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Non-Local Fields in the $Z(2)$ Higgs Model: The Global Gauge Symmetry Breaking and the Confinement Problem

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Abstract. Non-local order parameters are constructed in the $Z(2)$ Higgs model to probe the existence or non-existence of charged states. The non-local field corresponding to the Higgs field in a complete gauge fixing is found to be an order parameter for the locally unobservable global gauge symmetry breaking. This symmetry breakdown is shown to imply perfect screening for the bare charge.

1. Introduction

In gauge theories with matter fields the characterization of the confinement-deconfinement transition by means of an order parameter is a long-standing problem. Its difficulty is related to the fact that the charged states, if they exist, cannot be created from the vacuum locally, that is they do not belong to the vacuum sector. To deal with other sectors it is therefore indispensable to study the properties of the charged states. A proof of confinement would require even more, namely that one should have information about "all" the possible sectors of the theory. However this is quite an unusual task because there is no generally accepted method to construct and deal with representations of the quantum field algebra which are different from the vacuum representation. (But see [1–3] for an axiomatic treatment of the charged sectors in QED.) In the path integral formulation of the theory the problem manifests itself in the impossibility of finding a gauge invariant functional the expectation value of which would give any information about the charged states. Of course this does not exclude the possibility that the limit of an appropriate sequence of expectation values yet defines a charged state [4].

We propose a solution to these problems in the case of lattice regularized gauge field theories. The notion of the functional integral will be generalized maintaining its Gibbs state character in such a way that non-local fields with non-trivial infrared asymptotics will make sense. On the example of the $Z(2)$ Higgs model we can show that the non-local fields – though not integrable in any path integral measure – have all the physical properties to describe charged states. Our method also offers a framework to prove confinement because the non-local fields define a class of representations of the quantum field algebra which can be handled by the methods of classical statistical mechanics.

The charged sector of the $Z(2)$ Higgs model was at first constructed by Fredenhagen and Marcu [4]. There were proposals for an order parameter by Mack and Meyer [5] and by Bricmont and Fröhlich [6]. That non-local fields provide a method to analyse confinement and the structure of charged states in the $Z(2)$ Higgs model was announced in [7]. The present paper is but the detailed version of [7]. Some steps toward constructing the charged sector in the $U(1)$ Higgs model were made in [8].

Another important application of the non-local fields is the problem of global gauge symmetry breaking. It is well known that there is an apparent contradiction between the standard perturbative reasoning that the Higgs field must have a non-zero expectation value in order for the Higgs mechanism to work and the non-perturbative results related to Elitzur's theorem [9–11]. Certain non-local fields – corresponding to the Higgs field in a complete gauge fixing – will be shown to serve as order parameters for this symmetry breaking. However one must be careful when interpreting this as an observable transition because of the essential non-locality of the gauge. We have to emphasize also that we can single out a distinguished order parameter neither for the confinement-deconfinement transition nor for the global gauge symmetry breaking. The point is rather the construction of the class of fields which can "create" charges and/or can signal symmetry breaking.

It is intuitively clear what is the field corresponding to a one electron state in QED. Together with the bare electron one must create its Coulomb field too [12]:

$$\psi(x) = \Psi(x)\exp\{i\int d^3\mathbf{y}\mathbf{E}(\mathbf{y})\mathbf{A}(x^0,\mathbf{y})\}, \tag{1.1}$$

where Ψ and $\mathbf{A}$ are the usual electron and photon fields respectively. $\mathbf{E}$ is a parameter describing an electric field with charge density $\partial\mathbf{E}(\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$, in particular the Coulomb field. One expects that the Euclidean expectation value $\langle\bar{\psi}(x)\psi(y)\rangle$ is non-zero and that $\psi(x)$ really describes a state $\Phi(\psi(x))$ with finite energy. The precise connection between classical functionals and states is provided by the OS-construction [13–15]. This tells us that if reflection positivity is fulfilled for the expectation value $\langle\ \rangle$ then any field F supported in the $x^0>0$ half spacetime corresponds to a state $\Phi(F)$ with norm $\|\Phi(F)\| = \langle\theta(F)F\rangle^{1/2}$, where θ is the time reflection combined with complex (Dirac) conjugation. The matrix elements of local operators between $\Phi(\psi(x))$ and $\Phi(\psi(y))$ ($x^0>0$, $y^0>0$) can be obtained by calculating $\langle\bar{\psi}(-x^0,\mathbf{x})B\psi(y)\rangle$, where B is a local functional supported in the region $\{x'\in\mathbb{R}^4 | -x^0<x'^0<y^0\}$. Disregarding such delicate problems as the continuum limit these expectation values already define the one electron sector in QED. The representation on the subspace spanned by the vectors $\Phi(B\psi(x))$ (B is local, x is fixed) is supposedly translation covariant but not "Lorentz" [that is $O(4)$] covariant because of the necessary breaking of the boost symmetry [1]. A construction of the charged sector in this way presupposes, among others, the solution of the following technical problem. If the expectation value $\langle\dots\rangle$ is a functional integral $Z^{-1}\int\mathscr{D}A\mathscr{D}\bar{\psi}\mathscr{D}\psi\exp(-S)\dots$ then the field (1.1) has no meaning. Even on the lattice $\psi(x)$ is defined only on the zero measure set consisting of

fields A_μ vanishing faster than r^{-1} at infinity. Suppose that one somehow managed to extend $\psi(x)$ to all configurations A_μ maintaining of course its local but not global gauge invariance. Even then the sensitivity of $\psi(x)$ to the configuration at large distances – which is a consequence of Gauss' law – would contradict its integrability. This is because the notion of the path integral is specialized for the use of local fields which are insensitive to what is the configuration at far away regions of spacetime. In spite of this mathematical flaw the expectation value $\langle \ \rangle$ can be retained to be a Gibbs state on the (1.1)-type of fields too. The precise statements for the $Z(2)$ model can be found in Sect. 2.

On the lattice a generalization of (1.1) to arbitrary gauge groups and to arbitrary spacetime dimension d can be given [16]:

$$\psi^\alpha(x)=M^{\alpha\beta}(\mathbf{x}|U(x^0))\Psi^\beta(x)=\lim_{\Lambda\to\mathbb{Z}^{d-1}} Z_\Lambda^{-1}(U(x^0))\prod_{\mathbf{y}\in\Lambda\setminus\partial\Lambda}\int_G dg(\mathbf{y})D^{\alpha\beta}(g^{-1}(\mathbf{x}))$$
$$\times\exp\left\{\beta\sum_{\langle\mathbf{y},\mathbf{z}\rangle\subset\Lambda}\chi(g^{-1}(\mathbf{y})U((x^0,\mathbf{y}),(x^0,\mathbf{z}))g(\mathbf{z}))\right\}\Bigg|_{g|\partial\Lambda=1}\Psi^\beta(x). \qquad (1.2)$$

χ is a real character on the gauge group G, D is the matrix representation of G according to which Ψ transforms and dg is the Haar measure. That is, $M(\mathbf{x}|U(x^0))$ is the magnetization (at $\mathbf{x}$) in a $(d-1)$-dimensional nearest neighbour G-valued spin system with boundary condition $g=1$ and with frustrations given by the actual value of the gauge field U on the $x^0=\text{const}$ hyperplane. We will see in Sect. 2.3 that it is always enough to define a field on those gauge configurations U which satisfy $U=1$ on all but a finite number of links. So (1.2) is well defined. Equation (1.2) is locally gauge invariant but transforms according to the representation D under the action of the global gauge group. If β is large enough then $\psi(x)\neq 0$ in dimensions larger than a critical dimension determined by G and D. This already suggests that, for example, in $d=3$ the compact $U(1)$ Higgs model should confine because there is no magnetization in the planar model in 2-dimensions. But in $d=4$ it may have charged states. In the case of discrete gauge groups a similar argument suggests that there should be charged states in dimensions $d\geqq 3$.

This argument, of course, is very vague for two reasons. At first there can be charged states which are created by more sophisticated fields than (1.2). Secondly even if $\psi^\alpha(x)$ is non-zero the dynamics may force $\langle\theta(\psi^\alpha(x))\psi^\alpha(x)\rangle$ to vanish. In this case no non-zero vector in the Hilbert space can be associated to the field $\psi^\alpha(x)$. For instance in the $SU(2)_4$ gauge theory with a fundamental Higgs field (1.2) yields a non-zero charged field because there is spontaneous magnetization in the $O(4)_3$ spin model [17]. At the same time one expects that this model is confining.

Ending the discussion of (1.2) we mention that in the case of non-compact QED when G is the additive group $\mathbb{R}$, D is the representation $h\in\mathbb{R}\mapsto\exp(ih)$ and we choose $\chi(h)=-h^2/2$, then (1.2) reduces to the lattice version of (1.1) up to a constant.

Unfortunately the magnetization type ansatz (1.2) is not very practical for analytical calculations (but may be useful in numerical simulations). One can generalize Dirac's gauge invariant electron field (1.1) in another way which already leads to our second topic – the spontaneous breakdown of the global gauge symmetry. Notice that if $\mathbf{E}$ is the Coulomb field then $\psi(x)$ of (1.1) is nothing but $\Psi(x)$ in the Coulomb gauge. So it is natural to introduce the following type of charged fields:

$$\psi^{\alpha}(x) = \Omega^{\alpha\beta}(x|U)\varphi^{\beta}(x), \tag{1.3}$$

where φ is the original bosonic or fermionic matter field and $\Omega(\ |U)$ is the gauge transformation which transforms the actual U into a specific gauge. Equation (1.3) is locally gauge invariant but globally not just like (1.1) or (1.2). Besides the possibility that (1.3) may define a charged state it is interesting to see whether $\langle\psi^{\alpha}(x)\rangle$ is non-zero in certain gauges or – if $\langle\ \rangle$ is globally gauge symmetric – whether $\langle\psi^{\alpha}(x)\psi^{\beta}(y)\rangle$ fails to cluster, i.e. $\lim_{y\to\infty}\langle\psi^{\alpha}(x)\bar{\psi}^{\beta}(y)\rangle = h^2\delta^{\alpha\beta}$, $h \neq 0$. This happens e.g. in the 4-dimensional Stückelberg model if the gauge is chosen to be the Coulomb gauge [18]. Kennedy and King were able to demonstrate [19] similar phenomenon in the Abelian Higgs model in the Landau gauge. (See also [20, 21].)

What are the consequences of such a symmetry breakdown in quantum theory? At first we have to point out the essential non-locality of the functional (1.3) which forbids interpreting $\langle\psi(x)\rangle$ as the vacuum expectation value of any local or quasi-local operator. This fact may be concealed in a gauge fixed formalism but is explicitly seen in the gauge invariant formulation where the Higgs field of any complete gauge fixing appears as a non-local field (1.3). There remains the possibility that, nevertheless, the state $\Phi(\psi^{\alpha}(x))$ describes some new physical situation. What we mean is that the expectation values of local operators in the state $\Phi(\psi^{\alpha}(x))$ cannot be reproduced by any state from the vacuum sector. This would mean that $\Phi(\psi^{\alpha}(x))$ has a non-zero component lying in a new sector. [This is the expected behaviour when there is no symmetry breaking, but $\Phi(\psi^{\alpha}(x))$ is a charged state.] However this possibility is also excluded by the following argument. Because of the lack of clustering of the non-local fields in $\langle\ \rangle$ the limit $h^{-1}\lim_{n\to\infty}\mathbf{T}^n\Phi(\psi^{\alpha}(x)) = h^{-1}\lim_{n\to\infty}\Phi(\psi^{\alpha}(x+n\hat{0}))$ ($\mathbf{T}$ is the transfer matrix) exists and defines a unit vector Φ^{α}_{∞}. The expectation values of local operators in Φ^{α}_{∞} exactly agree with those in the vacuum because of clustering between local fields and the non-local ones:

$$(\Phi^{\alpha}_{\infty}, \mathfrak{a}\Phi^{\alpha}_{\infty}) = \langle A(\mathfrak{a})\rangle,$$

where $A(\mathfrak{a})$ is the classical functional corresponding to the operator $\mathfrak{a}$. Consequently Φ^{α}_{∞}, though it seems to be charged, generates the chargeless vacuum state. This means that the naive charge of Φ^{α}_{∞} is completely screened by dynamical effects and no observable transition can be associated to the breaking of the global gauge symmetry. This picture is supported by the fact that non-local fields signal a symmetry breaking transition even in the confinement-screening phase of the $Z(2)$ Higgs model where analyticity of the expectation values of all quasilocal fields is well known [14, 22].

We emphasize however that the non-observability of the global gauge symmetry breaking does not mean the non-observability of the Higgs mechanism in general. In more complicated models with larger gauge groups and containing more Higgs fields and/or fermions it may well happen that there exist local fields which can exhibit the presence of a Higgs mechanism. In these models the form of the gauge covariant local fields corresponding to the one particle states may change with the couplings and therefore yield an observable phase transition [11].

Summarizing from the point of view of non-local charged fields, three possibilities can be distinguished [18]: ($\langle\ \rangle$ is globally and locally gauge invariant and pure when restricted to the quasilocal fields.)

I. (Absolute confinement) Any charged field F corresponds to the zero norm state: $\langle\theta(F)F\rangle = 0$.

II. (Confinement by perfect screening) There exist charged fields F with $\langle\theta(F)F\rangle \neq 0$, but for all such F the clustering breaks down: $\lim_{n\to\infty}\langle\theta(F)\mathrm{T}^n F\rangle = h^2 > 0$.

III. (Existence of "free" charges) There exists a charged field F such that $\langle\theta(F)F\rangle \neq 0$ and $\lim_{n\to\infty}\langle\theta(F)\mathrm{T}^n F\rangle = 0$. Furthermore the physical charge measured in finite but arbitrary large volumes gives different results in the state $\Phi(F)$ and in the vacuum $\Phi(1)$.

One can add to III the conditions of finite energy, translation covariance and even a kind of gauge independence of the sector of $\Phi(F)$ if F is of type (1.3). These will be discussed in detail for the $Z(2)$ model.

As we have already mentioned the difference between I and II is not observable. In spite of this to distinguish them is conceptually interesting because it gives a natural definition of screening. (The screening itself is not observable because it relates an observable quantity, the physical charge, to an unobservable one, the bare charge.)

For the $Z(2)$ model with action $S = -\beta\sum_p U(\partial p) - \alpha\sum_\ell U(\ell)\varphi(\partial\ell)$ ($U(\ell)$, $\varphi(x)\in Z(2)$, $\alpha>0$, $\beta>0$) the regions corresponding to the cases I, II, and III are depicted in Fig. 1.

The paper is organized as follows. In Sect. 2 we introduce the notion of the non-local fields in the $Z(2)$ Higgs model and explain why the concept of functional integration has to be generalized. In Sect. 3 the consequences of such a generalized classical system to the quantum theory are outlined. We show how inequivalent representations of the quantum field algebra can emerge if non-local fields are included in the construction of the OS-Hilbert space. The remaining part of the

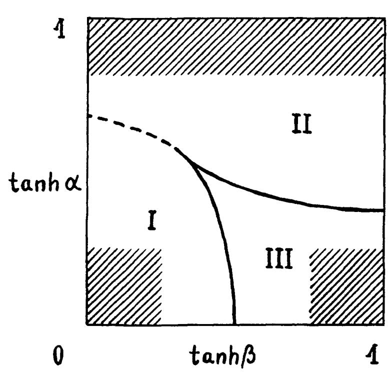

Fig. 1. The qualitative phase diagram of the $Z(2)$ Higgs model. Locally observable (solid lines) and unobservable (dashed line) transitions. The shaded regions are controlled by a cluster expansion for the non-local fields too

paper takes advantage of the several cluster expansions available in the $Z(2)$ Higgs model. In Sect. 4 we prove confinement for a large class of non-local fields when both α and β are small. The construction of the charged sector in the small α large β region is carried out in Sect. 5. We obtain a charged sector which is gauge independent and translation covariant without containing any translation invariant vector. Therefore no irreducible subrepresentation exists in this sector which would be unitarily equivalent to the vacuum representation. We also show that this charged representation contains, if not equal to, the representation constructed by Fredenhagen and Marcu. In Sect. 6 we prove that there is spontaneous symmetry breaking in the non-local classical system $(\mathscr{A}(\psi), \mathfrak{s}(\alpha,\beta))$ consisting of the non-local classical field algebra $\mathscr{A}(\psi)$ obtained from the quasilocal algebra by adjoining to it the fields ψ of the form (1.3) and from the Gibbs states $\mathfrak{s}(\alpha,\beta)$ constructed via different boundary conditions but with fixed couplings α and β. At last we show that this phase transition has no observable consequences in the local quantum system.

