs state corresponding to (3.51) and $\langle\cdot\rangle'_\Lambda$ the Gibbs state corresponding to (3.54), we have the inequalities [27]

$$\langle\cos m\theta\rangle_\Lambda \geqq \langle\cos m\theta\rangle'_\Lambda, \tag{3.55}$$

provided $K(m)\geqq 0$.

Remark. We can use, instead of the variables (3.49),

$$\tfrac{1}{2}(\sigma(i)-\sigma(i)')=\cos\theta(i), \qquad \tfrac{1}{2}(\sigma(i)+\sigma(i)')=\sin\theta(i). \tag{3.56}$$

3.6. Inequalities with Imaginary Angles

We consider the system defined by (3.51) with $K(m)\geqq 0$. In the second copy of the system we take the Hamiltonian H'_Λ

$$H'_\Lambda=-\sum_m K(m)\cos(m\theta+i\psi(m)) \tag{3.57}$$

with $\psi(m)$ a real number. Then we have the inequalities [27]

$$\langle\cos m\theta\rangle'_\Lambda \geqq \langle\cos m\theta\rangle_\Lambda. \tag{3.58}$$

Applications. The following results are taken from [28]. We consider the system in Λ defined by (3.1). We suppose that the external field h is *complex*, i.e. $h(j)=h_1(j)+ih_2(j)$ for all $j\in\Lambda$. Let $\mu(j)=(h_1(j)^2-h_2(j)^2)^{1/2}$, and let us suppose that the function μ, $j\in\Lambda\to\mu(j)\in\mathbb{R}$, is strictly positive.

Lemma 3.5. *Under the above hypotheses we have*

a) $$|Z_\Lambda(h)|\geqq Z_\Lambda(\mu), \tag{3.59}$$

b) $$\mathrm{Re}\langle\sigma(j)\rangle_\Lambda(h)\geqq\langle\sigma(j)\rangle_\Lambda(\mu), \tag{3.60}$$

c) $$\mathrm{Re}\langle\sigma(j)\sigma(k)\rangle_\Lambda(h)\geqq\langle\sigma(j)\sigma(k)\rangle_\Lambda(\mu), \tag{3.61}$$

d) $$\left|\left\langle\prod_{j\in A}\sigma(j)\right\rangle_\Lambda(h)\right|\geqq\left\langle\prod_{j\in A}\sigma(j)\right\rangle(\mu), \qquad A\subset\Lambda. \tag{3.62}$$

Proof. The proof is given in [28] and is a consequence of the inequalities (3.58). Let us consider the proof of d). We take a duplicate system where in the second copy the external field h is replaced by the complex conjugate field $\bar{h}$.

We have

$$\begin{aligned} h(j)\sigma(j)+\bar{h}(j)\sigma(j)' &= 2h_1(j)\cos\theta(j)+2ih_2(j)\sin\theta(j)\\ &=2\mu(j)(ch(\lambda(j))\cos\theta(j)+ish(\lambda(j))\sin\theta(j))\\ &=2\mu(j)\cos(\theta(j)+i\lambda(j)). \end{aligned} \tag{3.63}$$

[$\lambda(j)$ is defined by the second equality in (3.63).] Now

$$\sigma(j)\sigma(j)'=\cos^2\theta(j)-\sin^2\theta(j)=\cos 2\theta(j). \tag{3.64}$$

Therefore, the left-hand side of d) in Lemma 3.5 is expressed in the duplicate system with Hamiltonian

$$-2 \sum_{\{i,j\}\subset \Lambda} K(i,j)\cos(\theta(i)-\theta(j)) - 2\sum_{j\in\Lambda} \mu(j)\cos(\theta(j)+i\lambda(j)), \tag{3.65}$$

as

$$\left\langle \prod_{j\in\Lambda} \cos 2\theta(j) \right\rangle_{\Lambda} . \quad \square \tag{3.66}$$

Remarks. 1) Proofs of a), b), c) are given in [28].

2) a), b), and c) have been first proved by Dunlop [29].

3) The hypothesis on the external field can be written

$$\mathrm{Re}\,h(j) > |\mathrm{Im}\,h(j)| . \tag{3.67}$$

In our second application we prove a fluctuation-dissipation relation for the Ising model. This result can be extended to the XY model using the generalization of Lemma 3.5b) for this model (see [30]).

Let Λ be a finite cube, and let h_Λ be defined by

$$h_\Lambda(j) = h_2, \quad j \notin \Lambda; \quad h_\Lambda(j) = h_1, \quad j \in \Lambda . \tag{3.68}$$

We consider the function $\langle\sigma(0)\rangle^+(h_\Lambda)$, the expectation value being taken with respect to the infinite-volume Gibbs state. (This is well-defined for h_1 and h_2 real by F.K.G. inequalities.)

Lemma 3.6. a) *$\langle\sigma(0)\rangle^+(h_\Lambda)$ is an analytic function of h_1 and h_2 if* $\mathrm{Re}\,h_1 > |\mathrm{Im}\,h_1|$ *and* $\mathrm{Re}\,h_2 > |\mathrm{Im}\,h_2|$.

b)
$$\frac{d}{dh_1}\langle\sigma(0)\rangle^+(h_\Lambda) = \sum_{x\in\Lambda} \langle\sigma(0);\sigma(x)\rangle^+(h_\Lambda) . \tag{3.69}$$

c)
$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \langle\sigma(0)\rangle^+(h_\Lambda) = \langle\sigma(0)\rangle^+(h'),$$

locally uniformly, with $h'(j)\equiv h_1$.

d) *For the Ising model with constant magnetic field $h>0$ ($h(j)\equiv h$),*

$$\frac{d}{dh}\langle\sigma(0)\rangle^+(h) = \sum_{x\in\mathbb{Z}^d} \langle\sigma(0);\sigma(x)\rangle^+(h) .$$

Proof [28]. Let

$$D = \{(h_1,h_2)\in\mathbb{C}^2 : \mathrm{Re}\,h_i > |\mathrm{Im}\,h_i|,\ i=1,2\} .$$

Let $\Lambda_n \supset \Lambda$. The function $\langle\sigma(0)\rangle^+_{\Lambda_n}(h_\Lambda)$ is analytic on D [Lemma 3.5a)]. The function

$$g_{\Lambda,n}(h_1,h_2) = (1+\langle\sigma(0)\rangle^+_{\Lambda_n}(h_\Lambda))^{-1} \tag{3.70}$$

is well-defined on D, and locally uniformly bounded [Lemma 3.5b)],

$$|g_{\Lambda,n}(h_1,h_2)| \leqq \frac{1}{1+\mathrm{Re}\langle\sigma(0)\rangle^+_{\Lambda_n}(h_\Lambda)} \leqq 1 . \tag{3.71}$$

Since $\langle\sigma(0)\rangle^+_{\Lambda_n}(h_\Lambda)$ is analytic, $g_{\Lambda,n}(h_1,h_2)$ is nonzero on D. If h_1 and h_2 are real, the limit $n\to\infty$ exists (F.K.G. inequalities). The limit, $g_\Lambda(h_1,h_2)$, is also nonzero on D (Hurwitz theorem).

