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# A Generalization of the Momentum Mapping Construction for Quaternionic Kähler Manifolds

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**Abstract.** We present a method of reduction of any quaternionic Kähler manifold with isometries to another quaternionic Kähler manifold in which the isometries are divided out. Our method is a generalization of the Marsden-Weinstein construction for symplectic manifolds to the non-symplectic geometry of the quaternionic Kähler case. We compare our results with the known construction for Kähler and hyperKähler manifolds. We also discuss the relevance of our results to the physics of supersymmetric non-linear $\sigma$-models and some applications of the method. In particular, we show that the Wolf spaces can be obtained as the $U(1)$ and $SU(2)$ quotients of quaternionic projective space $\mathbb{H}P(n)$. We also construct an interesting example of compact riemannian $V$-manifolds *(orbifolds)* whose metrics are quaternionic Kähler and not symmetric.

## 1. Introduction

Quaternionic Kähler and hyperKähler manifolds are of increasing interest to both physicists and mathematicians. In quantum field theory nonlinear $\sigma$-model lagrangians with self-interacting scalar fields on these manifolds play a very special rôle: they admit supersymmetric extensions. It is very well known that in 4-dimensional spacetime $N=1$ $(N=2)$ globally supersymmetric interactions of bosons and fermions are determined by geometry of a Kähler (hyperKähler) manifold $M$ [1, 2]. Scalar $\sigma$-model fields $\phi(x)$ are then maps from 4-dimensional coordinate space (for instance Minkowski or Euclidean space) into $M$. In $N=2$ local supersymmetry the situation is different: The riemannian manifold $M$ is restricted to be quaternionic Kähler manifold of negative scalar curvature [3].

A more realistic picture must include also interacting gauge bosons. Thus, one would like to be able to couple fermionic and bosonic $\sigma$-model matter fields to the Yang-Mills vector multiplet without breaking supersymmetry. This issue was first

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investigated by Bagger and Witten [4, 5]. They showed that under certain assumptions it is possible to gauge holomorphic isometries of a Kähler manifold in an $N=1$ locally supersymmetric manner. Later, Hull et al. [6] presented a very detailed discussion of consistent, supersymmetric gaugings of isometries on Kähler manifolds. They also discussed $N=2$ supersymmetric $\sigma$-models and gauging isometries on hyperKähler manifolds. We refer the interested reader to that work for all the details.

In all cases when the gauging described in [6] is possible one can introduce the supersymmetrically and gauge invariant action. If one does not include the kinetic term for the Yang-Mills fields then they are auxiliary and, consequently, after solving their algebraic, non-propagating equation of motion, can be eliminated from the action. From the geometrical point of view this can be understood and interpreted as the symplectic reduction of a Kähler or a hyperKähler manifold with isometries. Hitchin et al. [7] have given a complete discussion of such symplectic quotients in both the Kähler and hyperKähler cases. This is a rather simple generalization of the Marsden-Weinstein reduction of symplectic manifolds with symmetries [8], since all Kähler manifolds are symplectic and all hyperKähler manifolds have three independent symplectic 2-forms. The above construction, formulated in the language of so-called momentum mappings, is of special interest also for mathematicians. It was implicitly used by authors of [9] to construct new hyperKähler metrics. There is quite a long list of examples of hyperKähler metrics that can be obtained through hyperKähler quotients of $\mathbb{R}^{4n}$. We give it in our paper in Table 1 (see also [10]).

As we already mentioned, in the case of $N=2$ local supersymmetry the situation is very different. A $\sigma$-model manifold $M$ must be quaternionic Kähler. Again, one would like to address the problem of gauging isometries of the $\sigma$-model manifold $M$ in a way which is consistent with $N=2$ local supersymmetry. The problem was investigated by de Wit et al. [11]. They coupled $N=2$ supergravity to an arbitrary number of scalar and vector multiplets. In our previous paper we tried to understand this coupling from the point of view of the geometry of the $\sigma$-model manifold $M$ [12]. We pointed out that there exists a very general construction that allows for a consistent reduction of the quaternionic Kähler manifold with isometries. Since there is a very beautiful mathematical description that corresponds to the gauging of isometries of $N=2$ supersymmetric hyperKähler $\sigma$-models given in terms of the momentum mappings, one should ask a natural question: Is there any generalization of the Marsden-Weinstein reduction to the case of quaternionic Kähler manifolds? From the point of view of field theory, such a formalism would correspond to the gauging of isometries of the $N=2$ locally supersymmetric $\sigma$-model.

Though a quaternionic Kähler manifold with non-zero scalar curvature does not have a symplectic structure, we show that a *quaternionic Kähler quotient* can indeed be consistently defined and that many examples of quaternionic Kähler metrics can be obtained as the quotient of quaternionic projective space $\mathbb{H}P(n)$. In general, when the isometry group does not act freely on the zero level set defined in Sect. 4, the above reduction leads to manifolds with singularities or orbifolds. However, these are special, quaternionic orbifolds with a well defined quaternionic Kähler metric everywhere away from singularities. We construct examples of such orbifolds and show that they are not quotients of Wolf spaces by finite groups. In principle, one can always write locally supersymmetric $\sigma$-model lagrangians on orbifolds. The metric defined locally on the non-singular part completely determines the interactions between the $\sigma$-model and the supergravity fields.

Our paper is organized as follows: In Sect. 2 we recall the original Marsden-Weinstein construction for symplectic manifolds. It can be applied to any Kähler manifold with holomorphic isometries. In Sect. 3 we review how the above reduction generalizes to the case of hyperKähler manifolds with triholomorphic isometries. We also give a simple example of the Calabi metric. In Sects. 4 and 5 we present our main result. We introduce *a quaternionic Kähler quotient* and we apply it to quaternionic projective space $\mathbb{H}P(n)$. As examples we show that the Wolf spaces $X(n)=U(n+2)/U(n)\times U(2)$ and $Y(n)=SO(n+4)/SO(n)\times SO(4)$ are just $U(1)$ and $SU(2)$ quotients of quaternionic projective space. We also discuss an example of the quaternionic orbifold that was first introduced in [12], showing that it is not a quotient of a Wolf space by some finite group. And finally, in Sect. 6, we briefly discuss our results and their possible applications. Since many statements are given without rigorous proofs, we refer the reader interested in details to [6–8, 13–15].

## 2. The Marsden-Weinstein Reduction of Kähler Manifolds with Holomorphic Isometries

In this section we review the discussion of [7] (see also [5]). Let $M$ be a smooth Riemannian manifold with a metric $h: TM\times TM\to\mathbb{R}$:

$$ds^2=h_{\alpha\beta}dx^\alpha\otimes dx^\beta\,, \tag{2.1}$$

where $\{x^\alpha\}$; $\alpha=1,\dots,\dim M$ are local coordinates on $M$. Let us introduce an almost complex structure on $M$, i.e., an endomorphism of $TM$; $J: TM\to TM$ such that $J^2=-1$. The manifold $M$ is complex when the almost complex structure $J$ is integrable or, equivalently, when

$$N(X,Y)\stackrel{\text{def}}{=}2(\mathscr{L}_{JX}JY-\mathscr{L}_XY-J\mathscr{L}_XJY-J\mathscr{L}_{JX}Y)=0 \tag{2.2}$$

for all $X, Y\in TM$. $N: TM\times TM\to TM$ is called the Nijenhuis tensor.

Let us suppose that the metric (2.1) is Hermitian with respect to the complex structure $J$:

$$h(X,Y)=h(JX,JY);\qquad \forall X,Y\in TM\,. \tag{2.3}$$

Then $M$ is called a Hermitian manifold. If the complex structure is covariantly constant with respect to the Levi-Civita connection $\nabla$

$$\nabla_X J=0;\qquad X\in TM\,, \tag{2.4}$$

then $M$ is called a Kähler manifold. Now, we can define a 2-form $\omega\in\Lambda^2M$,

$$\omega(X,Y)\stackrel{\text{def}}{=}h(JX,Y)\,. \tag{2.5}$$

Since both the metric $h$ and the complex structure $J$ are covariantly constant with respect to the metric connection $\nabla$, $\omega$ is also covariantly constant. Consequently, it is a closed, non-degenerate 2-form globally defined on $M$. It is usually called the Kähler form and it defines a symplectic structure on $M$. Thus any Kähler manifold $(M, h, J)$ is symplectic.