2. The Emergence of Non-Local Fields

2.1. Some Basic Definitions

Consider a d-dimensional cubic lattice $\mathbb{Z}^d$ and let $\sum$ denote the set of its simplexes: sites, links, plaquettes, ... etc. A simplex $\xi \in \sum$ can be regarded as a vector $\xi \in \frac{1}{2}\mathbb{Z}^d$. The set of p-dimensional simplexes is denoted by $\sum^p$ $p=0,1,\ldots,d$. The finite subsets of $\sum^p$, respectively of $\sum$ will be called *p-chains*, respectively *chains*. The set of p-chains $\mathbf{C}^p$ equipped with the symmetric difference operation Δ forms an Abelian group. One can introduce the usual *boundary* and *coboundary operations* ∂ *and* d as

$$H \in \mathbf{C}^p \mapsto \partial H = \mathop{\Delta}_{\xi \in H} \partial\xi \in \mathbf{C}^{p-1}, \qquad \partial\xi = \{\eta \in \textstyle\sum^{p-1} \,\|\, |\eta - \xi| = 1/2\},$$

$$H \in \mathbf{C}^p \mapsto dH = \mathop{\Delta}_{\xi \in H} d\xi \in \mathbf{C}^{p+1}, \qquad d\xi = \{\eta \in \textstyle\sum^{p+1} \,\|\, |\eta - \xi| = 1/2\}$$

with the property $\partial^2 = d^2 = 0$. The *distance* between simplexes is defined as $\mathrm{dist}(\xi,\eta) = |\xi - \eta| = \sum_{0 \le \mu \le d-1} |\xi^\mu - \eta^\mu|$.

We will frequently use the notion of open and closed subsets of $\sum$. $\Lambda \subset \sum$ is called *open (closed)* if $\xi \in \Lambda$ implies that $d\xi \subset \Lambda$ (respectively $\partial\xi \subset \Lambda$). An open or closed subset Λ is always a *subcomplex* of $\sum$, that is the boundary and coboundary operations restricted to Λ

$$\partial^\Lambda H = \Lambda \cap \partial(\Lambda \cap H), \qquad d^\Lambda H = \Lambda \cap d(\Lambda \cap H),$$

obey $(\partial^\Lambda)^2 = (d^\Lambda)^2 = 0$. The smallest closed set $\bar{\Lambda}$ containing a $\Lambda \subset \sum$ will be called the closure of Λ.

We introduce the following notations for special subsets of $\sum$:

$$\textstyle\sum(\geqq t) = \{\xi \in \sum \,|\, \xi^0 \geqq t\}, \ \sum(t) = \{\xi \in \sum \,|\, \xi^0 = t\} \ t \in \frac{1}{2}\mathbb{Z}.$$

$\sum(>t)$, $\sum^p(\geqq t)$, ... are defined analogously. Furthermore let $\sum_+ = \sum(>0)$, $\sum_- = \sum(<0)$, $\sum_0 = \sum(0)$, $\sum([t_1,t_2]) = \sum(\geqq t_1) \cap \sum(\leqq t_2)$.

For any set $\Lambda \subset \sum$, $\Lambda^p = \Lambda \cap \sum^p$, $\Lambda_t = \Lambda \cap \sum(t)$ and $\Lambda_t^p = \Lambda \cap \sum^p(t)$.

The solutions of the equation $\partial H = \emptyset$, $H \in \mathsf{C}^p$ are called *p-cycles* while that of $dH = \emptyset$, $H \in \mathsf{C}^p$ are called *p-cocycles*. A p-cycle (p-cocycle) H can always be written as $H = \partial K (H = dK)$ in our case.

Two simplexes $\xi, \eta \in \Sigma$ are called *connected (co-connected)* if $(\xi \cup \partial\xi) \cap (\eta \cup \partial\eta) \neq \emptyset$ (respectively $(\xi \cup d\xi) \cap (\eta \cup d\eta) \neq \emptyset$), and this relation is denoted by $\xi \vee \eta$ (respectively $\xi \wedge \eta$). These relations extend in a natural way to a relation $H \vee K$ and $H \wedge K$ on chains H, K. An important property is that each connected (co-connected) component of a p-cycle (p-cocycle) is again a p-cycle (p-cocycle).

We introduce the "bilinear form" $(H;K) = (-1)^{|H \cap K|}$ $H, K \in \mathsf{C}^p$, where $|H|$ denotes the cardinality of the set H. Then one has the relation $(H; \partial K) = (dH; K)$.

The group theoretical dual C_p of C^p consists of homomorphysms $\sigma : \mathsf{C}^p \to U(1)$ called p-cochains. $\sigma \in \mathsf{C}_p$ can be naturally identified with a $Z(2)$-valued function on $\sum^p$ by

$$\sigma(H) = \prod_{\xi \in H} \sigma(\xi), \qquad H \in \mathsf{C}^p.$$

The boundary and coboundary operations on C_p are defined as $\partial\sigma(H) = \sigma(dH)$ and $d\sigma(H) = \sigma(\partial H)$ respectively. They obey $d^2 = \partial^2 = 1$. The *support* of a cochain σ is $\mathrm{Supp}\,\sigma = \{\xi \in \sum | \sigma(\xi) = -1\}$.

In a $Z(2)$ gauge-matter system the *configuration space* is $\mathscr{C} = \mathsf{C}_0 \times \mathsf{C}_1$ consisting of $Z(2)$-valued functions $\sigma = (\varphi, U)$ on $\sum^0 \cup \sum^1$ ($\varphi \in \mathsf{C}_0$, $U \in \mathsf{C}_1$). The topology on $\mathscr{C}$ is the pointwise convergence topology with basis of neighbourhoods

$$\mathscr{V}(\sigma, \Lambda) = \{\sigma' \in \mathscr{C} | \sigma'(\xi) = \sigma(\xi)\, \xi \in \Lambda\} \tag{2.1}$$

with some finite subset Λ of $\sum$. In this topology $\mathscr{C}$ becomes a compact Haussdorf space.

The *local gauge transformation* on $\mathscr{C}$ is a continuous transformation defined by

$$\sigma \mapsto \sigma^\Omega = (\varphi^\Omega, U^\Omega) \quad \begin{matrix} (\varphi^\Omega)(x) = \varphi(x)\Omega(x) & x \in \sum^0, \\ (U^\Omega)(\ell) = U(\ell) d\Omega(\ell) & \ell \in \sum^1, \end{matrix}$$

where $\Omega \in \mathsf{C}_0$ has a finite support. The *global gauge transformation* is

$$\sigma \mapsto \sigma^C = (\varphi^C, U) \qquad (\varphi^C)(x) = -\varphi(x)$$

which is continuous too.

The *fields* are bounded complex functions $F : \mathscr{C} \to \mathbb{C}$ and are equipped with the sup-norm topology. The local and global gauge transformations on the fields defined by

$$F \mapsto \Omega F, (\Omega F)(\sigma) = F(\sigma^\Omega)$$
$$F \mapsto CF, (CF)(\sigma) = F(\sigma^C)$$

are isometries.

The *spacetime support* or *sensitivity region* $\mathrm{Sr}\, F$ of a field F is defined as the smallest subset $\Lambda \subset \sum$ which satisfies: $\sigma \restriction \Lambda = \sigma' \restriction \Lambda \Rightarrow F(\sigma) = F(\sigma')$ for any $\sigma, \sigma' \in \mathscr{C}$. The algebra containing all fields with $\mathrm{Sr}\, F \subset \Lambda$ is denoted by $\mathscr{A}(\Lambda)$. $\mathscr{A}_0 = \cup \{\mathscr{A}(\Lambda) | \Lambda \subset \sum, |\Lambda| < \infty\}$ is called the (classical) *local field algebra* and its closure $\mathscr{A} = \bar{\mathscr{A}}_0$ the *quasilocal algebra*. It is easy to see that $\mathscr{A}$ coincides with the C^*-algebra of continuous complex functions on $\mathscr{C}$. The closed $*$-subalgebra $\mathscr{A}^{\mathrm{inv}}$ of $\mathscr{A}$ consisting of locally gauge invariant fields plays the role of observables in the classical system. In this paper we will deal with only gauge invariant fields so, in order to economize the notation, let $\mathscr{A}(\Lambda)$, $\mathscr{A}_0$, $\mathscr{A}$ denote already the locally gauge invariant part of what they were above.

2.2. No-Go Theorems for Constructing Charged Fields in the Functional Integral

The fields "creating" charge in the functional integral must be locally gauge invariant on the one hand and must be charged, i.e. must obey $CF = -F$, on the other hand. These two requirements, however, have severe consequences on the mathematical properties of the field F, in particular it forbids F to be smooth. What is more the existence of such a field is incompatible with the notion of the functional integral in an infinite volume system. At first we examine the continuity properties of the charged fields.

Theorem 2.1. *Let* $F:\mathscr{C}\to\mathbb{C}$ be locally gauge invariant and charged, i.e. $CF = -F$. *Then F can be continuous only at those points $\sigma\in\mathscr{C}$ where it vanishes. In particular, each quasilocal field $F\in\mathscr{A}$ obeys $CF=F$.*

The proof is very elementary and left to the reader. We mention only that the fact that all $F\in\mathscr{A}$ have trivial charge expresses some kind of a superselection rule on the classical level. It shows also that there is no observable order parameter for the global gauge symmetry breaking.

Having been reconciled to nowhere continuity one still can search charged fields among the Borel measurable functions on $\mathscr{C}$. The following theorem, however, ruins such hopes.

Theorem 2.2. *Let $(\mathscr{C},\mathfrak{s})$ be the measurable space with the σ-algebra $\mathfrak{s}$ generated by the open sets $\mathscr{V}(\sigma,\Lambda)$ $\sigma\in\mathscr{C}$, $\Lambda\subset\sum$, $|\Lambda|<\infty$ (defined in* (2.1)*) as the algebra of measurable sets. Let $F:\mathscr{C}\to\mathbb{C}$ be measurable and locally gauge invariant. Suppose that $CF=-F$. Then F vanishes almost everywhere.*

Remark. It is this measurable space $(\mathscr{C},\mathfrak{s})$ on which one usually defines the a priori and the physical measures. We formulated the theorem for the $Z(2)$ model but the generalization to arbitrary gauge groups is fairly trivial.

Proof. It is enough to prove the statement for real F. Let $\mathscr{C}_+=F^{-1}((0,\infty))$, $\mathscr{C}_-=F^{-1}((-\infty,0))$ and $\mathscr{C}_0=F^{-1}(\{0\})$. The following implications hold:

$$\sigma\in\mathscr{C}_+ \Rightarrow F(\sigma^C)=CF(\sigma)=-F(\sigma)\in(-\infty,0) \Rightarrow \sigma^C\in\mathscr{C}_-,$$

$$\sigma\in\mathscr{C}_+ \Rightarrow F(\sigma^\Omega)=\Omega F(\sigma)=F(\sigma)\in(0,\infty) \Rightarrow \sigma^\Omega\in\mathscr{C}_+.$$

Now suppose that $\mathscr{V}(\sigma,\Lambda)\subset\mathscr{C}_+$. Then $\sigma\in\mathscr{C}_+$ and $\sigma^{C\Omega}\in\mathscr{C}_+$ if $\mathrm{Supp}\,\Omega\supset\Lambda$. On the other hand $\sigma^\Omega\in\mathscr{C}_+$, therefore $\sigma^{\Omega C}=(\sigma^\Omega)^C\in\mathscr{C}_-$, which is a contradiction. Therefore no set of the form $\mathscr{V}(\sigma,\Lambda)$ is contained in $\mathscr{C}_+$ and in $\mathscr{C}_-$ by symmetry. If F is measurable, then $\mathscr{C}_0\cup\mathscr{C}_-\in\mathfrak{s}$. Since the unique covering of $\mathscr{C}_0\cup\mathscr{C}_-$ with sets from the ring generated by $\mathscr{C}$ and the $\mathscr{V}(\sigma,\Lambda)$'s is $\mathscr{C}\supset\mathscr{C}_0\cup\mathscr{C}_-$ we have for any finite measure μ on $\mathfrak{s}$ that $\mu(\mathscr{C}_0\cup\mathscr{C}_-)=\mu(\mathscr{C})$ and analogously $\mu(\mathscr{C}_0\cup\mathscr{C}_+)=\mu(\mathscr{C})$. This proves that $\mu(\mathscr{C}_+)=\mu(\mathscr{C}_-)=0$. □

For illustration let us mention the charged field $\varphi(x)U(J_x)$, where $J_x \subset \sum^1$ and $\partial J_x = \{x\}$ which is locally gauge invariant but globally not. This field can be defined only on the zero measure set consisting of configurations (φ, U) with $\operatorname{Supp} U$ finite. Even on this subset of $\mathscr{C}$, $\varphi(x)U(J_x)$ is nowhere continuous.

2.3. Non-Local Fields and the Extended Gibbs States

The way out of the problems encountered above we are going to propose is very simple. What caused the trouble was our insistence on distinguishing configurations which are different only at very large distances. Pushing to extremes we can say that the problem is the contradiction between the Gauss law and the topology of $\mathscr{C}$ suggested by locality. However locality is a requirement on observables and not on the charged fields. So we will abandon the concept that a field is a function on $\mathscr{C}$. From now on a field will mean the following.

Definition 2.3. A field F is a bounded locally gauge invariant function $F: \mathscr{C}' \to \mathbb{C}$, where $\mathscr{C}'$ is the restricted configuration space consisting of configurations $\sigma = (\varphi, U) \in \mathscr{C}$ such that $|\operatorname{Supp} U| < \infty$ and either $|\operatorname{Supp} \varphi| < \infty$ or $|\operatorname{Supp}(-\varphi)| < \infty$. The topology on $\mathscr{C}'$ is the one inherited from $\mathscr{C}$.

Remark. $\mathscr{C}'$ is invariant under the group of local and global gauge transformations. At the same time, up to infinitely supported gauge transformations, it is the space of configurations which have finite actions relative to the action of (1,1). $\varphi(x)U(J_x)$ is now a well defined field.

The subspace of continuous fields on $\mathscr{C}'$ is homeomorphic to $\mathscr{A}$ by continuous extension and they will be identified in the sequel. The same remark holds for $\mathscr{A}(\Lambda)$ and $\mathscr{A}_0$. The fields which are not in $\mathscr{A}$ will be referred to as non-local fields.

Our next task is to define the averaging procedure on the fields, that is the extended Gibbs states. This extension does not mean more than that the boundary condition now affects the whole complement Λ^c of the finite volume Λ. Let Λ be a finite open subset in $\sum$ and define for any field F the following Gibbs states:

$$\langle F \rangle_\Lambda^+ = \frac{1}{Z_\Lambda^+} \sum_{\sigma \in \mathscr{C}(\Lambda)} e^{-S_\Lambda(\sigma)} F(\sigma), \tag{2.2}$$

$$\langle F \rangle_\Lambda^- = \langle CF \rangle_\Lambda^+, \tag{2.3}$$

$$\langle F \rangle_\Lambda = \tfrac{1}{2} \langle F \rangle_\Lambda^+ + \tfrac{1}{2} \langle F \rangle_\Lambda^-, \tag{2.4}$$

where

$$\mathscr{C}(\Lambda) = \{\sigma \in \mathscr{C} | \sigma(\xi) = 1 \ \xi \in (\Lambda^c)^0 \cup (\Lambda^c)^1\},$$

and

$$S_\Lambda(\sigma) = -\alpha \sum_{\ell \in \Lambda^1} U(\ell)\varphi(\partial \ell) - \beta \sum_{p \in \Lambda^2} U(\partial p).$$

The freezing of U to be 1 at least on the timelike links of $\partial\Lambda$ expresses the physisical situation that there are external charges in the wall $\partial\Lambda$ which can screen the electric flux created by a charged field. Without this the propagator of the charged fields to be introduced in Sect. 5 would be identically zero.

These Gibbs states are of course continuous positive linear functionals with norm 1, i.e. states, on the C^*-algebra of bounded functions on $\mathscr{C}'$. Therefore, if restricted to $\mathscr{A}$, the finite volume Gibbs states and their thermodynamical limits too (which exist on $\mathscr{A}$ as a consequence of Griffiths inequalities) can be represented as an integral with respect to a regular Borel probability measure. If the thermodynamical limit $\langle\ \rangle = \lim_{\Lambda\to\Sigma} \langle\ \rangle_\Lambda$ exists on a space larger than $\mathscr{A}$ and including charged fields then such an integral representation for $\langle\ \rangle$ does not exist any more because of Theorem (2.2). Throughout the paper the thermodynamical limit $\Lambda \to \sum$ will mean the pointwise convergence of the characteristic function of Λ to that of $\sum$.

