

# The Schrödinger Equation and Canonical Perturbation Theory

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**Abstract.** Let  $T_0(\hbar, \omega) + \varepsilon V$  be the Schrödinger operator corresponding to the classical Hamiltonian  $H_0(\omega) + \varepsilon V$ , where  $H_0(\omega)$  is the  $d$ -dimensional harmonic oscillator with non-resonant frequencies  $\omega = (\omega_1, \dots, \omega_d)$  and the potential  $V(q_1, \dots, q_d)$  is an entire function of order  $(d + 1)^{-1}$ . We prove that the algorithm of classical, canonical perturbation theory can be applied to the Schrödinger equation in the Bargmann representation. As a consequence, each term of the Rayleigh–Schrödinger series near any eigenvalue of  $T_0(\hbar, \omega)$  admits a convergent expansion in powers of  $\hbar$  of initial point the corresponding term of the classical Birkhoff expansion. Moreover if  $V$  is an even polynomial, the above result and the KAM theorem show that all eigenvalues  $\lambda_n(\hbar, \varepsilon)$  of  $T_0 + \varepsilon V$  such that  $n\hbar$  coincides with a KAM torus are given, up to order  $\varepsilon^\infty$ , by a quantization formula which reduces to the Bohr–Sommerfeld one up to first order terms in  $\hbar$ .

## I. Introduction and Statement of Results

Consider the formal Schrödinger operator acting in  $L^2(\mathbb{R}^d)$ :

$$T(\hbar, \varepsilon) = T_0(\hbar) + \varepsilon V. \tag{1.1}$$

Here  $q \equiv (q_1 \dots q_d) \in \mathbb{R}^d$ ,  $q \rightarrow V(q)$  is a real-valued function, and  $\varepsilon$  is a non-negative number. The operator  $T(\hbar, \varepsilon)$  is obtained through formal quantization (i.e., through the replacement  $p_i \rightarrow i\hbar(\partial/\partial q_i)$ ) of the classical Hamiltonian defined on  $\mathbb{R}^{2d}$

$$H(p, q; \varepsilon) = H_0(p, q) + \varepsilon V(q), \quad p \equiv (p_1 \dots p_d) \in \mathbb{R}^d, \quad \{p_i, q_j\} = \delta_{ij} \tag{1.2}$$

Let  $H_0(p, q)$  be canonically integrable over  $\mathbb{R}^{2d}$ , namely (see e.g. [4, p. 289]) let  $(\mathbb{R}^2 \setminus \{0\})^d$  be canonically foliated into  $(\mathbb{R}_+)^d \times \mathbb{T}^d$  through globally defined action-angle variable  $(A, \phi) = C(p, q)$ ,  $A \in \mathbb{R}_+^d$ ,  $\phi \in \mathbb{T}^d$ ,  $C$  being a completely canonical map of  $(\mathbb{R}^2 \setminus \{0\})^d$  onto  $\mathbb{R}_+^d \times \mathbb{T}^d$  such that  $H_0(C^{-1}(A, \phi)) \equiv f_0(A)$ . Accordingly, we rewrite (1.2) in the canonically equivalent form

$$H(C^{-1}(A, \phi), \varepsilon) = f_0(A) + \varepsilon V(A, \phi), \quad V(A, \phi) \equiv V(C^{-1}(A, \phi)). \tag{1.3}$$

Correspondingly, let  $T_0(\hbar)$  be self-adjoint in  $L^2(\mathbb{R}^d)$ , with purely discrete spectrum given by simple eigenvalues  $\lambda_n(\hbar) \uparrow \infty$  as  $n \equiv (n_1, \dots, n_d) \rightarrow \infty$ . Under appropriate assumptions on the pairs  $(T_0(\hbar), V)$ ;  $(H_0, V)$  the perturbation series in powers of  $\varepsilon$  exists in both cases. In the quantum case we have the Rayleigh–Schrödinger expansion, which yields a formal power series  $\sum_{k=0}^{\infty} \lambda_n^k(\hbar) \varepsilon^k$  of initial point any given eigenvalue  $\lambda_n(\hbar)$  of  $T_0(\hbar)$ . For conditions on  $(T_0(\hbar), V)$  ensuring convergence or summability to an eigenvalue of  $T(\hbar, \varepsilon)$  see e.g. [7, §VII. 1, 2] or [11, §XII. 1, 2]. In the classical case we have the canonical perturbation theory, which under some additional conditions [4, p. 472] yields a formal power series  $\sum_{k=0}^{\infty} \varepsilon^k f_k(A)$  of initial point  $f_0(A)$ , known as the Birkhoff expansion. The set of values of  $A$  (“tori”) such that this power series, which is in general divergent (see e.g. the discussion in [9, §3]) yields an asymptotic expansion to all orders of an Hamiltonian  $f_\infty(A, \varepsilon)$  canonically equivalent to (1.3), and hence to (1.2), is characterized by the KAM theorem (see e.g. Chierchia–Gallavotti [3], Gallavotti [5], Pöschel [10]).

It seems natural to raise the question of the convergence of the quantum algorithm to the classical one at the classical limit  $n \rightarrow \infty$ ,  $\hbar \rightarrow 0$ ,  $n\hbar \rightarrow A \in \mathbb{R}_+^d$ , i.e.  $n_i\hbar \rightarrow A_i$ ,  $i = 1 \dots d$ . Despite its obvious interest in semiclassical quantum mechanics (see below) this problem seems to have attracted so far little attention: to our knowledge, the only paper dealing explicitly with it is that of Turchetti [11].

The aim of this paper is to show that the answer to the above question is affirmative, at least in a particular but already significant case, namely  $H_0$  the harmonic oscillator with non-resonant frequencies, and  $V$  an entire holomorphic potential of order  $(d + 1)^{-1}$ .

More precisely let  $\omega_i > 0$ ,  $i = 1, \dots, d$ ,  $\omega = (\omega_1, \dots, \omega_d)$ , with  $\langle \omega, v \rangle \equiv \omega_1 v_1 + \dots + \omega_d v_d = 0$  iff  $v = 0$ , and let

$$H_0(p, q; \omega) = \frac{1}{2} \sum_{i=1}^d (p_i^2 + \omega_i^2 q_i^2) \quad (1.4)$$

so that the above mentioned canonical mapping is given by:

$$\begin{aligned} C(p, q) &:= \left\{ \begin{array}{l} A_i = (p_i^2 + \omega_i^2 q_i^2)/2\omega_i \\ \phi_i = -\operatorname{arctg}(p_i/\omega_i q_i) \end{array} \right\} \\ C^{-1}(A, \phi) &:= \left\{ \begin{array}{l} p_i = -\sqrt{2\omega_i A_i} \sin \phi_i \\ q_i = \sqrt{2A_i/\omega_i} \cos \phi_i \end{array} \right\} \end{aligned} \quad (1.5)$$

and

$$f_0(A) = \sum_{i=1}^d \omega_i A_i \equiv \langle \omega, A \rangle. \quad (1.6)$$

Let furthermore  $T_0(\hbar, \omega)$  be the self-adjoint realization of the differential expression  $\sum_{i=1}^d \left( -\hbar^2 \frac{d^2}{dq_i^2} + \omega_i^2 q_i^2 \right)$  in  $L^2(\mathbb{R}^d)$ . It is well known that  $T_0(\hbar, \omega)$  has discrete spectrum, consisting in simple eigenvalues  $\lambda_n(\hbar, \omega) = \lambda_n(\hbar) =$

$\hbar(\langle n, \omega \rangle + \frac{1}{2}|\omega|)$ ,  $n = (n_1, \dots, n_d)$ ,  $\langle n, \omega \rangle = n_1\omega_1 + \dots + n_d\omega_d$ ,  $|\omega| = \omega_1 + \dots + \omega_d$ ,  $n_i = 0, 1, \dots, i = 1, \dots, d$ . Let in addition  $q \rightarrow V(q)$  be entire holomorphic of order  $(d + 1)^{-1}$ , i.e.

$$\left. \begin{aligned} V(q) &= \sum_{n_1, \dots, n_d=2}^{\infty} v_{n_1 \dots n_d} q_1^{n_1} \dots q_d^{n_d} \equiv \sum_{|n|=2}^{\infty} v_n q^n, \quad \forall q \in \mathbb{C}^d \\ |V(q)| &\leq e^{L\|q\|^{1/(d+1)}}, \quad \forall q \in \mathbb{C}^d, \quad \|q\| = |q_1^2 + \dots + q_d^2|^{1/2}. \end{aligned} \right\} \quad (1.7)$$