Let $G$ be a connected Lie group and $G \times M \to M$ an action on $M$. If the above action preserves the 2-form $\omega$ we call it a symplectic (or holomorphic) action. Correspondingly, the infinitesimal action of $G$ on $M$ is given in terms of vector fields $X$ such that for any one-parameter subgroup of $G$ generated by $X$,

$$\mathcal{L}_X \omega = 0. \tag{2.6}$$

If the above action preserves the metric $h$ it is an isometry of $M$ and

$$\mathcal{L}_X h = 0. \tag{2.7}$$

Infinitesimal isometries are generated by Killing vectors; a Killing vector on a Kähler manifold which satisfies (2.6) is called a symplectic (or holomorphic) Killing vector. To each element of the Lie algebra $\mathcal{G}$ of the Lie group $G$ that acts holomorphically on $M$ there corresponds a holomorphic Killing vector field $X$ on $M$. With each symplectic Killing vector field $X$ we associate a Hamilton function $f^X$ on $M$, such that

$$i_X \omega = df^X. \tag{2.8}$$

The Hamilton function $f^X$ is defined only up to an arbitrary constant. Equation (2.8) defines a so-called momentum mapping $\Phi$ in the following way: For every element of the Lie algebra $\mathcal{G}$ we have a function on $M$ given by (2.8). With each point $m \in M$ we associate an element $\Phi(m)$ of the Lie co-algebra $\mathcal{G}^*$:

$$\langle \Phi(m), X \rangle = f^X(m). \tag{2.9}$$

Variation with respect to $m$ gives us a smooth mapping

$$\Phi : M \to \mathcal{G}^*. \tag{2.10}$$

Furthermore, the action of $G$ on $M$ is called Poisson if

$$f^{[X,Y]} = \{f^X, f^Y\} \equiv \omega(X, Y). \tag{2.11}$$

For a Poisson action of $G$ on $M$ the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{g \in G} & M \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{G}^* & \xrightarrow{\mathrm{Ad}_g^*} & \mathcal{G}^* \end{array}$$

or in other words the momentum mapping $\Phi$ is equivariant. In general the function $f^{[X,Y]}$ differs from the Poisson bracket of $f^X$ and $f^Y$ defined in (2.11) by a constant

$$f^{[X,Y]} = \{f^X, f^Y\} + C(X, Y). \tag{2.12}$$

$C(X, Y)$ is a bilinear, skew-symmetric function on the Lie algebra of $G$. Using the Jacobi identity we obtain the following property:

$$C([X, Y], Z) + C([Z, X], Y) + C([Y, Z], X) = 0. \tag{2.13}$$

If we choose different integration constants for $f^X$, then $C(X, Y)$ is replaced by:

$$C'(X, Y) = C(X, Y) + \varrho([X, Y]), \tag{2.14}$$

where $\varrho$ is a linear function on the Lie algebra. $C$ is called a two-dimensional cocycle of the Lie algebra $\mathscr{G}$. The relation (2.14) defines equivalence classes: $C'$ and $C$ are in the same cohomology class when (2.14) holds. One can introduce the cohomology group $H^2(\mathscr{G};\mathbb{R})$. For semi-simple, finite-dimensional Lie groups $H^1(\mathscr{G};\mathbb{R})=H^2(\mathscr{G};\mathbb{R})=0$. This means that all two-dimensional cocycles are cohomologous, and that the action of $G$ on $M$ can always be chosen to be Poisson just by adding constants to the Hamilton functions $f^X$. The notion of the momentum mapping is due to Souriau [16]. He also showed that the momentum mapping $\Phi$ is equivariant with respect to a certain affine action of $G$ on $\mathscr{G}^*$.

It was first noticed by Bagger and Witten [4] that the non-vanishing of $C(X, Y)$ is an obstruction to the consistent gauging of holomorphic isometries on a Kähler manifold. Hull et al. [7] gave an explicit method of calculating the obstructions $C(X, Y)$ in both the Kähler and hyperKähler cases.

Let us assume that our momentum mapping is equivariant. Consider a level set of the momentum $p$:

$$M_p \equiv \{m \in M\colon \Phi(m)=p\}\,; \qquad p \in \mathscr{G}^*\,. \tag{2.15}$$

In general, $M_p$ is not $G$-invariant. Only the isotropy group of $p$ in the co-adjoint representation leaves $M_p$ fixed. We call this subgroup $G \supset G_p$. If $p$ is a regular value of $\Phi$ (so that $M_p$ is a smooth submanifold of $M$) and if $G_p$ is compact and acts freely on $M_p$, then the orbit space:

$$\tilde{M}_p \stackrel{\text{def}}{=} M_p/G_p \tag{2.16}$$

is again a smooth Riemannian manifold of real dimension $\dim\tilde{M}_p = \dim M - 2\dim G_p$. The projection mapping $M_p$ onto $\tilde{M}_p$

$$\pi\colon M_p \to \tilde{M}_p \tag{2.17}$$

is a principal $G_p$-fiberation. Moreover (see [8]), there exists a unique symplectic 2-form $\tilde{\omega}_p$ on $\tilde{M}_p$ such that

$$\pi^*\tilde{\omega}_p = i^*\omega\,, \tag{2.18}$$

where $i$ is the inclusion mapping of $M_p$ into $M$. The same pullback defines a unique complex structure on $\tilde{M}_p$:

$$\pi^*\tilde{J}_p = i^*J\,, \tag{2.19}$$

and a unique Riemannian metric $\tilde{h}_p$. It can easily be shown using the O'Neil formulas that $\tilde{J}_p$ is a covariant constant, Hermitian complex structure on $(\tilde{M}_p, \tilde{h}_p)$ (see [7]). Thus the reduced manifold $\tilde{M}_p$ is not only symplectic but also Kähler.

As an illustrative example let us consider $2n$-dimensional complex vector space $\mathbb{C}^n$ with a flat Hermitian metric

$$ds^2 = d\bar{z}^\alpha \otimes dz^\alpha\,; \qquad \alpha=1,\dots,n \tag{2.20}$$

and the Kähler 2-form

$$\omega = i d\bar{z}^\alpha \wedge dz^\alpha\,. \tag{2.21}$$

We take the following holomorphic $U(1)$ action on $\mathbb{C}^n$:

$$\varphi_t(z) = e^{2\pi i t}z\,; \qquad t \in [0,1)\,. \tag{2.22}$$

It acts freely on $\mathbb{C}^n\backslash\{0\}$. In standard coordinates, the holomorphic Killing vector field has the form

$$X^\alpha(z)=iz^\alpha. \tag{2.23}$$

Now, one can calculate the Hamilton function for this $U(1)$ action from (2.8):

$$f^X(z,\bar{z})=\bar{z}z. \tag{2.24}$$

The $p$-momentum for the $p\neq 0$ level $M_p$ is just $S^{2n-1}$. $G_p=U(1)$ and it acts on the non-zero momentum level freely. The induced symplectic 2-form $\tilde{\omega}_p$

$$\tilde{\omega}_p=id\bar{z}^\alpha\wedge dz^\alpha-id\bar{z}^\alpha z^\alpha\wedge dz^\beta\bar{z}^\beta \tag{2.25}$$

is just symplectic $U(n)$-invariant 2-form on $\mathbb{C}P(n-1)$ with the standard Fubini-Study Kähler metric:

$$\tilde{h}_p=(\delta^{\alpha\beta}-z^\alpha\bar{z}^\beta)d\bar{z}^\alpha\otimes dz^\beta. \tag{2.26}$$

The possibility of obtaining new Kähler metrics through the symplectic quotient reduction is of rather little interest, since there are many other ways to generate interesting examples of them. However, the above method has proven to be extremely fruitful when generalized and applied to the hyperKähler case. Together with the Legendre transform method it has led to the discovery of many new hyperKähler metrics [9].