Since our primary interest is on the existence of charges it seems unnecessary to consider fields which are non-local in the Higgs field φ too. The largest space of fields we are going to study concretely (Sect. 4) is the C^*-algebra $\mathscr{B}$ of fields quasilocal in φ. More precisely $\mathscr{B}$ is the closure in the supnorm of $\mathscr{B}_0$ the algebra of fields F such that $|\mathrm{Sr}\, F \cap \sum^0| < \infty$. Especially $\mathscr{B}$ contains the string $\varphi(x)U(J_x)$ and also the Higgs field in any gauge where the gauge condition contains only the U's.

3. Representations of the Quantum Field Algebra on the OS-Hilbert Space

The aim of this section is to show that the extended non-local classical system after quantization is capable of describing various inequivalent representations of the quantum field algebra. The strategy we are going to outline is applicable not only to the $Z(2)$ Higgs model but to a large class of gauge matter systems as well.

Let $\mathscr{B}^g$ be the Banach space of all (non-local) fields on which the given Gibbs state $\langle\ \rangle$ exists. (The superscript g refers to the couplings and boundary conditions being implicit in $\langle\ \rangle$.) The Hilbert space $\mathscr{H}^g$ constructed à la Osterwalder and Seiler [14] from $\mathscr{B}^g$ carries a natural representation of the quantum field algebra which, for certain g's, may be reducible. Certain sectors, perhaps all sectors, of $\mathscr{H}^g$ can be characterized by the common asymptotic behaviour of the classical fields they are built from. Let us see what are the assumptions which lead to this picture.

Assumption 1. The finite volume Gibbs states $\langle\ \rangle_\Lambda$ are constructed from an action S_Λ describing a gauge matter system with nearest neighbour interaction and is supposed to be reflection positive to all $x^0 = \mathrm{const} \in \frac{1}{2}\mathbb{Z}$ hyperplanes: If f is a (locally gauge invariant) field with $\mathrm{Sr}\, f \subset \sum(>x^0)$ and $\theta(x^0)\Lambda = \Lambda$, then $\langle(\theta(x^0)f)f\rangle_\Lambda \geqq 0$. Here $\theta(x^0)$ is the reflection to the x^0-hyperplane multiplied by complex (Dirac) conjugation in the case of bosons (respectively fermions). We use the convention of ref. [14] that the lattice is embedded into $\mathbb{R}^d$ as $\mathbb{Z}^d + \frac{1}{2}\hat{0}$.

Assumption 2. $\mathscr{B}^g$ is closed under the transformations listed below and $\langle\ \rangle$ is symmetric (respectively conjugate symmetric) under them: Spacetime translations T_μ $\mu = 0, 1, \ldots, d-1$, rotations $\mathsf{R}_{\mu\nu}$ $1 \leqq \mu < \nu \leqq d-1$ leaving the time axis invariant, $\theta = \theta(0)$ and possibly some global internal symmetries of S_Λ. All elements of $\mathscr{B}^g$ are locally gauge invariant. Let $\mathscr{B}^g_\pm = \{F \in \mathscr{B}^g | \mathrm{Sr}\, F \subset \sum_\pm\}$, then $\mathscr{B}^g_- \mathscr{B}^g_+ \subset \mathscr{B}^g$. Let $\mathscr{A}$ be the quasilocal algebra, then $\mathscr{A}\mathscr{B}^g \subset \mathscr{B}^g$.

It is easy to see that the above assumptions admit the following construction. Define a positive semidefinite bilinear form on $\mathscr{B}^g_+$ as: $(B_1, B_2)=\langle\theta(B_1)B_2\rangle$. Then $\mathscr{H}^g=\overline{\mathscr{B}^g_+/\mathscr{N}}$, where $\mathscr{N}=\{N\in\mathscr{B}^g_+|(N,N)=0\}$, is a Hilbert space of locally gauge invariant states. The Euclidean dynamics is given by the transfer matrix **T**. **T** is defined on the dense set $\mathscr{B}^g_+/\mathscr{N}$ as

$$\mathbf{T}\Phi(B)=\Phi(\mathsf{T}(B)) \qquad \mathsf{T}\equiv\mathsf{T}_0, \tag{3.1}$$

where $\Phi(B)=B+\mathscr{N}$, $B\in\mathscr{B}^g_+$. From reflection positivity to the $x^0=1/2$ hyperplane **T** is positive. Due to translation invariance of $\langle\ \rangle$ **T** is symmetric and a repeated application of Schwartz inequality gives [23] that $\mathbf{T}\leqq\mathbb{1}$, so it can be extended to the whole $\mathscr{H}^g$ continuously. One can similarly define the space translation unitary group $\{\mathbf{U}(\mathbf{x})|\mathbf{x}\in\mathbb{Z}^{d-1}\}$ as

$$\mathbf{U}(\mathbf{x})\Phi(B)=\Phi(\mathsf{T}^{(0,\mathbf{x})}(B)), \tag{3.2}$$

where $\mathsf{T}^x=\prod_{\mu=0}^{d-1}\mathsf{T}_\mu^{x_\mu}$. If C is an internal global symmetry then the corresponding operator is

$$\mathbf{C}\Phi(B)=\Phi(C(B)). \tag{3.3}$$

The state $\Omega=\Phi(1)$ is invariant under all of these operations and is called the vacuum.

Assumption 3. $\mathsf{T}\mathscr{B}^g_+$ is dense in $\mathscr{B}^g_+$ in the topology provided by the physical norm: $|||B|||=\|\Phi(B)\|=\langle\theta(B)B\rangle^{1/2}$.

This assumption not only ensures the finiteness of energy (T has a densely defined inverse and no zero eigenvalue) but also admits a natural definition of the quantum field algebra.

For all $A\in\mathscr{A}(\Lambda)$ ($\Lambda\subset\sum_+, |\Lambda|<\infty$) we define at first an operator $\mathfrak{a}(A)$ on the dense set $\mathsf{T}^n\mathscr{B}^g_+$ as

$$\mathfrak{a}(A)\Phi(B)=\Phi(AB) \tag{3.4}$$

if n is so large that $\sum(>n)\cap\Lambda=\emptyset$. Equation (3.4) is a unique definition because $B\mapsto AB$ maps $\mathsf{T}^n\mathscr{B}^g_+\cap\mathscr{N}$ into $\mathscr{N}$. In order that we could extend $\mathfrak{a}(A)$ to the whole $\mathscr{H}^g$ we need boundedness of $\mathfrak{a}(A)$ which will follow from

Assumption 4. Every operator $\mathfrak{a}(A)$ if $A\in\mathscr{A}(\Lambda)$, $\Lambda\subset\Sigma_+$, $|\Lambda|<\infty$ can be expressed in terms of the canonical operators $\hat{\sigma}(\xi)$, $\xi\in\mathsf{V}_-(\Lambda)\cap\sum(1/2)$ and $\hat{\tau}(\eta)$, $\eta\in\mathsf{V}_-(\Lambda)\cap\sum(0)$, where

$$\hat{\sigma}(\xi)B=\sigma(\xi)B,$$
$$\hat{\tau}(\eta)B=\lim_{\Xi\to\Sigma_+}e^{S_\Xi}\tau(\eta+\tfrac{1}{2}\hat{0})e^{-S_\Xi}B, \qquad B\in\mathscr{B}^g_+.$$

$\sigma(\xi)$ denotes the, if necessary, compactified variables living on the simplex ξ and $\tau(\xi)$ runs over the elements of the transitive transformation group acting on the space of $\sigma(\xi)$ and leaving the a priori measure at ξ invariant. If $\sigma(\xi)$ is a Grassmann variable then $\tau(\xi)$ is simply the Grassmannian derivation $\delta/\delta\sigma(\xi)$. $\mathsf{V}_-(\Lambda)$ is the "causal" past of Λ, i.e.

$$\mathsf{V}_-(\Lambda)=\bigcup_{\xi\in\Lambda}\mathsf{V}_-(\xi),$$
$$\mathsf{V}_-(\xi)=\left\{\eta\in\Sigma\,\middle|\,\sum_{\mu=1}^{d-1}|\eta^\mu-\xi^\mu|\leqq\xi^0-\eta^0\right\}.$$

It is easy to see that $\hat{\sigma}(\xi)$ and $\hat{\tau}(\xi)$ leave $\mathcal{N}$ invariant. The corresponding operators on $\mathcal{B}^g_+/\mathcal{N}$ will be denoted by the same letters.

This Euclidean "causality" and Markov property of the time evolution can be verified for models with nearest neighbour interaction if the action is not degenerate with respect to the algebra $\mathcal{A}$. That is, if S_Λ cannot be expressed in terms of fields from $\mathcal{A}'(\xi)$ ($\xi \in \Lambda$) with a non-trivial subalgebra $\mathcal{A}'(\xi)$ of $\mathcal{A}(\{\xi\})$. The proof is based on Schwinger-Dyson equations. One sequentially cancels the dependence of A from variables at constant time hyperplanes – going backward in time – by adding to A zero norm vectors of the form $[D(\xi)-1]A'$. Here $D(\xi)$ is the Schwinger-Dyson operator,

$$D(\xi)= \lim_{\Lambda\to\Sigma} e^{S_\Lambda}\tau(\xi)e^{-S_\Lambda}. \tag{3.5}$$

If $\xi^0 \geqq 1$ then $[D(\xi)-1]:\mathcal{B}^g_+ \to \mathcal{N}$.

After this the boundedness of $\mathfrak{a}(A)$ follows from boundedness of $\sigma(\xi)$ and $\tau(\xi)$ on the one-simplex-Hilbert space spanned by functions Φ of $\sigma(\xi)$ with the scalar product

$$(\Phi_1, \Phi_2)_\eta = \int d\sigma(\eta_-)d\sigma(\eta_+)\overline{\Phi_1(\sigma(\eta_-))}e^{-S'_{\{\eta\}}}\Phi_2(\sigma(\eta_+)).$$

$\eta_\pm = \eta \pm \frac{1}{2}\hat{0}$, η is a timelike link or plaquette and $S'_{\{\eta\}}$ is the temporal gauge form of $S_{\{\eta\}}$.

In this way we have shown that $\mathfrak{U}(\Lambda)=\{\mathfrak{a}(A)|A\in\mathcal{A}(\Lambda)\}$, $\Lambda\subset\Sigma_+$ is a linear space of bounded operators on $\mathcal{H}^g$. The adjoint of $\mathfrak{a}\in\mathfrak{U}(\Lambda)$ can be naturally identified with an element of $\mathfrak{U}(\theta\Lambda)$. Then $\mathfrak{U}_0=\cup\{\mathfrak{U}(\Lambda)|\Lambda\subset\Sigma, |\Lambda|<\infty\}$ is an algebra of gauge invariant operators. As a matter of fact if $\mathfrak{a}_1=\mathfrak{a}(A_1)\in\mathfrak{U}(\Lambda_1)$ and $\mathfrak{a}_2=a(A_2)\in\mathfrak{U}(\Lambda_2)$, then using Schwinger-Dyson equations again $\mathfrak{a}_2$ can be rewritten as $\mathfrak{a}_2=\mathfrak{a}'_2=\mathfrak{a}(A'_2)\in\mathfrak{U}(\Lambda'_2)$, where Λ'_2 is already separated from Λ_1 by a hyperplane $\xi^0=t=\text{const}$ and lies in the $\xi^0>t$ halfspace. Then $\mathfrak{a}_1\mathfrak{a}_2=\mathfrak{a}_1\mathfrak{a}'_2=\mathfrak{a}(A_1A'_2)\in\mathfrak{U}(\Lambda_1 \cup\Lambda'_2)$.

The uniform closure $\mathfrak{U}^g$ of $\mathfrak{U}_0$ in $\mathcal{B}(\mathcal{H}^g)$ is then the quantum field algebra we are interested in. $\mathfrak{U}^g$ can be regarded also as a representation of the C^*-algebra $\mathfrak{U}$ defined in [4]. In the sequel we will identify $\mathfrak{U}^g$ with $\mathfrak{U}$ through this natural representation.

The mapping $A\in\mathcal{A}_0\mapsto\mathfrak{a}(A)\in\mathfrak{U}_0$ which produces a non-Abelian algebra $\mathfrak{U}_0$ from the abelian one $\mathcal{A}_0$ yields the correspondence between time ordered products and operators. These operators automatically obey the canonical commutation relations and the Heisenberg equation of motion.

Turning to the problem of sectors in $\mathcal{H}^g$, let us consider a special case when the subspace

$$\mathcal{H}(\psi)=\overline{\psi\mathcal{A}_+}=\overline{\psi\mathcal{A}_{0+}}=\overline{\{\Phi(\psi A)|A\in\mathcal{A}_{0+}\}} \tag{3.6}$$

is invariant not only under $\mathfrak{U}^g$ but the spacetime translations $\mathbf{U}(\mathbf{x})$, $\mathbf{x}\in\mathbb{Z}^{d-1}$, T^n, $n\geqq 0$ as well. In (3.6) ψ denotes an element of $\mathcal{B}^g_+$ and the above requirement means that ψ, though being a non-local field, still has some localization property (cf. [24]): Any translate of ψ can be arbitrarily approximated in the physical norm $|||\ \ |||$ by a field of the form ψA, where A is strictly local. This localization property is partly a classical notion and $\mathcal{H}(\psi)$ will be called a *classical sector*. It is intuitively clear that strong enough clustering of quasilocal fields in the presence of $\theta(\psi)\psi$ may lead to irreducibility of $\mathfrak{U}\upharpoonright\mathscr{H}(\psi)$, but so far we could not prove such a statement.

We mention that in the above scheme the vacuum sector is the classical sector $\mathscr{H}(1)$ and $\mathfrak{U}^g\upharpoonright\mathscr{H}(1)$ is always cyclic with cyclic vector Ω.

For the study of the problem of existence of charged states and the problem of gauge symmetry breaking the interesting case is when ψ is the matter field in a perfect gauge fixing.

In the $Z(2)$ Higgs model we will find all the above assumptions to hold instead of $\mathscr{B}^g$ for an algebra $\mathscr{A}(\psi)\subset\mathscr{B}^g$ generated by $\mathscr{A}$ and the gauge fixed form $\psi(x)\, x\in\mathbb{Z}^d$ of the Higgs field $\varphi(x)$. The less trivial Assumption 3 and 4 will follow from Theorem 5.11 and from the lemma and proposition below:

Lemma 3.1. *Let $\tau(\xi)$ denote the transformation which flips $\sigma(\xi)$ and let $D(\xi)$ be defined by* (3.5). *Let $M\subset\Sigma^0(n+1/2)$, $L\subset\Sigma^1(n+1/2)$ $n\in\mathbb{Z}$ be finite and B be an arbitrary bounded functional, then*

i) $$D(M\cup L)\tau(M\cup L)\prod_{p\in(dL)'} e^{2\beta U(\partial p)}\prod_{\ell\in(dM\Delta L)'} e^{2\alpha U(\ell)\varphi(\partial\ell)}B$$
$$=\prod_{\ell\in M-\frac{1}{2}\hat{0}} e^{-2\alpha U(\ell)\varphi(\partial\ell)}\prod_{p\in L-\frac{1}{2}\hat{0}} e^{-2\beta U(\partial p)}B,$$

where $\tau(H)=\prod_{\xi\in H}\tau(\xi)$, $D(H)=\prod_{\xi\in H}D(\xi)$ and $(H)'=H\cap\sum(\geqq n+1/2)$ for $H\subset\sum$.

ii) *Analogue expression with $(H)'=H\cap\sum(\leqq n+1/2)$ and $M-\frac{1}{2}\hat{0}$, $L-\frac{1}{2}\hat{0}$ replaced by $M+\frac{1}{2}\hat{0}$, $L+\frac{1}{2}\hat{0}$ respectively.*

Proof. Elementary algebra. □

Proposition 3.2. *Let $\psi(x)\in\mathscr{B}^g_+$ if $x^0>0$, $\psi(x)=\theta\psi(\theta x)$ if $x^0<0$ and $T^x\psi(y)=\psi(x+y)$ if $x^0\geqq0$, $y^0>0$. Let $\mathscr{A}(\psi)$ be the subspace spanned by $\{A\psi(x)|A\in\mathscr{A}_0,\ x\in\Sigma^0\}$. Suppose that $\{\Phi(\psi(x)A)|A\in\mathscr{A}_{0+}\}$ is dense in $\mathscr{H}(\psi)=\overline{\{\Phi(B)|B\in\mathscr{A}(\psi)_+\}}$ for any $x\in\Sigma^0_+$, i.e. $\mathscr{H}(\psi)=\mathscr{H}(\psi(x))$.*