Remark that, if we denote by  $V_v(A)$ ,  $v \in \mathbb{Z}^d$ , the Fourier coefficients of the function  $V(A, \phi)$  from  $\mathbb{R}_+^d \times \mathbb{T}^d$  to  $\mathbb{R}$ , then there is  $L_1 > 0$  such that

$$\max_{A \in \Omega} |V_v(A)| \leq L_1^{|v|} (v!)^{-1/(d+1)}, \quad v! = |v_1|! \dots |v_d|!, \quad |v| = v_1 + \dots + v_d, \quad (1.8)$$

$\Omega \in \mathbb{R}_+^d$  being compact. We also note that, by (1.7)–(1.5) and an elementary symmetry argument, the function  $A \rightarrow V_0(A)$  is entire in  $\mathbb{C}^d$ . Let finally  $\omega$  fulfill the following Diophantine condition: there exist  $B > 0$ ,  $\gamma > 0$  such that:

$$|\langle \omega, v \rangle|^{-1} < B e^{\gamma|v|}, \quad \forall v \in \mathbb{Z}^d, \quad v \neq 0. \quad (1.9)$$

Under these conditions it is well known that both the Rayleigh–Schrödinger and Birkhoff expansions exist to all order in  $\varepsilon$ . However, some familiarity with both classical and quantum perturbation theory shows at once that the two algorithms are generated in apparently unrelated ways. The main point of this paper consists in pointing out that, for perturbations of the harmonic oscillator, the Bargmann representation allows us to generate the quantum perturbation theory by the same algorithm of the canonical one. More specifically: by working in the Bargmann representation it is possible to rewrite the Schrödinger equation under the form of the classical Hamilton–Jacobi equation (written in canonical variables related to the standard ones by a linear complex canonical transformation) plus corrections under the form of a convergent power series in  $\hbar$ . Our first main result is thus obtained by recursively solving this equation by means of the Birkhoff transformation of canonical perturbation theory, up to the natural variant of working with the Laurent expansion instead of the Fourier one. Namely:

**Proposition 1.** *Let  $\omega, H_0, T_0(\hbar, \omega), V$  be as above. Let  $\sum_{k=0}^{\infty} \lambda_n^k(\hbar) \varepsilon^k$  be the formal Rayleigh–Schrödinger expansion of initial point  $\lambda_n^0(\hbar) = \lambda_n(\hbar) = \hbar(\langle n, \omega \rangle + \frac{1}{2}|\omega|)$ . Let  $\sum_{k=0}^{\infty} N_k(A) \varepsilon^k$  be the Birkhoff expansion for  $f_0(A) + \varepsilon V(A, \phi)$ ,  $N_0(A) = f_0(A)$ . Then for any  $k \in \mathbb{N}$  there are constants  $\beta > 0$ ,  $D(k) > 0$ , and a family of entire functions  $q \rightarrow Q_k^j(q)$ ,  $j = 1, 2, \dots$ , such that, if  $\Omega \subset \mathbb{C}^d$  is open and bounded:*

$$\sup_{q \in \Omega} |Q_k^j(q)| \leq \beta^j D(k)^j, \quad (1.10)$$

$$\lambda_n^k(\hbar) = N_k(n\hbar) + \sum_{j=1}^{\infty} \hbar^j Q_k^j(n\hbar). \quad (1.11)$$

The formulation of the Schrödinger equation as a classical, perturbed Hamilton–Jacobi equation plus explicit corrections in powers of  $\hbar$  is of course relevant to semiclassical quantum mechanics. It is well known that, for classically

integrable systems in  $\mathbb{R}^{2d}$ , the eigenvalues of the corresponding Schrödinger operator should tend to the classical Hamiltonian expressed as a function of the actions at the classical limit. Moreover, the Bohr–Sommerfeld formula should provide a quantization correct up to terms of order  $\hbar$  [8], and an outstanding problem is to determine all corrections in powers of  $\hbar$  beyond the first one, whose coefficient is known as the Maslov index. The above formulation of the Schrödinger equation enables us to directly apply KAM theory, in the version obtained by Chierchia and Gallavotti [3] in the analytic case (see also Gallavotti [5] for more detail), and consequently to take up these kind of questions, to all orders in perturbation theory, even for a class of non-integrable systems. The result is as follows.

**Proposition 2.** *Let  $\omega, H_0, T_0(\hbar, \omega)$  be as above. In addition, let  $V$  be a polynomial of degree  $2m$  such that:*

- (a) *For  $\varepsilon \geq 0$  the maximal operator  $T(\hbar, \varepsilon)$  in  $L^2(\mathbb{R}^d)$  defined by the action of  $T_0(\hbar, \varepsilon) + \varepsilon V$  has domain  $D(T_0(\hbar, \omega)) \cap D(V)$ , is self-adjoint and has discrete spectrum.*  
 (b) *Consider again  $V_0(A) = (2\pi)^{-d} \int_{\mathbb{T}^d} V(A, \phi) d\phi$ , and let  $M(A) = \|(\partial^2 V_0(A) / \partial A_i \partial A_j)\|_{i,j=1,\dots,d}$ . Let  $\Omega \subset \mathbb{R}^d$  be any bounded sphere,  $S_\rho(A_0) = \{A \in \mathbb{C}^d : \max_i |A_i - A_i^0| \leq \rho\}$ ,  $C(\rho, \Omega) = \bigcup_{A_0 \in \Omega} S_\rho(A_0)$ . Then there is  $\eta > 0$  such that*

$\sup |\det M(A)|^{-1} < \eta < +\infty$ , where the sup is taken over  $C(\rho, \Omega)$ .

*Let  $\Gamma^\infty(\varepsilon) \subset \Omega$  be the set of the invariant KAM tori, and  $f^\infty(A, \varepsilon)$  the integrable Hamiltonian canonically equivalent to  $H(p, q; \varepsilon)$  on  $\Gamma^\infty(\varepsilon)$ , which can be extended to a  $C^\infty$  function of  $A$  in  $\Omega$ . Then, for  $\delta < +\infty$ , and  $\bar{\varepsilon} > 0$  suitably small, there is  $(A, \hbar, \varepsilon) \rightarrow g^\infty(A, \hbar, \varepsilon) \in C^\infty(\Omega \times [0, \delta] \times [0, \bar{\varepsilon}])$  such that*

$$\lambda_n(\hbar, \varepsilon) = f^\infty(n\hbar, \varepsilon) + \hbar g(n\hbar, \hbar, \varepsilon) + \hbar |\omega|/2 + O(\varepsilon^\infty) \quad (1.12)$$

*whenever  $n\hbar \in \Gamma^\infty(\varepsilon)$ . Here  $\lambda_n(\hbar, \varepsilon)$  is any eigenvalue of  $T(\hbar, \varepsilon)$  close to  $\lambda_n(\hbar)$  for  $\varepsilon < \bar{\varepsilon}$ , and  $O(\varepsilon^\infty)$  is uniform with respect to  $\hbar \in [0, \delta]$ .*

*Remarks.*

(1) It is well known (see e.g. Gallavotti [5]) that  $\mu(\Gamma^\infty(\varepsilon)) = (1 - K\varepsilon)\mu(\Omega)$  for some  $K > 0$ . Here  $\mu(\cdot)$  denotes the Lebesgue measure.

(2) Formula (1.12) shows that the so-called Einstein–Brillouin–Keller quantization, which amounts to performing a Bohr–Sommerfeld quantization on the Birkhoff expansion truncated to any given order, is valid up to first order in  $\hbar$  had to all orders in  $\varepsilon$  for the “quantized KAM tori”, i.e. for those quantum numbers  $n$  such that  $n\hbar \in \Gamma^\infty(\varepsilon)$ .

The plan of the presentation is as follows: in the forthcoming Sect. II we describe how Rayleigh–Schrödinger perturbation theory can be generated by the classical, canonical algorithm, and in Sect. III we will give the proof of Propositions 1 and 2 which will be easy consequences of the Birkhoff transformation and of the Chierchia–Gallavotti [3] proof of the KAM theorem, respectively. Some useful estimates are collected in an Appendix.