Consequently,

$$\langle\sigma(0)\rangle^+(h_\Lambda)=\frac{1-g_\Lambda(h_1,h_2)}{g_\Lambda(h_1,h_2)} \tag{3.72}$$

is an analytic function of h_1 and h_2 on D. We have proved a); b) follows from the locally uniform convergence of $\langle\sigma(0)\rangle_{\Lambda_n}(h_\Lambda)$ as $n\to\infty$. If $\Lambda\uparrow\mathbb{Z}^d$, $g_\Lambda(h_1,h_2)$ converges locally uniformly on D, and the limit $g(h_1,h_2)$ is analytic on D, and nonzero on D. Thus $\langle\sigma(0)\rangle^+(h_\Lambda)$ converges locally uniformly on D as $\Lambda\uparrow\mathbb{Z}^d$. If $h_1\in\mathbb{R}$ and $h_2\in\mathbb{R}$ is large enough, correlation inequalities imply that the function $\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle\sigma(0)\rangle^+(h_\Lambda)$ does not depend on h_2. By analyticity, this is true for any h_2 in D, and we have proved c). The last point d) is a consequence of the locally uniform convergence. □

In the last application we consider a model with finite range interaction. Let μ_Λ be an external field defined by

$$\mu_\Lambda(j)=\mu \quad \text{if} \quad j\in\Lambda, \qquad \operatorname{dist}(j,\Lambda^c)\leqq R, \tag{3.73}$$

and $\mu_\Lambda(j)=0$ otherwise. R is the range of the interaction. Let Λ_n be a sequence of boxes, $\Lambda_n\subset\Lambda_{n+1}$, $\Lambda_n\uparrow\mathbb{Z}^d$. Let

$$\langle\cdot\rangle^\mu_{\Lambda_n}=\langle\cdot\rangle_{\Lambda_n}(h+\mu_{\Lambda_n}). \tag{3.74}$$

Lemma 3.7. *Let* $h(j)\geqq 0\ \forall j$ *and* $\mu>0$. *Then* $\lim_n\langle\cdot\rangle^\mu_{\Lambda_n}=\langle\cdot\rangle^+$.

Remarks. 1) This lemma is a straightforward modification of Lemma 3.6. We prove that the limit function is analytic in μ, $\operatorname{Re}\mu>|\operatorname{Im}\mu|$. But for $\mu\in\mathbb{R}$, large enough, the limit function is $\langle\cdot\rangle^+$. Thus this is true for any $\mu>0$ (see e.g. [28]).

2) The lemma has been proved for the first time by Lebowitz [31].

3) If $h(j)\geqq\delta>0$, then we can take $\mu>-\delta$.

4. Thermodynamic Functions

The organization of Sect. 4 is as follows.

In Sect. 4.1 we prove the existence of the surface free energy for different b.c. We derive an expression of this quantity in (4.8). Our tools are the F.K.G. inequalities. In the next section, 4.2, we prove analyticity properties, using correlation inequalities with imaginary angles. The crucial result is the bound (4.13). Once this bound is established, the proof is standard and is based on theorems of Vitali and Hurwitz. In Sect. 4.3, we prove a fluctuation-dissipation formula. The proof is based on analyticity properties and correlation inequalities. The proof of formula (4.23) is also valid for the XY model.

4.1. Existence of the Surface Free Energy

We prove the existence of F^+ [see (2.4) and (2.3)]. By definition,

$$F^+_{L,M}=-\frac{1}{2|\Sigma(L)|}\ln\frac{(Z^+_{L,M})^2}{Q^+_{L,M}}. \tag{4.1}$$

Let $i=(x,z)$, $x\in\mathbb{Z}^{d-1}$ and $z\in\mathbb{Z}$. We define the Hamiltonian

$$\Delta H = -h\sum_x(\sigma(x,0)+\sigma(x,1)) + K\sum_x \sigma(x,0)\sigma(x,1) - (J-K)\sum_{\langle x,y\rangle}(\sigma(x,0)\sigma(y,0)+\sigma(x,1)\sigma(y,1)). \tag{4.2}$$

Let $\mathbb{L}'=\{i\in\mathbb{Z}^d : i^d \leqq 0\}$, and let $\Sigma'=\{i\in\mathbb{Z}^d : i^d=0\}$. On $\mathbb{L}'$ we define

$$H'(J,K,h,\lambda) = -\sum_{\langle ij\rangle\subset\mathbb{L}'} K(i,j)\sigma(i)\sigma(j) - \sum_{i\in\mathbb{L}'}\lambda\sigma(i) - \sum_{i\in\Sigma'} h\sigma(i), \tag{4.3}$$

with $K(i,j)=J$ if $\langle ij\rangle\subset\Sigma'$, otherwise $K(i,j)=K$ [see (1.4)]. The definition of ΔH is such that the sum of the Hamiltonian (1.1) and ΔH gives the sum of $H(J,K,h,\lambda)$ and $H'(J,K,h,\lambda)$ defined by (1.4) and (4.3). We introduce $\mathscr{H}(t)$, which is given by the sum of Hamiltonians (1.1) and $t\cdot\Delta H$. Therefore, $\mathscr{H}(0)$ is Hamiltonian (1.1) and $\mathscr{H}(1)$ is $H(J,K,h,\lambda)+H'(J,K,h,\lambda)$. Let $\Xi^+_{L,M}(t)$ be the partition function in $\Omega(L,M)$ given by $\mathscr{H}(t)$ and + b.c. We can write

$$F^+_{L,M} = -\frac{1}{2|\Sigma(L)|}\ln\left(\frac{\Xi^+_{L,M}(1)}{\Xi^+_{L,M}(0)}\right) = -\frac{1}{2|\Sigma(L)|}\int_0^1 dt\left(\frac{d}{dt}\ln\Xi^+_{L,M}(t)\right). \tag{4.4}$$

If $\langle\cdot\rangle^t_{L,M}(t)$ is the Gibbs state in $\Omega(L,M)$ defined by $\mathscr{H}(t)$ and + b.c., we can write the integrand in (4.4) as

$$2h\sum_x\{\langle\sigma(x,1)\rangle^+_{L,M}(t) - K\langle\sigma(x,0)\sigma(x,1)\rangle^+_{L,M}(t)\} + 2(J-K)\sum_{\langle x,y\rangle}\langle\sigma(x,1)\sigma(y,1)\rangle^+_{L,M}(t). \tag{4.5}$$

[In (4.5) we have used symmetry properties of the state $\langle\cdot\rangle^+_{L,M}(t)$, and we have omitted boundary terms.] Using Lemma 3.1 it is easy to prove the existence of the thermodynamic limit for F^+. For example, if we fix $\varepsilon>0$, there is a box $\Delta(\varepsilon)$ such that if $\Omega(L,M)\supset\Delta(\varepsilon)+(x,0)$,

$$|\langle\sigma(x,1)\rangle^+_{L,M}(t) - \langle\sigma(x,1)\rangle^+(t)| \leqq \varepsilon. \tag{4.6}$$

In (4.6), $\langle\cdot\rangle^+(t) = \lim_{L,M}\langle\cdot\rangle^+_{L,M}(t)$. For any $\alpha>0$, if $M=L^\alpha$,

$$\lim_{L\to\infty}\frac{1}{|\Sigma(L)|}\sum_x\langle\sigma(x,1)\rangle^+_{L,L^\alpha}(t) = \langle\sigma(0,1)\rangle^+(t). \tag{4.7}$$

Using the dominated convergence theorem we obtain

$$F^+(J,K,h,\lambda) = \lim_{L\to\infty}F^+_{L,L^\alpha} = -h\int_0^1\langle\sigma(0,1)\rangle^+(t)dt + \frac{K}{2}\int_0^1\langle\sigma(0,1)\sigma(0,0)\rangle^+(t)dt - \frac{J-K}{2}\int_0^1\sum_{\substack{y:\\|y|=1}}\langle\sigma(0,1)\sigma(y,1)\rangle^+(t)dt. \tag{4.8}$$

Remarks. 1) The proof for F^- is the same. If we use free b.c. instead of + b.c., and if we take $h+\lambda\geqq 0$ and $\lambda\geqq 0$, we can define a surface free energy F^f. One arrives at a formula like (4.8) with $\langle\cdot\rangle^f(t)$ instead of $\langle\cdot\rangle^+(t)$.

