Commun. Math. Phys. 109, 397–415 (1987)

Communications in Mathematical Physics

Time-Delay and Lavine's Formula

Shu Nakamura

Department of Pure and Applied Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan

Abstract. Lavine's results on time-delay ([10]) is extended to higher dimensional Schrödinger operators.

1. Introduction

In [10], Lavine proved the existence of a quantity called time-delay and gave its representation formula which we call "Lavine's formula," for one-dimensional Schrödinger operators. The aim of this paper is to extend them to n-dimensional Schrödinger operators.

We consider Schrödinger operators:

$$H = H_0 + V(x); \quad H_0 = -\Delta$$

on $\mathscr{H} = L^2(\mathbb{R}^n)$, and we suppose that the potential V satisfies

Assumption (V). $V(x) = V_1(x) + V_2(x)$, and there exists a constant $\varepsilon > 0$ such that
(i) $V_1(x)$ is a C^∞-function and for any α,

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V_1(x)\right| \leqq C_\alpha(1+|x|)^{-1-\varepsilon-|\alpha|}; \tag{1.1}$$

(ii) the multiplication operator by $V_2(x)$ is compact from $H^2(\mathbb{R}^n)$ to $L^{2,2+\varepsilon}(\mathbb{R}^n)$.

$L^{2,\alpha}(\mathbb{R}^n) = \{\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : (1+|x|)^\alpha \phi \in L^2(\mathbb{R}^n)\}$ is the weighted L^2-space of order α. Then, as is well-known, H is self-adjoint; the wave operator defined by

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itH_0)$$

exists and is complete: Ran $W_\pm = \mathscr{H}^{ac}(H)$; hence the scattering operator defined by $S = W_+^* W_-$ is unitary.

For $R > 0$, let X_R be a multiplication operator defined by

$$X_R = X_R(x); \quad X_R(x) = X(|x|/R); \quad 0 \leqq X(x) \leqq 1;$$
$$X \in C_0^\infty(\mathbb{R}); \quad X(x) = 1 \quad \text{if} \quad |x| \leqq 1, = 0 \quad \text{if} \quad |x| \geqq 2. \tag{1.2}$$

For $\phi, \psi \in \mathscr{H}$, we set $\phi(t)$ and $\phi_0(t)$ as

$$\phi(t) = \exp(-itH) W_- \phi; \quad \phi_0(t) = \exp(-itH_0)\phi, \tag{1.3}$$

and $\psi(t), \psi_0(t)$ similarly. Note that if $\phi \in H^2(\mathbb{R}^n), \phi(t)$ is the unique solution of the Schrödinger equation: $i(\partial/\partial t)\phi(t) = H\phi(t)$ such that $\|\phi(t) - \phi_0(t)\| \to 0 \ (t \to -\infty)$.

We set $\mathscr{D}_0$ as

$$\mathscr{D}_0 = \{\phi \in \mathscr{H} : E_{H_0}(\Omega)\phi = \phi \quad \text{for some} \quad \Omega \subset (0, \infty); \ \Omega\text{:compact}; \quad \Omega \cap \sigma_{pp}(H)\text{:empty}\}.$$

Then T_R is defined by the following equation:

$$(\phi, T_R \psi) = \int_{-\infty}^{\infty} (\phi(t), X_R \psi(t))dt - \int_{-\infty}^{\infty} (\phi_0(t), X_R \psi_0(t))dt \tag{1.4}$$

for $\phi, \psi \in \mathscr{D}_0$. Since X_R is H_0-smooth in the sense of Kato, and is local H-smooth in the sense of Lavine (see XIII-7, [11]), $(\phi, T_R\psi)$ exists for such ϕ and ψ, and bounded from below. Hence, T_R is well-defined quadratic form on $\mathscr{D}_0$ and the Friedrichs extension exists. T_R represents approximately the difference of the sojourn time of an interacting particle in the ball of radius R, and that of a free particle.

We set $\mathscr{D}_1 = \mathscr{D}_0 \cap L^{2,3}(\mathbb{R}^n)$, and $A = (1/2i)(x\cdot(\partial/\partial x) + (\partial/\partial x)\cdot x)$ is the dilation generator. Our result is:

Theorem 1. *Suppose Assumption* (*V*), ϕ, ψ, $S\phi$ *and* $S\psi \in \mathscr{D}_1$, *then*

$$\lim_{R \to \infty} (\phi, T_R H_0 \psi) = \int_{-\infty}^{\infty} \left(\phi(t), \left\{ V + \frac{i}{2}[A, V] \right\} \psi(t)\right)dt. \tag{1.5}$$

Of course, Theorem 1 implies that the limit of the L.H.S. and the integral of the R.H.S. exist, and are equal. It also asserts that the limit: $\lim_{R \to \infty}(\phi, T_R\psi)$ exists for such ϕ and ψ. It is called time-delay.

On the other hand, in terms of the S-matrix $\{S(\lambda)\}$, the Eisenbud–Wigner time-delay operator is defined by

$$T = \left\{ -iS(\lambda)^* \frac{d}{d\lambda} S(\lambda) \right\} \tag{1.6}$$

on the spectral representation space for H_0. Jensen ([8, 9]) showed that under certain assumptions,

$$(\phi, TH_0\psi) = \int_{-\infty}^{\infty} \left(\phi(t), \left\{ V + \frac{i}{2}[A, V] \right\} \psi(t) \right) dt \tag{1.7}$$

holds for $\phi, \psi \in \mathscr{D}_0$. Combining (1.7) with Theorem 1, we can conclude the following.

Theorem 2. *If* $\phi, \psi, S\phi$ *and* $S\phi \in \mathscr{D}_1$, *then*

$$(\phi, T\psi) = \lim_{R \to \infty} (\phi, T_R \psi). \tag{1.8}$$

This formula gives a relation between the S-matrix and the sojourn times of particles.

Remark 1.1. By (1.6) and (1.7), the operator defined by the R.H.S. commutes with H_0. Hence if we set $\psi = H_0^{-1}\phi$ for $\phi\in\mathscr{D}_1\cap(S^{-1}\mathscr{D}_1)$, we have

$$\lim_{R\to\infty}(\phi, T_R\phi) = \int_{-\infty}^{\infty}\left(H^{-1/2}\phi(t), \left\{V+\frac{i}{2}[A,V]\right\}H^{-1/2}\phi(t)\right)dt.$$

As remarked by Lavine ([10]), if the quanity

$$V+\frac{i}{2}[A,V] = V+\frac{i}{2}x\cdot\nabla V$$

is non-positive everywhere, the time-delay is always non-positive, i.e. the interacting particles escape from every sufficiently large domain faster than the free particles.

Remark 1.2. We must consider how many ϕ's such that $\phi\in\mathscr{D}_1$ and $S\phi\in\mathscr{D}_1$ exist because if such ϕ's do not exist, Theorem 1 would be meaningless. But in many cases the set of such ϕ's is dense in $\mathscr{H}$. For example, (i) if V satisfies

$$V: H^2(\mathbb{R}^n)\to L^{2,4+\varepsilon}(\mathbb{R}^n)\text{: compact}$$

for $\varepsilon>0$, then $\phi\in\mathscr{D}_1$ implies $S\phi\in\mathscr{D}_1$ (see Jensen [7]); (ii) if $V_2=0$ (i.e. V is smooth and satisfies (1.1)), then $\phi\in\mathscr{D}_1$ and $\hat{\phi}\in C_0^\infty(\mathbb{R}^n)$ imply $S\phi\in\mathscr{D}_1$ and $(S\phi)\hat{}\in C_0^\infty(\mathbb{R}^n)$. This is a consequence of the result of Isozaki–Kitada [4]. It could also be proved that if V_2 satisfies (i) above, then $\phi\in\mathscr{D}_1$ and $\hat{\phi}\in C_0^\infty(\mathbb{R}^n)$ imply $S\phi\in\mathscr{D}_1$.

Time-delay has been studied by many physicists (see the introduction of Jensen [7] or Martin [13]) and mathematically rigorous treatment was initiated by Jauch and others ([5, 6], see also [1]). In particular the time-dependent formulation of time-delay such as (1.4) was introduced by Jauch and Marchand [5]. Lavine ([10]) showed that (1.5) holds for one-dimensional Schrödinger operators with V satisfying

$$|V(x)|+|x\cdot V'(x)|\leqq C(1+|x|)^{-1-\varepsilon}.$$

Later, Jensen ([7]) proved that for n-dimensional Schrödinger operators (1.8) holds if X_R is replaced by $X_R=E_A(\{\lambda:|\lambda|<R\})$. Jensen proved (1.7) also, which he called "Lavine's formula," under slightly weaker conditions than ours ([8, 9]). After this work was completed, the referee informed the author about papers of Wang ([14, 16]). He obtained similar results for smooth potential using a different method.