## 3. HyperKähler Quotients

In this section we review the generalization of the Marsden-Weinstein construction to hyperKähler manifolds with triholomorphic isometries. Let us recall that a hyperKähler manifold is a riemannian manifold $M$ with three independent complex structures $J^i$; $i=1,2,3$,

$$J^i\circ J^j=-\delta^{ij}id+\varepsilon^{ijk}J^k, \tag{3.1}$$

that are covariantly constant, and metric $h$ that is Hermitian with respect to all three complex structures. Consequently, $M$ is a Kähler manifold with respect to these three complex structures. Thus, as in the previous section, we can define three closed, non-degenerate, symplectic 2-forms on $M$:

$$\omega^i(X,Y)=h(J^iX,Y); \qquad X,Y\in TM; \qquad i=1,2,3. \tag{3.2}$$

Now, let us consider the group of isometries on $M$ generated infinitesimally by Killing vector fields. If the action of some subgroup $G$ of the isometry group preserves all three symplectic 2-forms then we call it a trisymplectic (or triholomorphic) action. In terms of the triholomorphic Killing vector field on $M$ we can express the above statement in the following way:

$$\mathcal{L}_X\omega^i=0 \quad \forall i. \tag{3.3}$$

We can exactly follow the method of the previous section to calculate the Hamilton functions of the Killing vector field $X$ with respect to all three symplectic 2-forms $\omega^i$. We can think of them as of three equivariant momentum mappings:

$$\Phi^i: M\to\mathcal{G}^* \tag{3.4}$$

or, equivalently, we can consider mapping $\Phi$:

$$\Phi : M \to \mathcal{G}^* \otimes R^3 . \tag{3.5}$$

Furthermore, let us introduce the $p^i$-momentum level for each mapping

$$M_{p^i} \stackrel{\text{def}}{=} \{m \in M;\ \Phi^i(m) = p^i\} ; \qquad p^i \in \mathcal{G}^* , \tag{3.6}$$

and let us take

$$M_p \stackrel{\text{def}}{=} M_{p^1} \cap M_{p^2} \cap M_{p^3} ,$$

where $p = (p^1, p^2, p^3) \in \mathcal{G}^* \otimes \mathbb{R}^3$. If $p$ is a regular value of the mapping (3.5) then $M_p$ is an algebraic smooth submanifold of $M$. Again. we can consider a subgroup of $G$ such that $p^i$'s are stationary points in the co-adjoint representation $\mathrm{Ad}^*_g p^i = p^i$. We denote it, as before, $G_p$. Assuming that $G_p$ is compact and that it acts freely on $M_p$ we can introduce a space of orbits $\tilde{M}_p = M_p / G_p$. $\tilde{M}_p$ has a uniquely defined quaternionic structure in terms of three closed 2-forms $\tilde{\omega}^i_p$ given by (2.18) or in terms of covariantly constant tensors $\tilde{J}^i_p$ as in (2.20). Consequently, $\tilde{M}_p$ is a hyperKähler manifold of real dimension $\dim \tilde{M}_p = \dim M - 4 \dim G_p$ (see [7]).

As an example (see [9]), we consider the $U(1)$ quotient of $\mathbb{H}^n \cong \mathbb{R}^{4n}$. Since $\mathbb{H}^n$ has an *integrable* global quaternionic structure we can work with global quaternionic coordinates $\{u^\alpha\}$; $\alpha = 1, \ldots, n$. We take the standard flat metric on $\mathbb{H}^n$:

$$ds^2 = \sum_\alpha d\bar{u}^\alpha \otimes du^\alpha , \tag{3.7}$$

where $\bar{u}^\alpha$ is the quaternionic conjugate of $u^\alpha$. We introduce the $\mathbb{H}$-valued 2-form $\omega$,

$$\omega = \sum_\alpha d\bar{u}^\alpha \wedge du^\alpha . \tag{3.8}$$

For any two $\mathbb{H}$-valued forms of degrees $p$ and $q$ we have

$$\overline{\Psi \wedge \Theta} = (-1)^{pq} \bar{\Theta} \wedge \bar{\Psi} . \tag{3.9}$$

We observe that $\omega$ is purely imaginary since $\omega + \bar{\omega} = 0$. The three quaternionic components of $\omega$, $\omega = \omega^i e_i$ correspond to the quaternionic structure in (3.2). $(e^1, e^2, e^3) \stackrel{\text{def}}{=} (i, j, k)$ are the imaginary quaternionic units. It is trivial to see that $\omega$ is closed. Thus $\mathbb{H}^n$ with the metric (3.7) and the quaternionic structure (3.8) is a trivial example of an hyperKähler manifold.

Now, let us consider the circle action on $\mathbb{H}^n$ defined as follows:

$$\varphi_t(u) = e^{2\pi i t} u ; \qquad t \in [0, 1) , \tag{3.10}$$

where $i$ is one of the quaternionic units. Infinitesimally, the above action is given by the $\mathbb{H}$-valued Killing vector field $X$:

$$X^\alpha(u) = i u^\alpha \tag{3.11}$$

which is triholomorphic.

Now, we want to calculate the Hamilton functions of this Killing vector field. It is trivial to see that

$$f^X = (u, \bar{u}) = \sum_\alpha \bar{u}^\alpha i u^\alpha = -\overline{f^X(u, \bar{u})} = \sum_i f^X_i(m) e^i ; \qquad m \in M . \tag{3.12}$$

**Table 1.** HyperKähler quotients of flat spaces

| Manifold $M$ | Quotient group $G_p$ | $\dim \tilde{M}_p$ | HyperKähler metric on $\tilde{M}_p$ |
|---|---|---|---|
| $\mathbb{H}^n$ | $U(1)^{n-1}$ | 4 | Multi-Eguchi-Hanson gravitational instanton |
| $\mathbb{H}^{n-1}\times\mathbb{R}^3\times S^1$ | $U(1)^{n-1}$ | 4 | Multi-Taub-NUT gravitational instanton |
| $\mathbb{H}^n$ | $U(1)$ | $4n-4$ | Calabi metrics on $T^*(\mathbb{C}P(n-1))$ |
| $\mathbb{H}^{n-1}\times\mathbb{R}^3\times S^1$ | $U(1)$ | $4n-4$ | Lindström-Roček metrics: new generalization with Taubian infinity |
| $\mathbb{H}^n$ | $U(1)^m$ | $4n-4m$ | Multi-Calabi spaces |
| $\mathbb{H}^{m(n+m)}$ | $U(m)$ | $4nm$ | Lindström-Roček spaces: hyperKähler metrics on cotangent bundles of complex Grassmannians |

$f^X$ is now a quaternionic valued, purely imaginary function on $M$. We consider the $p \stackrel{\text{def}}{=} p_i e^i$ momentum level in $\mathbb{H}^n$:

$$M_p=\left\{u^\alpha \in \mathbb{H}^n \colon \sum_\alpha \bar{u}^\alpha i u^\alpha = p;\ \bar{p}=-p\right\}. \tag{3.13}$$

For $p \neq 0$ it is a $(4n-3)$-dimensional algebraic submanifold of $M$. The circle action is free on $M_p$ so that $\tilde{M}_p = M_p/U(1)$ is well defined, smooth manifold with hyperKähler metric. Topologically it is the cotangent bundle of complex projective space $\tilde{M}_p = T^*(\mathbb{C}P(n-1))$ and the hyperKähler metric on it is called the Calabi metric. In 4 dimensions, the Calabi metric is simply the well known Eguchi-Hanson gravitational instanton. A very large class of hyperKähler manifolds with explicitly known metrics can be obtained as hyperKähler quotients of $\mathbb{R}^{4n}$. In Table 1 we list all such non-singular examples. They can be found in [9, 10]. One must specify the action of the quotient group $G_p$ in such a way that it is triholomorphic and it acts on the non-zero momentum level freely. As already mentioned, only then is the orbit space a smooth Riemannian manifold. Otherwise, in more general situations, one obtains manifold with singularities. If these singularities arise due to finite isotropy groups, then the quotient leads to interesting examples of $V$-manifolds or orbifolds with a hyperKähler metric everywhere away from singular points. Table 1 does not contain a complete list of hyperKähler spaces and hyperKähler metrics. It is, for instance, known that the $K3$-surface admits a hyperKähler metric, but its explicit form has not been found yet. Notice that all the manifolds listed in Table 1 have Killing vectors.

## 4. Reduction of Quaternionic Kähler Manifolds

In this section we generalize the results presented in Sect. 3 to quaternionic Kähler manifolds. Now, the situation is very different from the hyperKähler case because quaternionic Kähler manifolds are not symplectic. The Marsden-Weinstein construction cannot be applied trivially, but we can generalize it in such a way that a consistent reduction is possible. Let us start with some definitions and properties of quaternionic Kähler manifolds.