2) The fact that we can choose $\alpha < 1$ in taking the thermodynamic limit is important for the discussion of the wetting transition.

3) In the periodic case we have a formula like (4.8) already for $F^P_{L,M}$.

4) The right-continuity of F^+, in h and λ, follows from (4.8) and the right-continuity of $\langle \cdot \rangle^+$ [see (3.15)].

5) Using Lemma 3.7 we can prove that $F^- = F^+$ if $h+\lambda>0$ and $\lambda > K$.

4.2. Analyticity Properties

We consider $F^+_{L,M}$ as a function of λ, and we take for the moment $h=0$ and $J \leqq K$. We always suppose that $\mathrm{Re}\lambda > |\mathrm{Im}\lambda|$, so that we can apply Lemma 3.5. By Lemma 3.5a),

$$\frac{(Z^+_{L,M}(\lambda))^2}{Q^+_{L,M}(\lambda)} \tag{4.9}$$

is a holomorphic function without zeroes for $\mathrm{Re}\lambda > |\mathrm{Im}\lambda|$. There exists a unique holomorphic function $U_{L,M}(\lambda)$ such that

$$e^{-|\Sigma(L)|U_{L,M}(\lambda)} = \frac{Z^+_{L,M}(\lambda))^2}{Q^+_{L,M}(\lambda)}, \tag{4.10}$$

and on the real axis

$$U_{L,M}(\lambda) = F^+_{L,M}(\lambda). \tag{4.11}$$

If $h=0$ and $J \leqq K$, we write

$$\frac{Q^+_{L,M}(\lambda)}{(Z^+_{L,M}(\lambda))^2} = \langle \prod e^{K\sigma(x,0)\sigma(x,1)} \prod e^{(K-J)(\sigma(x,0)\sigma(y,0)+\sigma(x,1)\sigma(y,1))} \rangle^+_{L,M}(t=1). \tag{4.12}$$

[The Hamiltonian $\mathscr{H}(t=1)$ gives the Gibbs state in (4.12).] From Lemma 3.5d),

$$\left|\frac{Q^+_{L,M}(\lambda)}{(Z^+_{L,M}(\lambda))^2}\right| \geqq \frac{Q^+_{L,M}(\mu)}{(Z^+_{L,M}(\mu))^2} \geqq 1 \tag{4.13}$$

with $\mu = ((\mathrm{Re}\lambda)^2 - (\mathrm{Im}\lambda)^2)^{1/2}$. The last inequality in (4.13) is a consequence of G.K.S. inequalities.

From (4.13) we have

$$|e^{-U_{L,M}(\lambda)}| \leqq 1. \tag{4.14}$$

Since $\lim_{L,M} e^{-U_{L,M}(\lambda)} = e^{-F^+(\lambda)}$, $\lambda \in \mathbb{R}^+$, we can apply the theorem of Vitali, and we get a holomorphic function for $\mathrm{Re}\lambda > |\mathrm{Im}\lambda|$,

$$\lim_{L,M} e^{-U_{L,M}(\lambda)} = G(\lambda). \tag{4.15}$$

The functions $e^{-U_{L,M}(\lambda)}$ and $G(\lambda)$, with $\lambda \in \mathbb{R}$, have no zero. Therefore, $G(\lambda)$ has no zero on $\{\lambda\colon \mathrm{Re}\lambda > |\mathrm{Im}\lambda|\}$. We conclude that there exists a unique holomorphic function $U(\lambda)$ such that

$$G(\lambda) = e^{-U(\lambda)}, \tag{4.16}$$

and, on $\mathbb{R}^+$, $U(\lambda) = F^+(\lambda)$.

Let $J>K$ and $h>0$. We divide and multiply the quotient (4.9) by $(Z^+_{L,M}(K,K,0,\lambda))^2$. Therefore, it is sufficient to consider

$$-\frac{1}{|\Sigma(L)|}\ln\frac{Z^+_{L,M}(J,K,h,\lambda)}{Z^+_{L,M}(K,K,0,\lambda)}. \tag{4.17}$$

As above

$$\left|\frac{Z^+_{L,M}(J,K,h,\lambda)}{Z^+_{L,M}(K,K,0,\lambda)}\right| \geqq 1, \tag{4.18}$$

so that we conclude that

$$\lim_{L,M\to\infty}\frac{1}{|\Sigma(L)|}\ln\frac{Z^+_{L,M}(K,K,0,\lambda)}{Z^+_{L,M}(J,K,h,\lambda)} \quad \text{is holomorphic}.$$

4.3. Proof of (2.14) *to* (2.17)

We start with (2.14). We have

$$\lim_{L,M\to\infty} F^+_{L,M}(h)=F^+(h), \tag{4.19}$$

locally uniformly in h. We can write

$$\frac{dF^+}{dh}=\lim_{L,M\to\infty}\frac{dF^+_{L,M}}{dh}. \tag{4.20}$$

By Lemma 3.1 we get the result [using the shorthand $\sigma(0)$ for $\sigma(0,0\ldots0,1)$]

$$M_\Sigma(h)=\langle\sigma(0)\rangle^+(h). \tag{4.21}$$

Let h_L be the function

$$\begin{aligned} h_L(i)&=h \quad \text{if} \quad i\in\Sigma(L),\\ h_L(i)&=h' \quad \text{if} \quad i\in\Sigma\backslash\Sigma(L). \end{aligned} \tag{4.22}$$

We apply Lemma 3.6. We have that

$$\langle\sigma(0)\rangle^+(h)=\lim_{L\to\infty}\langle\sigma(0)\rangle^+(h_L)$$

and

$$\frac{d}{dh}\langle\sigma(0)\rangle^+(h)=\lim_{L\to\infty}\frac{d}{dh}\langle\sigma(0)\rangle^+(h_L)=\lim_{L\to\infty}\sum_{i\in\Sigma_L}\langle\sigma(0);\sigma(i)\rangle^+(h_L). \tag{4.23}$$

Since we can choose h' arbitrarily, we set $h'=h$. Then (4.23) is precisely

$$\frac{d}{dh}M_\Sigma(h)=X_\Sigma(h)=\sum_{i\in\Sigma}\langle\sigma(0);\sigma(i)\rangle^+. \tag{4.24}$$

This proves (2.15). Next we prove (2.17). We modify the external field λ as follows:

$$\lambda(j)=\lambda \quad \text{if} \quad 1\leqq j^d\leqq N, \tag{4.25}$$

and

$$\lambda(j)=\lambda' \quad \text{if} \quad j^d>N. \tag{4.26}$$

Let $M>N$. We have for $\lambda\geqq 0$ and $\lambda'\geqq 0$,

$$\lim_{N\to\infty}\lim_{L,M\to\infty} F^+_{L,M}(\lambda,\lambda')=F^+(\lambda). \tag{4.27}$$

Indeed, if $\mathrm{Re}\,\lambda>|\mathrm{Im}\,\lambda|$ and $\mathrm{Re}\,\lambda'>|\mathrm{Im}\,\lambda'|$, the functions $F^+_{L,M}(\lambda,\lambda')$ are analytic. The convergence in (4.27) is locally uniform, and for $\lambda\geqq 0$ and $\lambda'\geqq 0$ the limiting function does not depend on λ'. We can take the derivative with respect to λ before taking the limit, and we may choose $\lambda'=\lambda$,

$$\begin{aligned}-\frac{dF^+}{d\lambda}&=\lim_{N\to\infty}\sum_{z=1}^{N}(\langle\sigma(0,z)\rangle^+(\lambda)-\langle\sigma(0,z)\rangle^+_{\mathrm{Is}}(\lambda))\\ &=\sum_{z\geqq 1}(\langle\sigma(0,z)\rangle^+(\lambda)-\langle\sigma(0,z)\rangle^+_{\mathrm{Is}}(\lambda)).\end{aligned} \tag{4.28}$$

Finally, we prove in the same way that

$$\frac{dM_\Sigma}{d\lambda}=\sum_{i\in\mathbb{L}}\langle\sigma(0);\sigma(i)\rangle^+, \tag{4.29}$$

which is (2.16).

