Existence of Localized Solutions for a Classical Nonlinear Dirac Field

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Abstract. We prove the existence of stationary states for nonlinear Dirac equations of the form:

$$i\gamma^\mu \partial_\mu \psi - m\psi + F(\bar{\psi}\psi)\psi = 0.$$

We seek solutions which are separable in spherical coordinates and we use a shooting method for solving the associated problem of ordinary differential equations.

1. Introduction

In this paper we prove the existence of stationary states for nonlinear Dirac equations of the form

$$i \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu \psi - m\psi + F(\bar{\psi}\psi)\psi = 0 \tag{1.1}$$

under certain hypothesis on F.

The notation is the following: ψ is defined on $\mathbb{R}^4$ with values in $\mathbb{C}^4$, $\partial_\mu = \partial/\partial x_\mu$, m is a positive constant, γ^μ are 4×4 matrices given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad k = 1, 2, 3, \text{ where}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\bar{\psi}\psi = (\gamma^0\psi, \psi)$, where (,) is the usual scalar product in $\mathbb{C}^4$.

Nonlinear spinor fields giving rise to equations of the form (1.1) have been considered first by D. Ivanenko [7], H. Weyl [22], and by W. Heisenberg [6] in his unified theory of elementary particles. Later R. Finkelstein, C. F. Fronsdal and P. Kaus [4] considered the case of a spinor field with several types of fourth order self couplings. But it was M. Soler [16] who was the first to investigate the stationary

states of the nonlinear Dirac field with the scalar fourth order self coupling (corresponding to $F(x) = x$ in (1.1)) proposing them as a model of elementary extended fermions. Subsequently, the electromagnetic interaction was introduced [17, 13, 14] in order to construct a model of extended charged fermion, which in spite of its simplicity describes with a reasonable accuracy the properties of the nucleons [12]. To improve the model, the pseudoscalar fields were introduced in order to represent the cloud of pions [15, 5]. A summary of the above models, with the numerical computations and further developments are described by Rañada [10, 11]. Also the case $F(x) = x$ was considered by Rafelski [9], Takahashi [18] and Van der Merwe [23].

We are interested in stationary states, or localized solutions of (1.1), that is solutions of the form $\psi(t, x) = \exp(-i\omega t)\varphi(x)$ with $x = (x_1, x_2, x_3)$, $t = x_0$. Furthermore we want φ to be square integrable. Clearly, the equation for $\varphi \colon \mathbb{R}^3 \to \mathbb{C}^4$ is:

$$i \sum_{k=1}^{3} \gamma^k \partial_k \varphi - m\varphi + \omega\gamma^0\varphi + F(\bar\varphi\varphi)\varphi = 0. \qquad (1.2)$$

We shall prove the following result:

Theorem 1.1. *Assume that $F\colon [0, +\infty] \to [0, +\infty]$ is a C^1 function satisfying $F(0) = 0$, $F'(x) > 0$ for $x > 0$ and $\lim\limits_{x \to +\infty} F(x) = +\infty$. Then for any $\omega \in (0, m)$ there is $\varphi \in C^1(\mathbb{R}^3, \mathbb{C}^4)$ such that:*

    (i) *$\bar\varphi\varphi(x) > 0$ for $x \in \mathbb{R}^3$,*

    (ii) *φ and $\nabla\varphi$ have an exponential fall-off as $|x| \to +\infty$,*

    (iii) *φ satisfies Eq. (1.2).*

Following M. Wakano [20] and M. Soler [16] we seek solutions which are separable in spherical coordinates,

$$\varphi(x) = \begin{pmatrix} g(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r)\begin{pmatrix} \cos\theta \\ \sin\theta\, e^{i\varphi} \end{pmatrix} \end{pmatrix},$$

where f and g are real valued radial functions.

The equations for f and g are (compare [16, 20])

$$f' + \frac{2}{r}f = g(F(g^2 - f^2) - (m - \omega))$$

$$g' = f(F(g^2 - f^2) - (m + \omega)). \qquad (1.3)$$

We assume $f(0) = 0$ in order to eliminate solutions with singularities at the origin. Therefore, for any given value of $g(0)$ there is a local solution of (1.3).

The difficulty comes from the fact that we want the corresponding solution φ of (1.2) to be square integrable. Therefore we want a solution of (1.3) which exists globally and which goes to $(0, 0)$ as $r \to +\infty$.

However the numerical study of (1.3) shows, at least for certain F (compare

[10, 11]) that "most" of the solutions of (1.3) either blow-up or converge to $(0, \pm (F^{-1}(m-\omega))^{1/2})$ as $r \to +\infty$, those going to $(0,0)$ being exceptional. Therefore we have to solve a shooting problem for (1.3). Our main result is the following:

Theorem 1.2. *Assume that F satisfies the hypothesis of Theorem 1.1 and that $\omega \in (0, m)$. Then there is $(f, g) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ such that*

(i) $0 < f(r) \leqq g(r) \leqq \text{Cte} \exp\left(-\dfrac{m-\omega}{2} r \right)$ *for $r > 0$,*

(ii) (f, g) *is a solution of* (1.3).

It is immediate that Theorem 1.1 is a consequence of Theorem 1.2. Theorem 1.2 will be proved in a slightly more general form in Sect. 3 (see Theorem 3.1 and Remark 3.2).

Our argument is in a way similar to the one used by H. Berestycki, P. L. Lions and L. A. Peletier [2] for a second order equation, with the additional difficulty that we have to deal with a system and that solutions of (1.3) are not a priori bounded.