A quaternionic Kähler manifold is in some sense a quaternion analogue of a Kähler manifold. It has a quaternionic structure, i.e., three *locally* defined (1,1) tensors that in each neighborhood $U_{(\alpha)} \subset M$ satisfy the quaternionic algebra:

$$J^i_{(\alpha)} \circ J^j_{(\alpha)} = -\delta^{ij} id + \varepsilon^{ijk} J^k_{(\alpha)}; \qquad i, j, k = 1, \ldots, 3. \tag{4.1}$$

Transition functions $S_{ij}(m)$ on $U_{(\alpha)} \cap U_{(\beta)} \subset M$:

$$J^i_{(\alpha)}(m) = S_{ij}(m) J^j_{(\beta)}(m); \qquad m \in M \tag{4.2}$$

are local $SO(3)$ rotations. Thus we have a three-dimensional vector bundle $\mathscr{W}$ of endomorphisms over $M$. [From now on we shall omit the index $(\alpha)$, remembering that all geometrical objects with $SO(3)$ indices are defined *locally* on $M$.] If the manifold $M$ admits such a bundle we say that it is an almost quaternionic manifold. Since we also have a metric $h$ on $M$ we can construct a bundle $\mathscr{V}$ of 2-forms $\omega^i \in \wedge^2 M$ over $M$:

$$\omega^i(X, Y) = h(J^i X, Y); \qquad X, Y \in TM. \tag{4.3}$$

Then the 4-form:

$$\Omega = \sum_i \omega^i \wedge \omega^i \tag{4.4}$$

is defined *globally* on $M$. The manifold $M$ is said to be *quaternionic Kähler* if the 4-form $\Omega$ is parallel with respect to the metric connection. This in turn implies that $\Omega$ is also harmonic and closed:

$$\nabla \Omega = d\Omega = \Delta \Omega = 0. \tag{4.5}$$

Equation (4.5) implies the existence of three local 1-forms on $M$ such that

$$\nabla_X J^i = -\sum_{j,k} \varepsilon^{ijk} \alpha^j(X) J^k; \qquad \alpha^i \in \wedge^1 M \tag{4.6}$$

or, equivalently, in terms of the 2-forms $\omega^i$,

$$d\omega^i = -\sum_{j,k} \varepsilon^{ijk} \alpha^j \wedge \omega^k. \tag{4.7}$$

The $\mathscr{V}$-valued 1-form $\alpha$ is just the $Sp(1)$ part of the riemannian connection with the $Sp(1)$ curvature 2-form

$$F^i = d\alpha^i + \tfrac{1}{2} \sum_{j,k} \varepsilon^{ijk} \alpha^j \wedge \alpha^k. \tag{4.8}$$

Due to (4.5), the holonomy group of a quaternionic Kähler manifold is a subgroup of $Sp(n) \times Sp(1)$ $(n = \dim M)$. All quaternionic Kähler manifolds of dimension bigger than 4 are Einstein spaces, which implies that the $Sp(1)$ part of the riemannian curvature 2-form is proportional to $\omega^i$:

$$F^i = \lambda (n+2)^{-1} \omega^i, \tag{4.9}$$

where $\lambda$ is the proportionality constant between the Ricci tensor and the metric (see [17]).

In 4 dimensions Eq. (4.5) is meaningless: the volume 4-form of any 4-manifold is closed. Equation (4.9) then may be used to extend our definition of quaternionic Kähler manifold. It restricts $M$ to be an Einstein and self-dual manifold. Hitchin [18] proved that there are only two such manifolds: $\mathbb{H}P(1) \cong S^4$ and $\mathbb{C}P(2)$. As we already mentioned, the restricted holonomy group of a quaternionic Kähler manifold is a subgroup of $Sp(n) \times Sp(1)$. This implies the factorization of the Riemann curvature tensor $\mathcal{R}$ into $Sp(1)$ and $Sp(n)$ parts. But even more is true: Alekseevskii [19] showed that $\mathcal{R}$ can be written in the following form:

$$\mathcal{R} = \lambda \mathcal{R}_{\mathbb{H}P(n)} + \mathcal{R}_0 , \tag{4.10}$$

where $\lambda$ is a constant proportional to the scalar curvature, $\mathcal{R}_{\mathbb{H}P(n)}$ is the curvature tensor on quaternionic projective space, and $\mathcal{R}_0$ is the Ricci-flat part of the $Sp(n)$ curvature and behaves as a curvature tensor of a Riemannian manifold with the holonomy group contained in $Sp(n)$. Such manifolds are exactly hyperKähler manifolds discussed in the previous section. We clearly see that quaternionic Kähler manifolds with zero scalar curvature are hyperKähler. In the language of our 3-dimensional vector bundle $\mathcal{V}$, that means that the $Sp(1)$ curvature 2-form $F^i$ vanishes and the bundle $\mathcal{V}$ is trivial (the connection 1-form $\omega$ can be gauged everywhere to zero). The global existence of the covariantly constant 2-forms $\omega^i$ follows, and according to the definition of Sect. 3, the manifold $M$ is hyperKähler.

All homogeneous quaternionic Kähler manifolds were classified by Wolf [20] and Alekseevskii [19, 21]. For $\lambda > 0$ they are compact symmetric spaces:

$$\mathbb{H}P(n) = \frac{Sp(n+1)}{Sp(n) \times Sp(1)}; \quad X(n) = \frac{U(n+2)}{U(n) \times U(2)}; \quad Y(n) = \frac{SO(n+4)}{SO(n) \times SO(4)} \tag{4.11}$$

of dimension $4n$, $n > 2$. For $n = 1$, as was already mentioned, we have only two cases:

$$X(1) = \mathbb{C}P(2), \quad \mathbb{H}P(1) = Y(1) = S^4 . \tag{4.12}$$

Taking the isometry group to be an exceptional Lie group, we obtain five more examples

$$\frac{G_2}{SU(2) \times SP(1)}; \quad \frac{F_4}{Sp(3) \times Sp(1)}; \quad \frac{E_6}{SU(6) \times Sp(1)}; \quad \frac{E_7}{\mathrm{Spin}(12) \times Sp(1)}; \quad \frac{E_8}{E_7 \times Sp(1)} . \tag{4.13}$$

For $\lambda < 0$ there are non-compact analogues of (4.11, 13) and non-symmetric examples constructed by Alekseevskii [21] described by quaternionic representations of Clifford algebras, classified by Atiyah et al. [22]. Examples of non-homogeneous quaternionic Kähler manifolds are not known. Among their interesting properties, any quaternionic Kähler manifold has a naturally associated complex manifold or *twistor space* fibering over it [23]. It is also known that any quaternionic Kähler manifold of dimension $8n$ is spin. In general one cannot introduce quaternionic coordinates on $M$. In the compact case, the integrability condition is so restrictive that the only example of integrable structure exists on $\mathbb{H}P(n)$. Although quaternionic Kähler manifolds are in general not symplectic or Kähler, it may happen that there is a covariantly constant complex structure that is defined globally on $M$. This is, for instance, the case for $X(n)$. But $\mathbb{H}P(n)$ is not even almost complex.

Let us consider $\Gamma(\wedge^p(M)\otimes\mathscr{V})$: the space of differential, exterior $p$-forms on $M$ with values in the bundle $\mathscr{V}$. The $Sp(1)$ part of the riemannian connection on $\mathscr{V}$ gives us a "covariant derivative" $d^{\nabla}$,

$$\Gamma(\wedge^0(M)\otimes\mathscr{V}) \xrightarrow{d^{\nabla}} \Gamma(\wedge^1(M)\otimes\mathscr{V}) \xrightarrow{d^{\nabla}} \Gamma(\wedge^2(M)\otimes\mathscr{V}) \xrightarrow{d^{\nabla}} \dots . \tag{4.14}$$

In local coordinates of $\mathscr{V}$ we can write

$$d^{\nabla}\Theta^i = d\Theta^i + \sum_{j,k} \varepsilon^{ijk}\alpha^j \wedge \Theta^k, \tag{4.15}$$

where $\Theta \in \Gamma(\wedge^p(M)\otimes\mathscr{V})$ is an arbitrary $p$-form.

Let us introduce a Killing vector field $X$ on $M$. To carry out a reduction similar to that presented in the previous sections we first require $X$ to be a *quaternionic Kähler Killing vector field*, i.e. such that corresponding group action on $M$ preserves the 4-form $\Omega$. This means that

$$\mathscr{L}_X\Omega = 0. \tag{4.16}$$

Notice that the group action $G\times M\to M$ does not have to preserve each $\omega^i$ separately. It is sufficient that

$$\mathscr{L}_X\omega^i = \sum_{j,k} \varepsilon^{ijk} r^j \omega^k, \tag{4.17}$$

where $r^j$ are some functions locally defined on $M$ [i.e. $r \in \Gamma(\wedge^0(M)\otimes\mathscr{V})$].