The outline of the proof is as follows: at first we construct a pseudo-differential operator A_R such that

$$X_R H_0 = \frac{i}{2}[H_0, A_R] + (\text{small error terms}),$$

and that as $R\to\infty$, $A_R\to A+\text{constant}$ (Sect. 2); next, we introduce a operator $J_\pm$ such that

$$\begin{aligned}\|(J_\pm-1)e^{-itH_0}\phi\|&\to 0;\\ \|(HJ_\pm-J_\pm H_0)e^{-itH_0}\phi\|&=O(t^{-2-\varepsilon});\\ \|(J_\pm^* J_\pm-1)e^{-itH_0}\phi\|&=O(t^{-1-\varepsilon})\quad(t\to\pm\infty)\end{aligned}\tag{1.9}$$

(Sect. 3, cf. Isozaki–Kitada [4]); then minicking the proof of Lavine [10], we compute

$$(\phi(t), X_R H \psi(t)) - (\phi_0(t), X_R H_0 \psi_0(t))$$

and show that the error terms tend to zero as $R \to \infty$ (Sect. 4). For that purpose we employ the stationary phase method or the Enss method ([2]).

Notations. We shall use the following notations in the paper. We denote reals by $\mathbb{R}$ and Euclidean n-space by $\mathbb{R}^n$. $H^s(\mathbb{R}^n)$ is the Sobolev space of order s and $L^{2,\alpha}(\mathbb{R}^n)$ is the weighted L^2-space of order α. For Banach spaces X and Y, $B(X, Y)$ denotes the Banach space of all bounded operators from X to Y, and $B(X) = B(X, X)$.

We set $\langle x \rangle = (1 + |x|^2)^{1/2}$; $\hat{x} = x/|x|$ for $x \in \mathbb{R}^n$. We write any constant in the estimates by C or C_* denoting the dependence on $*$.

$\hat{\phi}$ denotes the Fourier transform of ϕ, and for a symbol $a(x, \xi)$, $x, \xi \in \mathbb{R}^n$, the operator $a(x, D_x)$ is defined by

$$(a(x, D_x)\phi)(x) = (2\pi)^{-n/2} \int e^{ix\xi} a(x, \xi) \hat{\phi}(\xi) d\xi$$

with $\phi \in \mathcal{S}$. About the theory of pseudo-differential operators, see e.g. Taylor [12] or Hörmander [3].

2. Construction of the Operator A_R

In this section, we construct the operator A_R such that

$$X_R H_0 \sim \frac{i}{2}[H_0, A_R] \tag{2.1}$$

in some sense. We set $A_R \sim a_R(x, D_x)$ with some symbol $a_R(x, \xi)$. Then, since

$$\left(\frac{i}{2}[H_0, a_R(x, D_x)]\phi\right)(x)$$
$$= (2\pi)^{-n/2} \int e^{ix\xi} \left\{ \xi \partial_x a_R(x, \xi) - \frac{i}{2} \Delta_x a_R(x, \xi) \right\} \hat{\phi}(\xi) d\xi,$$

(2.1) formally implies

$$\xi \partial_x a_R(x, \xi) - \frac{i}{2} \Delta_x a_R(x, \xi) \sim X_R(x) \xi^2. \tag{2.2}$$

We solve (2.2) as follows: let $a_R(x, \xi) = a_R^{(0)}(x, \xi) + a_R^{(1)}(x, \xi)$ and these are solutions of the equations

$$\xi \partial_x a_R^{(0)}(x, \xi) = X_R(x) \xi^2; \tag{2.3}$$

$$\xi \partial_x a_R^{(1)}(x, \xi) = \frac{i}{2} \Delta_x a_R^{(0)}(x, \xi). \tag{2.4}$$

Then the remainder term is

$$\xi \partial_x a_R - \frac{i}{2} \Delta_x a_R - X_R \xi^2 = -\frac{i}{2} \Delta_x a_R^{(1)}(x, \xi) \equiv b_R(x, \xi).$$

Transport equations (2.3), (2.4) can be easily solved if $\xi \neq 0$, and we choose the following solutions:

$$\begin{aligned} a_R^{(0)}(x,\xi) &= -\int_0^\infty \xi^2 X_R(x+t\xi)dt + \int_0^\infty \xi^2 X_R(t\xi)dt \\ &= |\xi|\left\{-\int_0^\infty X_R(x+t\hat{\xi})dt + \int_0^\infty X_R(t\hat{\xi})dt\right\} \qquad (2.5) \\ &= -|\xi|\int_0^\infty dt\int_0^1 d\theta\, x\cdot(\nabla X_R)(\theta x+t\hat{\xi}); \end{aligned}$$

$$\begin{aligned} a_R^{(1)}(x,\xi) &= \frac{i}{2}\int_0^\infty \left\{|\xi|\int_0^\infty (\Delta_x X_R)(x+t\hat{\xi}+s\hat{\xi})dt\right\}\frac{ds}{|\xi|} \qquad (2.6) \\ &= \frac{i}{2}\int_0^\infty (\Delta X_R)(x+s\hat{\xi})s\,ds; \end{aligned}$$

$$b_R(x,\xi) = \frac{1}{4}\int_0^\infty (\Delta^2 X_R)(x+s\hat{\xi})s\,ds. \qquad (2.7)$$

Lemma 2.1. *$a_R^{(0)}(x,\xi)$ and $a_R^{(1)}(x,\xi)$ are the unique solutions of (2.3) and (2.4) such that for $\xi \neq 0$, $a_R^{(0)}(0,\xi) = 0$ and*

$$a_R^{(0)}(x,\xi) = |\xi|\int_0^\infty X_R(t\hat{\xi})\,dt = |\xi| \times \text{constant} \times R,$$

$$a_R^{(1)}(x,\xi) = 0$$

if $|x| \geqq 2R$ and $x\cdot\xi \geqq 0$.

This can be verified directly. We next consider their asymptotic properties as $R\to\infty$.

Proposition 2.1. *For $(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n/\{0\})$, $a_R^{(0)}(x,\xi)\to x\cdot\xi$; $a_R^{(1)}(x,\xi)\to -(i/2)(n-2)$; $b_R(x,\xi)\to 0$, as $R\to\infty$, locally uniformly.*

Proof. We may suppose $\hat{\xi} = (1,0,0,\ldots,0)$, and we write

$$x = (x_1, x')\in\mathbb{R}\times\mathbb{R}^{n-1}; \quad \nabla' = \frac{\partial}{\partial x'}; \quad \Delta' = \nabla'\cdot\nabla'.$$

By (2.5), we have

$$\begin{aligned} a_R^{(0)}(x,\xi) &= -|\xi|\{(x,\hat{\xi})\hat{\xi} + (x-(x,\hat{\xi})\hat{\xi})\}\int_0^\infty ds\int_0^1 d\theta\,\nabla X_R(\theta x+s\hat{\xi}) \\ &= -x\cdot\xi\int_0^\infty ds\int_0^1 d\theta\frac{d}{ds}\{X_R(\theta x+s\hat{\xi})\} \\ &\quad -|\xi|x'\cdot\int_0^\infty ds\int_0^1 d\theta(\nabla' X_R)(\theta x+s\hat{\xi}). \qquad (2.8) \end{aligned}$$

Since

$$\int_0^\infty ds \int_0^1 d\theta \frac{d}{ds}\{X_R(\theta x + s\hat{\xi})\} = -\int_0^1 d\theta X_R(\theta x) = -1$$

If $R \geqq |x|$, combining this with (2.8), we obtain

$$a_R^{(0)}(x,\xi) = x\cdot\xi + |\xi| \int_0^1 d\theta \int_0^\infty ds x'\cdot\nabla' X_R(\theta x + s\hat{\xi}).$$

We shall show that the second term converges to zero as $R \to \infty$. Set $y = \theta x + s\hat{\xi}$, $r = |y|$. By definition of X_R, (1.2), we see

$$(\nabla X_R)(y) = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} X_R = \frac{y}{r}\frac{1}{R} X'(r/R).$$

On the other hand, on the support of $\nabla X_R(y), (y/r) = \hat{\xi} + O(R^{-1})$ for each x and θ. Hence

$$x'\cdot\nabla' X_R(y) = O(R^{-1})\cdot\frac{1}{R}\cdot X'(y/R) = O(R^{-2}),$$

$$\left|\int_0^1 d\theta \int_0^\infty ds x'\cdot\nabla' X_R(y)\right| \leqq R\cdot C\cdot R^{-2} \leqq C\cdot R^{-1}.$$

This completes the proof of $a_R^{(0)}(x,\xi) \to x\cdot\xi$.