Then $\mathscr{H}(\psi)$ is an invariant subspace of $\mathfrak{U}$, $\mathbf{T}$ and $\mathbf{U}(\mathbf{x})$ $\mathbf{x}\in\mathbb{Z}^{d-1}$. $\mathsf{T}\mathscr{A}(\psi)_+$ is $|||\ |||$-dense in $\mathscr{A}(\psi)_+$ and all matrix elements $(\varphi(B_1), \mathfrak{a}\varphi(B_2))$ of $\mathfrak{a}\in\mathfrak{U}(\Lambda)$ with a double cone $\Lambda=\mathsf{V}_-(\xi)\cap\theta\mathsf{V}_-(\xi)$ can be expressed as

$$(\varphi(B_1), a\varphi(B_2))=\langle\theta(B_1)AB_2\rangle,$$

where $A\in\mathscr{A}(\Lambda\cap\sum([-\frac{1}{2},\frac{1}{2}]))$. Let us denote by $A(\mathfrak{a})$ the functional so obtained. Then especially

$$A(\hat{\tau}(\ell))=e^{-2\alpha U(\ell)\varphi(\partial\ell)},\qquad \ell\in\textstyle\sum^1(0),$$
$$A(\hat{\tau}(p))=e^{-2\beta U(\partial p)},\qquad p\in\textstyle\sum^2(0).$$

Proof. The invariance of $\mathscr{H}(\psi)$ under $\mathfrak{U}$, $\mathbf{T}$ and $\mathbf{U}(\mathbf{x})$ is trivial. To see that $\mathsf{T}\mathscr{A}(\psi)_+$ is dense, let $B\in\mathscr{A}(\psi)_+$ and approximate it with $A\psi(x)$ $x^0\geqq3/2$ $A\in\mathscr{A}_{0+}$. Using the fact that A can be written as a finite sum

$$A=\sum_{L\subset\Sigma^1(1)}\sum_{P\subset\Sigma^2(1)}\prod_{\ell\in L}e^{-2\alpha U(\ell)\varphi(\partial\ell)}\prod_{p\in P}e^{-2\beta U(\partial p)}A_{L,P}\qquad(3.7)$$

with $\mathrm{Sr}\, A_{L,P} \subset \sum(\geqq 3/2)$ one can apply Lemma 3.1 i) with the replacement $n=1$, $M=L+\frac{1}{2}\hat{0}$, $L=P+\frac{1}{2}\hat{0}$, $B=A_{L,P}\psi(x)$ to show that each term of (3.7) multiplied by $\psi(x)$ is equal (mod $\mathcal{N}$) to a functional with sensitivity region lying in $\sum(\geqq 3/2)$.

To prove the last statement let $\mathfrak{a}=\mathfrak{a}(A_0)A_0 \in \mathscr{A}(\Lambda)$. Since $\mathsf{T}^n\mathscr{A}(\psi)_+$ is dense in $\mathscr{A}(\psi)_+$ too, it is sufficient to consider the case when $B_i \in \mathsf{T}^n\mathscr{A}(\psi)_+$ $i=1,2$, $n \geqq \xi^0$. Now we can apply Lemma 3.1 i) and ii) repeatedly to construct a finite sequence $A_0, A_1, \ldots, A_r$ such that $\langle \theta(B_1)(A_i - A_{i+1})B_2\rangle = 0$ and $A(\mathfrak{a}) = A_r \in \mathscr{A}(\Lambda \cap \sum([-\frac{1}{2}, \frac{1}{2}]))$. □

4. Confinement in the Small α-Small β-Region

Using the high temperature expansion we will prove that it is impossible to construct charged states in this region even taking into account all non-local fields of $\mathscr{B}$ defined at the end of Sect. 2. Before giving the proof let us consider another important aspect of the theory – the absence of symmetry breaking for small α.

Theorem 4.1. *If* $(2d-1)\tanh\alpha < (1-\tanh\alpha)^{2d-1}$, *then for all* $F \in \mathscr{B}$ *with* $CF = -F$,

$$\langle F\rangle^+ = \langle F\rangle^- = \langle F\rangle = 0.$$

Proof. At first we prove the statement for an F of the form $F = \varphi(x) f(U)$,

$$\begin{aligned}\langle F\rangle_\Lambda^+ &= \frac{1}{Z_\Lambda} \sum_{U \in \mathsf{C}_1(\Lambda)} e^{-S_\Lambda^G(U)} f(U) \sum_{L \subset \Lambda^1} (\tanh\alpha)^{|L|} \delta(\partial^\Lambda L \Delta \{x\}) U(L) \\ &= \sum_{J \in \mathrm{Conn}_\Lambda(\{x\})} (\tanh\alpha)^{|J|} \Big\langle f(U) U(J) \prod_{\substack{\ell \in \Lambda^1 \\ \ell \vee J}} [1 + \tanh\alpha\, U(\ell)\varphi(\partial\ell)]^{-1} \Big\rangle_\Lambda, \end{aligned} \tag{4.1}$$

where $\mathsf{C}_1(\Lambda) = \{U \in \mathsf{C}_1 | \mathrm{Supp}\, U \subset \Lambda\}$, S^G is the pure gauge part of the action and we introduced the notation,

$$\mathrm{Conn}_\Lambda(M) = \{J \subset \Lambda^1 | \partial^\Lambda J = M;\ J' \subset J \text{ and } J' \text{ is connected} \Rightarrow \partial J' \neq \emptyset\}.$$

Remember that Λ is open, therefore $\partial^\Lambda \neq \partial$. Now using the simple estimate $|\{J \in \mathrm{Conn}_\Lambda(\{x\}) \| J| = n\}| \leqq 2d(2d-1)^{n-1}$, and that it is zero if $n < \mathrm{dist}(x, \partial\Lambda)$, one obtains the bound

$$|\langle F\rangle_\Lambda^+| \leqq \sum_{n=\mathrm{dist}(x,\partial\Lambda)}^{\infty} 2\|f\|(1-\tanh\alpha)^{-2d} q^n \leqq 2^{2d+1}\|f\| \frac{q^{\mathrm{dist}(x,\partial\Lambda)}}{1-q} \tag{4.3}$$

if $q = (2d-1)\tanh\alpha(1-\tanh\alpha)^{1-2d} < 1$. This yields $\tanh\alpha < 0.111$ $(d=3)$ and $\tanh\alpha < 0.079$ $(d=4)$.

For fields of the form

$$F = \varphi(x) \prod_{\ell \in K} e^{-2\alpha U(\ell)\varphi(\partial\ell)} f(U), \qquad K \subset \sum^1, |K| < \infty, \tag{4.4}$$

we have

$$\langle F\rangle_\Lambda^+ = \langle \varphi(x) f(U)\rangle_\Lambda^+(K) \Big\langle \prod_{\ell \in K} e^{-2\alpha U(\ell)\varphi(\partial\ell)} \Big\rangle_\Lambda^+,$$

where $\langle\ \rangle_\Lambda^+(K)$ is the Gibbs state obtained from $\langle\ \rangle_\Lambda^+$ by changing the sign of the α coupling on the links of K. Since the bound (4.3) was obtained by an estimate uniform in the sign of α it descends to $\langle\ \rangle_\Lambda^+(K)$ too and we arrive at the inequality

$$|\langle F\rangle_\Lambda^+| \leqq 2^{2d+1}\|f\| e^{2\alpha|K|}(1-q)^{-1} q^{\mathrm{dist}(x,\partial\Lambda)}.$$

Taking finite linear combinations of (4.4) with different finite K's:

$$\lim_{\Lambda\to\Sigma}\langle F\rangle_\Lambda^+=0$$

for all such fields and consequently for all charged fields of $\mathscr{B}$ by continuity. □

Theorem 4.2. *Let $F\in\mathscr{B}_+$ be charged, i.e. $CF=-F$. Then $\langle\theta(F)F\rangle=0$ if α and β are sufficiently small.*

Proof. In the same way as in the proof of Theorem 4.1 one can reduce the problem to prove $\langle\theta(F_1)F_2\rangle=0$ when $F_i=\varphi(x_i)f_i(U)$ with f_i bounded and $\mathrm{Sr}\,f_i\subset\sum_+$ $i=1,2$. To estimate $\langle\theta(F_1)F_2\rangle_\Lambda^+$, expand $\exp(-S_\Lambda)$ in powers of $\tanh\alpha$ and $\exp(-S_{\Lambda_0})$ in powers of $\tanh\beta$, then integrate over $U(\ell)\,\ell\in\Lambda_0^1$ and $\varphi(x)\,(x\in\Lambda^0)$:

$$\langle\theta(F_1)F_2\rangle_\Lambda^+=\frac{1}{Z_\Lambda}\sum_{U\in C_1(\Lambda\setminus\Lambda_0)}e^{-S^G_{\Lambda\setminus\Lambda_0}(U)}[\theta(f_1)f_2](U)\sum_{P\subset\Lambda_0^2}(\tanh\beta)^{|P|}$$
$$\times\sum_{L\subset\Lambda^1}(\tanh\alpha)^{|L|}\delta(\partial^\Lambda L\Delta\{x_2,\theta x_1\})\delta(\partial^{\Lambda_0}P\Delta(L\cap\Lambda_0))U(\partial^\Lambda P\Delta L).$$

Decompose $P\cup L$ into connected components and let $J=J^1\cup J^2\subset\Lambda^1\cup\Lambda_0^2$ be the union of components connected to $\{x_2,\theta x_1\}$. Then we can write

$$\langle\theta(F_1)F_2\rangle_\Lambda^+=\sum_{J\in\mathrm{Conn}_\Lambda(\{x_2,\theta x_1\})}(\tanh\alpha)^{|J^1|}(\tanh\beta)^{|J^2|}\Big\langle\theta(f_1)f_2U(\partial^\Lambda J^2\Delta J^1)$$
$$\times\prod_{\substack{\ell\in\Lambda^1\\ \ell\vee J}}[1+\tanh\alpha U(\ell)\varphi(\partial\ell)]^{-1}\prod_{\substack{p\in\Lambda_0^2\\ p\vee J}}[1+\tanh\beta U(\partial p)]^{-1}\Big\rangle_\Lambda^+,$$

where now $\mathrm{Conn}_\Lambda(M)$ denotes the set of those $J\subset\sum^1\cup\sum_0^2$ which satisfy: 1) each connected component of J is connected to M, 2) $\partial^\Lambda J^1=M$ and 3) $\partial^{\Lambda_0}J^2=J^1\cap\Lambda_0$.

Now we prove that if $J\in\mathrm{Conn}_\Lambda(M)$ and $|M_+|$, $|M_-|$ are odd, then $J\vee\partial\Lambda$, and therefore $|J|\geqq\mathrm{dist}(M,\partial\Lambda)$.

Suppose that J^2 is not connected to $\partial\Lambda$. Then $|\partial^{\Lambda_0}J^2|=|J^1\cap\Lambda_0^1|=\text{even}$. Measuring the flux of J^1 through the coboundary of Λ_+ one finds

$$(d\Lambda_+^0;J^1)=((d\Lambda^0)_+;J^1)(\Lambda_0^1;J^1)=(-1)^{|M_+|}=-1.$$

Therefore $(d\Lambda^0)_+\cap J^1\neq\emptyset$ and $J\vee\partial\Lambda$ as promised.

We estimate the number of links and plaquettes connected to J from above as $2d+(2d-1)|J^1|+4|J^2|$ and $|J^1|+(4d-5)|J^2|$ respectively. The number of J's from $\mathrm{Conn}_\Lambda(\{x,y\})$ and having a fixed length can be estimated from above by the number of all connected sets $J\subset\Lambda^1\cup\Lambda^2$ having that length and connected to x, which in turn is estimated in the usual way using the solution of the Königsberg bridge problem. These considerations yield the following bound:

$$|\langle\theta(F_1)F_2\rangle_\Lambda^+|\leqq\sum_{\substack{n_1,n_2\\ n_1+n_2\geqq\mathrm{dist}(x,\partial\Lambda)}}\|f_1\|\,\|f_2\|\,2d(1-\tanh\alpha)^{-2d}q_1^{n_1}q_2^{n_2}\tag{4.5}$$

where

$$q_1=(6d-4)^2\tanh\alpha(1-\tanh\alpha)^{1-2d}(1-\tanh\beta)^{-1},$$
$$q_2=(8d-8)^2\tanh\beta(1-\tanh\alpha)^{-4}(1-\tanh\beta)^{5-4d}.$$

If $q_1<1$ and $q_2<1$ the sum in (4.5) converges and goes to zero with $\Lambda\to\sum$. In $d=3$ dimensions this corresponds to $\tanh\alpha<4.9\times 10^{-3}$, $\tanh\beta<3.9\times 10^{-3}$ and in $d=4$ to $\tanh\alpha<2.4\times 10^{-3}$, $\tanh\beta<1.6\times 10^{-3}$.

5. Construction of Charged States

5.1. λ-Regular Gauges

A gauge, that is a complete gauge fixing, is a mapping $\kappa: \mathsf{Z}^2\to\mathsf{C}^1$, where Z^2 is the set of 2-cocycles $P\subset\sum^2$ with $|P|<\infty$. κ is such that $d\circ\kappa$ is the identity on Z^2. Once such a gauge is given one can construct the locally gauge invariant field

$$[\psi(x)](\varphi, U)=\varphi(x)[\Omega(x)](dU), \tag{5.1}$$

where $\Omega(dU)\in C_0$ is the gauge transformation which transforms U into its κ-gauge form

$$V(\ell)=U(\ell)[d\Omega(dU)](\ell), \qquad \operatorname{Supp}V=\kappa(\operatorname{Supp}dU).$$

So $\psi(x)$ is nothing but $\varphi(x)$ in the gauge κ.

If $\operatorname{Supp}U=L$, $P=dL=\operatorname{Supp}dU$, then one can write

$$[\psi(x)](\varphi, U)=\varphi(x)U(J_x)V(J_x)=\varphi(x)(L;J_x)(\kappa(P);J_x), \tag{5.2}$$

where J_x is an arbitrary (infinite) 1-chain s.t. $\partial J_x=\{x\}$. The arbitrariness of J_x in (5.2) expresses the fact that the factor $V(J_x)$ realizes an ideal smearing of the naive string $\varphi(x)U(J_x)$.

In the sequel we will restrict our attention to those gauges κ which have certain regularity properties. These properties will be useful both in the large β and in the large α cluster expansions.

Definition 5.1. Let $\lambda:\mathbb{R}_+\to\mathbb{R}_+$ be a polynomial with non-negative coefficients then a gauge $\kappa:\mathsf{Z}^2\to\mathsf{C}^1$ is called λ-regular if the following two properties are satisfied:

i) $\kappa(P)=\kappa(P_1)\Delta\kappa(P_2)\Delta\ldots\Delta\kappa(P_n), \qquad P\in\mathsf{Z}^2$

if $P=P_1\Delta P_2\Delta\ldots\Delta P_n$ is the decomposition of P into connected components.

ii) $\operatorname{dist}(\ell,P)\leqq\lambda(|P|)\ \forall\ell\in\kappa(P), \qquad P\in\mathsf{Z}^2.$

λ-regularity of κ is already enough to control the thermodynamical limit, analyticity and exponential clustering of the correlation functions $\langle\psi(x)\psi(y)\rangle$ as we will see in the next paragraph. However this admits a good quantum interpretation only if κ obeys a further requirement:

Definition 5.2. A gauge $\kappa:\mathsf{Z}^2\to\mathsf{C}^1$ is called Markovian if for any two 2-cocycle P and P' and for any non-negative integers t and t',

$$[P\cap\textstyle\sum(>t)]+t'\hat{0}=P'\cap\sum(>t+t') \text{ implies}$$
$$[\kappa(P)\cap\textstyle\sum(>t)]+t'\hat{0}=\kappa(P')\cap\sum(>t+t'), \text{ and}$$
$$[P\cap\textstyle\sum(<-t)]-t'\hat{0}=P'\cap\sum(<-t-t') \text{ implies}$$
$$[\kappa(P)\cap\textstyle\sum(<-t)]-t'\hat{0}=\kappa(P')\cap\sum(<-t-t').$$

In words this means that $\kappa(P)\cap\sum(>t)$ (respectively $\kappa(P)\cap\sum(<-t)$) is uniquely determined by the geometrical figure of $P\cap\sum(>t)$ (respectively $P\cap\sum(<-t)$).

In addition to the above properties one can require κ to possess nearly all spacetime symmetries of the lattice. Rotations changing the time axis of course are not allowed.