Using $X$ and $\omega^i$ we can introduce 1-forms $\beta^i$ defined locally on $M$

$$\beta^i = i_X\omega^i. \tag{4.18}$$

It is easy to see that with each quaternionic Kähler Killing vector field we can associate a unique section $\mathbf{f}^X = \sum_i f^{i;X}\omega^i \in \Gamma(\wedge^0(M)\otimes\mathscr{V})$ of $\mathscr{V}$ in the following way [13]

$$\beta^i = d^{\nabla} f^{i;X}. \tag{4.19}$$

Let us apply $d^{\nabla}$ to both sides of (4.19). We have

$$d^{\nabla}\beta^i = d^{\nabla}d^{\nabla} f^{i;X}. \tag{4.20}$$

One can, however, easily check that for any $\Theta \in \Gamma(\wedge^p(M)\otimes\mathscr{V})$,

$$d^{\nabla}d^{\nabla}\Theta^i = \sum_{j,k} \varepsilon^{ijk} F^j \wedge \Theta^k, \tag{4.21}$$

where $F^i$ is the $Sp(1)$-curvature 2-form defined in (4.8). Now, using (4.9) we see that

$$d^{\nabla}d^{\nabla}\Theta^i = \lambda(n+2)^{-1} \sum_{j,k} \varepsilon^{ijk}\omega^j \wedge \Theta^k. \tag{4.22}$$

We see that, unlike in the hyperKähler case, the non-vanishing $Sp(1)$ curvature $F^i$ allows one to find the section $\mathbf{f}^X$ uniquely in terms of $\omega^i$, $\alpha^i$, $\beta^i$. Namely, using (4.19) we have

$$d\beta^i + \sum_{j,k} \varepsilon^{ijk}\alpha^j \wedge \beta^k = \lambda(n+2)^{-1} \sum_{j,k} \varepsilon^{ijk}\omega^j f^{k;X}. \tag{4.23}$$

Since the forms $\omega^i$ are non-degenerate and pointwise linearly independent, we can find a unique solution for $f^{i;X}$. It can be given in terms of $\mathscr{L}_X\omega^i$ as follows. Let us apply $i_X$ to both sides of (4.23). We obtain

$$i_X d(i_X\omega^i) + \sum_{j,k} \varepsilon^{ijk}(i_X\alpha^j)\beta^k = \lambda(n+2)^{-1} \sum_{j,k} \varepsilon^{ijk}(i_X\omega^j) f^{k;X}. \tag{4.24}$$

Since $i_X d(i_X\omega^i) = i_X \mathscr{L}_X \omega^i$, using (4.17) we get

$$\sum_{j,k} r^j\beta^k + \sum_{j,k} \varepsilon^{ijk}(i_X\alpha^j)\beta^k = -\lambda(n+2)^{-1} \sum_{j,k} \varepsilon^{ijk} f^{j;X}\beta^k. \tag{4.25}$$

Assuming that when $X \neq 0$ $i_X\omega^i \cong J^i X$ are pointwise linearly independent and non-degenerate, we can write

$$f^{i;X} = -(n+2)\lambda^{-1}(i_X\alpha^i + r^i), \tag{4.26}$$

where $r^i$ is the section of $\mathscr{V}$ defined in (4.17).

Now, let $\{\omega^1, \omega^2, \omega^3\}$ be a local frame field for $\mathscr{V}$:

$$\mathbf{f}^X = \sum_{i=1}^{3} f^{i;X}\omega^i. \tag{4.27}$$

$\mathbf{f}^X$ is a well defined, locally $SO(3)$ gauge invariant section of the bundle $\mathscr{V}$, uniquely determined by the Killing vector field $X$. By analogy with the previous section we can define a "momentum mapping" for the action of $G$ on $M$. At every point $m \in M$ with each quaternionic Kähler Killing vector field $X$ we can associate a Lie co-algebra $\mathscr{G}^*$, $\mathscr{V}$-valued element $\Phi^i(m)$,

$$\sum_i \langle \Phi^i(m), X_m \rangle \omega^i = \sum_i f^{i;X_m}(m)\omega^i. \tag{4.28}$$

Varying with respect to $m \in M$ we get a smooth mapping for the $G$-action,

$$\Phi: M \to \mathscr{G}^* \otimes \mathscr{V}. \tag{4.29}$$

Because of the uniqueness of the section $\mathbf{f}^X$ the map $X \to \mathbf{f}^X$ transforms naturally under $G$ and consequently our momentum mapping $\Phi$ is always $G$-equivariant.

Let us assume that

$$M_0 \overset{\text{def}}{=} \{m \in M;\ \Phi^i(m) = 0,\ i = 1, 2, 3\} \tag{4.30}$$

is a smooth algebraic submanifold in $M$ (i.e., the section $\Phi$ is transversal to the zero-section of $\mathscr{V}$). Since $\Phi$ is $G$-equivariant and the zero-section is $G$-invariant it follows that $M_0$ is $G$-invariant submanifold in $M$. Thus, if $G$ acts on $M_0$ freely, then one can introduce a quotient space

$$M_G = M_0/G. \tag{4.31}$$

One can prove [13] that $M_G$ is again a quaternionic Kähler manifold with a unique quaternionic Kähler structure defined by

$$M \overset{i}{\longleftarrow} M_0 \overset{\pi}{\longrightarrow} M_G, \tag{4.32}$$

$$\pi^*\Omega_G = i^*\Omega. \tag{4.33}$$

Notice that in the $\lambda \to 0$ limit (where $\lambda$ is a constant proportional to the scalar curvature of $M$), the vector bundle $\mathscr{V}$ trivializes to $\mathbb{R}^3$ and the reduction process described here approaches the one in the hyperKähler case. In the next section we shall discuss some examples of this reduction in the case of quaternionic projective space.

## 5. Quaternionic Reduction of $\mathbb{H}P(n)$ and Quaternionic Orbifolds

We want to illustrate the quaternionic reduction in the case of the geometry of quaternionic projective space. Quaternionic projective space is in some sense a model example of a quaternionic Kähler manifold. Moreover, it is the only compact quaternionic Kähler manifold with an integrable quaternionic structure. Thus it is possible to introduce standard quaternionic Fubini-Study coordinates on $\mathbb{H}P(n)$.

Let $\{u^\alpha\}_{\alpha=1,\dots,n+1}$ be the quaternionic global coordinates on $\mathbb{H}^{n+1}$. We introduce a $S^{4n+3}$ unit sphere in $\mathbb{H}^{n+1}$:

$$S^{4n+3}=\left\{u^\alpha \in \mathbb{H}^{n+1}: \sum_{\alpha=1}^{n+1} \bar{u}^\alpha u^\alpha = 1\right\}. \tag{5.1}$$

$Sp(1)$ act on $S^{4n+3}$ by multiplication from the right with unit quaternions $v$:

$$u^\alpha \sim u^\alpha v\,, \qquad \bar{v}v=1\,. \tag{5.2}$$

Quaternionic projective space $\mathbb{H}P(n)$ is defined as the $Sp(1)$ quotient of $S^{4n+3}$. [$S^{4n+3}$ is a $Sp(1)$ principle fiber bundle over $\mathbb{H}P(n)$.] The $u^\alpha$'s are usually called the homogeneous coordinates on $\mathbb{H}P(n)$. One can also introduce inhomogeneous coordinates inert under the action of $Sp(1)$ (the Fubini-Study coordinates). It is, however, more convenient to work with homogeneous coordinates for the moment.

In terms of $u^\alpha$, the $Sp(n+1)$-invariant metric on $\mathbb{H}P(n)$ is given,

$$ds^2=\frac{4}{c}\sum_\alpha d\bar{u}^\alpha \otimes du^\alpha - \frac{4}{c}\sum_{\alpha\beta}(\bar{u}^\alpha du^\alpha)\otimes(d\bar{u}^\beta u^\beta)\,. \tag{5.3}$$

This metric, just as in the $\mathbb{C}P(n)$ case, is associated to the fundamental $\mathbb{H}$-valued 2-form $\omega$,

$$\omega=\frac{4}{c}\sum_\alpha d\bar{u}^\alpha \wedge du^\alpha - \frac{4}{c}\sum_{\alpha\xi}(\bar{u}^\alpha du^\alpha)\wedge(d\bar{u}^\beta u^\beta)\,. \tag{5.4}$$

It is obvious that $\omega=-\bar{\omega}$, so that $\omega$ is purely imaginary, and we can write ·

$$\omega=\sum_{i=1}^{3}\omega^i e^i\,, \tag{5.5}$$

where $e^i=\{i,j,k\}$ are quaternionic units. $c$ is just a constant equal to the quaternionic sectional curvature. The $\mathbb{H}$-valued $Sp(1)$ 1-form $\alpha$ is given in quaternionic coordinates,

$$\alpha=\sum_i \alpha^i e^i = 2\sum_\alpha \bar{u}^\alpha du^\alpha\,, \tag{5.6}$$

and is also purely imaginary.

We can also write $d^V$ in quaternionic language. If $\Theta$ is an arbitrary but purely imaginary $\mathbb{H}$-valued $p$-form, then

$$d^V\Theta = d\Theta + \tfrac{1}{2}(\alpha \wedge \Theta - \Theta \wedge \alpha). \tag{5.7}$$

The 2-form $\omega$ is indeed covariantly constant with respect to $d^V$,

$$d^V\omega = 0. \tag{5.8}$$

The quaternionic Kähler 4-form $\Omega$ is then given by

$$\bar{\Omega} = \Omega = \omega \wedge \omega. \tag{5.9}$$

It is real and closed.