By (2.6), we have

$$a_R^{(1)}(x,\xi) = \frac{i}{2}\int_0^\infty \left(\frac{\partial}{\partial x_1}\right)^2 X_R(x + t\hat{\xi})\, t\, dt + \frac{i}{2}\int_0^\infty (\Delta' X_R(x + t\hat{\xi}) t\, dt. \tag{2.9}$$

By integration by parts, the first term is

$$\begin{aligned}\frac{i}{2}\int_0^\infty (\hat{\xi}\cdot\nabla)^2 X_R(x + t\hat{\xi}) t\, dt &= \frac{i}{2}\int_0^\infty \frac{d}{dt}\{\hat{\xi}\cdot\nabla X_R(x + t\hat{\xi})\}\, t\, dt \\ &= -\frac{i}{2}\int_0^\infty \hat{\xi}\nabla X_R(x + t\hat{\xi}) dt = -\frac{i}{2}\int_0^\infty \frac{d}{dt}\{X_R(x + t\hat{\xi})\} dt \\ &= \frac{i}{2} X_R(x).\end{aligned} \tag{2.10}$$

By elementary calculations, one can obtain

$$\begin{aligned}\Delta' X_R(y) &= \Delta' r \frac{d}{dr} X_R + |\nabla' r|^2 \frac{d^2}{dr^2} X_R \\ &= \left(\frac{n-1}{r} - \frac{|y'|^2}{r^3}\right) R^{-1} X'\left(\frac{|y|}{R}\right) + \frac{|y'|^2}{r^2} R^{-2} X''\left(\frac{|y|}{R}\right),\end{aligned}$$

where $y = x + t\hat{\xi}$ and $r = |y|$. Since on the support of ∇X_R

$$\frac{dr}{dt} = \frac{(x + t\hat{\xi})\hat{\xi}}{r} = \frac{t}{r} + \frac{x\cdot\hat{\xi}}{r} = \frac{t}{r} + O(R^{-1}),$$

and $|y'|=|x'|$, we have

$$\begin{aligned}\Delta' X_R(y)\cdot t &= (n-1)(t/r)R^{-1}X'(r/R)+O(R^{-3})\\ &= (n-1)\frac{dr}{dt}R^{-1}X'(r/R)+O(R^{-2})\\ &= (n-1)\frac{dr}{dt}\frac{d}{dr}X_R(r)+O(R^{-2})\\ &= (n-1)\frac{d}{dt}X_R(y)+O(R^{-2}).\end{aligned}$$

Hence, the second term of (2.9) is

$$\frac{i}{2}(n-1)\int_0^\infty \frac{d}{dt}X_R(y)dt+O(R^{-1}) = -\frac{i}{2}(n-1)X_R(x)+O(R^{-1}).$$

Combining this with (2.10), we conclude

$$a_R^{(1)}(x,\xi) = -\frac{i}{2}(n-2)X_R(x)+O(R^{-1}).$$

$b_R(x,\xi)\to 0$ can be shown easily from (2.7) since $(\Delta^2 X_R)(y)=O(R^{-4})$. □

Let $\rho\in C^\infty([-1,1])$ such that $0\leqq\rho(x)\leqq 1$; $\rho(x)=1$ if $x\leqq 1/4$, $=0$ if $x\leqq -1/4$. We set

$$\begin{aligned}Y_R(x,\xi) &= X_{2R}(x)+(1-X_{2R}(x))\rho(\hat{x}\cdot\hat{\xi})\\ &= X\left(\frac{|x|}{2R}\right)+\left(1-X\left(\frac{|x|}{2R}\right)\right)\rho\left(\frac{x}{|x|}\cdot\frac{\xi}{|\xi|}\right),\end{aligned}$$

and define $\tilde{a}_R(x,\xi)$ and $c_R(x,\xi)$ by

$$\tilde{a}_R(x,\xi)=a_R(x,\xi)Y_R(x,\xi);$$

$$c_R(x,\xi)=\xi\partial_x\tilde{a}_R(x,\xi)-\frac{i}{2}\Delta_x\tilde{a}_R(x,\xi)-X_R(x)\xi^2.$$

By easy computations, we obtain

$$c_R=b_RY_R+a_R\xi\cdot\partial_xY_R-i\partial_xa_R\cdot\partial_xY_R-\frac{i}{2}a_R(\Delta_xY_R). \tag{2.11}$$

Lemma 2.2. *For each $\alpha,\beta,\delta>0$,*

(i) $|\partial_x^\alpha\partial_\xi^\beta\tilde{a}_R(x,\xi)|\leqq C_{\alpha\beta\delta}\min(\langle x\rangle^{-|\alpha|},R)\langle\xi\rangle^{1-|\beta|}(|\xi|>\delta)$;

(ii) $|\partial_x^\alpha\partial_\xi^\beta c_R(x,\xi)|\leqq\begin{pmatrix}C_{\alpha\beta\delta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{1-|\beta|} & (|\xi|>\delta)\\ C_{\alpha\beta\delta}\langle x\rangle^{-2-|\alpha|}\langle\xi\rangle^{-|\beta|} & (|\xi|>\delta,\hat{x}\cdot\hat{\xi}>-\frac{1}{4}),\end{pmatrix}$

where $C_{\alpha\beta\delta}$'s are independent of R.

Proof. By (2.5) and (2.6), if $|x|\leqq 4R$ or $\hat{x}\cdot\hat{\xi}\geqq -1/4$,

$$|\partial_x^\alpha\partial_\xi^\beta a_R^{(0)}(x,\xi)|\leqq\begin{pmatrix}C_\beta\min(|x|,R)|\xi|^{1-|\beta|} & (\alpha=0)\\ C_{\alpha\beta}R^{1-|\alpha|}|\xi|^{1-|\beta|} & (\alpha\neq 0),\end{pmatrix} \tag{2.12}$$

$$|\partial_x^\alpha \partial_\xi^\beta a_R^{(1)}(x,\xi)| \leqq C_{\alpha\beta} R^{-|\alpha|} |\xi|^{-|\beta|}.$$

Since $\operatorname{supp}(\partial_x^\alpha a_R^{(i)}) \subset \{(x,\xi): |x| \leqq 2\sqrt{2} \text{ or } \hat{x}\cdot\hat{\xi} \leqq -1/\sqrt{2}\}$ if $\alpha \neq 0$ or $i = 1$, we have

$$|\partial_x^\alpha \partial_\xi^\beta a_R^{(i)}(x,\xi)| \leqq C_{\alpha\beta} \min(\langle x\rangle^{1-i-|\alpha|}, R) |\xi|^{1-i-|\beta|}$$

on supp Y_R, for $i = 0, 1$. By the definition of Y_R, we have also

$$|\partial_x^\alpha \partial_\xi^\beta Y_R(x,\xi)| \leqq \begin{pmatrix} C_{\alpha\beta} \langle x\rangle^{-|\alpha|} |\xi|^{-|\beta|} \\ 0 \quad (\hat{x}\cdot\hat{\xi} \geqq 1/4, |\alpha + |\beta| \neq 0). \end{pmatrix} \tag{2.13}$$

Then (i) follows easily from these estimates.

By (2.7), we obtain

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta b_R(x,\xi)| &\leqq C_{\alpha\beta} R^{-2-|\alpha|} |\xi|^{-|\beta|} \\ &\leqq C_{\alpha\beta} \langle x\rangle^{-2-|\alpha|} |\xi|^{-|\beta|}, \end{aligned}$$

similarly if $|x| \leqq 4R$ or $\hat{x}\cdot\hat{\xi} \geqq -1/4$. Hence

$$|\partial_x^\alpha \partial_\xi^\beta (b_R Y_R)(x,\xi)| \leqq C \langle x\rangle^{-2-|\alpha|} |\xi|^{-|\beta|};$$

$$|\partial_x^\alpha \partial_\xi^\beta (a_R \xi\cdot\partial_x Y_R)(x,\xi)| \leqq \begin{pmatrix} C\langle x\rangle^{-|\alpha|} |\xi|^{2-|\beta|} \\ 0 \quad (\hat{x}\cdot\hat{\xi} \geqq 1/4); \end{pmatrix}$$

$$|\partial_x^\alpha \partial_\xi^\beta (\partial_x a_R \cdot \partial_x Y_R)(x,\xi)| \leqq \begin{pmatrix} C\langle x\rangle^{-1-|\alpha|} |\xi|^{1-|\beta|} \\ 0 \quad (\hat{x}\cdot\hat{\xi} \geqq 1/4); \end{pmatrix}$$

$$|\partial_x^\alpha \partial_\xi^\beta (a_R \Delta_x Y_R)(x,\xi)| \leqq \begin{pmatrix} C\langle x\rangle^{-1-|\alpha|} |\xi|^{1-|\beta|} \\ 0 \quad (\hat{x}\cdot\hat{\xi} \geqq 1/4) \end{pmatrix}$$

using (2.13) again. Equation (2.11) and these estimates imply (ii). □

Since our symbols have singularities at $\xi = 0$, we must introduce a suitable cut-off. We set Z_R as

$$\begin{aligned} &Z_R = Z_R(D_x); \quad Z_R(\xi) = Z(R|\xi|); \\ &Z \in C^\infty(\mathbb{R}); \quad 0 \leqq Z(\xi) \leqq 1 \, (\xi \in \mathbb{R}); \\ &Z(\xi) = 0 \quad \text{if} \quad |\xi| \leqq 1, = 1 \quad \text{if} \quad |\xi| \geqq 2. \end{aligned}$$

We define A_R and C_R by

$$\begin{aligned} A_R(x,\xi) &= \tilde{a}_R(x,\xi) Z_R(\xi); \quad A_R = A_R(x, D_x), \\ C_R(x,\xi) &= c_R(x,\xi) Z_R(\xi); \quad C_R = C_R(x, D_x). \end{aligned}$$

Then, by Lemma 2.3 and the L^2-boundedness theorem (Ch. 13 of [12]), A_R is in $B(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$, and C_R is in $B(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ for each R. Moreover, we can prove their uniform boundedness in R.