*Notation.* We use the abbreviations:  $z = (z_1, \dots, z_d)$ ,  $z^2 = \sum_{i=1}^d z_i^2$ ,  $(n_1\hbar, \dots, n_d\hbar) =$

$n\hbar$ ,  $(\omega_1 z_1, \dots, \omega_d z_d) = \omega z$ ,  $\sum_{i=1}^d \omega_i^2 z_i^2 = \omega^2 z^2$ ,  $|z| = |z_1| + \dots + |z_d$ . If  $z \rightarrow f(z)$  from  $\mathbb{C}^d$  to  $\mathbb{C}$  is analytic at  $z$ , we denote its gradient by  $\nabla_z f$ , and by  $(D_z^\mu f)(z)$  its partial derivatives  $(D_z^\mu f)(z) = ((\partial^{\mu_1 + \dots + \mu_d} / \partial z_1^{\mu_1} \dots \partial z_d^{\mu_d}) f)(z_1 \dots z_d)$ . We refer to Gallavotti [3, 5] for all notation on canonical perturbation theory in the analytic case not explicitly recalled in what follows.

## II. Perturbation Theory in the Bargmann Representation

The starting point of our analysis is represented by some well known results on the quantum harmonic oscillator and on their classical counterpart, stated for the sake of convenience under the form of three lemmas.

**2.1. Lemma (Bargmann).** *Let  $\mathcal{F}_d$  be the Hilbert space of all entire functions  $z \rightarrow f(z)$  from  $\mathbb{C}^d$  to  $\mathbb{C}$  such that  $\int_{\mathbb{R}^{2d}} |f(z)|^2 e^{-\langle z, \bar{z} \rangle} dz d\bar{z} < +\infty$ . Let  $q \rightarrow \psi(q) \in L^2(\mathbb{R}^d)$ . Then the map  $\psi \rightarrow U\psi \equiv f(z)$ ,  $z \in \mathbb{C}^d$ , defined as:*

$$(U\psi)(z) = \int_{\mathbb{R}^d} A(x, q) \psi(q) dq, \tag{2.1}$$

$$A(z, q) = (\sqrt{\pi \hbar})^{-d/2} (\omega_1 \dots \omega_d)^{1/2} \cdot e^{-[(z^2 + \omega q^2) + 2\sqrt{2}\langle z, \omega q \rangle]/2\hbar} \tag{2.2}$$

is unitary between  $L^2(\mathbb{R}^d)$  and  $\mathcal{F}_d$ . If  $T_0(\hbar, \omega)$  is the Schrödinger operator of the harmonic oscillator in  $L^2(\mathbb{R}^d)$ , with frequencies  $\omega = (\omega_1, \dots, \omega_d)$ , then:

$$UT_0(\hbar, \omega)U^{-1} = P_0(\hbar, \omega) + \frac{1}{2}\hbar|\omega|, \tag{2.3}$$

where  $P_0(\hbar, \omega)$  is the maximal operator in  $\mathcal{F}_d$  generated by the differential expression

$$\tilde{P}_0(\hbar, \omega) = \hbar \sum_{i=1}^d \omega_i z_i D_{z_i} \equiv \hbar \langle \omega z, \nabla_z \rangle.$$

As remarked by Bargmann himself [1], the spectral analysis of  $P_0(\hbar, \omega)$  is a triviality. However, following Voros [13], let us reobtain it in a different way, which is much closer to the integration procedure of the Hamilton–Jacobi equation for the classical oscillator and thus introduces a convenient quantum analogue of the action-angle variables.

Consider the Schrödinger equation in  $\mathcal{F}_d$ .

$$P_0(\hbar, \omega)\psi(z, E(\hbar)) = E(\hbar)\psi(z, E(\hbar)). \tag{2.4}$$

To find  $\psi$ ,  $E$  we tentatively set:

$$\psi(z, E(\hbar)) = e^{[W_0(z, E(\hbar)) - z^2/2]/\hbar}. \tag{2.5}$$

Substitution in (2.4) yields

$$\langle \omega z, \nabla_z W_0(z, E(\hbar)) \rangle - \omega z^2 = E(\hbar). \tag{2.6}$$

Look now for solutions of (2.6) in separated variables:

$$E(\hbar) = \sum_{i=1}^d E_i(\hbar), \quad W_0(z, E(\hbar)) = \sum_{i=1}^d W_0^i(z_i, E_i(\hbar)), \quad i = 1, \dots, d. \tag{2.7}$$

Then (2.6) yields:

$$\partial_i W_0^i(z_i, E_i(\hbar)) = (E_i(\hbar) + \omega_i z_i^2)/\omega_i z_i, \quad i = 1, \dots, d. \quad (2.8)$$

Analyticity of  $z \rightarrow \psi(z)$  yields the quantization condition

$$(2\pi i)^{-1} \int_{\Gamma} \frac{\partial_i \psi_i(z_i, E_i(\hbar))}{\psi_i(z_i, E_i(\hbar))} dz_i = \frac{(2\pi i)^{-1}}{\hbar} \int_{\Gamma} \partial_i W_0^i(z_i, E_i(\hbar)) dz_i = n_i, \quad (2.9)$$

where  $\Gamma$  is any circumference in  $\mathbb{C}$  avoiding the zeros of  $\psi_i(\cdot)$  and  $\psi_i = \exp\{[W_0^i(z_i, E_i(\hbar)) - z^2/2]/\hbar\}$ . By (2.8):

$$(2\pi i)^{-1} \int_{\Gamma} \partial_i W_0^i(z_i, E_i(\hbar)) dz_i = E_i(\hbar)/\omega_i, \quad i = 1, \dots, d.$$

Therefore:

$$\begin{cases} E_i(\hbar) = \hbar n_i \omega_i, & E(\hbar) = \hbar \langle n, \omega \rangle, & W_0^i(z_i, E_i(\hbar)) \\ = \hbar \log(z_i)^{n_i} + \frac{1}{2} z_i^2, & \psi(z, E(\hbar)) = c_{n_1 \dots n_d} z_1^{n_1} \dots z_d^{n_d}, \end{cases} \quad (2.10)$$

which are of course the familiar eigenvalues and eigenvectors (in the Bargmann representation) of the harmonic oscillator;  $c_{n_1 \dots n_d}$  are the usual normalization constants.

Since the converse direction is trivial, we have:

**2.2. Lemma.**  *$E(\hbar) = \hbar \langle n, \omega \rangle$  is an eigenvalue of  $P_0(\hbar, \omega)$  if and only if its eigenvectors admit the representation (2.5), with  $W_0(z, E(\hbar))$  determined by (2.6)–(2.9).*

To see that the variables  $\{\partial_i W_0^i, z_i\}$  are a sort of quantum, complex action-angle variables, let us now examine the classical problem.

Consider again the Hamiltonian (1.4) and the canonical transformation (1.5). Set now:

$$(z, R) \equiv C_1(A, \phi) := \begin{cases} z_i = \sqrt{A_i} e^{i\phi_i} \\ R_i = 2\sqrt{A_i} \cos \phi_i \end{cases} \quad (2.11)$$

inverted as

$$(A, \phi) = C_1^{-1}(z, R) := \begin{cases} A_i = z_i R_i - z_i^2 \\ \phi_i = \arg z_i = i \operatorname{Log} z_i \end{cases} \quad (2.12)$$

and generated by

$$W(A, z) = \sum_{i=1}^d (A_i \operatorname{Log} z_i + \frac{1}{2} z_i^2), \quad (2.13)$$

where  $\operatorname{Log} z$  denotes the principal branch of  $z \rightarrow \log z$ ,  $z \in \mathbb{C}$ . Under the natural identification  $\mathbb{R}_+^d \times \mathbb{T}^d \simeq (\mathbb{C} \setminus \{0\})^d$ , (2.11), (2.12) define a holomorphic bijection of  $(\mathbb{C} \setminus \{0\})^d$  onto itself, which is completely canonical because

$$\{R_j, z_k\} = i\delta_{jk}, \quad \{R_j, R_k\} = \{z_j, z_k\} = 0. \quad (2.14)$$

The canonical image of  $f_0(A) = \langle \omega, A \rangle$  under  $C_1$  is

$$F_0(z, R) \equiv f_0(C_1^{-1}(z, R)) = \sum_{i=1}^d \omega_i (z_i R_i - z_i^2). \quad (2.15)$$

Equivalently, we may look at  $C_1$  as the composition  $C \circ C_0$ , where  $C_0$  is the linear complex canonical transformation on  $\mathbb{C}^d \simeq \mathbb{R}^{2d}$  defined as follows:

$$C_0:(z_i, \bar{z}_i) \equiv ((\omega_i q_i + ip_i)/\sqrt{2\omega_i}, (\omega_i q_i - ip_i)/\sqrt{2\omega_i}) \leftrightarrow (z_i, z_i + \bar{z}_i) \quad (2.16)$$

that is,  $z_i = (\omega_i q_i + ip_i)/\sqrt{2\omega_i}$ ,  $R_i = q_i \sqrt{2\omega_i}$ .