Definition 5.3. A gauge $\kappa:\mathbf{Z}^2\to\mathbf{C}^1$ is called maximally covariant if

i) it is Markovian,

ii) $g\circ\kappa=\kappa\circ g$, where g stands for θ, T_μ $\mu=1,\dots,d-1$ and $R_{\mu\nu}$ $1\leqq\mu<\nu\leqq d-1$.

That λ-regular maximally covariant gauges exist is shown in the Appendix constructively.

Proposition 5.4. *If κ is a maximally covariant gauge and ψ is defined by Eq. (5.2), then*

$$T^x\psi(y)=\psi(x+y)\qquad \operatorname{sgn}x^0=\operatorname{sgn}y^0$$

$$\theta\psi(x)=\psi(\theta x)$$

$$R_{\mu\nu}\psi(x)=\psi(R_{\mu\nu}x)$$

$$\operatorname{Sr}\psi(x)\subset\textstyle\sum_{\operatorname{sgn}x^0}.$$

The proof is a simple consequence of the above definitions and is left to the reader.

5.2. *Correlation Functions of Non-Local Fields*

The small α-large β-region of the free charge phase possesses a convergent cluster expansion as it was shown by Marat and Miracle-Solé [25]. Their proof demonstrates that the expectation value of a quasilocal field can be expressed as the sum of its perturbation series, each term being an analytic function of α and $e^{-2\beta}$ and converging uniformly in a neighbourhood of $\alpha=0$, $\beta=\infty$. We will see that the same is true for expectation values of quasilocal fields multiplied by any number of $\psi(x)$'s, provided ψ is derived from a λ-regular gauge. This result can be interpreted also as a proof of convergence of the cluster expansion for the correlation functions in a λ-regular gauge – where the interaction may be fairly non-local due to the complete gauge fixing.

We will adapt the finite volume Gibbs state for the use of Marat-Miracle-Solé clusters by slightly modifying the boundary condition. All the proofs could be carried out for the Gibbs state (2.4) too (at the price of encountering some technical complication with clusters ending on the boundary) and would yield the same result as far as the infinite volume limit is concerned. Our new boundary condition is of a mixed type in the sense that the gauge field is fixed to be 1 outside the *open* set Λ but the matter field is free outside it. It is important also that we allow the matter field to propagate on $\partial\Lambda$ so let $\bar{\Lambda}=\Lambda\cup\partial\Lambda$ be the closure of Λ and define

$$\langle\dots\rangle_\Lambda=\frac{1}{Z_\Lambda}\int d\varphi\sum_{U\in\mathbf{C}_1(\Lambda)}e^{-S_{\bar{\Lambda}}(\varphi,U)}\dots,\tag{5.3}$$

where $\mathbf{C}_1(\Lambda)$ was defined after (4.1) and $d\varphi$ is the infinite volume a priori measure for φ.

Theorem 5.5. *Let $\psi(x)$ be constructed as in* (5.1) *from a λ-regular gauge. Then there exists a real number b_0 depending on λ and the dimension d such that in the region* $\max\{|\tanh\alpha|, |e^{-2\beta}|\} < e^{-b_0}$ *the thermodynamical limit*

$$\langle\psi(x)\psi(y)\rangle = \lim_{\Lambda\to\Sigma} \langle\psi(x)\psi(y)\rangle_\Lambda$$

of the finite volume Gibbs state (5.3) *exists, it is an analytic function of* $\tanh\alpha$ *and* $e^{-2\beta}$ *and for positive values of α and β obeys the bounds*

$$C_1 e^{-m_1|x-y|} < \langle\psi(x)\psi(y)\rangle < C_2 e^{-m_2|x-y|}$$

with some positive constants C_1, C_2, m_1, m_2 depending only on λ and d.

Before proceeding to the proof of Theorem 5.5 let us introduce some notations and definitions. The winding number of a 2-cocycle P and a 1-cycle L is

$$v(P,L) = (\kappa(P); L). \tag{5.4}$$

In fact v is independent of the gauge κ. Sometimes we will use (5.4) also when L is an arbitrary 1-chain in which case of course κ is fixed. We can write in this way that $\psi(x) = \varphi(x)U(J_x)v(P, J_x)$, where $P = \operatorname{Supp} dU$.

Let $P \sim P'(P, P' \subset \sum^2)$ denote the situation that P and P' are not co-connected and $L \sim L'(L, L' \subset \sum^1)$ the situation that L and L' are not connected. We write $L \sim P(L \in \mathrm{Z}_1, P \in \mathrm{Z}^2)$ if $v(P,L) = 1$. The relation $\sim$ is called compatibility.

A Marat-Miracle-Solé cluster γ is a collection of co-connected 2-cocycles $P_1, \ldots, P_n$ and connected 1-cycles $L_1, \ldots, L_m$ such that $P_i \sim P_j$ $i \neq j$, $L_i \sim L_j$ $i \neq j$ and the graph with vertex set $\{P_1, \ldots, P_n, L_1, \ldots, L_m\}$ and with a bond drawn between L_i and P_j whenever $P_j \nsim L_i$ is connected. The set of all clusters γ is denoted by $\mathscr{P}$. The family of clusters belonging to the open set Λ is defined as

$$\mathscr{P}_\Lambda = \{\gamma \in \mathscr{P} | P \in \gamma \Rightarrow P \subset \Lambda^2, L \in \gamma \Rightarrow L \subset \bar{\Lambda}^1\}.$$

Let Z denote either a connected 1-cycle L or a co-connected 2-cocycle P. We say that $\gamma \in \mathscr{P}$ and $\gamma' \in \mathscr{P}$ are compatible and write $\gamma \sim \gamma'$ if $Z \sim Z'$ for all $Z \in \gamma$ and $Z' \in \gamma'$. The length of a cluster $|\gamma| = \sum_{Z\in\gamma} |Z|$. The activity of γ is defined as

$$z(\gamma) = \prod_{L\in\gamma} (\tanh\alpha)^{|L|} \times \prod_{P\in\gamma} e^{-2\beta|P|} \times \prod_{L\in\gamma}\prod_{P\in\gamma} v(P,L).$$

Obviously $|z(\gamma)| < e^{-b_0|\gamma|}$ if α and β are in the domain $\max\{|\tanh\alpha|, |e^{-2\beta}|\} < e^{-b_0}$.

Proof of Theorem 5.5. If Λ already contains x and y then

$$\langle\psi(x)\psi(y)\rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{\substack{L\subset\bar{\Lambda}^1\\ \partial L=\{x,y\}}} (\tanh\alpha)^{|L|} \sum_{\substack{P\subset\Lambda^2\\ dP=\emptyset}} e^{-2\beta|P|} v(P, L\Delta J_{xy}) v(P, J_{xy}), \tag{5.5}$$

where – because of the arbitrariness of J_x in the expression (5.2) – $J_{xy} = J_x \Delta J_y$ is an arbitrary 1-chain with boundary $\partial J_{xy} = \{x,y\}$. So we may choose different J_{xy}'s for each value of L in (5.5). Let J_{xy} be the component of L connected to $\{x,y\}$. Then

$$\langle\psi(x)\psi(y)\rangle_\Lambda = \sum_{J\in\mathrm{Conn}_{\bar{\Lambda}}(\{x,y\})} (\tanh\alpha)^{|J|} \frac{Z_\Lambda(J)}{Z_\Lambda(\emptyset)}, \tag{5.6}$$

where $\mathrm{Conn}_\Lambda(M)$ was defined in (4.2) and we introduced the notation

$$Z_\Lambda(J) = \sum_{\Gamma \in \mathscr{G}_\Lambda} \prod_{\gamma \in \Gamma} z(\gamma|J), \tag{5.7}$$

where $\mathscr{G}_\Lambda$ is the set of compatible cluster families $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, $\gamma_i \in \mathscr{P}_\Lambda$, $\gamma_i \sim \gamma_j$ $i \neq j$. $z(\gamma|J)$ denotes a modified activity

$$\begin{aligned} z(\gamma|J) &= z(\gamma) u(\gamma|J), \\ u(\gamma|J) &= \prod_{P \in \gamma} v(P, J) \prod_{L \in \gamma} \delta(L \sim J). \end{aligned} \tag{5.8}$$

With the usual trick one obtains:

$$\ln \frac{Z_\Lambda(J)}{Z_\Lambda(\emptyset)} = \int_0^1 dt \frac{d}{dt} \ln Z(t) = \int_0^1 dt \sum_{\gamma \in \mathscr{P}_\Lambda} z(\gamma)[u(\gamma|J) - 1] \varrho_t(\{\gamma\}), \tag{5.9}$$

where $Z(t)$ and $\varrho_t(\Gamma)$ are the partition function and correlation function respectively of a system with activity $z_t(\gamma) = t\, z(\gamma|J) + (1-t) z(\gamma)$:

$$\varrho_t(\Gamma) = \frac{1}{Z(t)} \sum_{\substack{\Gamma' \in \mathscr{G}_\Lambda \\ \Gamma' \sim \Gamma}} \prod_{\gamma \in \Gamma'} z_t(\gamma), \qquad Z(t) = \sum_{\Gamma \in \mathscr{G}_\Lambda} \prod_{\gamma \in \Gamma} z_t(\gamma).$$

Notice that $|z_t(\gamma)| \leqq |z(\gamma)| < e^{-b_0|\gamma|} \forall t \in [0,1]$. Now if we introduce a norm indexed by a parameter b_1,

$$\|\varrho\| = \sup_{\Gamma \in \mathscr{G} \setminus \{\emptyset\}} |\varrho(\Gamma)| \exp\{-b_1 \|\Gamma\|\}$$

$\left(\mathscr{G} = \bigcup_\Lambda \mathscr{G}_\Lambda, \|\Gamma\| = \sum_{\gamma \in \Gamma} |\gamma|\right)$ on the space of ϱ's one can apply standard methods [26] to prove that if

$$K_0 \equiv \exp\{-4[b_1 - F(b_0 - b_1)]\} < 1, \tag{5.10}$$

then

$$|\varrho_t(\Gamma)| \leqq (1 - K_0)^{-1} e^{b_1(\|\Gamma\| - 4)}. \tag{5.11}$$

Here K_0 is an upper bound on the norm of the Kirkwood-Salsburg operator $K(t)$ which has the matrix elements

$$[K(t)](\Gamma, \Gamma') = \delta(\Gamma \in \mathscr{G}_\Lambda) \delta(\Gamma' \in \mathscr{G}_\Lambda) \delta(\Gamma' \subset [\Gamma])(-1)^{|\Gamma'|} \prod_{\gamma \in \Gamma'} z_t(\gamma),$$

where $[\Gamma] = \{\gamma \in \mathscr{P} | \gamma \nsim \Gamma\}$. F in (5.10) denotes a function $F: (c_0, \infty) \to \mathbb{R}_+$ which occurs in the estimate

$$\sum_{\substack{\gamma \in \mathscr{P} \\ \gamma \nsim \gamma'}} e^{-b|\gamma|} \leqq F(b)|\gamma'|, \qquad b > c_0. \tag{5.12}$$

$F(b)$ together with its derivatives $\left(-\frac{d}{db}\right)^n F(b)$ $n = 1, 2, \ldots$ is monotone decreasing. The concrete form of F will not be specified here. (We refer the reader to the Appendix of [4] for more details.)

Substituting (5.11) into (5.9) one obtains the estimate

$$\ln \frac{Z_\Lambda(J)}{Z_\Lambda(\emptyset)} \leqq 2 \frac{e^{-4b_1}}{1-K_0} \sum_{\gamma\in\mathscr{P}(J)} e^{-(b_0-b_1)|\gamma|}, \tag{5.13}$$

where $\mathscr{P}(J)=\{\gamma\in\mathscr{P}|u(\gamma|J)\neq 1\}$.

From the definition of $u(\gamma|J)$ and from λ-regularity of κ it follows that $u(\gamma|J)$ can be different from 1 only if $\mathrm{dist}(\gamma, J)\leqq\lambda(|\gamma|)$. Thus

$$\sum_{\gamma\in\mathscr{P}(J)} e^{-b|\gamma|} \leqq \sum_{n=2}^{\infty} e^{-2bn} C_J(\lambda(2n))\mathscr{N}(2n) = C_J\left(\lambda\left(-\frac{d}{db}\right)\right) f(b),$$

where $C_J(r)=|\{p\in\sum^2|\mathrm{dist}(p,J)\leqq r\}|\leqq|J|\binom{d}{2}(2r)^d$ and $\mathscr{N}(k)$ is the number of clusters γ with length $|\gamma|=k$ and such that $\gamma\nsim\{\partial p\}$ for a fixed plaquette p. $f(b)$ is by definition

$$f(b)=\sum_{n=2}^{\infty} e^{-2nb}\mathscr{N}(2n),$$

and therefore can be estimated using (5.12) as $f(b)\leqq 4F(b)$. Because $-d/db$ can be applied arbitrary many times to both sides of this inequality one obtains that

$$\sum_{\gamma\in\mathscr{P}(J)} e^{-b|\gamma|} \leqq R(b)|J|, \tag{5.14}$$

where

$$R(b)=\binom{d}{2} 2^{d+2}\left[\lambda\left(-\frac{d}{db}\right)\right]^d F(b)$$

is monotone decreasing.

Substituting (5.14) into (5.13) and the latter into (5.6), one obtains

$$|\langle\psi(x)\psi(y)\rangle_\Lambda| \leqq \sum_{J\in\mathrm{Conn}_\Lambda(\{x,y\})} \left\{|\tanh\alpha|\exp\left[2\frac{e^{-4b_1}}{1-K_0}R(b_0-b_1)\right]\right\}^{|J|}$$
$$\leqq \sum_{N=|x-y|}^{\infty} \tfrac{1}{2}q^N = \frac{1}{2(1-q)} q^{|x-y|}, \tag{5.15}$$

where

$$\ln q < \ln(2d-1) - b_0 + \frac{2e^{-4b_1}}{1-K_0} R(b_0-b_1). \tag{5.16}$$

In the second inequality of (5.15) we used the bound $\frac{1}{2}(2d-1)^N$ on the number of paths connecting x and y and having length $|J|=N$. The uniform estimate (5.15) establishes the analyticity of $\langle\psi(x)\psi(y)\rangle$ and the exponential upper bound if we can show that $q<1$ and $K_0<1$ can be satisfied in a region $\max\{|\tanh\alpha|, |e^{-2\beta}|\}<e^{-b_0}$. The exponential lower bound is simply

$$\langle\psi(x)\psi(y)\rangle > (\tanh\alpha)^{|x-y|} \exp\left\{-\frac{2e^{-4b_1}}{1-K_0} R(b_0-b_1)|x-y|\right\} \quad \alpha\geqq 0, \quad \beta\geqq 0.$$

For demonstrating $q<1$ substitute the value (5.10) of K_0 into (5.16). Then $q<1$ is equivalent to the inequality

$$e^{4b_1} \geqq \frac{2R(b_0-b_1)}{b_0-\ln(2d-1)} + e^{4F(b_0-b_1)},$$

and after this $K_0<1$ is automatic. Remember that both R and F are monotone decreasing positive functions. Let b_2 be in the domain of F (therefore of R too) and take for b_1 the larger one among the solutions of the equation,

$$e^{4b_1} = \frac{2R(b_2)}{b_2+b_1-\ln(2d-1)} + e^{4F(b_2)}.$$

Then $b_0=b_2+b_1$ defines a region where the exponential bounds and analyticity hold. So we can choose the smallest one among those for b_0. □

The quantum interpretation of the above theorem is the following:

Corollary 5.6. *If in addition to λ-regularity the gauge κ is Markovian then*

$$T^n\Phi(\psi(x)) = \Phi(\psi(x+n\hat{0})) \quad \textit{and} \quad (\Phi(\psi(x)), T^n\Phi(\psi(x))) = \langle \psi(\theta x)\psi(x+n\hat{0})\rangle .$$

Therefore in the region defined by Theorem 5.5, the energy of the state $\Phi(\psi(x))$ is finite and positive.

The proof is trivial taking into account Definition 5.2 and Theorem 5.5.

The next two lemmas deal with clustering properties of ψ's and of quasilocal fields in the presence of ψ's.