Proposition 2.2.

(i) $\sup_{R \geqq 1} \| A_R \|_{B(H^1(\mathbb{R}^n), L^{2,-1}(\mathbb{R}^n))} < \infty$,

(ii) $\sup_{R \geqq 1} \| A_R^* \|_{B(H^1(\mathbb{R}^n), L^{2,-1}(\mathbb{R}^n))} < \infty$,

(iii) $\sup_{R \geqq 1} \| C_R \|_{B(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} < \infty$,

(iv) $\sup_{R \geqq 1} \| C_R^* \|_{B(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} < \infty$.

Proof. We define $d_{R,k}(x, \xi)$ by

$$d_{R,k}(x, \xi) = -\int_0^\infty ds \int_0^1 d\theta \frac{\partial}{\partial x_k} X_R(\theta x + s\hat{\xi})\, Y(x, \xi)$$

for $k = 1, \ldots, n$, then by (2.5) we see

$$a_R^{(0)}(x, \xi)\, Y_R(x, \xi) = \sum_k x^k d_{R,k}(x, \xi) |\xi|. \tag{2.14}$$

By a change of coordinates, one immediately obtains $d_{R,k}(x, \xi) Z_R(\xi) = d_{1,k}(x/R, R\xi) Z_1(R\xi)$. Hence, if we set $\rho = \log R$ and $U(\sigma)$ be the dilation operator defined by $(U(\sigma)\phi)(x) = \exp(n\sigma/2)\phi(e^\sigma x)$ $(\sigma \in \mathbb{R})$, we have

$$\begin{aligned} d_{R,k}(x, D_x) Z_R(D_x) &= d_{1,k}(x/R, R \cdot D_x) Z_1(R \cdot D_x) \\ &= U(-\rho) d_{1,k}(x, D_x) Z_1(D_x) U(\rho). \end{aligned}$$

Since $U(\rho)$ is unitary, $\| d_{R,k}(x, D_x) Z_R \|_{B(\mathscr{H})} = \text{constant}$ (we remarked that $d_{1,k}(x, D_x) Z_1(D_x)$ is bounded in L^2 $(\mathbb{R}^n)$). This and (2.14) yield

$$\| (a_R^{(0)} Y_R Z_R)(x, D_x) \|_{B(H^1, L^{2, -1})} \leqq C.$$

On the other hand, $a_R^{(1)}(x, \xi) = a_1^{(1)}(x/R, R\xi)$ and an analogous argument can be carried out to show the uniform boundedness of $(a_R^{(1)} Y_R Z_R)(x, D_x)$ in $B(\mathscr{H})$. These imply (i).

Next, by the definition and (2.14), we see

$$\begin{aligned} &((a_R^{(0)} Y_R Z_R)(x, D_x))^* \phi(x) \\ &\quad = (2\pi)^{-n} \sum_k \int e^{i(x-y)\xi} d_{R,k}(y, \xi) |\xi| Z_R(\xi) y_k \phi(y) dy\, d\xi \\ &\quad = (2\pi)^{-n} \sum_k \int \frac{\partial}{\partial \xi_k} \{e^{i(x-y)\xi}\} d_{R,k}(y, \xi) |\xi| Z_R(\xi) \phi(y)\, dy\, d\xi \\ &\qquad - (2\pi)^{-n} \sum_k i x_k \int e^{i(x-y)\xi} d_{R,k}(y, \xi) |\xi| Z_R(\xi) \phi(y)\, dy\, d\xi \\ &\quad = \mathrm{I}\phi + \mathrm{II}\phi, \end{aligned}$$

and by integration by parts,

$$\begin{aligned} \mathrm{I}\phi &= -i(2\pi)^{-n} \sum_k \int e^{i(x-y)\xi} \frac{\partial}{\partial \xi_k} \{d_{R,k}(y, \xi) |\xi| Z_R(\xi)\} \phi(y)\, dy\, d\xi \\ &= -i(2\pi)^{-n} \sum_k \int e^{i(x-y)\xi} \frac{\partial}{\partial \xi_k} \{d_{R,k}(y, \xi) |\xi|\} Z_R(\xi) \phi(y)\, dy\, d\xi \\ &\quad - (2\pi)^{-n} \sum_k \int e^{i(x-y)\xi} d_{R,k}(y, \xi) |\xi| \frac{\partial}{\partial \xi_k} \{Z_R(\xi)\} \phi(y)\, dy\, d\xi \\ &= \mathrm{I}_1\phi + \mathrm{I}_2\phi. \end{aligned}$$

For I_1, one can prove the uniform boundedness in $B(\mathscr{H})$ by the same method as above. Since

$$\mathsf{I}_2\phi = -i\sum_k \left\{ |D_x| \left(\frac{\partial}{\partial \xi_k} Z_R \right)(D_x) \right\} d_{R,k}(x, D_x)^* \phi,$$

and the symbol of $\{|D_x|(\partial/\partial\xi_k Z_R)(D_x)\}$ is bounded uniformly in ξ and R, I_2 is uniformly bounded in $B(\mathscr{H})$.

$$\begin{aligned}
\mathbb{I}\phi &= -i(2\pi)^{-n}\sum_{k,j} x_k \int e^{i(x-y)\xi} d_{R,k}(y,\xi) Z_R(\xi)\xi_j \hat{\xi}_j \phi(y)\,dy\,d\xi \\
&= (2\pi)^{-n}\sum_{k,j} x_k \int \frac{\partial}{\partial y_j}\{e^{i(x-y)\xi}\} d_{R,k}(y,\xi) Z_R(\xi)\hat{\xi}_j \phi(y)\,dy\,d\xi \\
&= -(2\pi)^{-n}\sum_{k,j} x_k \int e^{i(x-y)\xi} \frac{\partial}{\partial y_j} d_{R,k}(y,\xi) Z_R(\xi)\hat{\xi}_j \phi(y)\,dy\,d\xi \\
&\quad -(2\pi)^{-n}\sum_{k,j} x_j \int e^{i(x-y)\xi} d_{R,k}(y,\xi) Z_R(\xi)\hat{\xi}_j \left\{ \frac{\partial}{\partial y_j}\phi(y) \right\} dy\,d\xi \\
&= -\sum_{k,j} x_k \left\{ \left(\frac{\partial}{\partial x_j} d_{R,k} \right)(x, D_x) Z_R(\hat{D}_x)_j \right\}^* \phi \\
&\quad -\sum_{k,j} x_k \{ d_{R,k}(x, D_x) Z_R(\hat{D}_x)_j \}^* \frac{\partial}{\partial x_j}\phi \\
&= \mathbb{I}_1\phi + \mathbb{I}_2\phi.
\end{aligned}$$

Similar argument shows that $\{((\partial/\partial x_j)d_{R,k})(x, D_x)Z_R\hat{D}_x\}^*$ and $\{a_{R,k}^{(0)}(x, D_x)Z_R\hat{D}_x\}^*$ are uniformly bounded in $B(\mathscr{H})$, again. Hence $\mathbb{I}_1$ is uniformly bounded in $B(\mathscr{H}, L^{2,-1})$, and $\mathbb{I}_2$ is uniformly bounded in $B(H^1, L^{2,1})$. These prove (ii).

The next estimates can be shown in the same way:

$$\begin{aligned}
\|(b_R Y_R Z_R)(x, D_x)\|_{B(\mathscr{H})} &\leqq CR^{-2}; \\
\|(a_R \xi\cdot\partial_x Y_R)(x, D_x)\|_{B(H^2,L^2)} &\leqq C; \\
\|(\partial_x a_R\cdot\partial_x Y_R)(x\cdot D_x)\| E_{B(H^1,L^2)} &\leqq CR^{-1}; \\
\|(a_R \Delta_x Y_R)(x, D_x)\|_{B(H^1,L^2)} &\leqq CR^{-1},
\end{aligned}$$

and (iii) follows. (iv) can be proved by the standard method using integration by parts. □

Remark 2.1. Using the Calderon–Lions interpolation theorem (Th. IX-20 of [11]), one can prove that $A_R(C_R$ respectively) is uniformly bounded in $B(H^{s,\alpha}(\mathbb{R}^n), H^{s-1,\alpha-1}(\mathbb{R}^n))$ $(B(H^{s+2}, H^s)$ respectively) for $s \in \mathbb{R}, 0 \leqq \alpha \leqq 1$, where $H^{s,\alpha}(\mathbb{R}^n)$ is the weighted Sobolev space.