Now, let us consider a circle action on $\mathbb{H}P(n)$ defined by the following infinitesimal transformations,

$$\delta_\lambda u^\alpha = i\lambda u^\alpha = i\lambda X^\alpha(u)\,; \qquad \lambda \in \mathbb{R}, \tag{5.10}$$

or globally,

$$\varphi_t^\alpha(u) = e^{2\pi i t} u^\alpha, \qquad t \in [0, \tfrac{1}{2}). \tag{5.11}$$

Equation (5.10) defines a Killing vector since the above $U(1)$ is a subgroup of the isometry group $Sp(n+1)$. Moreover, the action (5.11) preserves the 2-form $\omega$, and consequently the 4-form $\Omega$. Hence, we have

$$\mathscr{L}_X\Omega = \mathscr{L}_X h = \mathscr{L}_X\omega = 0, \tag{5.12}$$

and $X$ is a quaternionic Kähler Killing vector field on $\mathbb{H}P(n)$. Notice that, exactly as for complex projective geometry, where all isometries of $\mathbb{C}P(n)$ are holomorphic, here all isometries of $\mathbb{H}P(n)$ are quaternionic Kähler.

We can compute the section $\mathbf{f}^X$ associated with the $U(1)$ action. $\mathbf{f}^X$ is again an $\mathbb{H}$-valued function $\mathbb{H}P(n)$

$$\mathbf{f}^X(\bar{u}, u) = \sum_\alpha \bar{u}^\alpha i u^\alpha = -\overline{\mathbf{f}^X}. \tag{5.13}$$

The zero momentum level set $M_0$ is then defined by

$$M_0 \overset{\mathrm{def}}{=} \left\{ u^\alpha \in \mathbb{H}P(n)\colon \sum_{\alpha=1}^{n+1} \bar{u}^\alpha i u^\alpha = 0 \right\}. \tag{5.14}$$

$M_0$ is a $(4n-3)$-dimensional algebraic submanifold of $\mathbb{H}P(n)$. It is $U(1)$-invariant and the circle action on $M_0$ is free [although it is obviously not free on $\mathbb{H}P(n)$]. Hence, the orbit space is a smooth quaternionic Kähler manifold. One can easily see that $M_G = M_0/U(1)$ is indeed the homogeneous symmetric space with isometry $SU(n+1)$:

$$M_G = X(n-1) = \frac{SU(n+1)}{SU(n-1) \times SU(2) \times U(1)}. \tag{5.15}$$

We would like to mention that the above construction was first used by Breitenlohner and Sohnius [24] to construct a locally $N=2$ supersymmetric coupling of non-linear $\sigma$-model with scalar fields parametrizing the complex Wolf space $X(n)$.

Another example of the quaternionic reduction is provided by the $SU(2)$ action on $\mathbb{H}P(n)$ given infinitesimally by the following transformations

$$\delta_\lambda^{(i)} u^\alpha = \lambda e^i u^\alpha = \lambda e^i X^{(i),\alpha}(u), \tag{5.16}$$

where $\{e^i\}_{i=1,2,3}=\{i,j,k\}$ are quaternionic units. Naturally, we have

$$[\delta_\lambda^{(i)},\delta_\mu^{(j)}]u^\alpha=\sum_k \varepsilon^{ijk}\delta_{\lambda\mu}^{(k)}u^\alpha. \tag{5.17}$$

The corresponding sections of the vector bundle $\mathscr{V}$ are

$$f^{(i);X}(\bar{u},u)=\sum_\alpha \bar{u}^\alpha e^i u^\alpha. \tag{5.18}$$

We introduce the zero level set for the action of $SU(2)$,

$$\tilde{M}_0 \stackrel{\text{def}}{=} \{u^\alpha\in \mathbb{H}P(n)\colon f^{(i);X}(\bar{u},u)=0,\ i=1,2,3\}. \tag{5.19}$$

The manifold $\tilde{M}_0$ is an $SU(2)$ invariant algebraic submanifold of $\mathbb{H}P(n)$. $SU(2)$ acts on $\tilde{M}_0$ freely. Consequently, $\tilde{M}_G=\tilde{M}_0/SU(2)$ is a smooth riemannian manifold with a unique quaternionic Kähler structure. Its isometry group is $SO(n+1)$ and it is a homogeneous space. In fact, it is again a Wolf space:

$$\tilde{M}_G=Y(n-3)=\frac{SO(n+1)}{SO(n-3)\times SO(4)}. \tag{5.20}$$

These two cases of quaternionic Kähler reduction are the only ones we know that give a smooth complete riemannian manifold with holonomy $Sp(n)\times Sp(1)$. Unlike in the hyper Kähler case, it is rather difficult to find actions of isometries which are free on the "zero momentum level set" $M_0$. It would be, however, very interesting to look for such actions: especially on spaces with exceptional isometry groups, such as those listed in (4.13). The question of the existence of non-symmetric, compact, quaternionic Kähler manifolds with positive scalar curvature is still open. It is known that in four dimensions one has only two examples of self-dual Einstein metrics: $\mathbb{C}P^2$ and $S^4$, each of which is a symmetric space. This result is due to Hitchin [18]. But in dimension larger than four we do not know the answer. If such spaces do indeed exist, one might expect that our method could be used to construct examples.

Although we have not found any new examples of non-symmetric compact quaternionic Kähler manifolds, our reduction does yield interesting examples of compact quaternionic orbifolds with quaternionic Kähler metrics away from singular points.

As before, we consider an $S^1$-action on the quaternionic projective space $\mathbb{H}P(n)$, defined infinitesimally in terms of the Killing vector field $X^\alpha(u)$,

$$\delta_\lambda u^\alpha=i\lambda\sum_\beta T^{\alpha\beta}u^\beta=i\lambda X^\alpha(u),\qquad \lambda\in\mathbb{R}, \tag{5.21}$$

where $\{u^\alpha\}_{\alpha=1,\dots,n+1}$ are homogeneous coordinates on $\mathbb{H}P(n)$, $i$ is one of the quaternionic units, and $T^{\alpha\beta}$ is some real, symmetric matrix. (In the previous case we took $T^{\alpha\beta}=\delta^{\alpha\beta}$.) Let us notice that $T^{\alpha\beta}$ can be always diagonalized by an $Sp(n+1)$ rotation of coordinates. We also assume that it is non-degenerate, i.e., $\det T\neq 0$. The zero-momentum level set for the above $S^1$-action is given by the following constraints

$$M_0 \stackrel{\text{def}}{=} \left\{u^\alpha\in\mathbb{H}P(n)\colon \mathbf{f}^X(\bar{u},u)=\sum_{\alpha,\beta=1}^{n+1}\bar{u}^\alpha i T^{\alpha\beta}u^\beta=0\right\}. \tag{5.22}$$

From the assumption that $\det T \neq 0$, it can be easily shown that the Killing vector field $X^\alpha(u)$ in (5.21) is non-zero everywhere on $M_0$. Thus the $U(1)$-action on $M_0$ is locally free and the space of orbits $M_0/S^1$ is an orbifold with a quaternionic Kähler metric at all non-singular points (see [13]). Before we examine its structure, let us recall some general facts about orbifolds.

Let $G$ be a compact group of transformations of a riemannian manifold $M$. Every orbit passing through a point $m \in M$,

$$G(m) \stackrel{\text{def}}{=} \{m' \in M\colon m' = gm,\ g \in G\}, \tag{5.23}$$

is then a compact submanifold of $M$. In general $G$ does not have to act freely on $M$ and we have different isotropy groups $H_m$,

$$H_m \stackrel{\text{def}}{=} \{g \in G\colon gm = m\} \tag{5.24}$$

at different points on $M$. When $m_1, m_2$ belong to the same orbit $H_{m_1} = gH_{m_2}g^{-1}$ for some $g \in G$, i.e., $H_{m_1}$ and $H_{m_2}$ are conjugate. But it is, in general, not true for the points that belong to the different orbits.

If $G$ and $M$ are compact there exists a principal isotropy group $H \subset G$ such that

$$\forall m \in M, \quad \exists g_m \in G, \quad H \subset g_m H_m g_m^{-1}. \tag{5.25}$$

The manifold $M$ stratifies with respect to the group action. Two points belong to the same stratum if their isotropy groups are conjugate. The stratum consisting of orbits with the principal isotropy group is an open, dense submanifold in $M$ [25, 26]. If all isotropy groups are conjugate to the principal one, there is only one stratum and the orbit space $M/G$ is again a compact, smooth, riemannian manifold with a metric given by the riemannian submersion. In particular, such is the case when $G$ acts freely on $M$, i.e., the principal isotropy group is trivial. If, however, there is more than one stratum the orbit space $M/G$ cannot be given a smooth riemannian structure.

If all isotropy groups are of the same dimension (i.e. they differ from the principal one by some discrete subgroups) the quotient space is an orbifold. An orbifold is always locally $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a discrete subgroup of $O(n)$. It can be viewed as a riemannian manifold with singularities (or upon removal of the singular set – as a riemannian manifold with an incomplete metric).