5. Correlation Lengths and Susceptibility

The first part of this section contains a study of the transverse correlation length. The main inequalities which are used are the G.K.S. and G.H.S. inequalities. The main result is Lemma 5.5. In the second part of the section we analyze the parallel correlation length and prove that the finiteness of the susceptibility in the high-temperature region implies the finiteness of the parallel correlation length. The proof is based on [13]. However, it is more complicated, because we do not have translation invariance. It is given in Lemmas 5.9, 5.11, and 5.12. Lemmas 5.6 and 5.7 are technical results. Lemmas 5.8 and 5.10 give continuity properties of X_Σ. Finally, Lemma 5.13 gives a lower bound on the critical exponent γ.

5.1. Correlation Lengths

In this section $\lambda, h\geqq 0$. We study the correlation lengths ξ_Σ and $\xi_\perp$. It is convenient to introduce

$$m_\Sigma=\xi_\Sigma^{-1}=\lim_{L\to\infty}-\frac{1}{L}\ln\langle\sigma(0,1);\sigma(x_L,1)\rangle^+, \tag{5.1}$$

where in (5.1) we choose $x_L=(L,0,\ldots,0)\in\mathbb{Z}^{d-1}$. Similarly,

$$m_\perp=\xi_\perp^{-1}=\lim_{z\to\infty}-\frac{1}{z}\ln\langle\sigma(0,1);\sigma(0,z)\rangle^+. \tag{5.2}$$

If $\lambda=0$ and $K\leqq K_c(d)$, we also introduce

$$m'_\perp=\lim_{z\to\infty}-\frac{1}{z}\ln\langle\sigma(0,1)\sigma(0,z)\rangle^+. \tag{5.3}$$

Notice that we can replace, in (5.2) and (5.3), $\sigma(0,1)$ by $\sigma(0,z')$, with z' fixed, without changing the result. Indeed,

$$\langle\sigma(0,z')\sigma(0,z)\rangle^+ \geqq \langle\sigma(0,1)\sigma(0,z')\rangle^+ \langle\sigma(0,1)\sigma(0,z)\rangle^+ , \tag{5.4}$$

and

$$\langle\sigma(0,1)\sigma(0,z)\rangle^+ \geqq \langle\sigma(0,1)\sigma(0,z')\rangle^+ \langle\sigma(0,z')\sigma(0,z)\rangle^+ . \tag{5.5}$$

For (5.2) we use the inequality of Graham,

$$\langle\sigma(i);\sigma(j)\rangle^+ \geqq \langle\sigma(i);\sigma(k)\rangle^+ \langle\sigma(k);\sigma(j)\rangle^+ . \tag{5.6}$$

Finally, we introduce, for $J \geqq K$ and $h \geqq K$, or when $M_{\mathrm{Is}}=0$ and $\langle\sigma(0,1)\rangle^+ > 0$,

$$\alpha = \lim_{L\to\infty} -\frac{1}{L}\ln(\langle\sigma(0,L)\rangle^+ - M_{\mathrm{Is}}), \tag{5.7}$$

where $M_{\mathrm{Is}} = \langle\sigma(0,0)\rangle^+_{\mathrm{Is}}$ is the magnetization of the Ising model on $\mathbb{Z}^d$ in the + phase.

Lemma 5.1. *Let $h \geqq 0$ and $\lambda \geqq 0$. Then $m_\Sigma(h_1) \leqq m_\Sigma(h_2)$ if $0 \leqq h_1 \leqq h_2$, and $\lim_{h_2 \downarrow h_1} m_\Sigma(h_2) = m_\Sigma(h_1)$.*

Proof. We use the G.H.S. inequality,

$$\langle\sigma(i);\sigma(j)\rangle^+(h_1,\lambda_1) \geqq \langle\sigma(i);\sigma(j)\rangle^+(h_2,\lambda_2), \tag{5.8}$$

if $\lambda_1 \leqq \lambda_2$ and $h_1 \leqq h_2$. By (5.6) and the Σ-invariance of the state $\langle\cdot\rangle^+$, the function $f(L) = -\ln\langle\sigma(0,1);\sigma(x_L,1)\rangle^+$ is subadditive, $f(L_1+L_2) \leqq f(L_1)+f(L_2)$. Therefore,

$$m_\Sigma = \inf_L \frac{1}{L}\cdot f(L), \tag{5.9}$$

and

$$m_\Sigma(h_1) \leqq m_\Sigma(h_2) \leqq \frac{f(L)(h_2)}{L}. \tag{5.10}$$

The lemma is proved using the continuity property of $\langle\cdot\rangle^+$ [see (3.15)]. □

Remarks. 1) We have a similar result for $\lambda_2 \downarrow \lambda_1$.

2) If $\langle\sigma(i)\rangle^+ = 0$, we can use the state $\langle\cdot\rangle^f$ with free b.c. instead of $\langle\cdot\rangle^+$, since there is a unique Gibbs state. By G.K.S. inequalities we have $m_\Sigma(J_1) \geqq m_\Sigma(J_2)$, if $J_1 \leqq J_2$, and as above

$$\lim_{J_1 \uparrow J_2} m_\Sigma(J_1) = m_\Sigma(J_2). \tag{5.11}$$

Lemma 5.2. *Let $h \geqq 0$ and $\lambda \geqq 0$. Let $m_{\mathrm{Is}} = \lim_{|x|\to\infty} -\frac{1}{|x|}\ln\langle\sigma(0);\sigma(x)\rangle^+_{\mathrm{Is}}$. Then $m_\perp(J,K,h,\lambda) \leqq m_{\mathrm{Is}}(K,\lambda)$.*

Proof. Let $z_i = i\cdot L$, $i=0,\dots,p$. From (5.6)

$$\langle\sigma(0,1);\sigma(0,z_p)\rangle^+ \geqq \prod_{i=1}^{p} \langle\sigma(0,z_{i-1});\sigma(0,z_i)\rangle^+ . \tag{5.12}$$

Let $z \geqq 2$ and $z' \geqq 2$. Let $h \uparrow \infty$. The G.H.S. inequality gives

$$\langle \sigma(0,z); \sigma(0,z') \rangle^+(J,K,h,\lambda) \geqq \langle \sigma(0,z-1); \sigma(0,z'-1)\rangle^+(K,K,K,\lambda). \quad (5.13)$$

Using (5.13) and Lemma 3.1, we get

$$\lim_{p\to\infty} -\frac{1}{pL} \ln \langle \sigma(0,1); \sigma(0,z_p)\rangle^+ \leqq -\frac{1}{L} \ln \langle \sigma(0,1); \sigma(0,L)\rangle^+_{\mathrm{Is}}(K,\lambda). \quad (5.14)$$

This proves $m_\perp \leqq m_{\mathrm{Is}}$. □

Lemma 5.3. *Let $h \geqq 0$ and $\lambda \geqq 0$.*

a) $m_\perp(K,K,h,\lambda) = m_{\mathrm{Is}}(K,\lambda)$ *if* $h \geqq K$.