The next lemma follows easily from the definitions, (2.11) and Proposition 2.1:

Lemma 2.3. *For* $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n/\{0\}), A_R(x,\xi) \to x\cdot\xi - (i/2)(n-2); C_R \to 0$ *as* $R \to \infty$, *locally uniformly.*

3. Modifier $J_\pm$

Here we introduce a pseudo-differential operator $J_\pm$ such that it satisfies (1.9). $J_\pm$ we shall define is approximately the same as that in Isozaki–Kitada [4] (for short range potentials), and their modifier is more precise than ours. But our construction is slightly easier to handle and enough for our purpose.

Let $p_\pm(x,\xi)$ be a solution of

$$2i\xi\cdot\partial_x p_\pm(x,\xi) = V_1(x): \tag{3.1}$$

$$\sigma_\pm(x,\xi) = -\frac{1}{2i}\int_0^{\pm\infty} V_1(x+t\xi)\,dt = -\frac{1}{2i}\frac{1}{|\xi|}\int_0^{\pm\infty} V_1(x+t\hat{\xi})\,dt.$$

It satisfies

$$|\partial_x^\alpha \partial_\xi^\beta p_\pm(x,\xi)| \leqq C_{\alpha\beta}|\xi|^{-1-|\beta|}\langle x\rangle^{-\varepsilon-|\alpha|}, \tag{3.2}$$

If $\pm\hat{x}\cdot\hat{\xi} \geqq -1/4$. We set $j_\pm(x,\xi) = j_\pm(\delta,\Delta;x,\xi)(0<\delta<\Delta<\infty)$ by

$$j_\pm(x,\xi) = \exp\{p_\pm(x,\xi)(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\} f(\delta,\Delta;|\xi|^2),$$

where $f(\delta,\Delta;\lambda)\in C_0^\infty((O,\infty)); 0\leqq f(\delta,\Delta;\lambda)\leqq 1; f(\delta,\Delta;\lambda)=1$ if $\lambda\in[\delta,\Delta]$, $=0$ if $\lambda\notin(\delta/2,2\Delta)$. Let $t_\pm(x,\xi) = t_\pm(\delta,\Delta;x,\xi)$ be

$$t_\pm(x,\xi) = 2i\xi\cdot\partial_x j_\pm(x,\xi) + \Delta_x j_\pm(x,\xi) + V_1(x) j_\pm(x,\xi). \tag{3.3}$$

Lemma 3.1. *For any* α,β,

(i) $|j_\pm(x,\xi)| \leqq 1$;

(ii) $|\partial_x^\alpha \partial_\beta^\alpha j_\pm(x,\xi)| \leqq C_{\alpha\beta}\langle x\rangle^{-\varepsilon-|\alpha|}$ $\quad(|\alpha|+|\beta|\neq 0)$;

(iii) $|\partial_x^\alpha \partial_\beta^\alpha t_\pm(x,\xi)| \leqq \begin{cases} C_{\alpha\beta}\langle x\rangle^{-1-\varepsilon-|\alpha|} \\ C_{\alpha\beta}\langle x\rangle^{-2-\varepsilon-|\alpha|}(\pm\hat{x}\cdot\hat{\xi}\geqq 1/4). \end{cases}$

Proof. (i) is immediate since $p_\pm(x,\xi)$ is pure imaginary. (ii) follows from (3.3) and the fact that $\rho(\pm\hat{x}\cdot\hat{\xi})$ is homogeneous in x. By (3.3),

$$\begin{aligned} t_\pm(x,\xi) &= 2i\xi\cdot\partial_x\{p_\pm(x,\xi)(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\} j_\pm(x,\xi) \\ &\quad + \Delta_x\{p_\pm(x,\xi)(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\} j_\pm(x,\xi) \\ &\quad + |\partial_x\{p_\pm(x,\xi)(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\}|^2 j_\pm(x,\xi) \\ &\quad - V_1(x) j_\pm(x,\xi) \\ &= \{2i\xi\cdot\partial_x p_\pm(x,\xi) - V_1(x)\} j_\pm(x,\xi) \\ &\quad + 2i\xi\cdot\partial_x\{(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\} p_\pm(x,\xi) j_\pm(x,\xi) \\ &\quad + \Delta_x\{p_\pm(x,\xi)(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\} j_\pm(x,\xi) \\ &\quad + |\partial_x\{p_\pm(x,\xi)(1-X_1(x))\rho(\pm\hat{x}\cdot\hat{\xi})\}|^2 j_\pm(x,\xi). \end{aligned}$$

The first term vanishes by (3.1), and it is easily seen that the second term satisfies the former property of (iii), and vanishes outside $\{x:|x|\leqq 2\}$ if $\pm\hat{x}\cdot\hat{\xi}\geqq 1/4$. (iii) follows since the third term is $O(\langle x\rangle^{-2-\varepsilon})(O(\langle x\rangle^{-2-\varepsilon-|\alpha|})$ after differentiation $\partial_x^\alpha\partial_\beta^\alpha)$, and the last term is $O(\langle x\rangle^{-2-2\varepsilon})$ $(O(\langle x\rangle^{-2-2\varepsilon-|\alpha|}$, after $\partial_x^\alpha\partial_\xi^\beta)$. □

We define $J_\pm = J_\pm(\delta, \Delta)$ by

$$J_\pm = f(\delta/2, 2\Delta; H_0)\, j_\pm(x, D_x).$$

Then the symbol of $J_\pm$ concides with $j_\pm$ modulo $O(\langle x\rangle^{-\infty})$ since $f(\delta/2, 2\Delta; \xi^2) = 1$ on supp $j_\pm(\delta, \Delta; \cdot, \cdot)$.

Lemma 3.2. *Let $q_\pm(x, \xi)$ be a symbol such that*

$$|\partial_x^\alpha \partial_\xi^\beta q_\pm(x, \xi)| \leqq \begin{cases} C_{\alpha\beta}\langle x\rangle^{-|\alpha|} \\ C_{\alpha\beta}\langle x\rangle^{-\mu-|\alpha|}(\pm \hat{x}\cdot\hat{\xi} \geqq \gamma) \end{cases}$$

for any α, β, with $\gamma \in (-1, 1)$ and $\mu \in [0, 3]$. Then for $\phi \in \mathscr{D}_1$,

$$\| q_\pm(x, D_x) e^{-itH_0}\phi \| \leqq C_\phi \langle t\rangle^{-\mu}(\pm t \geqq 0).$$

C_ϕ depends only on ϕ and finite number of constants in the assumption.

Proof. We prove the $(+)$-case only. Let $\tilde{\rho} \in C_0^\infty((-1, 1))$ such that $\tilde{\rho}(\theta) = 1$ if $\theta \geqq (1+\gamma)/2$, $= 0$ if $\theta \leqq \gamma$: $0 \leqq \tilde{\rho}(\theta) \leqq 1$, and set

$$q_1(x, \xi) = q_+(x, \xi)\{X_1(x) + (1 - X_1(x))\cdot\tilde{\rho}(\hat{x}\cdot\hat{\xi})\};$$

$$q_2(x, \xi) = q_+(x, \xi) - q_1(x, \xi) = q_+(x, \xi)(1 - X_1(x))(1 - \tilde{\rho}(\hat{x}\cdot\hat{\xi})).$$

As is easily seen, they satisfy

$$|\partial_x^\alpha \partial_\xi^\beta q_1(x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle^{-\mu-|\alpha|}; \tag{3.4}$$

$$|\partial_x^\alpha \partial_\xi^\beta q_2(x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle^{-|x|};$$

$$\operatorname{supp} q_2 \subset \{(x, \xi) : \hat{x}\cdot\hat{\xi} \leqq \gamma\}. \tag{3.5}$$

At first we consider $q_1(x, D_x)$:

$$q_1(x, D_x) e^{-itH_0}\phi = (q_1(x, D_x)\langle x\rangle^\mu)(\langle x\rangle^{-\mu} e^{-itH_0}\phi).$$

and by Lemma 4.3 of [7], we have

$$\| \langle x\rangle^{-\mu} e^{-itH_0}\phi \| \leqq C\langle t\rangle^{-\mu}, \tag{3.6}$$

since $\phi \in \mathscr{D}_1 \subset D(A^3)$. By virtue of (3.4), $q_1(x, D_x)\langle x\rangle^\mu$ is bounded in $L^2(\mathbb{R}^n)$, and the claim has been proved for q_1.