Remark that  $z_i \bar{z}_i = A_i$ , so that

$$R_i = z_i + \bar{z}_i = \partial_{z_i} W(A, z) = (A_i + z_i^2)/z_i, \quad (2.17)$$

and  $W(A, z)$  fulfills identically the Hamilton–Jacobi equation:

$$F_0(z, \nabla_z W(A, z)) = f_0(A), \quad (2.18)$$

which is the analogue of (2.6) under the correspondence  $A \rightarrow n\hbar, R = (R_1, \dots, R_d) \rightarrow \nabla_z W_0$ .

Let us sum up these simple remarks:

**2.3. Lemma.** *The harmonic-oscillator Hamiltonian  $H_0(p, q; \omega)$ , considered in  $\mathbb{C}^d$ , admits the canonically equivalent form  $F_0(z, R)$ . The analytic, completely canonical transformation mapping  $F_0(z, R)$  into  $f_0(A)$  is given by (2.11)–(2.12). Its generating function is given by (2.13), and solves the Hamilton–Jacobi equation (2.18).*

Next, let us proceed to write the perturbed Schrödinger equation in the Bargmann representation. Let  $T(\hbar, \varepsilon)$  once more be the maximal operator in  $L^2(\mathbb{R}^d)$  generated by the differential expression (1.1). Since the unitary image under the Bargmann transform  $U$  of the maximal multiplication operator by  $q_i$  in  $L^2(\mathbb{R}^d)$  is the maximal operator generated by  $(z_i + \hbar(d/dz_i))/\sqrt{2\omega_i}$  in  $\mathcal{F}_d$ , the unitary image of  $T(\hbar, \varepsilon)$  in  $\mathcal{F}_d$  is  $P(\hbar, \varepsilon) + \frac{1}{2}\hbar|\omega|$ , where  $P(\hbar, \varepsilon)$  is the maximal operator in  $\mathcal{F}_d$  generated by:

$$\tilde{P}(\hbar, \varepsilon) = \tilde{P}_0(\hbar, \omega) + \varepsilon V(z + \hbar \nabla_z)/\sqrt{2\omega}, \quad (2.19)$$

where, as usual,  $z/\sqrt{2\omega} = (z_1/\sqrt{2\omega_1}, \dots, z_d/\sqrt{2\omega_d})$ .

Our purpose now is to generate the perturbation expansion in powers of  $\varepsilon$  of initial point any given eigenvalue  $\lambda_n(\hbar) = \hbar \langle n, \omega \rangle$  of  $P_0(\hbar, \omega)$ . The following preliminary result is proved in Appendix.

**2.4. Lemma.** *Let  $E \in \mathbb{C}$ ,  $\varepsilon > 0$  and let the family of functions  $z \rightarrow W(z; E, \varepsilon)$  indexed by  $(E, \varepsilon)$  be locally holomorphic in  $\mathbb{C}^d$  for any fixed  $(E, \varepsilon)$ . Set, for  $l = 1, 2, \dots$*

$$R_l(W(z; E, \varepsilon)) = \sum_{|t|=l+1}^{2l} D_q^{|t|} V(\nabla_z W(\cdot)/\sqrt{2\omega}) \sum_{2 \leq |\mu| \leq l+1}^* \prod \left( \frac{D_z^\mu W(\cdot)}{\mu!} \Big/ \sqrt{2\omega} \right)^{a_\mu} \frac{1}{a_\mu!}, \quad (2.20)$$

where  $t = (t_1, \dots, t_d)$ ,  $\mu = (\mu_1, \dots, \mu_d)$  are multiindices,  $t_i, \mu_i = 0, 1, \dots, i = 1, \dots, d$ ,  $|t| = t_1 + \dots + t_d$ ,  $|\mu| = \mu_1 + \dots + \mu_d$ ,  $t! = t_1! \dots t_d!$ ,  $\mu! = \mu_1! \dots \mu_d!$ , and  $\sum^*$  means summation over all non-negative integers  $a_\mu$  such that

$$\sum_{|\mu|=2}^{l+1} a_\mu = |t| - l, \quad \sum_{|\mu|=2}^{l+1} \mu_i a_\mu = t_i, \quad i = 1, \dots, d.$$

Then, if  $W(z; E, \varepsilon)$  is holomorphic in  $\Omega(E, \varepsilon) \subset \mathbb{C}^d$ , continuous in  $\bar{\Omega}$ ,  $\Omega$  open and

bounded, there are  $K_1(E, \varepsilon) > 0$ ,  $K_2(E, \varepsilon) > 0$  such that

$$\max_{z \in \Omega} |R_l(W(z; E, \varepsilon))| \leq K_1 K_2^l, \quad l = 1, 2, \dots \quad (2.21)$$

Moreover,  $K_1(E, \varepsilon)$ ,  $K_2(E, \varepsilon)$  can be chosen independently of  $(E, \varepsilon)$  whenever  $\Omega(E, \varepsilon)$  is independent of  $(E, \varepsilon)$  on the compacts of  $\mathbb{C} \times [0, a]$  and the family  $z \rightarrow W(z; E, \varepsilon)$  is equibounded.

We can now state and prove the extension of 2.1 to the perturbed case.

**2.5. Lemma.** *The Schrödinger equation*

$$P(\hbar, \varepsilon)\psi(z; E, \varepsilon) = E(\hbar, \varepsilon)\psi(z; E, \varepsilon) \quad (2.22)$$

admits for each fixed  $(E, \varepsilon)$  as above a locally holomorphic solution  $z \rightarrow \psi(z; E, \varepsilon)$  under the form:

$$\psi(z; E, \varepsilon) = e^{[W(z; E, \varepsilon) - z^2/2]\hbar} \quad (2.23)$$

if and only if  $z \rightarrow W(z; E, \varepsilon)$  is a locally holomorphic solution of

$$\begin{aligned} & \langle \omega z, \nabla_z \rangle W(z; E, \varepsilon) - \omega z^2 + \varepsilon \left[ V(\nabla_z W(z; E, \varepsilon)/\sqrt{2\omega}) \right. \\ & \left. + \sum_{l=1}^{\infty} \hbar^l R_l(W(z; E, \varepsilon)) \right] = E(\hbar, \varepsilon). \end{aligned} \quad (2.24)$$

*Remark.* Making  $\hbar = 0$  in (2.24), and taking  $E = f(A, \varepsilon)$ , we formally recover the Hamilton–Jacobi equation, written out of the  $(R, z)$  coordinates. I.e., one looks for a solution  $W(A, z, \varepsilon)$  of (2.24) for  $\hbar = 0$ , parametrized by  $(A, \varepsilon)$ , which represents the generating function of the canonical transformation mapping  $F_0(z, R) + \varepsilon V(R/\sqrt{2\omega})$  into the new Hamiltonian  $f(A, \varepsilon)$ . It will be seen later on that this remark can be rigorously implemented in perturbation theory.

*Proof.* Of course we have:

$$P_0(\hbar, \omega)\psi/\psi = \langle \omega z, \nabla_z \rangle W(z, \cdot) - \omega z^2. \quad (2.25)$$

To determine the action of  $V((z + \hbar \nabla_z)/\sqrt{2\omega})$  on  $\psi$ , let us first recall the formula (see e.g. Voros [12]):

$$V((z + \hbar \nabla_z)/\sqrt{2\omega})e^{[W(z, \cdot) - z^2/2]/\hbar} = e^{-z^2/2\hbar} V(\hbar \nabla_z/\sqrt{2\omega})e^{W(z, \cdot)/\hbar}. \quad (2.26)$$

Next, we recall the Faà di Bruno formula in  $d$  variables (see e.g. Bolley–Camus [2]): if  $p = (p_1 \cdots p_d)$  is a multiindex taking values in  $(\mathbb{N} \cup \{0\})^d$ ,  $g: \mathbb{C}^d \rightarrow \mathbb{C}$  is analytic at  $z$ , and  $f: \mathbb{C} \rightarrow \mathbb{C}$  is analytic at  $g(z)$ , we have:

$$(D_z^p f \circ g)(z) = \sum_{m=1}^{|p|} f^{(m)}(g(z)) \sum_1^* \prod_{1 \leq |\mu| \leq p} \frac{p!}{(\mu!)^{a_\mu} a_\mu!} (D_z^\mu g(z))^{a_\mu}, \quad (2.27)$$

where the notation is the same as in (2.20), except for  $\sum_1$  which now means summation over all non-negative integers  $a_\mu$  such that

$$\sum_{|\mu|=1}^{|p|} a_\mu = m, \quad \sum_{|\mu|=1}^{|p|} \mu_i a_\mu = p_i, \quad i = 1, \dots, d.$$



Therefore:

$$e^{-W(z, \cdot)/\hbar} \hbar^p D_z^p e^{W(z, \cdot)/\hbar} = \sum_{l=0}^{|p|-1} \hbar^l \sum_2^* \prod_{1 \leq |\mu| \leq |p|} \frac{p!}{(\mu!)^{a_\mu} a_\mu!} (D_z^\mu W(z, \cdot))^{a_\mu}, \quad (2.28)$$

where  $\sum_2^*$  means summation over all  $a_\mu$  such that  $\sum_{|\mu|=1} a_\mu = |p| - l$ , and  $\sum_{|\mu|=1} \mu_i a_\mu = p_i$ .