Lemma 5.7. *Let $A, B \in \mathscr{A}_0$ and $\psi(x)$ be what was in Theorem 5.5. Then in the region of the couplings determined in Theorem 5.5 there exist an $M>0$ independent of A and B such that*

$$|\langle A\psi(x)\mathrm{T}^a(B\psi(y))\rangle| \leqq C(A,B)e^{-M|a|}.$$

Proof. The family $\left\{A_L = \prod_{\ell\in L} e^{-2\alpha U(\ell)\varphi(\partial\ell)} \middle| L\subset\sum^1, |L|<\infty\right\}$ of fields forms a basis for $\mathscr{A}_0$,

$$\langle A_L\psi(x)\mathrm{T}^a(A_K\psi(y))\rangle_\Lambda = \langle\psi(x)\psi(y+a)\rangle_\Lambda(L\Delta K^a)\langle A_{L\Delta K^a}\rangle_\Lambda, \tag{5.17}$$

where $\langle\ \rangle_\Lambda(L)$ is the Gibbs state with α replaced by $-\alpha$ on the links of L and K^a is the translate of K by the lattice vector a. Since the exponential upper bound of Theorem 5.5 was uniform in the phase of $\tanh\alpha$ the first factor in the RHS of (5.17) decays exponentially while the second is bounded uniformly in a. Therefore the choice $M=m_2$ yields the required bound. □

Lemma 5.8. *Let A, B, $\psi(x)$ and the couplings be the same as in Lemma 5.7. Then*

$$\lim_{a\to\infty} \langle A\psi(x)\psi(y)\mathrm{T}^a(B)\rangle = \langle A\psi(x)\psi(y)\rangle\langle B\rangle .$$

Proof. It is enough to verify the statement for $A=A_L$ and $B=A_K$ L and K being finite sets of links,

$$\langle A_L\psi(x)\psi(y)A_{K^a}\rangle_\Lambda = \sum_{J\in\mathrm{Conn}_\Lambda(\{x,y\})} \prod_{\ell\in J} \tanh\tilde{\alpha}_\ell \frac{Z_\Lambda(J, L\Delta K^a)}{Z_\Lambda(J,L)} \frac{Z_\Lambda(J,L)}{Z_\Lambda(\emptyset,\emptyset)}, \tag{5.18}$$

where $\tilde{\alpha}_\ell = -\alpha$ if $\ell \in L\Delta K^a$ and $\tilde{\alpha}_\ell = \alpha$ otherwise. $Z_\Lambda(J, L\Delta K^a)$ is the same as $Z_\Lambda(J)$ in (5.7) except that α is replaced by $\tilde{\alpha}_\ell$. Since the sum in the RHS of (5.18) is uniformly convergent both in Λ and in a we can take the $\Lambda \to \infty$ and the $a \to \infty$ limit term by term. Now because

$$\lim_{a\to\infty} \prod_{\ell \in J} \tanh \tilde{\alpha}_\ell = (\tanh \alpha)^{|J|}(-1)^{|L\cap J|}$$

and

$$\lim_{a\to\infty} \lim_{\Lambda\to\Sigma} \frac{Z_\Lambda(J, L\Delta K^a)}{Z_\Lambda(J,L)} = \langle A_K \rangle,$$

the lemma is proven. □

The discussion of the consequences of these lemmas for the quantum theory is postponed until Sect. 5.4 when translation covariance of the classical sector of a single $\psi(x)$ will be at our disposal.

5.3. Translation Covariance and Gauge Independence

Our aim in this paragraph is to demonstrate that the classical sector $\mathscr{H}(\psi(x)) = \overline{\psi(x)\mathscr{A}_+}$ is independent of x ($x^0 > 0$) and also of the gauge κ taking into account the whole class of λ-regular gauges which are convergent at the given coupling. This means that the single new physical entity which can emerge from the use of the non-local fields ψ is a translation invariant and gauge independent one: presumably the asymptotics of a $Z(2)$-electric field with total charge -1.

The validity of the following lemmas are not restricted to the large β, small α region. They apply as well to the large α cluster expansion discussed in Sect. 6. Let $\mathscr{P}_\Lambda$ denote the set of clusters in the volume Λ, $\mathscr{P} = \bigcup_\Lambda \mathscr{P}_\Lambda$ and $\sim$ be the compatibility relation. Let the activity $z: \mathscr{P} \to \mathbb{C}$ be such that the Mayer-Montroll type equation $\varrho_\Lambda = 1 + K_\Lambda \varrho_\Lambda$ for the correlation functions

$$\varrho_\Lambda(\Gamma) = \frac{1}{Z_\Lambda} \sum_{\substack{\Gamma' \in \mathscr{G}_\Lambda \\ \Gamma' \sim \Gamma}} z(\Gamma'), \qquad \Gamma \in \mathscr{G}_\Lambda \backslash \{\emptyset\}, \qquad z(\Gamma) = \prod_{\gamma \in \Gamma} z(\gamma)$$

have a uniformly convergent solution in the region $|z(\gamma)| < e^{-b_0|\gamma|}$ and

$$|\varrho_\Lambda(\Gamma)| \leqq C\, e^{b_1 \|\Gamma\|}$$

like in (5.11). We suppose also that the clusters $\gamma \in \mathscr{P}$ can be uniquely characterized by, and will be identified with, certain finite subsets of $\sum$.

Lemma 5.9. *Let $\mathscr{G} = \bigcup_\Lambda \mathscr{G}_\Lambda$ and $F: \mathscr{G} \to \mathbb{C}$ be a functional bounded by* 1 *and multiplicative:*

$$F(\Gamma) = \prod_{\gamma \in \Gamma} F(\gamma), \qquad \Gamma \in \mathscr{G}.$$

Suppose that there exists a finite subset Ξ of $\sum$ such that

$$F(\Gamma) = F(\Gamma \cap \mathscr{P}_f(\Xi)),$$

where $\mathscr{P}_f(\Xi)=\{\gamma\in\mathscr{P}|\mathrm{dist}(\gamma,\Xi)\leqq f(|\gamma|)\}$. *The function* $f:\mathbb{R}_+\to\mathbb{R}_+$ *is monotone increasing and supposed to obey*

$$\sum_{\substack{\gamma\in\mathscr{P}\\|\gamma|>r\\\gamma\vee dx_0}} e^{-(b_0-b_1)|\gamma|}[f(|\gamma|)]^d\to 0 \quad \text{if} \quad r\to\infty \ (x_0\in\textstyle\sum^0 \text{ is fixed}).$$

Then there exists a sequence $\{F_r|r=1,2,\ldots\}\subset\mathscr{A}_0$ *such that*

$$\lim_{r\to\infty}\langle|F-F_r|^2\rangle=0.$$

Proof. Let $\mathscr{P}_f(\Xi,r)=\{\gamma\in\mathscr{P}_f(\Xi)||\gamma|\leqq r\}$. We define

$$F_r(\Gamma)=F(\Gamma\cap\mathscr{P}_f(\Xi,r)),\qquad \Gamma\in\mathscr{G}.$$

Let $\Lambda_r=\{\xi\in\sum|\mathrm{dist}(\xi,\Xi)\leqq f(r)+r\}$, then $\gamma\in\mathscr{P}_f(\Xi,r)$ implies that $\gamma\subset\Lambda_r$.

Suppose that Γ and Γ' are such that $s(\Gamma)\cap\Lambda_r=s(\Gamma')\cap\Lambda_r$, where $s(\Gamma)=\cup\{\gamma|\gamma\in\Gamma\}$ is the "shadow" of Γ on Σ. In this case each $\gamma\in\Gamma\cap\mathscr{P}_f(\Xi,r)$ must be an element of Γ' too and vice versa. That is $\Gamma\cap\mathscr{P}_f(\Xi,r)=\Gamma'\cap\mathscr{P}_f(\Xi,r)$, consequently $F_r(\Gamma)=F_r(\Gamma')$. This proves that the sensitivity region of F_r is finite, namely $\mathrm{Sr}\,F_r\subset\Lambda_r$.

Let $F(\Gamma)-F_r(\Gamma)\neq 0$. Then $\Gamma\cap\mathscr{P}_f(\Xi)\neq\Gamma\cap\mathscr{P}_f(\Xi,r)$ therefore $\exists\gamma\in\Gamma$ s.t. $|\gamma|>r$ and $\mathrm{dist}(\gamma,\Xi)\leqq f(|\gamma|)$. The set of these γ's is denoted by $\mathscr{P}(r)$.

Now we make the following estimation:

$$\begin{aligned}\langle|F-F_r|^2\rangle_\Lambda&=\frac{1}{Z_\Lambda}\sum_{\Gamma\in\mathscr{G}_\Lambda}w(\Gamma)\Big|\prod_{\gamma\in\Gamma\cap\mathscr{P}(r)}F(\gamma)-1\Big|^2\\&=\langle|F_r|^2\rangle_\Lambda\Big\langle\Big|\prod_{\gamma\in\Gamma\cap\mathscr{P}(r)}F(\gamma)-1\Big|^2\Big\rangle_\Lambda^{(r)}\\&\leqq 4\langle\delta(\Gamma\cap\mathscr{P}(r)\neq\emptyset)\rangle_\Lambda^{(r)},\end{aligned}\tag{5.19}$$

where $\langle\ \rangle_\Lambda^{(r)}=\langle\ \rangle_\Lambda|_{z\to w}$ and $w(\gamma)=z(\gamma)|F_r(\gamma)|^2$.

Since $|w(\gamma)|\leqq|z(\gamma)|$ the Gibbs state $\langle\ \rangle_\Lambda^{(r)}$ enjoys the same convergence properties as the original one:

$$\begin{aligned}\langle\delta(\Gamma\cap\mathscr{P}(r)\neq\emptyset)\rangle_\Lambda^{(r)}&=\sum_{\substack{\Gamma_0\subset\mathscr{P}(r)\\\Gamma_0\in\mathscr{G}_\Lambda\setminus\{\emptyset\}}}(-1)^{|\Gamma_0|-1}w(\Gamma_0)\varrho_\Lambda^{(r)}(\Gamma_0)\\&\leqq C\left\{\exp\Big[\sum_{\gamma\in\mathscr{P}(r)}e^{-(b_0-b_1)|\gamma|}\Big]-1\right\}.\end{aligned}\tag{5.20}$$

Now the lemma is proven by virtue of (5.19), (5.20) and the simple estimate

$$\sum_{\gamma\in\mathscr{P}(r)}e^{-(b_0-b_1)|\gamma|}\leqq\mathrm{const}|\Xi|\sum_{\substack{\gamma\in\mathscr{P}\\|\gamma|>r\\\gamma\vee dx_0}}[f(|\gamma|)]^d e^{-(b_0-b_1)|\gamma|}\to 0.\quad\square$$

Lemma 5.10. *Change the conditions of Lemma 5.9 in two places. Replace the definition of* $\mathscr{P}_f(\Xi)$ *by*

$$\mathscr{P}_f(\Xi)=\{\gamma\in\mathscr{P}|\gamma_H\neq\emptyset \quad \text{and} \quad \mathrm{dist}(\gamma_H,\Xi)\leqq f(|\gamma_H|)\}$$

and add the requirement that $\mathrm{Sr}\,F\subset H$. *H denotes a closed half space in* $\sum$ *and* $\gamma_H=\gamma\cap H$. *In this case the conclusion is that there exists a sequence* $\{F_r|r=1,2,\ldots\}\subset\mathscr{A}_0$ *such that*

$$\mathrm{Sr}\,F_r\subset H\,,\quad r=1,2,\ldots \quad \textit{and} \quad \lim_{r\to\infty}\langle|F-F_r|^2\rangle=0\,.$$

Proof. Let $\mathscr{P}_f(\Xi,r)=\{\gamma\in\mathscr{P}_f(\Xi)||\gamma_H|\leqq r\}$ and $\Lambda_r=\{\xi\in H|\mathrm{dist}(\xi,\Xi)\leqq f(r)+r\}$. Then $\gamma\in\mathscr{P}_f(\Xi,r)$ implies that $\emptyset\neq\gamma_H\subset\Lambda_r$, i.e. $\gamma\cap H=\gamma\cap\Lambda_r$.

For each $\gamma\in\mathscr{P}$ define $\bar{\Gamma}(\gamma)$ as the family of clusters in $\gamma_H\cup\hat{r}(\gamma_H)$, where $\hat{r}$ is the reflection to the $(d-1)$-dimensional hyperplane ∂H. Clearly $\bar{\Gamma}(\gamma_1)\sim\bar{\Gamma}(\gamma_2)$ if $\gamma_1\sim\gamma_2$. Because $\mathrm{Sr}\,F\subset H$ we have

$$F(\Gamma)=F(\bar{\Gamma})\,,\qquad \bar{\Gamma}=\bigcup_{\gamma\in\Gamma}\bar{\Gamma}(\gamma)\,.$$

Now let Γ, $\Gamma'\in\mathscr{G}$ be such that $s(\Gamma)\cap\Lambda_r=s(\Gamma')\cap\Lambda_r$. Then $s(\bar{\Gamma})\cap\Lambda_r=s(\bar{\Gamma}')\cap\Lambda_r$ too.

$$\gamma\in\bar{\Gamma}\cap\mathscr{P}_f(\Xi,r)\;\Rightarrow\;\gamma_H\subset\Lambda_r\;\Rightarrow\;\gamma\in\bar{\Gamma}'\cap\mathscr{P}_f(\Xi,r)$$

and vice versa. Thus $\bar{\Gamma}\cap\mathscr{P}_f(\Xi,r)=\bar{\Gamma}'\cap\mathscr{P}_f(\Xi,r)$, and we obtain that the (multiplicative) function

$$\Gamma\mapsto F_r(\Gamma):=F(\bar{\Gamma}\cap\mathscr{P}_f(\Xi,r))$$

is strictly local with sensitivity region $\mathrm{Sr}\,F_r\subset\Lambda_r$.

To prove $\lim_{r\to\infty}\langle|F-F_r|^2\rangle=0$ take a $\Gamma\in\mathscr{G}$ such that $F(\Gamma)-F_r(\Gamma)\neq0$. Then $F(\bar{\Gamma})-F_r(\bar{\Gamma})\neq0$ too and therefore $\exists\bar{\gamma}\in\bar{\Gamma}$ with $|\bar{\gamma}_H|>r$ and $\mathrm{dist}(\bar{\gamma}_H,\Xi)\leqq f(|\bar{\gamma}_H|)$. Since there exists a cluster $\gamma\in\Gamma$ such that $\bar{\gamma}$ is a cluster of $\bar{\Gamma}(\gamma)$, this γ satisfies $|\gamma|>r$ and $\mathrm{dist}(\gamma,\Xi)\leqq f(|\gamma|)$. Hence formulae (5.19) and (5.20) are applicable. □

Theorem 5.11. *Let* κ *and* κ' *be* λ*-regular and Markovian gauges,* ψ *and* ψ' *the corresponding non-local fields. Let* x, $y\in\sum_+$, $A\in\mathscr{A}_{0+}$ *be arbitrary. Then there exists a sequence* $\{A_r|r=1,2,\ldots\}\subset\mathscr{A}_{0+}$ *such that*

$$\lim_{r\to\infty}\|\Phi(A_r\psi(x))-\Phi(A\psi'(y))\|=0$$

whenever the couplings are either in the region determined in Theorem 5.5 or in the one determined in Theorem 6.2.

Proof. Since the bilinear form $(B_1,B_2)_G=\langle\bar{B}_1B_2\rangle$ is positive semidefinite

$$\begin{aligned}\|\Phi(A_r\psi(x))-\Phi(A\psi'(y))\|^2&=(\theta(A_r\psi(x)-A\psi'(y)),A_r\psi(x)-A\psi'(y))_G\\&\leqq\langle|A_r\psi(x)-A\psi'(y)|^2\rangle\,.\end{aligned}$$

Let us take the ansatz $A_r=A\varphi(x)U(J_{xy})\varphi(y)A'_r$ with some $J_{xy}\subset\sum_+^1$, $\partial J_{xy}=\{x,y\}$. Since

$$\langle|A_r\psi(x)-A\psi'(y)|^2\rangle\leqq\|A\|^2\langle|A'_r-\psi(x)\varphi(x)U(J_{xy})\varphi(y)\psi'(y)|^2\rangle$$

we are ready if we can show that $F=\psi(x)\varphi(x)U(J_{xy})\varphi(y)\psi'(y)$ satisfies the conditions of Lemma 5.10 with $H=\sum_+$.