Now, (3.5) implies that q_2 has support in the in-coming subspace, and the Enss method can be applied to obtain

$$\| q_2(x, D_x) e^{-itH_0} f(\delta, \Delta; H_0)\chi_{\{x:|x|<ct\}} \| \leqq C_N\langle t\rangle^{-N}(t > 0) \tag{3.7}$$

for any N and sufficiently small $c > 0$ (cf. Enss [2]). We take $0 < \delta < \Delta < \infty$ so that $f(\delta, \Delta; H_0)\phi = \phi$. Then

$$q_2(x, D_x) e^{-itH_0}\phi = (q_2(x, D_x) e^{-itH_0} f(\delta, \Delta; H_0)\chi_{\{x:|x|<ct\}})\phi + (q_2(x, D_x) e^{-itH_0} f(\delta, \Delta; H_0))(\chi_{\{x:|x|\geqq ct\}}\phi),$$

hence we have

$$\| q_2(x, D_x)e^{-itH_0}\phi \| \leqq \| q_2(x, D_x)e^{-itH_0}f(\delta, \Delta; H_0)\chi_{\{x:|x|<ct\}} \| \, \| \phi \| + \| q_2(x, D_x) \| \, \| \chi_{\{x.|x| \geqq ct\}}\phi \| . \tag{3.8}$$

Since $\phi \in L^{2,3}(\mathbb{R}^n)$,

$$\| \chi_{\{x:|x| \geqq ct\}}\phi \| \leqq \langle ct \rangle^{-3} \| \phi \|_{L^{2,3}(\mathbb{R}^n)}, \tag{3.9}$$

and (3.7), (3.8), (3.9) complete the proof. □

Proposition 3.1 *For* $\phi \in \mathscr{D}_1$,

(i) $\| (HJ_{\pm} - J_{\pm}H_0)e^{-itH_0}\phi \| \leqq C_\phi \langle t \rangle^{-2-\varepsilon}(t \to \pm\infty)$;
(ii) $\| (J_{\pm}^* J_{\pm} - f(\delta, \Delta; H_0)^2)e^{-itH_0}\phi \| \leqq C_\phi \langle t \rangle^{-1-\varepsilon}(|t| \to \infty)$;
(iii) $\| (J_{\pm} - f(\delta, \Delta; H_0))e^{-itH_0}\phi \| \leqq C_\phi \langle t \rangle^{-\varepsilon}(|t| \to \infty)$.

Proof. As remarked after the definition of $J_{\pm}$, the symbol of $J_{\pm}$ coincides $j_{\pm}(x, \xi)$ modulo $O(\langle x \rangle^{-\infty})$, and we may consider $j_{\pm}(x, \xi)$ as the symbol of $J_{\pm}$. Then the symbol of $\{(H_0 + V_1)J_{\pm} - J_{\pm}H_0\}$ is $-t_{\pm}(x, \xi)$, hence by Lemmas 3.1-(iii) and 3.2, we see

$$\| \{H_0 + V_1)J_{\pm} - J_{\pm}H_0\}e^{-itH_0}\phi \| \leqq C\langle t \rangle^{-2-\varepsilon}(\pm t \geqq 0).$$

$$V_2 J_{\pm} e^{-itH_0}\phi = \{V_2\langle x \rangle^{2+\varepsilon}(H_0 + 1)\}\{(H_0 + 1)\langle x \rangle^{-2-\varepsilon}J_{\pm}\langle x \rangle^{2+\varepsilon}\}\{\langle x \rangle^{-2-\varepsilon}e^{-itH_0}\phi\}.$$

The first factor is bounded by Assumption (V). The symbol of the second factor, say $r(x, \xi)$, satisfies

$$|\partial_x^\alpha \partial_\xi^\beta r(x, \xi)| \leqq C_{\alpha\beta N}\langle x \rangle^{-|\alpha|}\langle \xi \rangle^{-N}$$

for any α, β and N, hence the second factor is bounded. The last factor can be estimated by (3.6) to conclude

$$\| V_2 J_{\pm} e^{-itH_0}\phi \| \leqq C\langle t \rangle^{-2-\varepsilon}.$$

This completes the proof of (i).

By the asymptotic expansion theorem ([12], §2.3) and Lemma 3.1-(ii), the symbol of $\{J_{\pm}^* J_{\pm} - f(\delta, \Delta; H_0)^2\}$ is in $S_{1,0}^{-1-\varepsilon}(\mathbb{R}_\xi^n)$, so $(J_{\pm}^* J_{\pm} - f(\delta, \Delta; H_0)^2)\langle x \rangle^{1+\varepsilon}$ is bounded in $L^2(\mathbb{R}^n)$. Thus (ii) follows from (3.6) again. (iii) follows similarly from Lemma 3.1-(ii). □

Corollary 3.1. *For* $\phi \in \mathscr{H}$,

$$W_{\pm} f(\delta, \Delta; H_0)\phi = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} J_{\pm} e^{-itH_0}\phi.$$

Proof. If $\phi \in \mathscr{D}_1$, this follows easily from Proposition 3.1-(iii), and the density argument yields the assertion. □

Corallary 3.2 *For* $\phi\in\mathscr{D}_1$,

(i) $\|(W_\pm f(\delta,\Delta;H_0)-J_\pm)e^{-itH_0}\phi\|\leqq C_\phi\langle t\rangle^{-1-\varepsilon}(\pm t\geqq 0)$;

(ii) $\int_0^{\pm\infty}\|(W_\pm f(\delta,\Delta;H_0)-J_\pm)e^{-itH_0}\phi\|\,dt<\infty$.

Proof. We prove the (+)-case only. By Corollary 3.1,

$$(W_\pm f(\delta,\Delta;H_0)-J_+)e^{-itH_0}\phi=\left(\operatorname*{s-lim}_{s\to+\infty}e^{isH}J_+e^{-isH_0}-J_+\right)e^{-itH_0}\phi$$
$$=\int_0^\infty ie^{isH}(HJ_+-J_+H_0)e^{-itH_0}e^{-itH_0}\phi\,ds.$$

Hence we have

$$\|(W_+f(\delta,\Delta;H_0)J_+)e^{-itH_0}\phi\|\leqq\int_0^\infty\|(HJ_+-J_+H_0)e^{-i(s+t)H_0}\phi\|\,ds$$
$$\leqq\int_0^\infty C|s+t|^{-2-\varepsilon}ds=\frac{C}{2+\varepsilon}|t|^{-1-\varepsilon}\quad(t>0)$$

by Proposition 3.1-(i). (ii) follows immediately from (i). □

4. Proof Theorem 1.

At first, we sum up the remainder terms of (2.1).

Lemma 4.1. *As forms on* $H^2(\mathbb{R}^n)$,

$$X_RH_0=\frac{i}{2}[H_0,A_R]+X_RH_0(1-Z_R)-C_R;\tag{4.1}$$

$$X_RH=\frac{i}{2}[H,A_R]+X_RV-\frac{i}{2}[V,A_R]+X_RH_0(1-Z_R)-C_R.\tag{4.2}$$

Equation (4.1) follows from the definitions of A_R and C_R. Equation (4.2) follows immediately from (4.1).

We fix $\phi,\psi\in\mathscr{D}_1\cap(S^{-1}\mathscr{D}_1)$ and $0<\delta<\Delta<\infty$ so that $f(\delta,\Delta;H_0)\phi=\phi$, $f(\delta,\Delta;H_0)\psi=\psi$. Then we obtain by Lemma 4.1,

$$\begin{aligned}&(e^{-itH}W_-\phi,X_Re^{-itH}W_-H_0\psi)-(e^{-itH_0}\phi,X_Re^{-itH_0}H_0\psi)\\&=(\phi(t),X_RH\psi(t))-(\phi_0(t),X_RH_0\psi_0(t))\\&=\frac{i}{2}\{(\phi(t),[H,A_R]\psi(t))-(\phi_0(t),[H_0,A_R]\psi_0t))\}\\&\quad+(\phi(t),\left\{X_RV+\frac{i}{2}[A_R,V]\right\}\psi(t))+(\phi(t),X_RH_0(1-Z_R)\psi(t))\\&\quad-(\phi_0(t),X_RH_0(1-Z_R)\psi_0(t))-\{(\phi(t),C_R\psi(t))-(\phi_0(t),C_R\psi_0(t))\}.\end{aligned}\tag{4.3}$$

We shall estimate the integrals of these terms.