If $\lambda = 0$, $K \leqq K_c(d)$ *and* $\langle \sigma(0,1)\rangle^+(J,K,h) > 0$, *then we have:*

b) $\alpha(K,K,K) \leqq m'_\perp(J,K,h) \leqq m_\perp(J,K,h)$,

c) $\alpha(K,K,K) \leqq m'_\perp(J,K,h) \leqq \alpha(J,K,h)$, *with* α *as in* (5.7).

Proof. From the G.H.S. inequality, and the G.K.S. inequalities, and since $h \geqq K$,

$$\begin{aligned}\langle \sigma(0,1); \sigma(0,L)\rangle^+(K,K,h,\lambda) &\leqq \langle \sigma(0,1); \sigma(0,L)\rangle^+_{\mathrm{Is}}(K,\lambda) \\ &\leqq \langle \sigma(0,1)\sigma(0,L)\rangle^+_{\mathrm{Is}}(K,\lambda) \\ &\leqq \langle \sigma(0,1)\sigma(0,L)\rangle^+(K,K,h,\lambda). \end{aligned} \quad (5.15)$$

Lemma 5.2 and (5.15) give a). From G.K.S. inequalities,

$$\begin{aligned}\langle \sigma(0,L-1)\rangle^+(K,K,K) &\geqq \langle \sigma(0,1)\sigma(0,L)\rangle^+(J,K,h) \\ &\geqq \langle \sigma(0,1)\rangle^+(J,K,h)\langle \sigma(0,L)\rangle^+(J,K,h). \end{aligned} \quad (5.16)$$

This proves c). Finally,

$$\langle \sigma(0,1)\sigma(0,L)\rangle^+(J,K,h) \geqq \langle \sigma(0,1); \sigma(0,L)\rangle^+(J,K,h). \quad (5.17)$$

This proves b). □

Lemma 5.4. *Let $J = K = h$. Let $\lambda \geqq 0$. Then $\alpha \geqq m_{\mathrm{Is}}$.*

This lemma is implicitly proved in [32].

Proof. Let us consider the Ising model with + b.c. in the box $\Omega = \Omega(L,M) \subseteqq \mathbb{Z}^d$,

$$\Omega(L,M) = \{x \in \mathbb{Z}^d : |x^i| \leqq L,\ i=1,\dots,d-1,\ |x^d| \leqq M\}.$$

We add an external field $h'\sigma(x)$ for all x with $x^d = 0$ and the corresponding expectation value is $\langle \cdot \rangle^+_\Omega(h)$. Let $j = (0,z) \in \Omega$, $z > 0$. We have

$$\begin{aligned}&\langle \sigma(0,z)\rangle^+_{L,M}(K,K,K,\lambda) - \langle \sigma(0,z)\rangle^+_\Omega(0) \\ &= \sum_{x:x^d=0} \int_0^\infty dh' \langle \sigma(j); \sigma(x)\rangle^+_\Omega(h') \\ &= \sum_{x:x^d=0} \int_0^\infty dh' (\langle \sigma(j); \sigma(x)\rangle^+_\Omega(h'))^{1-\varepsilon} (\langle \sigma(j); \sigma(x)\rangle^+_\Omega(h'))^{\varepsilon} \\ &\leqq \sum_{x:x^d=0} (\langle \sigma(j); \sigma(x)\rangle^+_{\mathrm{Is}})^{1-\varepsilon} \int_0^\infty dh' e^{-\varepsilon c h'} \cdot \mathrm{const}. \end{aligned} \quad (5.18)$$

To prove (5.18) we have used the G.H.S. inequality and the existence of $c>0$, such that

$$\langle\sigma(j);\sigma(x)\rangle_{\Omega}^{+}(h')\leqq\exp(-ch')\cdot\text{const}. \tag{5.19}$$

To prove (5.19) it is sufficient to show the following inequality for $\text{Prob}\{\sigma(x)=1\}$, uniformly in Ω:

$$\text{Prob}\{\sigma(x)=1\}=\left\langle\frac{1+\sigma(x)}{2}\right\rangle_{\Omega}^{+}(h')\geqq 1-2e^{-ch'}. \tag{5.20}$$

Inequality (5.20) is an easy consequence of the G.K.S. inequality, since we have a lower bound for $\text{Prob}\{\sigma(x)=1\}$ by considering the special case $\Omega=\{x\}$ which gives a system of one spin at x in an external magnetic field h'.

Let us suppose that $m_{\text{Is}}(K,\lambda)\equiv m>0$. Then for any $\delta>0$, if $|j|=z$ is large enough,

$$\sum_{\substack{x\in\mathbb{Z}^d\\ x^d=0}}\langle\sigma(j);\sigma(x)\rangle_{\text{Is}}^{+}\leqq\text{Const}\cdot e^{-(m-\delta)z}. \tag{5.21}$$

From (5.18) and (5.21) we have

$$\alpha(K,K,K,\lambda)\geqq(1-\varepsilon)(m-\delta). \quad \square$$

Remark. If $\alpha(K,K,K,\lambda)>0$, then $m_{\text{Is}}(K,\lambda)>0$. This follows from

$$\langle\sigma(0,-z)\sigma(0,z)\rangle_{\text{Is}}^{+}-(M_{\text{Is}})^2\leqq(\langle\sigma(0,z-1)\rangle^{+}(K,K,K,\lambda))^2-M_{\text{Is}}^2. \tag{5.22}$$

One obtains (5.22) by putting an external field $h\sigma(x)$ for all x with $x^d=0$, and by letting $h\uparrow\infty$.

Lemma 5.5. *Let $\lambda=0$, $h\geqq 0$.*

a) *If $K\leqq K_c(d)$, $\alpha(K,K,K)=m'_{\perp}(J,K,h)=m_{\perp}(J,K,h)=m_{\text{Is}}(K)$.*

b) *If $K=K_c(d)$, $J\geqq K$ and $h\geqq K$, then the asymptotic behaviours of $\langle\sigma(0,L)\rangle^{+}$, $\langle\sigma(0,1)\sigma(0,L)\rangle^{+}$, $\langle\sigma(0,1);\sigma(0,L)\rangle^{+}$, and $\langle\sigma(0,1)\sigma(0,L)\rangle_{\text{Is}}^{+}$ are the same as $L\to\infty$.*

Proof. The first part follows from Lemmas 5.2, 5.3b), and 5.4. It is easy to verify, using G.K.S. inequalities, that the behaviour of $\langle\sigma(0,L)\rangle^{+}(J,K,h)$ as $L\to\infty$ is independent of J and h if $J\geqq K$ and $h\geqq K$. By (5.16) this behaviour is the same as the behaviour of

$$\langle\sigma(0,1)\sigma(0,L)\rangle^{+}(J,K,h),\qquad J\geqq K\quad\text{and}\quad h\geqq K.$$

Therefore, this is true for $\langle\sigma(0,1);\sigma(0,L)\rangle^{+}(J,K,h)$. From (5.15) this behaviour is the same as the behaviour of $\langle\sigma(0,0)\sigma(0,L)\rangle_{\text{Is}}^{+}(K)$. $\square$