Consider now the  $d$  multiindices  $\bar{\mu}_1, \dots, \bar{\mu}_d$  of length 1, i.e. such that  $|\mu| = 1$ , denoted as  $(\bar{\mu}^i)_j = \delta_{ij}$ ,  $i, j = 1 \dots d$ , and rewrite the right-hand side of (2.28) in the following way:

$$\sum_{l=0}^{|p|-1} \hbar^l \sum_{a_{\bar{\mu}_i}=0}^{p_i} \frac{p!}{(a_{\bar{\mu}_1})! \dots (a_{\bar{\mu}_d})!} \left( \frac{\partial W}{\partial z_i} \right)^{a_{\bar{\mu}_1}} \dots \left( \frac{\partial W}{\partial z_d} \right)^{a_{\bar{\mu}_d}} \sum_3^* \prod_{2 \leq |\mu| \leq |p|} \left( D_z^\mu \frac{W(z, \cdot)}{\mu!} \right)^{a_\mu} \frac{1}{a_\mu!}, \quad (2.29)$$

where  $\sum_3^*$  means summation over all  $a_\mu$  such that

$$\sum_{|\mu|=2} a_\mu = |p| - l - \sum_{i=1}^d a_{\bar{\mu}_i}; \quad \sum_{|\mu|=2} \mu_i a_\mu = p_i - \sum_{j=1}^d \bar{\mu}_i^j a_{\bar{\mu}_j}. \quad (2.30)$$

Now the condition  $\sum_{|\mu|=1} a_\mu = |p| - l$  yields  $\sum_{i=1}^d a_{\bar{\mu}_i} \leq |p| - l$ , i.e.  $|p| - \sum_{i=1}^d a_{\bar{\mu}_i} \geq l$ , and furthermore we have:

$$\left| \sum_{|\mu| \geq 2} \mu a_\mu \right| = \sum_{|\mu| \geq 2} |\mu| a_\mu \geq 2 \sum_{|\mu| \geq 2} a_\mu. \quad (2.31)$$

Since  $\sum_{|\mu|=1} \mu a_\mu = p$ , and  $\sum_{|\mu|=1} a_\mu = |p| - l$ , (2.31) yields

$$|p| - \sum_{i=1}^d a_{\bar{\mu}_i} \leq 2l. \quad (2.32)$$

Now equality in  $\sum_{i=1}^d a_{\bar{\mu}_i} \leq |p| - l$  implies  $a_\mu = 0$  for  $|\mu| \geq 2$ , and then  $\sum_{|\mu|=1} \mu a_\mu = p$ , that is  $a_{\bar{\mu}_i} = p_i$  and thus  $l = 0$  by (2.30). Hence we have:

$$\left. \begin{aligned} l + 1 &\leq |p| - \sum_{|\mu|=1} a_\mu \leq 2l, \quad l \neq 0 \\ |p| &= \sum_{|\mu|=1} a_\mu, \quad l = 0. \end{aligned} \right\} \quad (2.33)$$

Set now  $t_i = p_i - a_{\bar{\mu}_i}$ . Then, by (2.33), (2.29) can be rewritten as:

$$\sum_{i=1}^d \left( \frac{\partial W}{\partial z_i} \right)^{p_i} + \sum_{l=1}^{|p|-1} \hbar^l \sum_{|t|=l+1}^{2l} \frac{p!}{(p-t)!} \prod_{i=1}^d \left( \frac{\partial W}{\partial z_i} \right)^{p_i - t_i} \cdot \sum_4^* \prod_{2 \leq |\mu| \leq |p|} \left( D_z^\mu \frac{W(z, \cdot)}{\mu!} \right)^{a_\mu} \frac{1}{a_\mu!}, \quad (2.34)$$

where  $\sum_4^*$  means summation over all  $a_\mu$  such that

$$\sum_{|\mu|=2} a_\mu = |t| - l, \quad \sum_{|\mu|=2} \mu a_\mu = t. \quad (2.35)$$

Now the second of (2.35) yields  $\sum_{|\mu|=2}^{|\mu|} |\mu| a_\mu = |t|$ , and thus by the first

$\sum_{|\mu|=2}^p a_\mu (|\mu| - 1) = l$ . Recalling that  $V(q) = \sum_{|\mu| \geq 2} v_p q^\mu$ , we get:

$$e^{-W(z, \cdot)/\hbar} V(\hbar \nabla_z / \sqrt{2\omega}) e^{W(z, \cdot)/\hbar} = \sum_{|\mu| \geq 2} v_p \prod_{i=1}^d \left( \frac{\partial W}{\partial z_i} / \sqrt{2\omega_i} \right)^{\mu_i} \\ + \sum_{l=1}^{\infty} \hbar^l \sum_{|t|=l+1}^{2l} \sum_{|\mu| \geq l+1} v_p \frac{p!}{(p-t)!} \prod_{i=1}^d \left( \frac{\partial W}{\partial z_i} \right)^{p_i - t_i} \sum_{\substack{* \\ 2 \leq |\mu| \leq |\mu|}} \prod \left( \frac{D_z^\mu W}{\mu!} \right)^{a_\mu} \frac{1}{a_\mu!} \quad (2.36)$$

where  $\sum_{\substack{* \\ 2 \leq |\mu| \leq |\mu|}}$  means summation over all  $a_\mu$  such that  $\sum_{|\mu|=2}^{l+1} a_\mu = |t| - l$ ,  $\sum_{|\mu|=2}^{l+1} \mu a_\mu = t$ .

We can thus conclude:

$$e^{-W(z, \cdot)/\hbar} V(\hbar \nabla_z / \sqrt{2\omega}) e^{W(z, \cdot)/\hbar} = V(\nabla_z W / \sqrt{2\omega}) + \sum_{l=1}^{\infty} \hbar^l R_l(W), \quad (2.37)$$

where  $R_l(W)$  is given by (2.20). Then (2.37), (2.25), (2.26) and Lemma 2.5 yield the assertion. ■

We are now in a position to generate the Rayleigh–Schrödinger perturbation theory by applying to Eq. (2.24) the Birkhoff transformation, as described e.g. in [4, §5.10, Proposition 17]. We have:

**2.6. Proposition.** *Let  $\Omega$  be any bounded open sphere in  $\mathbb{C}^d$ . Then there is  $\delta(k, \Omega) > 0$  such that, if  $n\hbar \in \Omega$  and  $\hbar \in C_\delta = \{z \in \mathbb{C} : |z| \leq \delta\}$ , the equations*

$$\langle \omega z, \nabla_z \rangle W_k(n, \hbar; z) + \frac{1}{(k-1)!} \frac{d^{k-1}}{d\varepsilon^{k-1}} \left[ V \left( \sum_{j=0}^{k-1} \varepsilon^j \nabla_z W(n, \hbar; z) / \sqrt{2\omega} \right) \right. \\ \left. + \sum_{l=1}^{\infty} \hbar^l R_l \left( \sum_{j=0}^{k-1} \varepsilon^j W_j(n, \hbar; z) \right) \right] \Big|_{\varepsilon=0} = \lambda_k(n; \hbar), \quad k \geq 1 \\ \lambda_0(n, \hbar) = \hbar \langle n, \omega \rangle; \quad \nabla_z W_0(n, \hbar; z) = \nabla_z W_0(n\hbar; z) \\ = ((n_1 \hbar + z_1^2)/z_1, \dots, (n_d \hbar + z_d^2)/z_d) \quad (2.38)$$

are recursively solved by a family of functions  $z \rightarrow W_k(n\hbar, \hbar; z)$ , parametrized by  $(n\hbar, \hbar)$ , and a family of functions  $(n\hbar, \hbar) \rightarrow \lambda_k(n\hbar, \hbar)$ ,  $\hbar = 1, 2, \dots$ , such that:

(1)  $W_k(n\hbar, \hbar, z)$  is holomorphic with respect to

$$(n\hbar, \hbar, z) \in \Omega \times C_\delta \times \mathbb{C}^d \setminus \{0\}, \quad k \geq 0.$$

(2)  $\lambda_k(n\hbar, \hbar)$  is holomorphic in  $\Omega \times C_\delta$  and admits the representation:

$$\lambda_k(n\hbar, \hbar) = P_k(\hbar n) + \sum_{l=1}^{\infty} \hbar^l Q_k^l(\hbar n), \quad (2.39)$$

where the functions  $x \rightarrow P_k(x)$ ,  $x \rightarrow Q_k^l(x)$ ,  $(l, k) = 1, 2, \dots$ , are holomorphic in  $\Omega$  and the series is convergent for  $\hbar \in C_\delta$ .