F depends only on $P=\mathrm{Supp}\,dU$, namely

$$F(P)=(\kappa(P);J_{xy})(\kappa(P)\Delta\kappa'(P);J_y)\,,\tag{5.21}$$

where $J_y \subset \sum_+^1$, $\partial J_y = \{y\}$. So F is seemingly multiplicative, $|F| \leqq 1$ and $\mathrm{Sr}\, F \subset \sum_+$. Now let P be a co-connected 2-cocycle and suppose that $F(P) = -1$. Then either the first or the second factor in (5.21) is -1. In both cases $P_+ \neq \emptyset$. In the first case, because of the Markov property, $\kappa(\bar{P}) \cap J_{xy} \neq \emptyset$, where $\bar{P} = P_+ \cup \theta P_+$. Therefore

$$\mathrm{dist}(J_{xy}, P_+) = \mathrm{dist}(J_{xy}, \bar{P}) \leqq \lambda(|\bar{P}|) = \lambda(2|P_+|). \tag{5.22}$$

In the second case consider the unique finite $B \subset \sum^0$ which solves the equation $dB = \kappa(P) \Delta \kappa'(P)$. Then one finds that $y \in B$. Because of λ-regularity of κ and κ', any link $\ell \in dB$ has a distance from P not larger than $\lambda(|P|)$. Thus

$$\mathrm{dist}(\ell, P_+) \leqq \lambda(2|P_+|), \quad \ell \in (dB)_+$$

from the Markov property. So one has for any $p_0 \in P_+$ that the set

$$\{\xi \in \textstyle\sum_+ | \mathrm{dist}(\xi, p_0) \leqq |P_+| + \lambda(2|P_+|)\}$$

contains dB_+ and therefore y too. Consequently

$$\mathrm{dist}(y, P_+) \leqq |P_+| + \lambda(2|P_+|). \tag{5.23}$$

Comparing (5.22) and (5.23) we obtain that $F(P) \neq 1$ implies that $P_+ \neq \emptyset$ and $\mathrm{dist}(\Xi, P_+) \leqq f(|P_+|)$, where $\Xi = J_{xy} \cup \{y\}$ and $f(r) = r + \lambda(2r)$. So we can apply Lemma 5.10 substituting (5.21) into F and the output $F_r \in \mathscr{A}_{0+}$ into A_r'. □

This theorem establishes the promised translation covariance and gauge independence of the sector $\mathscr{H}(\psi(x))$. As a matter of fact the closure of $\psi(x)\mathscr{A}_{0+}$ does not depend any more on $x \in \sum_+^0$ and on the gauge κ whatever λ-regular and Markovian gauge κ is, provided it is countrolled by the given cluster expansion. The transfer matrix and the unitary space translations therefore leave the subspace $\mathscr{H}(\psi(x))$ invariant even if the gauge is not maximally covariant.

Remark. The proof of Theorem 5.11 shows that the approximation of $\psi(x)\psi(y)$ by local fields is possible in an even stronger sense. Namely in the topology defined by the norm $\|B\|_G = \langle |B|^2 \rangle^{1/2}$. Hence the algebra $\mathscr{A}(\psi)$ generated by $\psi(x)$ $x \in \sum^0$ and by the quasilocal fields is in reality a one element extension of $\mathscr{A}$:

$$\mathscr{A}(\psi) = \overline{\mathscr{A} + \psi(x)\mathscr{A}}^G,$$

some kind of a "square root" of the quasilocal algebra. In the quantum theory we have to divide $\mathscr{A}(\psi)$ into $\mathscr{A}_+(\psi)$ and $\mathscr{A}_-(\psi)$, therefore $\psi(x)\psi(y)$ $x^0 < 0$, $y^0 > 0$ has a significance in this case. But multiple products $A\psi(x_1)\ldots\psi(x_n)$ $n \geqq 3$, $A \in \mathscr{A}$ carry no new information.

5.4. The Charged Sector

Theorem 5.12. *Let $\psi(x)$ be constructed from a Markovian and λ-regular gauge. If the couplings are in the region determined in Theorem 5.5 then the subspace $\mathscr{H}(\psi(x))$ of the big Hilbert space $\mathscr{H}^g$ has the following properties.*

i) *The charge operator $\mathbf{C}$ restricted to $\mathscr{H}(\psi(x))$ is $-\mathbb{1}$.*

ii) *$\mathscr{H}(\psi(x))$ is invariant under the action of the quantum field algebra $\mathfrak{U}$, the transfer matrix $\mathbf{T}$ and the space translations $\mathbf{U}(\mathbf{x})$.*

iii) *All vector states* $\Phi \in \mathscr{H}(\psi) \equiv \mathscr{H}(\psi(x))$ $\Phi \neq 0$ *have finite energy, i.e.* $(\Phi, \mathbf{T}\Phi) > 0$. *There exists an energy gap* $M > 0$ *separating* $\mathscr{H}(\psi)$ *from the vacuum* $\Omega = \Phi(1) \in \mathscr{H}(1)$, *i.e.*

$$(\Phi, \mathbf{T}^n \Phi) \leqq e^{-Mn} \quad \textit{if} \quad \|\Phi\| = 1 .$$

iv) *There exists no translation invariant vector in* $\mathscr{H}(\psi)$.

v) *At large distances the sector* $\mathscr{H}(\psi)$ *is indistinguishable from the vacuum sector for local measurements:*

$$\lim_{\mathbf{x} \to \infty} (\Phi, \mathbf{U}(\mathbf{x}) \mathfrak{a} \mathbf{U}(-\mathbf{x}) \Phi) = (\Omega, \mathfrak{a} \Omega) \; \forall \Phi \in \mathscr{H}(\psi), \qquad \|\Phi\| = 1, \qquad \mathfrak{a} \in \mathfrak{U} .$$

Proof. i) is trivial. ii) follows from Proposition 3.2 and Theorem 5.11. iii): The finite energy property follows from the fact that by virtue of Proposition 3.2 again $\mathsf{T}(\psi \mathscr{A}_{0+})$ is dense in $\psi \mathscr{A}_{0+}$, therefore $\mathbf{T}$ has a densely defined inverse. To prove the existence of an energy gap use Lemma 5.7 and the Schwartz inequality m times:

$$\begin{aligned}(\Phi(A\psi(x)), \mathbf{T}^n \Phi(A\psi(x))) &\leqq \|\Phi(A\psi(x))\|^{1+2^{-1}+\dots+2^{-m+1}} \\ &\times (\Phi(A\psi(x)), \mathbf{T}^{n2^m} \Phi(A\psi(x)))^{2^{-m}} < \|\Phi(A\psi(x))\|^{2-2^{-m+1}} C_A^{2^{-m}} e^{-Mn} \\ &\to \|\Phi(A\psi(x))\|^2 e^{-Mn} .\end{aligned}$$

Now the statement for all $\Phi \in \mathscr{H}(\psi)$ follows from continuity. iv): Applying Lemma 5.7 to space translations, one obtains for all $\mathbf{p} \in [-\pi, \pi]^{d-1}$ that

$$\lim_{\Lambda \to \Sigma} \frac{1}{|\Lambda|} \sum_{\mathbf{x} \in \Lambda} e^{-i\mathbf{p}\mathbf{x}} (\Phi, \mathbf{U}(\mathbf{x}) \Phi) = 0$$

at first for the dense set $\Phi \in \psi(x) \mathscr{A}_{0+}$ then by continuity for all $\Phi \in \mathscr{H}(\psi)$. So there is no eigenvector of $\mathbf{U}(\mathbf{x})$ in $\mathscr{H}(\psi)$. v) follows from Lemma 5.8 in the special case when $B \in \mathscr{A}_0$ is such that $\operatorname{Sr} B \subset \sum([-\frac{1}{2}, \frac{1}{2}])$ because every $\mathfrak{a} \in \mathfrak{U}_0$ can be represented by such a B (Proposition 3.2). The extension to $\mathfrak{a} \in \mathfrak{U}$ and to all Φ is straightforward. □

Remark. The statement under v) that $\mathscr{H}(\psi)$ is locally indistinguishable from the vacuum does not exclude the possibility that measuring the electric flux through an infinitely far removed $(d-2)$-sphere in space gives different results in $\mathscr{H}(\psi)$ and in $\mathscr{H}(1)$. To decide whether this really happens is desirable anyway because until now we have seen only for the charge operator $\mathbf{C}$ that it is different in the two sectors. However $\mathbf{C}$ was defined in terms of C, the global gauge transformation, therefore it is a purely algebraic and very non-local quantity and is not expected to be measurable.

The physically measurable charge must be determined by local measurements. Let us define for $\Phi \in \mathscr{H}^g$ the quantity

$$\mathbb{Q}[\Phi] = \frac{1}{\|\Phi\|^2} \lim_{\Lambda_0 \to \Sigma_0} \frac{(\Phi, Q(\Lambda_0)\Phi)}{(\Omega, Q(\Lambda_0)\Omega)} \; (\Lambda_0 \subset \Sigma_0, |\Lambda_0| < \infty) , \tag{5.24}$$

where $Q(\Lambda_0) = \prod_{\ell \in \Lambda_0^1} \hat{\tau}(\ell)$ is the charge operator of the finite volume $\mathbf{\Lambda} = \Lambda_0 + \frac{1}{2}\hat{0} \subset \sum(1/2) \equiv \mathbf{\Sigma}$.

The numerator and the denominator of (5.24) separately would go to zero. This is a property of the multiplicative and compact charge that the charge density fluctuations in a neighbourhood of the surface $\partial\Lambda$ completely destroy the information about the charge in the middle of Λ. However because of Theorem 5.12 v) these fluctuations are common in the two sectors so one expects (5.24) to be a sensible quantity.

Theorem 5.13. *For Φ in the dense subspace $\psi(x)\mathscr{A}_{0+}+\mathscr{N}$ $\mathbb{Q}[\varphi]=-1$, while if $\Phi\in\mathscr{A}_{0+}+\mathscr{N}$ then $\mathbb{Q}[\Phi]=1$.*

Proof. The action of $Q(\Lambda_0)$ is represented by the classical field A_L where $L=\Lambda_0^1$. Let $\Phi=\Phi(\psi(x)A_K)$ and $\Phi'=\Phi(\psi(x)A_{K'})$ with some $K, K'\subset\sum_+^1$, then

$$\frac{(\Phi,Q(\Lambda_0)\Phi')}{(\Omega,Q(\Lambda_0)\Omega)}=\langle\psi(\theta x)\psi(x)A_{\theta K\Delta K'}\rangle(L)\,, \tag{5.25}$$

where $\langle\ \rangle(L)$ denotes the Gibbs state with a changed sign in the matter coupling α on the links of L. The cluster expansion analogous to (5.6) is

$$\langle\psi(\theta x)\psi(x)A_{\theta K\Delta K'}\rangle=\sum_{J\in\mathrm{Conn}(\{x,y\})}(\tanh\alpha)^{|J|}(L\Delta\theta K\Delta K';J)\,e^{-\mathscr{F}(J,\theta K\Delta K'|L)}, \tag{5.26}$$

where $\mathscr{F}(J,K|L)$ is the free energy excess produced by the replacement [cf. (5.8)]

$$z_L(\gamma)\to z_{L\Delta K}(\gamma)u(\gamma|J)$$

in the system with activity $z_L(\gamma)=z(\gamma)\prod_{L'\in\gamma}(L;L')$. The series (5.26) is absolutely and uniformly convergent in L. On the other hand for a given J if L is large enough $|J\cap L|$ must be odd. Similarly the cluster expansion for $\mathscr{F}(J,\theta K\Delta K'|L)$ cut off at a maximal length is identical with the cut off cluster expansion of $\mathscr{F}(J,\theta K\Delta K'|\emptyset)$ if L is large enough because $z_L(\gamma)=z(\gamma)$ in that case. Therefore

$$\lim_{L\to\Sigma_0}\langle\psi(\theta x)\psi(x)A_{\theta K\Delta K'}\rangle(L)=-\langle\psi(\theta x)\psi(x)A_{\theta K\Delta K'}\rangle=-(\Phi,\Phi')\,. \tag{5.27}$$

Now taking finite linear combinations of (5.27) $\mathbb{Q}[\Phi(A\psi(x))]=-1$ follows. After this the proof of $\mathbb{Q}[\Phi(A)]=1$ is straightforward. □

To establish connection between our construction and that of Fredenhagen and Marcu let us mention the following theorem without proof:

Theorem 5.14. *Let $\Phi_n=\Phi(\varphi(x)U(J_{x\,x+n\hat{0}})\mathrm{T}^n(\varphi(x)\psi(x)))$, where $x=(1/2,0)$ and $J_{xx+n\hat{0}}$ is the straight line connecting x and $x+n\hat{0}$. Then the following limit converges in norm for sufficiently small α and large β:*

$$\Phi=\lim_{n\to\infty}\frac{\Phi_n}{\|\Phi_n\|}\,.$$

Hence Φ is a charged vectorstate from $\mathscr{H}(\psi)$. Furthermore

$$(\Phi,\mathfrak{a}\Phi)=\omega(\mathfrak{a})\,,\qquad \mathfrak{a}\in\mathfrak{U}\,,$$

where ω is the charged state defined by Fredenhagen and Marcu.

6. The Problem of Global Gauge Symmetry Breaking

6.1. Symmetry Breaking in the Classical System

In this section we return to our original definition (2.2–4) of the Gibbs state and will study the expectation values of the non-local field $\psi(x)$ when α is large and β is arbitrary.

The cluster expansion in this region is obtained by going to the unitary gauge $\varphi(x)=1$ and expanding in powers of $e^{-2\alpha}$. The clusters will be co-connected sets of links L with activity

$$z(L)=e^{-2\alpha|L|}e^{-2\beta|dL|} \tag{6.1}$$

and compatibility $\sim$ will mean the opposite of co-connectedness.

The advantageous property of $\psi(x)$ is that it factorizes in terms of these clusters too. If $\operatorname{Supp}U=L=L_1\cup\ldots\cup L_n$, $L_i\sim L_j$ $i\neq j$ then

$$\begin{aligned}\psi(x)&=\prod_{i=1}^{n}(L_i;J_x)(\kappa(dL_i);J_x)=\prod_{i=1}^{n}u(L_i)\\&=\frac{1}{Z_\Lambda}\sum_{L\subset\Lambda^1}e^{-2\alpha|L|-2\beta|dL|}[\psi(x)](L)\\&=\frac{1}{Z_\Lambda}\sum_{\Gamma\in\mathscr{G}_\Lambda}\prod_{L\in\Gamma}z(L)u(L)=\frac{Z_\Lambda(J_x)}{Z_\Lambda(\emptyset)}.\end{aligned}\tag{6.2}$$

The remarkable property of u is that if one considers L if $dL\neq\emptyset$ as the image of $P=dL$ under a (multiplicative in the sense of Definition 5.1 i)) gauge κ' then

$$u(L)=(J_x;dB)=(-1)^{\delta(x\in B)}, \tag{6.3}$$

where $B\subset\sum^0$ is the unique finite solution of the equation

$$dB=\begin{cases}\kappa'(P)\Delta\kappa(P) & \text{if } P\neq\emptyset\\ L & \text{if } P=\emptyset.\end{cases}$$

Now we could argue like in the proof of Theorem 5.11 that $u(L)=-1$ implies $\operatorname{dist}(x,P)\leqq\operatorname{const}\lambda(|P|)$ if κ' were λ-regular. The lack of λ-regularity of κ' is, however, compensated by the fact that now the activity $z(L)$ will supress the contribution from large co-surfaces $L=\kappa'(dL)$. So one conjectures that $\langle\psi(x)\rangle^+$ will be non-zero if α is large enough.

Lemma 6.1. *Let $B\subset\sum^0$ be finite and $x\in B$. Then*

$$\operatorname{dist}(x,dB)\leqq\frac{d}{2}\left[\left(\frac{|dB|}{2d}\right)^{\frac{1}{d-1}}+1\right].$$

Proof. If $\operatorname{dist}(x,dB)>R\geqq d$ then the cube $C=\{y\in\sum^0||y^\mu-x^\mu|\leqq R/d\ \mu=0,\ldots,d-1\}$ lies inside B. Since each straight line starting from a site of one of the faces of ∂C and perpendicular to it must intersect dB, it follows that

$$|dB|>2d\left(\frac{2R}{d}-1\right)^{d-1}.$$

If R is chosen to be the RHS of Lemma 6.1, then $\operatorname{dist}(x,dB)>R$ leads to contradiction. If $\operatorname{dist}(x,dB)<d$ the statement is trivial. $\square$

Theorem 6.2. *There exists a region* $|e^{-2\alpha}|<e^{-b_0}$, $\mathrm{Re}\beta\geqq 0$ *of the coupling constant plane where the correlation functions of the Higgs field in any* λ*-regular gauge satisfy:*

i) $\langle\psi(x)\rangle^{\pm}$ *and* $\langle\psi(x)\psi(y)\rangle$ *are analytic functions of* $e^{-2\alpha}$ *and* $e^{-2\beta}$.