In some cases it is known how to repair these singularities by removing singular points and gluing in manifolds with appropriate boundaries. This construction is called blowing up or resolving singularities. One such interesting example is $K3$ metric. As suggested by Page [27], it can be obtained by gluing in 16 Eguchi-Hanson metrics to the 4-dimensional torus with 16 isolated singularities. In general, however, and this also includes our case, it is known how to repair these singularities on orbifolds.

Now, we want to examine the singular set of our orbifold $M_0/S^1$. We first introduce the following inhomogeneous Fubini-Study coordinates on $\mathbb{H}P(n)$:

$$u^\alpha = (u^i; u^{n+1}) \equiv \left( \frac{w^i \sigma}{\sqrt{1 + \sum_{i=1}^{n} \bar{w}^i w^i}}; \frac{\sigma}{\sqrt{1 + \sum_{i=1}^{n} \bar{w}^i w^i}} \right), \tag{5.26}$$

where $\{w^i\}_{i=1,\dots,n}$ are $n$ quaternions and $\sigma$ is a unit quaternion. Notice that $(w^i;\sigma)$ parametrize the $S^{4n+3}$ sphere [i.e. the constraint (5.1) is solved]. The $Sp(1)$ action on $S^{4n+3}$ in terms of the new local coordinates reads

$$(w^i;\sigma)\xrightarrow{v\in Sp(1)}(w^i;\sigma v)\,. \tag{5.27}$$

Thus $\{w^i\}_{i=1,\dots,n}$ are inert under the $Sp(1)$ action and $\sigma$ gets multiplied by a unit quaternionic from the right. It is then the action of $Sp(1)$ in the fiber. The projection

$$\pi(w^i;\sigma)=w^i \tag{5.28}$$

is the canonical projection from the bundle to the base space $\mathbb{H}P(n)$. Now, let us describe $S^1$ action on $\mathbb{H}P(n)$ in $\{w^\alpha\}$ coordinates. We take the matrix $T^{\alpha\beta}$ in the following form,

$$T=\mathrm{diag}(1,\dots,1,q/p)\,;\qquad q,p\in Z_+\,;\qquad q<p\,, \tag{5.29}$$

where $q$ and $p$ are relatively prime integers so that $q/p$ is rational. Thus the global action in the homogeneous coordinates may be written

$$\varphi_t^i(u)=e^{2\pi it}u^i\,;\qquad i=1,\dots,n\,,\qquad \varphi_t^{n+1}(u)=e^{2\pi i\frac{q}{p}t}u^{n+1}\,, \tag{5.30}$$

where $t\in\left[0,\frac{p}{2}\right)$ for even $(p+q)$ and $t\in[0,p)$ for odd $(p+q)$. First, let us assume that $u^{n+1}\neq 0$, and let us go to the projective coordinates. Then (5.30) becomes

$$\varphi_t(w^i;\sigma)=\left(e^{2\pi it}w^ie^{-2\pi i\frac{q}{p}t};\,e^{2\pi i\frac{q}{p}t}\sigma\right) \tag{5.31}$$

as the circle action in the bundle $S^{4n+3}$. The above action projects to the base space [i.e. $\mathbb{H}P(n)$] as follows:

$$\varphi_t^i(w)=e^{2\pi it}w^ie^{-2\pi i\frac{q}{p}t}\,. \tag{5.32}$$

The zero level set $M_0$ in (5.22) is thus given by the following algebraic submanifold in $\mathbb{H}P(n)$:

$$\mathbb{H}P(n)\supset M_0=\left\{w^i\in\mathbb{H}P(n)\colon\ \sum_{i=1}^{n}\bar{w}^iiw^i=-i\frac{q}{p}\right\}. \tag{5.33}$$

Let us introduce the following notation:

$$w^i\overset{\text{def}}{=}w_+^i+jw_-^i\,, \tag{5.34}$$

where $\{i,j,ij\}$ are three quaternionic units. (We simply split our quaternionic coordinates into two complex pieces: one commuting with $i$ and another anticommuting.) As it is easy to see,

$$M_0=\left\{(w_+^i,w_-^i)\in\mathbb{H}P(n)\colon\ \sum_{i=1}^{n}(\bar{w}_+^iw_+^i-\bar{w}_-^iw_-^i)=-\frac{q}{p};\ \sum_{\alpha=1}^{n}w_-^\alpha w_+^\alpha=0\right\}, \tag{5.35}$$

and (5.32) gives

$$\varphi_t^i(w_\pm)=e^{\pm 2\pi i\frac{p\mp q}{p}t}w_\pm^i\,. \tag{5.36}$$

Define

$$\mathbb{H}P(n)\supset M_0\supset M'\stackrel{\text{def}}{=}\{(w^i_+,w^i_-)\in M_0:\ w^i_+=0\ \forall i\}\,. \tag{5.37}$$

It is trivial to see that there is a discrete subgroup of $S^1$ that acts as the isotropy group on $M_0$. This is the cyclic group

$$Z_{p+q}:\ \left\{t=0,\frac{p}{p+q},\frac{2p}{p+q},\ldots,\frac{p(p+q-1)}{p+q}\right\}\quad \text{for } (p+q) \text{ odd} \tag{5.38}$$

and

$$Z_{\frac{p+q}{2}}:\ \left\{t=0,\frac{p}{p+q},\frac{2p}{p+q},\ldots,\frac{(p+q-2)p}{2(p+q)}\right\}\quad \text{for } (p+q) \text{ even}. \tag{5.39}$$

(Notice that $w^i_-=0$ is excluded from $M_0$ by the constraints.) However, $M'$ is not the only set with the non-trivial isotropy group. Let us consider the $u^{n+1}=0$ case,

$$\mathbb{H}P(n)\supset M_0\supset M''\stackrel{\text{def}}{=}\{u^\alpha\in M_0:\ u^{n+1}=0\}\,. \tag{5.40}$$

Again, $S^1$ does not act freely on $M''$. There are cyclic subgroups $Z_{2p}$ and $Z_p$ for $(p+q)$ odd and even respectively that leave $M''$ fixed. The action of the circle is free outside $M_0\backslash\{M'\cup M''\}$. Thus our manifold for any $q/p$ stratifies into three distinct strata. Consequently, on the orbifold, there are two disjoint singular sets $M'/S^1$ and $M''/S^1$ of the real dimensions $2(n-1)$ and $4(n-2)$, respectively. $M'/S^1$ has the topology of $\mathbb{C}P(n-1)$ and $M''/S^1$ has the topology of the Wolf space $X(n-2)$. In four dimensions, for instance, the corresponding singular sets are $S^2$ and a single point. It strongly resembles the bolt and the nut of $\mathbb{C}P^2$. However, in the case of $\mathbb{C}P^2$ those are only coordinate singularities, whereas in our orbifold case these are real singular sets.

Now, we would like to know if our examples of riemannian, quaternionic orbifolds are different than those which can be obtained as quotients of the Wolf spaces $X(n-1)$ by finite groups. We shall demonstrate that such is indeed the case. Let us begin with explicit computation of the metric at all regular points. The Fubini-Study metric (5.2) in the projective coordinates $\{w^i\}_{i=1,\ldots,n}$ reads

$$ds^2=\frac{4}{c}\,\frac{\sum_{i=1}^{n} d\bar{w}^i\otimes dw^i}{1+\sum_{i=1}^{n}\bar{w}^iw^i}-\frac{4}{c}\,\frac{\sum_{i,j=1}^{n}(\bar{w}^idw^i)\otimes(d\bar{w}^jw^j)}{1+\sum_{i=1}^{n}\bar{w}^iw^i}\,. \tag{5.41}$$

The riemannian metric on the orbifold can be obtained by the riemannian submersion, and one can easily see that

$$d\hat{s}^2=ds^2-\frac{4}{c}\,\frac{\left(\alpha^2+\sum_{i=1}^{n}\bar{w}^iw^i\right)}{\left(1+\sum_{i=1}^{n}\bar{w}^iw^i\right)}V^2\,, \tag{5.42}$$

where

$$V=\frac{1}{2}\left(\alpha^2+\sum_{i=1}^{n}\bar{w}^iw^i\right)^{-1}\sum_{i=1}^{n}(d\bar{w}^iiw^i-\bar{w}^iidw^i)=\bar{V}\,, \tag{5.43}$$

and $\alpha \equiv q/p$. Thus

$$d\hat{s}^2 = ds^2 - \frac{1}{c} \frac{\sum_{i=1}^{n} (d\bar{w}^i i w^i - \bar{w}^i i dw^i) \otimes \sum_{i=1}^{n} (d\bar{w}^i i w^i - \bar{w}^i i dw^i)}{\left(\alpha^2 + \sum_{i=1}^{n} \bar{w}^i w^i\right)\left(1 + \sum_{i=1}^{n} \bar{w}^i w^i\right)}, \tag{5.44}$$

where

$$w = w_+ + jw_- = f(\phi_+^i, i\sqrt{\sum \phi_+^i \phi_-^i}) + jf(\phi_-^i, i\sqrt{\sum \phi_+^i \phi_-^i}) \tag{5.45}$$

and

$$f(\phi_+, \phi_-) = \sqrt{\frac{\alpha}{\sum_{i=1}^{n-1} (\bar{\phi}_-^i \phi_-^i - \bar{\phi}_+^i \phi_+^i)}}. \tag{5.46}$$

Here $\{\phi_+^i, \phi_-^i\}_{i=1,\dots,n-1}$ are local coordinates on the regular part of the orbifold $M_\alpha$ ($\phi_\pm^i < \infty$, $\bar{\phi}_+ \phi_+ \neq 0$). Let us take $\alpha = c\Lambda$. The constraints (5.35) are solved and the circle action is fixed.