5.2. The Layer Susceptibility

We set $h\geqq 0$, $\lambda\geqq 0$, and $\langle\cdot\rangle_L^{+}\equiv\langle\cdot\rangle_{L,L}^{+}$, where this last expectation value is the expectation value with respect to the Gibbs state in $\Lambda(L)\equiv\Lambda(L,L)$ with $+$ b.c. To simplify the notation we write $\langle i;j\rangle_L^{+}$ for $\langle\sigma(i);\sigma(j)\rangle_L^{+}$. If $i\notin\Lambda(L,L)$ or $j\notin\Lambda(L,L)$ we set $\langle\sigma(i);\sigma(j)\rangle_L^{+}=0$. Finally, X_Σ is *by definition*

$$X_\Sigma=\sum_{i\in\Sigma}\langle 0;i\rangle^{+}. \tag{5.23}$$

Lemma 5.6. *If $\lambda \geqq 0$ and $h \geqq 0$, then*

$$X_\Sigma = \lim_{L\to\infty} \sum_{i\in\Sigma(L)} \langle 0; i\rangle_L^+ = \lim_{L\to\infty} \frac{1}{|\Sigma(L)|} \sum_{i,j\in\Sigma(L)} \langle i; j\rangle_L^+ .$$

Proof. By the G.H.S. inequality,

$$\langle 0; i\rangle^+ \geqq \langle 0; i\rangle_L^+ \geqq 0 \tag{5.24}$$

and

$$\langle 0; i\rangle_{L_1}^+ \geqq \langle 0; i\rangle_{L_2}^+, \qquad L_1 \geqq L_2 . \tag{5.25}$$

If $X_\Sigma < \infty$, we get by the dominated convergence theorem,

$$\lim_{L\to\infty} \sum_{i\in\Sigma} \langle 0; i\rangle_L^+ = \sum_{i\in\Sigma} \langle 0; i\rangle^+ . \tag{5.26}$$

If $\lim_{L\to\infty} \sum_{i\in\Sigma} \langle 0; i\rangle_L^+ < \infty$, then (5.26) is true by the monotone convergence theorem. To prove the second equality we notice that

$$\frac{1}{|\Sigma(L)|} \sum_{i,j\in\Sigma} \langle i; j\rangle_L^+ \leqq \sum_{i\in\Sigma(2L)} \langle 0; i\rangle^+ , \tag{5.27}$$

by (5.24) and translation invariance. On the other hand, let $\Sigma_j(L)$ be the box obtained from $\Sigma(L)$ by the translation $j\in\Sigma$. For fixed L, let $L_2 = L_1 + L$. Then

$$\begin{aligned} \frac{1}{|\Sigma(L_2)|} \sum_{i,j} \langle i; j\rangle_{L_2} &\geqq \frac{1}{|\Sigma(L_2)|} \sum_{i\in\Sigma(L_1)} \sum_{j\in\Sigma_i(L)} \langle i; j\rangle_{L_2}^+ \\ &\geqq \frac{|\Sigma(L_1)|}{|\Sigma(L_2)|} \sum_j \langle 0; j\rangle_L^+ . \end{aligned} \tag{5.28}$$

From this we get the result. □

Lemma 5.7. *Let $\lambda = 0$ and $h = 0$. If $\langle\sigma(0)\rangle^+ = 0$, then*

$$X_\Sigma = \lim_{L\to\infty} \sum_{i\in\Sigma(L)} \langle 0i\rangle^f = \lim_{L\to\infty} \sum_{i\in\Sigma(L)} \langle 0i\rangle_L^f = \lim_{L\to\infty} \frac{1}{|\Sigma(L)|} \sum_{i,j\in\Sigma(L)} \langle ij\rangle_L^f . \tag{5.29}$$

Proof. We have unicity of the state, and since

$$\langle 0i\rangle_L^f \leqq \langle 0i\rangle^f , \qquad \forall i\in\Sigma(L), \tag{5.30}$$

and

$$\langle 0i\rangle_{L_1}^f \leqq \langle 0i\rangle_{L_2}^f , \qquad L_1 \leqq L_2 , \qquad i\in\Lambda(L_2), \tag{5.31}$$

we can repeat the proof of Lemma 5.6. □

Lemma 5.8. *Let $\lambda = 0$ and $h = 0$. If $\langle\sigma(0)\rangle^+ = 0$, and $X_\Sigma(J,K) < \infty$, then for any (J_n, K_n), $J_n \uparrow J$ and $K_n \uparrow K$,*

$$\lim_{n\to\infty} X_\Sigma(J_n, K_n) = X_\Sigma(J, K) .$$

Proof. By Lemma 5.7 we know that $X_\Sigma = \sum_i \langle 0i\rangle^f$. For the state with free b.c. [see (3.20)],

$$\lim_n \langle 0i\rangle^f(J_n, K_n) = \langle 0i\rangle^f(J, K) . \tag{5.32}$$

Thus

$$\sum_i \langle 0i\rangle^f(J,K) = \sum_i \liminf_n \langle 0i\rangle^f(J_n,K_n) \leqq \liminf_n \sum_i \langle 0i\rangle^f(J_n,K_n)$$
$$\leqq \limsup_n \sum_i \langle 0i\rangle^f(J_n,K_n) \leqq \sum_i \langle 0i\rangle^f(J,K). \quad \square \tag{5.33}$$

We set

$$f(L) = \sup_{x\in\Sigma(L)} \sum_{y\in\Sigma(L)} \langle xy\rangle_L^f. \tag{5.34}$$

Lemma 5.9. *Let* $h \geqq 0$ *and* $\lambda \geqq 0$. *Then*

a) $\frac{df(L)}{dJ} \leqq 2(d-1)(f(L))^2$.

b) *If* $X_\Sigma^f < \infty$, *then* $\lim_{L\to\infty} f(L) = X_\Sigma^f$.

Proof. Using the U_4-inequality (3.36) we get

$$\frac{d}{dJ}\sum_{z\in\Sigma(L)} \langle xz\rangle_L^f = \sum_{\langle yy'\rangle}\sum_{z\in\Sigma(L)} \langle xz; yy'\rangle_L^f$$
$$\leqq \sum_{\langle yy'\rangle}\sum_{z\in\Sigma(L)} (\langle xy\rangle_L^f\langle y'z\rangle_L^f + \langle xy'\rangle_L^f\langle yz\rangle_L)$$
$$\leqq \sum_{\langle yy'\rangle} (\langle xy\rangle_L^f f(L) + \langle xy'\rangle_L^f f(L)). \tag{5.35}$$

From (5.35) we get a). To prove b) we notice that, as in Lemma 5.6,

$$\sum_{i\in\Sigma(L)} \langle 0i\rangle_L^f \leqq f(L) \leqq \sup_{x\in\Sigma(L)} \sum_{i\in\Sigma_x(2L)} \langle xi\rangle^f = \sum_{i\in\Sigma(2L)} \langle 0i\rangle^f. \quad \square \tag{5.36}$$

Lemma 5.10. *Let* $h=\lambda=0$, *and let* $X_\Sigma^f = \sum_{i\in\Sigma} \langle 0i\rangle^f \leqq \infty$. *If* $J_2 > J_1$, *then*

$$0 \leqq \frac{1}{X_\Sigma^f(J_1)} - \frac{1}{X_\Sigma^f(J_2)} \leqq 2(d-1)(J_2-J_1).$$

Proof. From Lemma 5.9 we have

$$(f(L))^{-2}\frac{df(L)}{dJ} \leqq 2(d-1). \tag{5.37}$$

By integration we get

$$\frac{1}{f(L)(J_1)} - \frac{1}{f(L)(J_2)} \leqq 2(d-1)(J_2-J_1). \tag{5.38}$$

Taking the limit $L\to\infty$, and using Lemma 5.9b), we get the result. $\square$

Lemma 5.11. *Let* $\lambda=h=0$, $K<K_c(d)$ *and* $X_\Sigma^f(J)<\infty$. *Then* $\sum_{i\in\mathbb{L}} \langle xi\rangle^f(J) < \infty$ *for all* $x\in\mathbb{L}$.