On the physical sheet $\alpha\geqq 0$, $\beta\geqq 0$:

ii) $\langle\psi(x)\rangle^{+}=-\langle\psi(x)\rangle^{-}=h>0,$

iii) $\lim_{y\to\infty}[\langle\psi(x)\psi(y)\rangle-\langle\psi(x)\rangle\langle\psi(y)\rangle]=h^2,$

iv) $\lim_{y\to\infty}[\langle\psi(x)\psi(y)\rangle^{\pm}-\langle\psi(x)\rangle^{\pm}\langle\psi(y)\rangle^{\pm}]=0.$

Proof. Using (6.2) we write

$$\ln\frac{Z_\Lambda(J_x)}{Z_\Lambda(\emptyset)}=\int_0^1 dt\sum_{L\in\mathscr{P}_\Lambda}z(L)[u(L)-1]\varrho_t(\{L\}),\tag{6.4}$$

where $\varrho_t(\Gamma)$ for Γ a set of mutually compatible clusters satisfies the bound

$$|\varrho_t(\Gamma)|\leqq\frac{e^{b_1(\|\Gamma\|-1)}}{1-K_0}\quad\text{if}\quad|z(L)|<e^{-b_0|L|}\tag{6.5}$$

like in (5.11), where now

$$K_0=e^{-b_1+F(b_0-b_1)},$$

and we have to find the smallest b_0 for which there exists a $b_1\in\mathbb{R}$ with $K_0<1$. The function F comes from an estimate on the number $\mathscr{N}(n)$ of clusters L with length $|L|=n$ and incompatible with an elementary cluster $\{\ell\}$:

$$\mathscr{N}(n)\leqq C_d^{\,n},\qquad C_d=[6(d-1)]^2.$$

Thus

$$\sum_{\substack{L\in\mathscr{P}\\ L\not\sim L_0}}e^{-b|L|}\leqq F(b)|L_0|,\qquad F(b)=\frac{C_de^{-b}}{1-C_de^{-b}},\qquad b>\ln C_d.$$

Substituting (6.5) into (6.4) one finds

$$\ln\frac{Z_\Lambda(J_x)}{Z_\Lambda(\emptyset)}\leqq\frac{2e^{-b_1}}{1-K_0}\sum_{\substack{L\in\mathscr{P}\\ u(L)=-1}}e^{-(b_0-b_1)|L|}.\tag{6.6}$$

So to verify analyticity and positiveness of $\langle\psi(x)\rangle^{+}$ it is enough to show that the sum in (6.6) converges for all λ-regular gauges.

If $u(L)=-1$ then there are two possibilities:

1) $dL=\emptyset$. Then $dB=L$ and from Lemma 6.1

$$\mathrm{dist}(x,L)\leqq\frac{d}{2}\left[\left(\frac{|L|}{2d}\right)^{\frac{1}{d-1}}+1\right].$$

2) $dL\neq\emptyset$. Then $dB=L\Delta\kappa(dL)$. For $\ell\in\kappa(dL)\,\mathrm{dist}(\ell,dL)\leqq\lambda(|dL|)$, thus $\mathrm{dist}(\ell,L)\leqq\lambda(|dL|)+\frac{1}{2}$.

For $\ell\in L$ $\mathrm{dist}(\ell,L)=0$. Consequently

$$\mathrm{dist}(\ell,L)\leqq\lambda(|dL|)+\tfrac{1}{2},\qquad\ell\in dB.$$

Let $\ell_0 \in L$, then the "ball"

$$\{\xi \in \sum | |\xi - \ell_0| \leqq \lambda(|dL|) + |L|\}$$

contains dB and therefore x too. Hence

$$\operatorname{dist}(x, L) \leqq \lambda(|dL|) + |L|.$$

In both cases $\operatorname{dist}(x, L) \leqq \operatorname{const}|L|^k$ according to our definition of λ-regularity. The sum in (6.6) therefore converges for all $b_0 - b_1 > 0$. The minimal b_0 for which $K_0 < 1$ can be satisfied with some b_1 is

$$b_0 = \ln C_d + \ln \frac{3+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}.$$

This corresponds to the value $\alpha = 3.27$ ($d=3$) and $\alpha = 3.68$ ($d=4$).

We turn to the proof of iv).

$$\ln \frac{\langle \psi(x)\psi(y)\rangle_A^+}{\langle \psi(x)\rangle_A^+ \langle \psi(y)\rangle_A^+} = \sum_{X:\mathscr{P}_A \to \mathbb{N}_0} \frac{a(X)}{X!} z^X [u_{xy}^X - 1 - (u_x^X - 1) - (u_y^X - 1)]$$
$$= \sum_{X:\mathscr{P}_A \to \mathbb{N}_0} \frac{a(X)}{X!} z^X (u_x^X - 1)(u_y^X - 1),$$

where $\mathbb{N}_0 = \{0, 1, \ldots\}$ so X is a multiindex on the set of clusters. $a(X)$ is a combinatorical factor [23], $X! = \prod_L X(L)!$, $z^X = \prod_L z(L)^{X(L)}, \ldots$ etc. u_{xy}, u_x, u_y are obtained from the u of (6.3) substituting $J_x \Delta J_y$, J_x, J_y respectively. Since there exists [23] a constant K such that $|a(X)/X!| \leqq \exp K \|X\|$, where $\|X\| = \sum_L X(L)|L|$, one finds that

$$\left| \ln \frac{\langle \psi(x)\psi(y)\rangle_A^+}{\langle \psi(x)\rangle_A^+ \langle \psi(y)\rangle_A^+} \right| \leqq; \sum_{\substack{X:\mathscr{P} \to \mathbb{N}_0 \\ \operatorname{Supp} X \cap \mathscr{P}_x \neq \emptyset \\ \operatorname{Supp} X \cap \mathscr{P}_y \neq \emptyset}} \frac{a(X)}{X!} e^{-2\alpha\|X\|}$$
$$\leqq \operatorname{const} \sum_{n \geqq \operatorname{const}|x-y|^{1/k}} n^{kd} \left(\frac{C_d}{e^{2\alpha - K} - 1} \right)^n, \quad (6.7)$$

where $\mathscr{P}_x = \{L \in \mathscr{P} | u_x(L) = -1\}$. For sufficiently large α the RHS of (6.7) goes to zero when $y \to \infty$ and iv) is proven. Then iii) follows from $\langle \psi(x)\psi(y)\rangle = \langle \psi(x)\psi(y)\rangle^+$ and from $\langle \psi(x)\rangle = 0$. □

The above theorem proves that there exist gauges in the $Z(2)$ model in which the Higgs field acquires a non-zero expectation value if the Higgs coupling α is large (cf. Theorem 4.1). Furthermore if the Gibbs states are considered on the classical algebra $\mathscr{A}(\psi)$ generated by $\mathscr{A}$ and the $\psi(x)$'s ($x \in \sum^0$) then the global gauge symmetry C is spontaneously broken for large α because there exist pure states, namely $\langle\ \rangle^+$ and $\langle\ \rangle^-$ which are not C-symmetric. We must emphasize, however, that we can speak about symmetry breaking only if we enlarge the classical quasilocal algebra $\mathscr{A}$ to become a non-local field algebra $\mathscr{A}(\psi)$. So this result is not in contradiction with the analyticity proven by Osterwalder and Seiler [14] for all α if β is small.

6.2. No Symmetry Breaking in the Quantum System

The cluster expansion discussed previously allows us to use Theorem 5.11 which in turn proves that the sector $\mathscr{H}(\psi(x))$ is an invariant subspace of the quasilocal algebra $\mathfrak{U}$ as well as of the spacetime translations $\mathbf{T}$, $\mathbf{U}(\mathbf{x})$. Moreover the sector $\mathscr{H}(\psi)=\mathscr{H}(\psi(x))$ has C-charge -1. At this point two questions emerge:

1) Does the failure of clustering in $\langle\ \rangle$ imply that there exist vectors in $\mathscr{H}(\psi)$ degenerated with the vacuum?

2) If yes, can a local quantum measurement distinguish a "charged" vacuum from a chargeless one?

The answer is affirmative to the first question but denying to the second:

Theorem 6.3. *If α is large enough then the sequence $\Phi(\mathsf{T}^n\psi(x))/\|\Phi(\psi(x))\|$ $n=1,2,\ldots$ converges in norm and defines a unit vector $\Phi_\infty\in\mathscr{H}(\psi)$ which is space and time translation invariant. However for any observable $\mathfrak{a}\in\mathfrak{U}$*

$$(\Phi_\infty, \mathfrak{a}\Phi_\infty)=(\Omega, \mathfrak{a}\Omega).$$

Moreover $\mathbb{Q}[\Phi_\infty]=1$ while $\mathbf{C}\Phi_\infty=-\Phi_\infty$.

Proof.

$$\begin{aligned}\|\Phi(\mathsf{T}^n\psi(x))-\Phi(\mathsf{T}^m\psi(x))\|^2 &= \langle\psi(\theta x-n\hat{0})\psi(x+n\hat{0})\rangle+\langle\psi(\theta x-m\hat{0})\psi(x+m\hat{0})\rangle\\ &\quad -\langle\psi(\theta x-n\hat{0})\psi(x+m\hat{0})\rangle-\langle\psi(\theta x-m\hat{0})\psi(x+n\hat{0})\rangle\\ &\to 0\end{aligned}$$

when $n,m\to\infty$ and $\lim\limits_{n\to\infty}\|\Phi(\mathsf{T}^n\psi(x))\|=h\neq 0$ both because of the iii) part of Theorem 6.2. This proves the existence of Φ_∞. To prove $(\Phi_\infty, \mathfrak{a}\Phi_\infty)=(\Omega, \mathfrak{a}\Omega)\mathfrak{a}\in\mathfrak{U}$ we have only to show that

$$\lim_{n\to\infty}\langle\psi(\theta x-n\hat{0})U(K)\varphi(\partial K)\psi(x+n\hat{0})\rangle=h^2\langle U(K)\varphi(\partial K)\rangle \tag{6.8}$$

for $K\subset\sum^1$ finite. The cluster expansion for the LHS of (6.8) differs from that of $\langle\psi(\theta x-n\hat{0})\psi(x+n\hat{0})\rangle$ in that the activities are changed according to $z(L)\to z(L)(L;K)$. Therefore the clustering (6.8) can be proven in the usual way.

To see $\mathbb{Q}[\Phi_\infty]=1$ it is enough to refer to the previous statement; since $Q(\Lambda_0)$ is a special local operator, therefore $(\Phi_\infty, Q(\Lambda_0)\Phi_\infty)/(\Omega, Q(\Lambda_0)\Omega)=1$. □

Summarizing the breaking of the global gauge symmetry in the non-local classical system is unobservable in the corresponding local quantum theory: $\mathfrak{U}\restriction\mathscr{H}(\psi)$ is unitarily equivalent to $\mathfrak{U}\restriction\mathscr{H}(1)$. Now new sector was found in this region by means of λ-regular gauges.

Though not an observable one, there is still a difference between the small α-small β and the large α regions of the confinement phase. This is the presence of states with $\mathbf{C}=-1$ for large α which are absent for small α. The difference between $\mathbf{C}$ and $\mathbb{Q}$ can be interpreted as screening. In this respect $\mathbf{C}$ plays the role of a bare charge.

Appendix

At first we construct a λ-regular Markovian gauge κ. Let us divide $\sum$ into the following disjoint subcomplexes: $\sum = \ldots \cup \Xi_{-1} \cup \Xi_0 \cup \Xi_1 \cup \ldots$. Here $\Xi_n = \sum([n-\frac{1}{2}, n])$ if $n=1,2,\ldots$, $\Xi_n = \theta\Xi_{-n}$ if $n=-1,-2,\ldots$ and $\Xi_0 = \sum(0)$. Any co-connected 2-cocycle P can be decomposed accordingly as $P(t_-)\Delta P(t_-+1)\Delta\ldots\Delta P(t_+)$; $P(n)\subset\Xi_n$, $t_-, t_+\in\mathbb{Z}$, $t_-\leqq t_+$. Define $L=\kappa(P)$ recursively as follows: If $t_+>0$ then let

$$L_{t_+-1/2}\equiv L\cap\sum(t_+-1/2)=(P(t_+)-\tfrac{1}{2}\hat{0})_{t_+-1/2}.$$

Since $P'=P\Delta dL_{t_+-1/2}=P(t_-)\Delta\ldots\Delta P(t_+-2)\Delta P'(t_+-1)$, where $P'(t_+-1)=P(t_+-1)\Delta(P_{t_+}-\hat{0})\subset\Xi_{t_+-1}$ one can repeat this procedure for P' instead of P, so defining $L_{t_+-3/2}\ldots$ etc. until when all plaquettes in $P\cap\sum(>\max\{0,t_-\})$ are cancelled by dL_+ where $L_+=L_{t_+-1/2}\Delta\ldots\Delta L_{\max\{0,t_-\}+1/2}$. If $t_-<0$ then one constructs $L_-=L_{t_-+1/2}\Delta\ldots\Delta L_{\min\{0,t_+\}-1/2}$ analogously. Eventually one arrives to that

$$P=dL_-\Delta\mathbf{P}\Delta dL_+,$$

where $\mathbf{P}\subset\sum(\mathbf{t})$ and

$$\mathbf{t}=\begin{cases} t_- & \text{if } t_->0 \\ 0 & \text{if } t_-\leqq 0\leqq t_+ \\ t & \text{if } t_+<0. \end{cases}$$

Using the fact that $\sum(\mathbf{t})$ is isomorphic to the $(d-1)$-dimensional complex $\mathbf{\Sigma}$ and $\mathbf{P}$ becomes a 1-cocycle in $\mathbf{\Sigma}$ there exists a unique finite $\mathbf{L}\subset\sum^1(\mathbf{t})$ which obeys $d\mathbf{L}=\mathbf{P}$. So let $\kappa(P)=L=L_-\Delta\mathbf{L}\Delta L_+$.

This gauge κ is Markovian and θ-symmetric by construction. In order to verify λ-regularity with as small λ as possible, consider the following general lemma.

Lemma A. *If Z is a connected p-cycle $1\leqq p\leqq d'-1$ on the d'-dimensional lattice and is such that the smallest closed d'-rectangle R containing Z has size $s_1\times s_2\times\ldots\times s_{d'}$ with $s_\mu\geqq 1$ $\mu=1,\ldots,d'$, then for all $\xi\in R$*

$$\mathrm{dist}(\xi,\mathsf{Z})\leqq\frac{d'}{2}\left(1-\frac{1}{p}\right)+|\mathsf{Z}|\Big/4\binom{d'-1}{p-1}.$$

Proof.

$$|\mathsf{Z}|=\sum_{\substack{I\subset\{1,\ldots,d'\}\\|I|=p}}|\mathsf{Z}_I|,$$

where Z_I is the projection of Z onto the p-dimensional coordinate hyperplane indexed by I and $|\mathsf{Z}_I|=\sum_{\xi\in\Sigma}\mathsf{Z}_I(\xi)$. Z_I is an integer valued connected p-chain with mod2 vanishing boundary. Since the smallest rectangle containing Z_I has size $\bigtimes_{\mu\in I} s_\mu$

$$|\mathsf{Z}|\geqq\sum_{\substack{I\subset\{1,\ldots,d'\}\\|I|=p}}2\left[1+\sum_{\mu\in I}(s_\mu-1)\right]=2\binom{d'-1}{p-1}\sum_\mu s_\mu-2(p-1)\binom{d'}{p}.$$

From this and from $\mathrm{dist}(\xi,\mathsf{Z})\leqq\frac{1}{2}\sum_\mu s_\mu(\xi\in R)$ the statement follows. □

Corollary A. *If* Z *is embedded in a* $d \geqq d'$ *dimensional lattice then*

$$\operatorname{dist}(\xi, \mathsf{Z}) \leqq \frac{|\mathsf{Z}|}{4p} + \frac{d}{2}\left(1 - \frac{1}{p}\right).$$

Taking the dual of this statement for $p = d-2$ we obtain – since $\kappa(P)$ was contained in the smallest open rectangle containing P – that

$$\operatorname{dist}(\ell, P) \leqq \frac{|P|}{4(d-2)} + \frac{d(d-3)}{2(d-2)} \qquad \forall \ell \in \kappa(P).$$

So the gauge κ is λ-regular with $\lambda(r) = \dfrac{r}{4(d-2)} + \dfrac{d(d-3)}{2(d-2)}$.

For the construction of a maximally covariant λ-regular gauge $\bar{\kappa}$ from κ let us choose a representing element P_Γ from each orbit $\Gamma \subset \mathscr{E}$ of the transformation group $(G, \mathscr{E})$, where G is the group generated by T_μ $\mu = 1, \ldots, d-1$ and $\mathsf{R}_{\mu\nu}$ $1 \leqq \mu < \nu \leqq d-1$ and $\mathscr{E}$ is the set of co-connected 2-cocycles. Then for $P = gP_\Gamma$ $(g \in G)$ let $\bar{\kappa}(P) = g\kappa(P_\Gamma)$. The multiplicative extension of $\bar{\kappa}$ to all 2-cocycles is then the required gauge.

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