(3) The formal power series  $\sum_{k=0}^{\infty} \lambda_k(n\hbar, \hbar) \varepsilon^k$  and  $\sum_{k=0}^{\infty} W_k(n\hbar, \hbar; z) \varepsilon^k$  represent the

perturbation expansion to all orders near  $\lambda_n(\hbar)$ ,  $W_0(n\hbar, z)$ , respectively, for Eq. (2.24), and thus for the original Schrödinger equation (2.22), i.e. for all  $N = 1, 2, \dots$  we have:

$$\begin{aligned} & \sum_{k=0}^N \varepsilon^k \langle \omega z, \nabla_z \rangle W_k(n\hbar, \hbar; z) - \omega z^2 + \varepsilon \left[ V \left( \sum_{k=0}^{N-1} \varepsilon^k \nabla_z W_k(n\hbar, \hbar; z) / \sqrt{2\omega} \right) \right. \\ & \left. + \sum_{l=1}^{\infty} \hbar^l R_l \left( \sum_{k=0}^{N-1} \varepsilon^k W_k(n\hbar, \hbar; z) \right) \right] = \sum_{k=0}^N \lambda_k(n\hbar, \hbar) \varepsilon^k + O(\varepsilon^{N+1}). \end{aligned} \quad (2.40)$$

*Remark.* By unitary equivalence and the uniqueness of the perturbation expansion the numbers  $\lambda_k(n\hbar, \hbar)$  are just the Rayleigh–Schrödinger coefficients. Then formula (2.39), together with (2.57) below, yields their full, explicit semiclassical expansion.

*Proof.* Set, heuristically:

$$W(n, \hbar, \varepsilon; z) = \sum_{k=0}^{\infty} W_k(n, \hbar; z) \varepsilon^k, \quad (2.41)$$

$$E(n, \hbar, \varepsilon) = \sum_{k=0}^{\infty} \lambda_k(n, \hbar) \varepsilon^k. \quad (2.42)$$

Insertion in (2.24), Taylor expansion of both sides near  $\varepsilon = 0$  and equality of the  $k$ -th order coefficients of both sides yields (2.38), with (see [4, p. 477])

$$\begin{aligned} & \frac{1}{(k-1)!} \frac{d^{k-1}}{d\varepsilon^{k-1}} V \left( \sum_{j=0}^{k-1} \varepsilon^j \nabla_z W_j(z, \cdot) / \sqrt{2\omega} \right) \Big|_{\varepsilon=0} \\ & = \sum_{|\mu| \leq k-1} D_q^\mu V(\nabla_z W_0(z, \cdot) / \sqrt{2\omega}) \sum_{a_{p_1}, \dots, a_{p_d}}^* \prod_{j=1}^d \left( \prod_{s=1}^{p_j} \partial_{z_j} W_{a_s^j}(z, \cdot) / \sqrt{2\omega_j} \right), \end{aligned} \quad (2.43)$$

where  $a_{p_1}, \dots, a_{p_d}$  are multiindices:  $a_{p_j} = (a_{p_j}^1, \dots, a_{p_j}^{p_j})$ , and  $\sum^*$  means summation over all non-negative integers  $a_s^j$  such that  $\sum_{j=1}^d \sum_{s=1}^{p_j} a_s^j = k-1$ ;

$$\frac{1}{(k-1)!} \frac{d^{k-1}}{d\varepsilon^{k-1}} R_l \left( \sum_{j=0}^{k-1} \varepsilon^j W_j(z, \cdot) \right) \Big|_{\varepsilon=0} = \sum_{\alpha=0}^{k-1} R_{l,\alpha}^{(1)} R_{l,k-1-\alpha}^{(2)}, \quad (2.44)$$

$$R_{l,\alpha}^{(1)} = \sum_{|l|=l+1}^{2l} \sum_{|\mu|=0}^{\alpha-1} \frac{D_q^{|\mu|+p}}{p!} V(\nabla_z W_0 / \sqrt{2\omega}) \cdot \sum_{a_{p_1}, \dots, a_{p_d}}^* \prod_{j=1}^d \left( \prod_{s=1}^{p_j} (\partial_{z_j} W_{a_s^j}(\cdot) / \sqrt{2\omega_j}) \right), \quad (2.45)$$

$$\begin{aligned} R_{l,\beta}^{(2)} &= \frac{1}{\beta!} \frac{d^\beta}{d\varepsilon^\beta} \sum_{|\mu|=2}^* \prod_{i=2}^{i+1} \left[ \sum_{j=0}^{k-1} \varepsilon^j \sum_{n_1, \dots, n_{k-1}}^{(k,j)} \frac{1}{n_1! \dots n_{k-1}!} \right. \\ & \quad \left. \cdot \left( \frac{D_z^\mu W_0}{\mu!} \right)^{n_1} \dots \left( \frac{D_z^\mu W_{k-1}}{\mu!} \right)^{n_{k-1}} \right] \Big|_{\varepsilon=0}, \end{aligned} \quad (2.46)$$

where  $\sum^*$  has the same meaning as in (2.20), and  $\sum^{(k,j)}$  means summation over all non-negative integers  $n_1, \dots, n_{k-1}$  such that  $n_1 + \dots + n_{k-1} = a_\mu$ , and  $n_2 + 2n_3 + \dots + (k-2)n_{k-1} = j$ .

We then see that the procedure is triangular, because for each  $k$  the right-hand side of both (2.43) and (2.44) only depends on  $W_0, \dots, W_j$  up to  $j = k-1$ , and on their derivatives. Therefore for each  $k$  (2.35) becomes an inhomogeneous, linear first order

equation for  $W_k$ . Denoting by  $Y_k(W_0, \dots, W_{k-1})$ ,  $Z_k^l(W_0, \dots, W_{k-1})$  the right-hand side of (2.43), (2.44) respectively, the infinite hierarchy of Eq. (2.38) can be rewritten in abbreviated form:

$$\langle \omega z, \nabla_z \rangle W_k(n, \hbar; z) + Y_k(W_0, \dots, W_{k-1}) + \sum_{l=1}^{\infty} Z_k^l(W_0, \dots, W_{k-1}) \hbar^l = \lambda_k(n, \hbar). \quad (2.47)$$

To find  $W_k$  and  $\lambda_k$ ,  $k \geq 1$ , we look for the Laurent expansion of  $W_k$ :

$$W_k(n, \hbar; z) = \sum_{q \in \mathbb{Z}^d} w_q^{(k)}(n, \hbar) z^q, \quad z^q = z_1^{q_1} \dots z_d^{q_d}. \quad (2.48)$$

Inserting this in (2.47) and requiring the identity of the Laurent expansion of both sides we get:

$$\langle \omega, q \rangle w_q^{(k)}(n, \hbar) + y_q^{(k)}(n, \hbar) + \sum_{l=1}^{\infty} \hbar^l \theta_q^{(k), l}(n, \hbar) = 0, \quad q \neq 0, \quad (2.49)$$

$$y_0^{(k)}(n, \hbar) + \sum_{l=1}^{\infty} \hbar^l \theta_0^{(k), l}(n, \hbar) = \lambda_k(n, \hbar), \quad (2.50)$$

where  $y_q^{(k)}(\cdot)$ ,  $\theta_q^{(k), l}(\cdot)$  stand for the Laurent coefficients of  $Y_k(W_0(n, \hbar, z), W_1(n, \hbar; z), \dots, W_{k-1}(n, \hbar; z))$  and  $Z_k^l(W_0(\cdot), \dots, W_{k-1}(\cdot))$ , respectively.