Proof. The proof is similar to the proofs of Lemmas 5.9 and 5.10. We set

$$b(L) = \sup_{x\in\Lambda(L)} \sum_{y\in\Sigma(L)} \langle xy\rangle_L^f, \tag{5.39}$$

and

$$c(L)=\sup_{x\in\Lambda(L)}\sum_{y\in\Lambda(L)}\langle xy\rangle_L^f. \tag{5.40}$$

As above we find

$$\frac{db(L)}{dJ}\leqq 2(d-1)f(L)b(L), \tag{5.41}$$

and similarly,

$$\frac{dc(L)}{dJ}\leqq 2(d-1)b(L)c(L). \tag{5.42}$$

By integration we find

$$b(L)(J_2)\leqq b(L)(J_1)\exp\left(2(d-1)\int_{J_1}^{J_2} f(L)(J')dJ'\right), \tag{5.43}$$

and

$$c(L)(J_2)\leqq c(L)(J_1)\exp\left(2(d-1)\int_{J_1}^{J_2} b(L)(J')dJ'\right). \tag{5.44}$$

$K<K_c(d)$ implies $X_{\mathrm{Is}}(K)<\infty$ [11]. If $J_1=K$,

$$\begin{aligned} b(L)(K)\leqq c(L)(K)&\leqq \sup_{x\in\Lambda(L)}\sum_{y\in\Lambda(L)}\langle xy\rangle^f \\ &\leqq \sup_{x\in\Lambda(L)}\sum_{y\in\mathbb{Z}^d}\langle xy\rangle_{\mathrm{Is}}^f=X_{\mathrm{Is}}(K)<\infty. \end{aligned} \tag{5.45}$$

Therefore, if $J_1=K$ and $J_2=J$, the result follows from (5.43), (5.44), and Lemma 5.9b), since

$$\sup_{x\in\Lambda(L)}\sum_{y\in\Lambda(L)}\langle xy\rangle_L^f\geqq\sum_{y\in\Lambda(L)}\langle x'y\rangle_L^f \tag{5.46}$$

for any $x'\in\Lambda(L)$. □

Lemma 5.12. *Let $h=\lambda=0$, $K<K_c(d)$, $X_\Sigma^f(J)<\infty$, and $\langle\sigma(0)\rangle^+(J,K)=0$. Then the correlation length $\xi_\Sigma<\infty$.*

Proof. For the infinite d-dimensional Ising model, with coupling constant K, we choose R so that

$$\max(J,K)\sum_{y:\,\|y\|=R}\langle 0y\rangle_{\mathrm{Is}}^f=\tfrac{1}{2}, \tag{5.47}$$

where $\|y\|=\max\{|y^i|,\ i=1,\dots,d\}$. For any $x\in\mathbb{L}$ we define

$$a(x;R)=a(x)=\sum_{\substack{y\in\mathbb{L}\\ \|y-x\|=R}}\langle xy\rangle^f. \tag{5.48}$$

By correlation inequalities we have

$$\langle\bar{x}\bar{y}\rangle^f(K,K)\leqq\langle xy\rangle^f(J,K)\leqq\langle\bar{x}\bar{y}\rangle^f\quad(K,K,h=K), \tag{5.49}$$

where $\bar{x}=(x^1,\dots,x^{d-1},x^d-1)$, $\bar{y}=(y^1,\dots,y^{d-1},y^d-1)$. If, in (5.48), we replace $\langle xy\rangle^f$ by $\langle\bar{x}\bar{y}\rangle^f(K,K)$, respectively, $\langle\bar{x}\bar{y}\rangle^f(K,K,K)$, we get functions which we denote by $a_0(x)$ and $a_+(x)$. Clearly, $a_0(x)\leqq a(x)$ and $a_+(x)\geqq a(x)$. The functions $a_0(x)$

and $a_+(x)$ are monotone increasing, decreasing, respectively, and since $K<K_c(d)$, we have (uniqueness of the Gibbs state)

$$\lim_{x^d\to\infty} a_0(x)=\lim_{x^d\to\infty} a_+(x)=\sum_{\|y\|=R}\langle 0y\rangle^f_{\text{Is}}. \tag{5.50}$$

Thus, there exists an $L'<\infty$, independent of J, such that

$$\max(J,K)\cdot a(x;R)\leqq 3/4, \quad \forall x \quad \text{with} \quad x^d\geqq L'. \tag{5.51}$$

For each x with $x^d=j$ and $j<L'$, we choose R_j such that $\max(J,K)\cdot a(x;R_j)\leqq 3/4$. This is possible by Lemma 5.11. We can now apply the argument of [13]. Let $L>R_1$. Since there is a unique Gibbs state, $\langle 0x_L\rangle^+=\langle 0x_L\rangle^f$ with $X_L=(L,0,\ldots,0)$; the inequality of Simon gives

$$\langle 0x_L\rangle^f\leqq \sum_{y\in\Lambda(R_1)}\sum_{z\notin\Lambda(R_1)} K(y,z)\langle 0y\rangle^f\langle zx_L\rangle^f. \tag{5.52}$$

Here $K(y,z)$ is the coupling constant between the spins at y and z. By our choice of K, we have only contributions from y,z such that $|z-y|=1$. Hence

$$\langle 0x_L\rangle^f\leqq (a(0;R_1)\max(J,K)\langle x_L\hat{z}\rangle^f\leqq 3/4\langle x_L\hat{z}\rangle^f, \tag{5.53}$$

where $\hat{z}$ is the maximum of $\langle x_L z\rangle^f$ for z at distance one from $\Lambda(R_1)$. If $L=n\bar{R}$, $\bar{R}=\max\{R,R_j,\ j=1,\ldots,L'-1\}$, then we can repeat this argument at least n times. Thus

$$\langle 0x_L\rangle^f\leqq (3/4)^n. \quad \square \tag{5.54}$$

For $K<K_c(d)$ and $h=\lambda=0$, we define

$$\bar{J}(K)=\sup\{J\colon X^f_\Sigma(J,K)<\infty\}. \tag{5.55}$$

$\bar{J}$ is a monotone decreasing function of K.