Thus we obtain a one-parameter family of metrics $g(\alpha)$. As pointed out in [12] we can take the limit

$$g(0) \stackrel{\text{def}}{=} \lim_{\alpha \to 0} g(\alpha). \tag{5.47}$$

It turns out that $g(0)$ is the Calabi metric [28]. $g(\alpha)$ is a regular function of $\alpha \in (0, 1]$. Our orbifold picture makes sense only for rational $\alpha$, but away from singular points this does not matter. We want to know if $g(\alpha)$ is locally symmetric. Since we have calculated it explicitly we could compute the Riemann curvature tensor and check if it is parallel with respect to the metric connection. This is a straightforward but rather tedious calculation. Instead we give a very simple argument that $g(\alpha)$ cannot be locally symmetric for an infinite number of values of the parameter $\alpha$. Since the metric $g(\alpha)$ is quaternionic Kähler, the Riemann curvature tensor $\mathcal{R}(\alpha)$ must be of the form

$$\mathcal{R}(\alpha) = \alpha \mathcal{R}_{\mathbb{H}P(n)} + \mathcal{R}_0(\alpha). \tag{5.48}$$

Let us apply the Levi-Civita connection $\nabla^\alpha$ to both sides of (5.48), and let us take the value of $\nabla \mathcal{R}$ at some point $m \in M_\alpha$ on the orbifold $M_\alpha$,

$$\nabla^\alpha \mathcal{R}(\alpha)|_m = \alpha \nabla^\alpha \mathcal{R}_{\mathbb{H}P(n)}|_m + \nabla^\alpha \mathcal{R}_0(\alpha)|_m. \tag{5.49}$$

Now, assume that for any $\alpha \in (0, 1]$ the metric $g(\alpha)$ on $M_\alpha$ is locally symmetric, i.e.

$$\nabla^\alpha \mathcal{R}(\alpha)|_m \stackrel{\text{def}}{=} f(\alpha, m) \equiv 0 \tag{5.50}$$

as a function of $\alpha$. We see that $f(\alpha, m)$ must be continuous and smooth for $\alpha \in (0, 1]$. Furthermore,

$$\lim_{\alpha \to 0} f(\alpha, m) = \lim_{\alpha \to 0} \nabla^\alpha \mathcal{R}_0(\alpha)|_m = \nabla^0 \mathcal{R}_0^{\text{Calabi}}|_m \neq 0, \tag{5.51}$$

because the Calabi metrics are <u>not</u> locally symmetric. Thus $f(\alpha, m)$ cannot be identically zero and it is some non-vanishing function of $\alpha$,

$$f(\alpha, m) = (\alpha - 1)\tilde{f}(\alpha, m), \tag{5.52}$$

which is zero for $\alpha=1$ when we obtain the complex Wolf space $X(n-1)$ metric. Consequently, $f(\alpha, m)=\nabla^\alpha \mathscr{R}(\alpha)|_m$ must not vanish for infinitely many rational parameters $\alpha$. For these $\alpha$ the metric $g(\alpha)$ is not locally symmetric, and thus according to the Berger theorem [29] the holonomy is $Sp(n)\times Sp(1)$. This has a very important consequence: For all $\alpha$'s for which the metric is not locally symmetric, our orbifold is not a quotient of $X(n-1)$ by some discrete subgroup of $U(n+1)$.

We do not know if the quaternionic riemannian orbifolds described in this section are quotients of some compact quaternionic Kähler manifolds by a finite group. If so, the manifolds in question would provide the first examples of compact, quaternionic Kähler manifolds with not locally symmetric riemannian metrics. In four dimensions the answer seems to be negative. Our orbifolds cannot be globally obtained by such a quotient because there are only two self-dual and Einstein metrics with positive scalar curvature: $\mathbb{C}P^2$ and $S^4$ – both locally symmetric and homogeneous [18]. Thus no quotients of $\mathbb{C}P^2$ or $S^4$ can produce non-symmetric metrics on resulting orbifolds. This indicates that our orbifolds are only locally some manifolds divided by a discrete group. It would also be interesting to see if orbifolds described by different parameters $\alpha=q/p$ have a distinct geometry.

## 6. Conclusions

Our paper describes a new generalization of the old Marsden-Weinstein symplectic reduction: The quaternionic reduction of quaternionic Kähler manifold with isometries. We compare our quotient with the hyperKähler quotient of Hitchin et al. [7]. There are two main motivations for studying such quotients. The first one comes from field theory of $N=2$ supersymmetric $\sigma$-models. The minimal coupling of the Yang-Mills multiplets in $N=2$ globally supersymmetric $\sigma$-models can be very well understood from the point of view of the hyperKähler quotient of hyperKähler manifolds by their triholomorphic isometries. Hull et al. [6] analyze supersymmetric gauging of these models in great detail. Much less, however, is understood in the case of $N=2$ local supersymmetry. We believe that our approach gives a new insight in that problem. We clearly see that the correspondence between quaternionic and hyperKähler quotients, which allows for seeing the latter as the $\lambda\to 0$ scalar curvature limit of the former, is precisely the correspondence between $N=2$ local and global supersymmetric gaugings of a $\sigma$-model manifold. The $\lambda\to 0$ limit is then just the flat superspace limit (or decoupling limit) since the scalar curvature $\lambda$ of a $\sigma$-model manifold must be proportional to Newton's constant [3, 12].

The second motivation for our work was connected with the great success of the hyperKähler reduction method in constructing new examples of hyperKähler metrics. It is indeed remarkable that a great many hyperKähler metrics can be obtained as simple quotients of $\mathbb{H}^n$ by the unitary groups. Our quaternionic reduction can be carried out in a very similar way. We saw that the Wolf spaces $X(n)$ and $Y(n)$ are such quotients of the model quaternionic Kähler manifold – quaternionic projective space $\mathbb{H}P(n)$. Unfortunately, it seems to be rather difficult to find appropriate regular actions of many different isometry groups which would act freely on the zero level set of their momentum mappings. We were not able to apply our reduction process to construct any new compact smooth riemannian quaternionic Kähler manifold with non-symmetric metric. (All symmetric spaces were classified by Wolf [20].) However, we have constructed examples of quaternionic Kähler riemannian orbifolds with non-symmetric metrics at all regular points. Not locally symmetric metric implies that our orbifolds are not the Wolf spaces divided by a discrete group. We discussed the geometry of these orbifolds, their metrics away from singular sets. In the $\alpha \to 0$ limit the metric $g(\alpha)$ approaches the Calabi metric. It would be interesting to see and understand this limit from a topological point of view. Another important question is again connected with $\sigma$-models. Is it, for instance, possible to define a consistent $\sigma$-model action when the scalar $\sigma$-model fields are differentiable maps from coordinate space to some orbifold rather than manifold? How can one extend the usual definition to include singularities, and what, if any, is the significance of these singularities in the physics of such a field theory? What is the change in geometry of orbifolds upon renormalization? There are many more questions like these to be answered. Dixon et al. [30] have shown that string propagation on orbifolds is perfectly consistent even without the requirement that singularities could be removed by blowing up. Since the string action can be viewed as a two-dimensional $\sigma$-model, it is then a $\sigma$-model on an orbifold.

Lately it has been shown [31] that in two dimensions $N=4$ local supersymmetry allows for the coupling of both hyperKähler and quaternionic $\sigma$-models with positive or negative scalar curvature. The action is conformally invariant and corresponds to the $SU(2)$ spinning string. It would also be interesting to see if one can replace a smooth manifold by our riemannian quaternionic orbifold in this case. We hope to address some of these problems in the feature.

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