The formal solution of (2.49) is of course recursively provided by:

$$w_0^{(k)}(n, \hbar) = 0, w_q^{(k)}(n, \hbar) = -\frac{1}{\langle \omega, q \rangle} \left[ y_q^{(k)}(n, \hbar) + \sum_{l=1}^{\infty} \hbar^l \theta_q^{(k), l}(n, \hbar) \right], \quad q \neq 0. \quad (2.51)$$

Let us now prove that

$$y_q^{(k)}(n, \hbar) = y_q^{(k)}(n\hbar), \quad \theta_q^{(k), l}(n, \hbar) = \theta_q^{(k), l}(n\hbar) \quad (2.52)$$

and that there are  $C_1(k) > 0$ ,  $C_2(k) > 0$  such that

$$\sup_{n\hbar \in \Omega} |y_q^{(k)}(n\hbar) / \langle \omega, q \rangle| \leq C_1(k) e^{-\alpha|q|}, \quad (2.53)$$

$$\sup_{n\hbar \in \Omega} |\theta_q^{(k), l}(n\hbar) / \langle \omega, q \rangle| \leq C_2(k) D^l e^{-\alpha|q|} \quad (2.54)$$

for some  $D > 0$  and any  $\alpha > 0$ . In fact, by the initial condition in (2.38) and (2.43) we immediately see that  $y_q^{(1)}(n, \hbar)$  has the form (2.52) and  $n\hbar \rightarrow y_q^{(1)}(n\hbar)$  is holomorphic in  $\Omega$ . Moreover, it fulfills (2.53) by (1.7)–(1.9). Looking now at (2.44)–(2.46), we see that also  $\theta_q^{(k), l}(n, \hbar)$  has the form (2.52) and the same holomorphy property.

Proceeding as in the Appendix, it is not difficult to show that there are  $C_3 > 0$ ,  $D > 0$  such that

$$\sup_{n\hbar \in \Omega} |\theta_q^{(1), l}(n\hbar) / \langle \omega, q \rangle| \leq C_3 D^l e^{-\alpha|q|}, \quad \forall \alpha > 0. \quad (2.55)$$

By (2.51) we can thus conclude that the function  $n\hbar \rightarrow w_q^{(1)}(n\hbar)$  is holomorphic in  $\Omega$ , and that there is  $C_4 > 0$  such that, for all  $\alpha > 0$ :

$$\sup_{n\hbar \in \Omega} |w_q^{(1)}(n\hbar)| \leq C_4 e^{-\alpha|q|}. \quad (2.56)$$

Therefore the function  $(n\hbar, \hbar, z) \rightarrow W_1(n\hbar, \hbar, z)$  defined by (2.48) and (2.51) with  $k = 1$  and by the initial condition in (2.38) is holomorphic in  $\Omega \times C_\delta \times \mathbb{C}^d \setminus \{0\}$ .

The whole argument can now be iterated to all  $k > 1$ , yielding the existence of  $n\hbar \rightarrow y_q^{(k),l}(n\hbar)$ ,  $n\hbar \rightarrow \theta_q^{(k),l}(n\hbar)$  holomorphic in  $\Omega$ , and of constants  $C_1(k) > 0$ ,  $C_2(k) > 0$ ,  $D_1(k) > 0$ ,  $D_2(k) > 0$ , such that for any  $\alpha > 0$ ,

$$\sup_{n\hbar \in \Omega} |y_q^{(k),l}(n\hbar) / \langle \omega, q \rangle| \leq C_1(k) D_1(k)^l e^{-\alpha|q|}, \quad (2.57)$$

$$\sup_{n\hbar \in \Omega} |\theta_q^{(k),l}(n\hbar) / \langle \omega, q \rangle| \leq C_2(k) D_2(k)^l e^{-\alpha|q|}, \quad (2.58)$$

$$y_q^{(k)}(n\hbar) = \sum_{l=0}^{\infty} \hbar^l y_q^{(k),l}(n\hbar), \quad \theta_q^{(k),l}(n\hbar) = \theta_q^{(k),l}(n\hbar). \quad (2.59)$$

This proves all assertions, and furthermore yields the explicit expressions:

$$P_k(n\hbar) = y_0^{(k),0}(n\hbar), \quad Q_k^l(n\hbar) = \theta_0^{(k),l}(n\hbar) + y_0^{(k),l}(n\hbar). \quad (2.60)$$

### III. Proof of the Main Results

By Proposition 2.6, we have:

$$\lim_{n\hbar \rightarrow A} \lambda_k(n, \hbar) = P_k(A), \quad k = 0, 1, 2, \dots \quad (3.1)$$

Therefore Proposition 1 is a direct consequence of the following statement:

**3.1. Lemma.** *Let  $A \in \Omega$ ,  $\Omega$  as in Proposition 2.6. Let  $N_k(A)$ ,  $k = 0, 1, 2, \dots$ , be the  $k$ -th order term of the Birkhoff expansion for  $H(p, q; \varepsilon) = H_0(p, q; \omega) + \varepsilon V(q)$ . Then:*

$$P_k(A) = N_k(A), \quad k = 0, 1, 2, \dots \quad (3.2)$$

*Proof.* The assertion is true for  $k = 0$  as recalled in Sect. II. To prove it for all  $k$ , consider the classical Hamiltonian  $H(p, q; \varepsilon)$  and write it in the  $(R, z)$  canonical variables (Sect. II):

$$H_2(R, z; \varepsilon) \equiv H(C_1^{-1}(R, z); \varepsilon) = F_0(z, R) + \varepsilon V(R/\sqrt{2\omega}) \quad (3.3)$$

the notation being as in (2.11)–(2.18). Now, according to canonical perturbation theory (see e.g. [4, §5.10]), look for a completely canonical bijection  $C_\varepsilon(R, z) \equiv (A, \phi)$  of  $(\mathbb{C} \setminus \{0\})^d$  such that  $H_2(C_\varepsilon^{-1}(A, \phi))$  has a formal expansion in powers of  $\varepsilon$  with coefficients independent of  $\phi$ . To this end, look for the generating function  $\Phi(A, z; \varepsilon)$  of  $C_\varepsilon$ :

$$\begin{cases} R = \nabla_z \Phi(A, z; \varepsilon) \\ \phi = i\nabla_A \Phi(A, z; \varepsilon) \end{cases} \quad (3.4)$$

under the form of a formal power series in  $\varepsilon$ :

$$\Phi(A, z; \varepsilon) = \sum_{k=i}^{\infty} \Phi_k(A, z) \varepsilon^k \quad (3.5)$$

where  $\Phi_k(A, z)$ ,  $k = 1, 2, \dots$ , have to be recursively determined, because by (2.17) we

have:

$$\nabla_z \Phi_0(A, z) = ((A_1 + z_1^2)/z_1, \dots, (A_d + z_d^2)/z_d). \quad (3.6)$$

Look now for the Laurent expansion of  $\Phi_k(A, z)$ :

$$\Phi_k(A, z) = \sum_{q \in \mathbb{Z}^d} \Phi_k^{(q)}(A) z^q. \quad (3.7)$$

Then, upon insertion of (3.7), (3.5), (3.4) in (3.3), and after a universal expansion in powers of  $\varepsilon$ , the request that all the resulting coefficients be  $z$ -independent yields

$$\langle \omega, q \rangle \Phi_k^{(q)}(A) + y_k^{(q)}(A) = 0, \quad k \geq 1, \quad (3.8)$$

where  $y_k^{(q)}(A)$  are the Laurent coefficients of  $Y_k(\Phi_0, \dots, \Phi_{k-1})$ , defined by (2.43) with  $\Phi_0, \dots, \Phi_{k-1}$  in place of  $W_0, \dots, W_{k-1}$ . Therefore the recursive equations (3.8) are identical to the recursive equations (2.49) with  $\hbar = 0$  and have the same initial condition by (3.6). Thus proves the lemma. ■

Proposition 2 is now an immediate consequence of Proposition 1, given the KAM theorem and the  $C^\infty$  version of the Borel summability method (see e.g. Hörmander [6]).

*Proof of Proposition 2.* Since  $V$  is a polynomial of degree  $2m$ , by (2.20), (2.38), (2.60) we have  $Q_k^l(n\hbar) = 0$  for  $l \geq 2km$ .