Remark. The function $\bar{J}(K)$ can be defined as

$$\bar{J}(K)=\sup\{J\colon \langle\sigma(0,1)\rangle^+(J,K)=0,\ X_\Sigma(J,K)<\infty\}. \tag{5.56}$$

Indeed, if $\langle\sigma(0,1)\rangle^+(J,K)=0$, we have only one Gibbs state, and therefore, $X_\Sigma=X^f_\Sigma$. On the other hand, if $X^f_\Sigma<\infty$, we have $\langle\sigma(0,1)\rangle^+(J,K)=0$. The proof is given in [11]. We can write

$$\langle\sigma(0,1)\rangle^+(J,K)=-\lim_{h\downarrow 0}\frac{1}{h}(F^+(J,K,h)-F^+(J,K,0)). \tag{5.57}$$

For $\lambda=0$ and $h\geqq 0$, we prove in the Appendix that $F^+=F^f$. Thus, we can use free b.c. in (5.57), and we write

$$-F^f(J,K,h)=\lim_{L\to\infty}\frac{1}{|\Sigma(L)|}\ln\left\langle\exp\left(h\sum_{i\in\Sigma(L)}\sigma(i)\right)\right\rangle^f_L(J,K,0)+F^f(J,K,0). \tag{5.58}$$

We have the upper bound

$$\left\langle\exp\left(\sum_{i\in\Sigma(L)}h\sigma(i)\right)\right\rangle^f_L(J,K,0)\leqq\exp\left(\tfrac{1}{2}h^2X^f_{\Sigma(L)}(J,K,0)\cdot|\Sigma(L)|\right). \tag{5.59}$$

If $X^f_\Sigma<\infty$, we get from (5.57), (5.58), and (5.59), $\langle\sigma(0,1)\rangle^+(J,K)=0$.

Let (J_0, K_0) be given in such a way that $K_0 < K_c(d)$, and $J_0 = \bar{J}(K_0)$. Let

$$X_\Sigma(\beta) = X_\Sigma(\beta J_0, \beta K_0)\,. \tag{5.60}$$

By the above remark, if $\beta < 1$, we also have $X_\Sigma(\beta) = X_\Sigma^f(\beta)$.

Lemma 5.13. *Let $\lambda = h = 0$, K_0 and J_0 as above. If the function $K \to \bar{J}(K)$ is differentiable at K_0, then $X_\Sigma(\beta)$ diverges at least like $(1-\beta)^{-1}$ as $\beta \uparrow 1$.*

Proof.

$$\begin{aligned} X_\Sigma^{-1}(\beta) &= (X_\Sigma^f)^{-1}(\beta J_0, \beta K_0) - (X_\Sigma^f)^{-1}(\bar{J}(\beta K_0), \beta K_0) \\ &\leqq 2(d-1)(\bar{J}(\beta K_0) - \beta J_0) = 2(d-1)(\bar{J}(\beta K_0) - J_0 + J_0(1-\beta))\,. \end{aligned} \tag{5.61}$$

By hypothesis

$$\lim_{\beta \to 1} \frac{J_0 - \bar{J}(\beta K_0)}{1-\beta} = \alpha \leqq 0\,.$$

Therefore, the lemma is proved. □

Finally, we show that the behaviour of X_Σ is the same for all $0 \leqq J \leqq K$, when $h = \lambda = 0$. Indeed,

$$\begin{aligned} \langle \sigma(i)\sigma(j)\rangle^+(K,K,0,0) &\geqq \langle \sigma(i)\sigma(j)\rangle^+(J,K,0,0) \geqq \langle \sigma(i)\sigma(j)\rangle^+(0,K,0,0) \\ &= (\tanh K)^2 \langle \sigma(i)\sigma(j)\rangle^+(K,K,0,0)\,. \end{aligned} \tag{5.63}$$

In particular, the critical exponents of $\langle \sigma(0)\rangle^+$, X_Σ are the same for all $0 \leqq J \leqq K$, when $K = K_c(d)$ and $h = \lambda = 0$.

Appendix. Proof of $F^+ = F^p = F^f$, for $\lambda \geqq 0$ and $h \geqq 0$

Let $h \geqq 0$ and $\lambda \geqq 0$. If $\lambda > 0$, Lemma 3.7 implies that $F^f = F^+$. Since $\langle \sigma_A \rangle^+_{L,M}(t) \geqq \langle \sigma_A \rangle^p_{L,M}(t) \geqq \langle \sigma_A \rangle^f_{L,M}(t)$, for $A \subset \Lambda(L,M)$, formula (4.8) and Remarks 1 and 3 following it give that $F^f = F^p = F^+$. We prove $F^+ = F^p$ for $\lambda = 0$ and $h \geqq 0$. The same proof holds for $F^+ = F^f$.

The functions F^+ and F^p are bounded, concave in h and hence continuous in h, $h \in \mathbb{R}$. If $\mathrm{Re}\,h > 0$, they are analytic in h, and

$$\frac{dF^+}{dh} = \langle \sigma(0)\rangle^+\,, \qquad \frac{dF^p}{dh} = \langle \sigma(0)\rangle^p\,.$$

We use now a result proved in our second paper: for $h \geqq K$ and $J \geqq K$, there is a unique Gibbs state. Since $\langle \sigma(0)\rangle^+$ and $\langle \sigma(0)\rangle^p$ are analytic, they coincide when $h > 0$ and $J \geqq K$: $\langle \sigma(0)\rangle^+(J,K,h,0) = \langle \sigma(0)\rangle^p(J,K,h,0)$. Similarly, we have

$$\langle \sigma(0)\sigma(i)\rangle^+(J,K,h,0) = \langle \sigma(0)\sigma(i)\rangle^p(J,K,h,0)\,.$$

Let $J \leqq K$. If $h = 0$, (4.8) and Remark 3 following it, give $F^+(J,K,0,0) \geqq F^p(J,K,0,0)$. On the other hand, if $h \geqq K + 2(d-1)|K-J|$, the same formula (4.8) can be written in terms of the decreasing functions, $-\sigma(i)$, $-f_{ij}^{\pm} = -(\sigma(i) + \sigma(j) \pm \sigma(i)\sigma(j))$. By F.K.G. inequalities, for $h \geqq K + 2(d-1)|K-J|$, $F^p(J,K,h,0) \geqq F^+(J,K,h,0)$. By continuity of $F^p - F^+$ in h, there exists $h^*(J,K)$, $0 \leqq h^* \leqq K + 2(d-1)|K-J|$, such that $F^+(J,K,h^*,0) = F^p(J,K,h^*,0)$.

Using these preliminary results we prove that $F^+ = F^p$ for the three cases $J = K$, $J \geqq K$, and $J \leqq K$.

Let $J = K$. From the above results we have that $F^+(K,K,h,0) = F^p(K,K,h,0)$ for all $h \geqq 0$, since

$$F^+(K,K,h,0) = F^+(K,K,h^*,0) + \int_{h^*}^{h} \langle\sigma(0)\rangle^+(h')dh' . \tag{A.1}$$

Let $J \geqq K$ and $h \geqq K$. In this case there is a unique Gibbs state. By concavity of F^+ in J, we can write

$$F^+(J,K,h,0) = F^+(K,K,h,0) - (d-1) \int_K^J \langle\sigma(i)\sigma(j)\rangle^+(J')dJ' , \tag{A.2}$$

where $i,j \in \Sigma$ and $|i-j| = 1$. Thus $F^+(J,K,h,0) = F^p(J,K,h,0)$ when $J \geqq K$ and $h \geqq K$. By analyticity in h, and continuity in h, this holds for $h \geqq 0$.

Let $J \leqq K$. There exists $h^*(J,K)$ such that $F^+ = F^p$. The difference $F^p - F^+$ is increasing in h and increasing in J. We have, for $J_1 \geqq J$ and $h_1 \geqq h \geqq h^*$,

$$\begin{aligned} 0 &= F^p(J,K,h^*,0) - F^+(J,K,h^*,0) \leqq F^p(J,K,h,0) - F^+(J,K,h,0) \\ &\leqq F^p(J_1,K,h_1,0) - F^+(J_1,K,h_1,0) = 0 , \end{aligned} \tag{A.3}$$

provided that $J_1 \geqq K$ and $h_1 \geqq K$. Therefore, if $h \geqq h^*(J,K)$, $F^p = F^+$; by analyticity and continuity this is true for $h \geqq 0$.

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