Lemma 4.2. *For sufficiently large R,*

$$\lim_{\substack{T\to\infty\\ T'\to-\infty}} \int_{T'}^{T} \{(\phi(t),[H,A_R]\psi(t))-(\phi_0(t),[H_0,A_R]\psi_0(t))\}\,dt=0.$$

Proof. By (1.3), we have

$$\begin{aligned} &i\{(\phi(t),[H,A_R]\psi(t))-(\phi_0(t),[H_0,A_R]\psi_0(t))\}\\ &\quad=\frac{d}{dt}\{(\phi(t),A_R\psi(t))-(\phi_0(t),A_R\psi_0(t))\},\end{aligned}$$

hence

$$\begin{aligned} &i\int_{T'}^{T}\{(\phi(t),[H,A_R]\psi(t))-(\phi_0(t),[H_0,A_R]\psi_0(t))\}\,dt\\ &\quad=\{(\phi(T),A_R\psi(T))-(\phi_0(T),A_R\psi_0(T))\}\\ &\qquad-\{(\phi(T'),A_R\psi(T'))-(\phi_0(T'),A_R\psi_0(T'))\}. \end{aligned}\tag{4.4}$$

Again in by (1.3),

$$\begin{aligned} &(\phi(t),A_R\psi(t))-(\phi_0(t),A_R\psi_0(t))\\ &\quad=(W_-e^{-itH_0}\phi,A_RW_-e^{-itH_0}\psi)-(e^{-itH_0}\phi,A_Re^{-itH_0}\psi)\\ &\quad=((W_--1)e^{-itH_0}\phi,A_RW_-e^{-itH_0}\psi)+(A_R^*e^{-itH_0}\phi,(W_--1)e^{-itH_0}\psi). \end{aligned}\tag{4.5}$$

Since $A_R, A_R^*\in B(H^1(\mathbb{R}^n),L^2(\mathbb{R}^n))$ for each R (see Lemma 2.2) and $\|(W_--1)\exp(-itH_0)\phi\|\to 0(t\to-\infty)$ by definition of W_-, we obtain

$$\begin{aligned} &|(\phi(t),A_R\psi(t))-(\phi_0(t),A_R\psi_0(t))|\\ &\quad\leqq\|(W_--1)e^{-itH_0}\phi\|\,\|A_R\|_{B(H^1,L^2)}\|W_-\|_{B(H^2)}\|\psi\|_{H^2}\\ &\qquad+\|A_R^*\|_{B(H^1,L^2)}\|\phi\|_{H^1}\|(W_--1)e^{-itH_0}\psi\|\\ &\qquad\longrightarrow 0\qquad(t\to-\infty). \end{aligned}\tag{4.6}$$

We will show

$$(\phi(t),A_R\psi(t))-(\phi_0(t),A_R\psi_0(t))\to 0\quad(t\to\infty).\tag{4.7}$$

Let $\phi_1(t)$ and $\psi_1(t)$ be

$$\phi_1(t)=\exp(-itH_0)S\phi;\quad \psi_1(t)=\exp(-itH_0)S\psi.$$

Then similarly to (4.5) and (4.6), one can see

$$(\phi(t),A_R\psi(t))-(\phi_1(t),A_R\psi_1(t))\to 0\quad(t\to\infty).\tag{4.8}$$

If R is so large that $2/R\leqq\delta$, then $\xi\in\operatorname{supp}\hat{\psi}$, $|x|\geqq 2R$ and $\hat{x}\cdot\hat{\xi}\geqq 1/4$ imply $a_R(x,\xi)Z_R(\xi)=R'|\xi|$ by Lemma 2.1, where $R'=\text{constant}\times R$ in Lemma 2.1. Therefore, using Lemma 3.2, one can show

$$\|(A_R-R'|D_x|)e^{-itH_0}\psi\|\to 0(t\to\infty).$$

It follows that

$$\begin{aligned}(\phi_0(t), A_R\psi_0(t)) &= (\phi_0(t), R'|D_x|\psi_0(t)) + (\phi_0(t), (A_R - R'|D_x|)e^{-itH_0}\psi) \\ &= R'(\phi, |D_x|\psi) + (\phi_0(t), (A_R - R'|D_x|)e^{-itH_0}\psi) \\ &\longrightarrow R'(\phi, |D_x|\psi) \quad (t \to \infty). \end{aligned} \tag{4.9}$$

In the same way, we have

$$\begin{aligned}&(\phi_1(t), A_R\psi_1(t)) \\ &\longrightarrow R'(S\phi, |D_x|S\psi) = R'(\phi, |D_x|\psi) \quad (t \to \infty). \end{aligned} \tag{4.10}$$

Combining (4.8) with (4.9) and (4.10), we obtain (4.7). The lemma follows from (4.4), (4.6) and (4.7). □

Lemma 4.3. *For sufficiently large* R,

$$\int_{-\infty}^{\infty} (\phi_0(t), X_R H_0(1 - Z_R)\psi_0(t))dt = 0.$$

Proof. This is immediate since $(1 - Z_R)\psi = 0$ if $2/R \leqq \delta$. □

Lemma 4.4

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} (\phi(t), X_R H_0(1 - Z_R)\psi(t))dt = 0,$$

where the integral converges absolutely.

Proof. Since the integrand clearly converges to zero for each t, it is sufficient to show that the integral is dominated uniformly.

Let $M_R = X_R H_0(1 - Z_R)$. Then, similarly to (4.4) and (4.6), we see

$$|(\phi(t), M_R\psi(t))| \leqq \begin{cases} |(\phi_0(t), J_-^* M_R J_- \psi_0(t))| + \|(W_- - J_-)e^{-itH_0}\phi\| \, \|M_R\| \, \|\psi\| \\ + \|J_-^* M_R^*\| \, \|\phi\| \, \|(W_- - J_-)e^{-itH_0}\psi\| \\ |(\phi_1(t), J_+^* M_R J_+ \psi_1(t))| + \|(W_+ - J_+)e^{-itH_0}S\phi\| \, \|M_R\| \, \|\psi\| \\ + \|M_R^* J_+\| \, \|\phi\| \, \|(W_+ - J_+)e^{-itH_0}S\psi\|. \end{cases} \tag{4.11}$$

If $2/R \leqq \delta/4$, $M_R J_\pm = 0$ and by Corollary 3.2, we have

$$|(\phi(t), M_R\psi(t))| \leqq C\langle t\rangle^{-1-\varepsilon},$$

to conclude the assertion. □

Lemma 4.5

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \{(\phi(t), C_R\psi(t)) - (\phi_0(t), C_R\psi_0(t))\} \, dt = 0,$$

where the integral converges absolutely.

Proof. By Lemma 2.3, C_R weakly converges to zero and it is sufficient to show the dominated convergence, again.

On $(-\infty, 0)$, the integrand is

$$(\phi(t), C_R\psi(t)) - (\phi_0(t), C_R\psi_0(t)) = ((W_- - J_-)e^{-itH_0}\phi, C_R W_- e^{-iTH_0}\psi) + (C_R^* J_- e^{-itho}\phi, (W_- - J_-)e^{-itH_0}\psi) + (e^{-itH_0}\phi, (J_-^* C_R J_- - C_R)e^{-itH_0}\psi),$$

and the former two terms can be dominated in the same way as the last lemma using Proposition 2.2. We have

$$J_-^* C_R J_- - C_R = J_-^* [C_R, J_-] + (J_-^* J_- - 1)C_R$$

and by Lemmas 2.2, 3.1 and the asymptotic expansion theorem, the symbol of $[C_R, J_-]$ satisfy the assumption of Lemma 3.2 with $\mu = -1-\varepsilon$ ((−)-case) uniformly in R, if $|\xi| \geqq \delta$. Hence, by Lemma 3.2, Proposition 3.1-(ii) and Proposition 2.2-(iii),

$$|(e^{-itH_0}\phi, (J_-^* C_R J_- - C_R)e^{-itH_0}\psi)| \leqq \|J_- e^{-itH_0}\phi\| \, \|[C_R, j_-]e^{-itH_0}\psi\| + \|(J_-^* J_- - 1)e^{-itH_0}\phi\| \, \|C_R e^{-itH_0}\psi\| \leqq C\langle t\rangle^{-1-\varepsilon} \quad (t \leqq 0).$$

It follows immediately from Lemma 2.2 that the symbol of $C_R\ f(\delta, \Delta; H_0)$ satisfies the assumption of Lemma 3.2 with $\mu = 2$ ((+)-case) uniformly in R and

$$\|C_R e^{-itH_0}\psi\| \leqq C\langle t\rangle^{-2} (t \geqq 0).$$

Since

$$\begin{aligned}(\phi(t), C_R\psi(t)) &= ((W_+ - J_+)e^{-itH_0}S\phi, C_R W_+ \psi_1(t)) \\ &+ (C_R^* J_+ \phi_1(t), (W_+ - J_+)e^{-itH_0}S\psi) \\ &+ (J_+ \phi_1(t), (C_R J_+)e^{-itH_0}S\psi),\end{aligned}$$

and the symbol of $(C_R J_+)$ satisfies the same estimates as Lemma 2.2-(ii), we can conclude by Lemma 3.2 and Corollary 3.2,

$$\begin{aligned}|(\phi(t), C_R\psi(t))| &\leqq \|(W_+ - J_+)e^{-itH_0}S\phi\| \, \|C_R W_+ \psi_1(t)\| \\ &+ \|C_R^* J_+ \phi_1(t)\| \, \|(W_+ - J_+)e^{-itH_0}S\psi\| + \|J_+\phi(t)\| \, \|(C_R J_+)e^{-itH_0}S\psi\| \\ &\leqq C\langle t\rangle^{-1-\varepsilon} \quad (t \geqq 0).\end{aligned}$$

Thus the integrand is dominated uniformly in R. □

Lemma 4.6.

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \left(\phi(t), \left\{X_R V + \frac{i}{2}[A_R, V]\right\}\psi(t)\right) dt = \int_{-\infty}^{\infty} \left(\phi(t), \left\{V + \frac{i}{2}[A, V]\right\}\psi(t)\right) dt,$$

where the integrals converge absolutely.