Set now:

$$\beta_k = \sup_{x \in \Omega, 0 \leq y \leq \delta} \sum_{l=0}^{2mk-1} |Q_k^{l+1}(x)y^l|, \quad k = 1, 2, \dots, \quad (3.9)$$

$$F_k(x, y) = \sum_{l=0}^{2mk-1} Q_k^{l+1}(x)y^l, \quad k = 1, 2, \dots \quad (3.10)$$

so that  $(x, y) \rightarrow F_k(x, y) \in C^\infty(\Omega \times [0, \delta])$ , and  $\sup_{x \in \Omega, 0 \leq y \leq \delta} |F_k(x, y)| \leq \beta_k$ . Let  $x \rightarrow \chi(x) \in C_0^\infty(\mathbb{R})$ ,  $\chi(x) = 1$  for  $|x| \leq 1$ ,  $\chi(x) = 0$  for  $|x| > 2$ , and let  $\{\gamma_k\}_{k=1}^\infty$  be a positive sequence increasing monotonically to  $+\infty$ . Set:

$$g_N(x, y; \varepsilon) = \sum_{k=1}^{N-1} F_k(x, y) \chi(\varepsilon \gamma_k), \quad N = 1, 2, \dots \quad (3.11)$$

Then we can directly apply [6, Theorem 1.2.6], choosing a sequence  $\{\gamma_k\}$  suitably large, depending on  $\{\beta_k\}$  (and thus not on  $y$ ) to conclude the existence of  $(x, y, \varepsilon) \rightarrow g^\infty(x, y; \varepsilon) \in C^\infty(\Omega \times [0, \delta] \times [0, \bar{\varepsilon}])$  such that

$$g^\infty(x, y; \varepsilon) - \sum_{k=1}^{N-1} F_k(x, y) \varepsilon^k = O(\varepsilon^N), \quad N = 1, 2, \dots, \quad (3.12)$$

uniformly with respect to  $(x, y) \in \Omega \times [0, \delta]$ .

Next we recall that, for  $n\hbar = A \in \Gamma^\infty(\varepsilon)$ ,

$$f^\infty(n\hbar, \varepsilon) - \sum_{k=0}^{N-1} N_k(n\hbar) \varepsilon^k = O(\varepsilon^N), \quad N = 1, 2, \dots \quad (3.13)$$

Since under assumption (a) it is well known that, for any fixed  $\hbar > 0$ , one has:

$$\lambda_n(\hbar, \varepsilon) - \sum_{k=0}^{N-1} \lambda_n^k(\hbar) \varepsilon^k = O(\varepsilon^N), \quad N = 1, 2, \dots \quad (3.14)$$

equations (3.12), (3.13), (3.10) and Proposition 1 prove the assertion. ■

### Appendix

We have to prove (2.21). To this end, consider first the entire function  $q \rightarrow V(q)$ . Let  $\Gamma \subset \mathbb{C}^d$  be compact. Integrating the Cauchy formula on  $S_1(R_1) \times S_d(R_d)$ ,  $S_i(R_i)$  the circle in  $\mathbb{C}$  of radius  $R_i$  centered at the origin, for  $\min_{1 \leq i \leq d} R_i$  suitably large there is  $K(\Gamma) > 0$  such that

$$\max_{q \in \Gamma} |D_q^t V(q)| \leq K(\Gamma) R^{-t} t! \bar{V}(R_1, \dots, R_d), \quad (A.1)$$

where  $t! = t_1! \dots t_d!$ ,  $R^{-t} = R_1^{-t_1} \dots R_d^{-t_d}$ ,  $\bar{V}(R_1, \dots, R_d) = \max_{0 \leq \theta_i \leq 2\pi, 1 \leq i \leq d} |V(R_1 e^{i\theta_1}, \dots, R_d e^{i\theta_d})|$ .

Let now  $z \rightarrow W(z)$  be holomorphic in  $\Omega \subset \mathbb{C}^d$ ,  $\Omega$  open, bounded and connected, and let  $\bar{\Omega} \subset \subset \Omega$  be compact. Then there are  $A > 0$ ,  $B_i > 0$ ,  $i = 1, \dots, d$ , such that

$$\max_{z \in \bar{\Omega}} |D_z^\mu W(z)| \leq AB^\mu \mu!, \quad B^\mu = B_1^{\mu_1} \dots B_d^{\mu_d}, \quad \mu! = \mu_1! \dots \mu_d!. \quad (A.2)$$

We have to estimate the maximum over  $\bar{\Omega}$  of the right-hand side of (2.20). By (A.2):

$$\max_{z \in \bar{\Omega}} \left| \prod_{2 \leq |\mu| \leq l+1} \left( \frac{(D_z^\mu W)(z)}{\mu!} \right)^{a_\mu} \frac{1}{a_\mu!} \right| \leq \prod_{|\mu|=2}^{l+1} \frac{1}{a_\mu!} \left( \prod_{i=1}^d B_i^{\mu_i a_\mu} \right). \quad (A.3)$$

Since  $\sum^*$  in (2.20) means summation over all non-negative integers  $a_\mu$  such that  $\sum_{|\mu|=2}^{l+1} \mu_i a_\mu = t_i$ ,  $i = 1, \dots, d$ , and  $\sum_{|\mu|=2}^{l+1} a_\mu = |t| - l$ , we have:

$$\max_{z \in \bar{\Omega}} \left| \sum^* \prod_{|\mu|=2}^{l+1} \left( \frac{(D_z^\mu W)(z)}{\mu!} \right)^{a_\mu} \frac{1}{a_\mu!} \right| \leq A \bar{B}^{|t|} \sum^* \prod_{|\mu|=2}^{l+1} \frac{1}{a_\mu!}, \quad (A.4)$$

where  $\bar{B} = \max(B_1, \dots, B_d)$ , and  $\sum^*$  has the same meaning as in (2.20). Now the number of multiindices  $\mu$  such that  $0 \leq |\mu| \leq |t|$  is  $|t|^d$ . Since  $l + 1 \leq |t|$ , we have:

$$A \bar{B}^{|t|} \sum^* \prod_{|\mu|=2}^{l+1} (a_\mu)^{-1} \leq A \bar{B}^{|t|} (|t| - l)!^{-1} (|t|^d)^{|t| - l} \leq A (2\bar{B}^2)^{|t|} \frac{l^{|t|}}{(|t| - l)!}, \quad (A.5)$$

the second inequality being implied by  $|t| \leq 2l$ .

Therefore we can estimate (2.20) as follows:

$$\max_{z \in \bar{\Omega}} |R_l(W(z))| \leq A \sum_{|t|=l+1}^{2l} \max_{z \in \bar{\Omega}} |D_q^t V(\nabla_z W(z))| (2\bar{B}^2)^{|t|} \frac{l^{|t|}}{(|t| - l)!}. \quad (A.6)$$

Now we can always choose  $\Gamma$  such that  $(\nabla_z W)(z) \in \Gamma$ , whence:

$$\max_{z \in \bar{\Omega}} |R_l(W(z))| \leq AK(\Gamma) \bar{V}(R_1, \dots, R_d) l^{|t|} R^{-(l+1)} \sum_{|t|=l+1}^l \frac{t!}{(|t| - l)!}, \quad (A.7)$$

where  $R = \min(R_1, \dots, R_d)$ . Since  $|t|! \geq t!$ , we have:

$$\sum_{|t|=l+1}^{2l} \frac{t!}{(|t|-l)!} \leq \sum_{|t|=l+1}^{2l} \frac{|t|!}{(|t|-l)!} \leq (2l)^d \sum_{p=1}^l \frac{(l+p)!}{p!} \leq (2l)^{d+1} \frac{(2l)!}{l!},$$

so that:

$$\max_{z \in \Omega} |R_t(W(z))| \leq AK(\Gamma) \bar{V}(R_1, \dots, R_d) (2l)^{d+1} \frac{(2l)!}{l!} l^d R^{-(l+1)}. \quad (\text{A.8})$$

Choosing  $R = l^{d+1}$ , by our assumption on  $V$  (formula (1.7)), we have  $V(R_1, \dots, R_d) \leq e^{Al}$ , whence the assertion for  $(E, \varepsilon)$  fixed. If now  $W(z, E, \varepsilon)$  represents an equibounded family of holomorphic functions on  $\Omega$ , then the constants  $A, B, K(\Gamma)$  can be chosen independently of  $(E, \varepsilon)$  and the stated uniformity holds. ■

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