Proof. For each t, clearly

$$\lim_{R\to\infty} (\phi(t), X_R V\psi(t)) = (\phi(t), V\psi(t)),$$

and by Proposition 2.2 and Lemma 2.3,

$$\begin{aligned}(\phi(t), [A_R, V]\psi(t)) &= (A_R^*\phi(t), V\psi(t)) - (V\phi(t), A_R\psi(t)) \\ &= ((\langle x\rangle^{-1} A_R^*)\phi(t), (\langle x\rangle V)\psi(t)) \\ &\quad - ((\langle x\rangle V)\phi(t), (\langle x\rangle^{-1} A_R)\psi(t))\end{aligned}$$

$$\xrightarrow[R\to\infty]{} \left(\langle x\rangle^{-1}\left(-\frac{1}{i}x\cdot\nabla-\frac{i}{2}(n-2)\right)\phi(t),(\langle x\rangle V)\psi(t)\right)$$

$$-\left(\langle x\rangle V\phi(t),\langle x\rangle^{-1}\left(-\frac{1}{i}x\cdot\nabla+\frac{i}{2}(n-2)\right)\psi(t)\right)$$

$$=(\phi(t),[A,V]\psi(t)).$$

It remains to prove the dominated convergence.

Since V is locally H-smooth, we have

$$\int_{-\infty}^{\infty}|(\phi(t),X_R V\psi(t))|dt\leqq C_{\phi\psi}<\infty.$$

Similarly to (4.11), we obtain

$$\langle x\rangle V_2\phi(x)=\{\langle x\rangle V_2(H+i)^{-1}\}e^{-itH}W_-(H_0+i)\phi$$
$$=\begin{cases}\{\langle x\rangle V_2(H+i)^{-1}\}\{J_-e^{-itH_0}(H_0+i)\phi+(W_--J_-)e^{-itH_0}(H_0+i)\phi\}\\ \{\langle x\rangle V_2(H+i)^{-1}\}\{J_+e^{-itH_0}S(H_0+i)\phi+(W_+-J_+)e^{-itH_0}S(H_0+i)\phi\}.\end{cases}$$

$$\|\langle x\rangle V\phi(t)\|$$
$$\leqq\begin{cases}\|\langle x\rangle V_2(H+i)^{-1}\langle x\rangle^{1+\varepsilon}\|\,\|\langle x\rangle^{-1-\varepsilon}Je^{-itH_0}(H_0+i)\phi\|\\ \qquad+\|\langle x\rangle V_2(H+i)^{-1}\|\,\|(W_--J_-)e^{-itH_0}(H_0+1)\phi\|\\ \|\langle x\rangle V_2(H+i)^{-1}\langle x\rangle^{1+\varepsilon}\|\,\|\langle x\rangle^{-1-\varepsilon}J_+e^{-itH_0}S(H_0+i)\phi\|\\ \qquad+\|\langle x\rangle V_2(H+i)^{-1}\|\,\|(W_+-J_+)e^{-itH_0}S(H_0+i)\phi\|.\end{cases}$$

Lemma 3.2 can be applied to $(\langle x\rangle^{-1-\varepsilon}J_\pm)$, and combining this with Corollary 3.2 we conclude

$$\|\langle x\rangle V_2\phi(t)\|\leqq C\langle t\rangle^{-1-\varepsilon}\qquad(t\in\mathbb{R}).$$

This implies

$$|(\phi(t),[A,V_2]\psi(t))|\leqq C\langle t\rangle^{-1-\varepsilon}\quad(t\in\mathbb{R})\tag{4.13}$$

by virtue of (4.12);

$$(\phi(t),[A_R,V_1]\psi(t))=((W_\pm-J_\pm)\phi_i(t),[A_R,V_1]\psi(t))$$
$$+(J_\pm\phi_i(t),[A_R,V_1](W_\pm-J_\pm)\psi_i(t))$$
$$+(J_\pm\phi_i(t),[A_R,V_1]J_\pm\psi_i(t)).$$

where $i=1/0$ for $(+)/(-)$ respectively. The former two terms can be dominated as above (we remark that $[A_R,V_1]=A_RV_1-V_1A_R$ is uniformly bounded in $B(H^1,L^2)$).

$$[A_R,V_1]J_\pm=[A_RJ_\pm,V_1]+A_R[V_1,J_\pm].$$

The symbol of $[A_RJ_\pm,V_1]$ ($[V_1,J_\pm]$ respectively) is in $S_{1,0}^{-1-\varepsilon}(\mathbb{R}^n_\xi)(S_{1,0}^{-2-2\varepsilon}(\mathbb{R}^n_\xi)$ respectively), and is bounded in R by Lemmas 2.2, 3.1 and Assumption (V). Hence

$$|(J_\pm\phi_i(t),[A_R,V_1]J_\pm\psi_i(t))|\leqq\|J_\pm\phi_i(t)\|\;\|[A_RJ_\pm,V_1]\psi_i(t)\|$$
$$+\|A_R^*\|_{B(H^1,L^{2,-1})}\|J_\pm\phi_i(t)\|_{H^1}\|\langle x\rangle[V_1,J_\pm]\psi_i(t)\|$$
$$\leqq C\langle t\rangle^{-1-\varepsilon}(t\in\mathbb{R}),$$

and these estimates follow

$$|(\phi(t),[A_R, V_1]\psi(t))| \leqq C\langle t\rangle^{-1-\varepsilon}(t\in\mathbb{R}). \tag{4.14}$$

Equations (4.13) and (4.14) prove the dominated convergence. □

Proof of Theorem 1. Combining (4.3) with Lemmas 4.2–4.6, we obtain

$$\lim_{R\to\infty}(\phi, T_R H_0\psi) = \lim_{R\to\infty}\left\{\int_{-\infty}^{\infty}(\phi(t), X_R H\psi(t))dt - \int_{-\infty}^{\infty}(\phi_0(t), X_R H_0\psi_0(t))dt\right\}$$

$$= \int_{-\infty}^{\infty}\left(\phi(t),\left\{V+\frac{i}{2}[A,V]\right\}\psi(t)\right)dt. \qquad \square$$

Acknowledgements. The author wishes to thank Professor K. Yajima and Professor H. Kitada for valuable discussions. In particular, Professor Kitada informed about the time-delay operator and Professor Yajima suggested to employ pseudo-differential operators. The author also thanks Professor S. T. Kuroda for various support.

References

1. Amrein, W. O., Jauch, J. M., Sinha, K. B.: Scattering theory in quantum mechanics. Reading, MA: W. A. Benjamin 1977
2. Enss, V.: Geometric methods in spectral and scattering theory. In: Velo G., Wightman, A. S. (eds.). Rigorous atomic and molecular physics. pp. 1–69. New York: Plenum Press 1981
3. Hörmander, L.: The analysis of linear partial differential operators. Vol. **I–IV**. Berlin, Heidelberg, New York: Springer 1983–1985
4. Isozaki, H., Kitada, H.: Scattering matrices for two-body Schrödinger operators. Sci. Pap. Col. Arts and Sci., Univ. Tokyo **35**, 81–107 (1985)
5. Jauch, J. M., Marchand, J. P.: The delay time operator for simple scattering systems. Helv. Phys. Acta **40**, 217–229 (1967)
6. Jauch, J. M., Sinha, K. B., Misra, B. N.: Time-delay in scattering processes. Helv. Phys. Acta **45**, 398–426 (1972)
7. Jensen, A.: Time-delay in potential scattering theory. Some "geometric" results. Commun. Math. Phys. **82**, 435–456 (1981)
8. Jensen, A.: On Lavine's formula for time-delay. Math. Scand. **54**, 253–261 (1984)
9. Jensen, A.: A stationary proof of Lavine's formula for time-delay. Lett. Math. Phys. **7**, 137–143 (1983)
10. Lavine, R.: Commutators and local decay. In: Lavita, J. A., Marchand J. P. (eds.). Scattering theory in mathematical physics. pp. 141–156. Dordrecht: Reidel 1974
11. Read, M., Simon, B.: Methods of modern mathematical physics, Vol. **I–IV**. New York: Academic Press 1971–1978
12. Taylor, M.: Pseudodifferential operators. Princeton Math. Series, New Jersey: Princeton University Press 1981
13. Martin, Ph.: Time-delay of quantum scattering process. Acta Phys. Austr. Suppl. **23**, 157–208 (1981)
14. Wang, X P.: Opérateurs de temps-retards dans la théorie de la diffuson. CR. Acad. Sci. Paris. Sér. I, **301**, 789–791 (1985)
15. Wang, X. P.: Low energy resolvent estimates and continuity of time-delay operators. preprint, University de Rennes I
16. Wang, X. P.: Phase-space description of time-delay in scattering theory. Preprint, University de Nantes

Communicated by B. Simon

Received July 25, 1986, in revised form September 23, 1986