The paper is organized as follows:

—In Sect. 2 we establish preliminary results concerning systems like (1.3). Some of those results rely on sharp estimates related to the particular structure of (1.3) and their proofs are somewhat technical.

—In Sect. 3, we state and prove our main result.

—Section 4 is devoted to further remarks and comments.

2. Analysis of a System of O.D.E.'s

Throughout Sects. 2 and 3 we shall make the following assumptions and notations:

—m and ω are real numbers such that $0 < \omega < m$.

—α is a real number, $\alpha > 0$.

—$g \in C^1(\mathbb{R}, \mathbb{R})$ is an increasing function satisfying

$$g(0) = 0, \quad \lim_{x \to +\infty} g(x) > m + \omega \quad \text{and} \quad g'(g^{-1}(m-\omega)) > 0.$$

—For convenience we define $l = (g^{-1}(m-\omega))^{1/2}$.

—We also define the functions G and H by

$$G(x) = \int_0^x g(s)ds, \quad x \in \mathbb{R},$$

$$H(u, v) = \tfrac{1}{2}[G(v^2 - u^2) - m(v^2 - u^2) + \omega(v^2 + u^2)] \quad \text{for } (u, v) \in \mathbb{R}^2.$$

We shall study the system:

$$u' + \frac{\alpha}{r}u = v(g(v^2 - u^2) - (m - \omega))$$

$$v' = u(g(v^2 - u^2) - (m + \omega)). \tag{2.1}$$

First we consider the associated conservative system:

$$u' = v(g(v^2 - u^2) - (m - \omega))$$

$$v' = u(g(v^2 - u^2) - (m + \omega)). \tag{2.2}$$

Clearly (2.2) is the Hamiltonian system associated with H. We begin with:

Lemma 2.1. *H has the following properties*:
  (i) $\nabla H(u,v) = 0$ *if and only if* $(u,v) = (0,0)$ *or* $(u,v) = (0, \pm l)$.
  (ii) $H(u,v) \to +\infty$, $|u| + |v| \to +\infty$.
  (iii) *The minimum of H is negative and is achieved for* $(u,v) = (0, \pm l)$.

Proof.
  (i) is obvious.
  (ii) Let $x_0 > 0$ and $\varepsilon > 0$ be such that $g(x_0) = m - \omega + \varepsilon$. Let $x \geqq x_0$. Then:

$$G(x) - mx = \int_0^{x_0} (g(s) - m)ds + \int_{x_0}^x (g(s) - m)ds \geqq (\varepsilon - \omega)x + (\omega - m - \varepsilon)x_0.$$

For $x \in [0, x_0]$ we have $G(x) - mx \geqq -mx$, and for $x \leqq 0$ we have $G(x) - mx \geqq 0$. Therefore there is C such that $G(x) - mx \geqq (\varepsilon - \omega)x - C$ for $x \in \mathbb{R}$. Then $H(u,v) \geqq \varepsilon(u^2 + v^2) - C$; hence (ii).
  (iii) Applying (i), (ii) we only have to check that Min $H(u,v) < 0$. Actually we shall prove the stronger property:

$$\text{Let } 0 < v \leqq l; \quad \text{then} \quad H(0,v) < 0. \tag{2.3}$$

Indeed, $H(0,v) = \frac{1}{2} \int_0^{v^2} (g(s) - (m - \omega))ds$. However, for $0 \leqq s < l^2$ we have $g(s) - (m - \omega) < 0$; hence (2.3).

Lemma 2.2. *Let* $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$. *If* $H(u_0, v_0) \neq 0$ *and* $H(u_0, v_0) \neq H(0, l)$, *then the solution* (u,v) *of* (2.2) *with initial data* (u_0, v_0) *is periodic.*

Proof. Consider $\Gamma = \{(u,v), H(u,v) = H(u_0, v_0)\}$. It follows from Lemma 2.1 that Γ is compact and that $|\nabla H| \geqq \alpha > 0$ on Γ for some $\alpha > 0$. Therefore Γ is a curve and (u,v) must cover the whole of the connected component of Γ containing (u_0, v_0) in a finite time. Therefore, (u,v) is periodic.

Lemma 2.3. *Let* $(u_0, v_0) \in \mathbb{R}^2$ *and let* (u,v) *be the solution of* (2.2) *with initial data* (u_0, v_0). *Then we have*

$$\begin{aligned} &\text{If there is } k > 0 \text{ such that } v(r) \equiv k \text{ on some interval,}\\ &\text{then } (u(r), v(r)) \equiv (0, l) \text{ on } \mathbb{R}. \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\text{If } 0 < u_0 \leqq v_0 \text{ and } H(u_0, v_0) > 0, \text{ then there is } r_0 > 0\\ &\text{such that } \inf_{r \in [0, r_0]} u(r) > 0, \quad \inf_{r \in [0, r_0]} v(r) > 0 \text{ and } v(r_0) < u(r_0). \end{aligned} \tag{2.5}$$

Proof. If v' vanishes of some interval, then considering the second equation of (2.2) we see that u' also vanishes on the same interval. Hence (2.4) applying (i) of Lemma 2.1.

For proving (2.5) consider $r_1 = \text{Sup}\{r \geqq 0, u > 0 \text{ and } v > 0 \text{ on } [0,r]\}$. We shall first establish that $r_1 < \infty$.

Indeed, if $r_1 = +\infty$, then since (u,v) is periodic there is $\delta > 0$ such that $u(r)v(r) \geqq \delta$ for $r \geqq 0$.

Consider now $h(r) = \frac{1}{2}(u^2(r) - v^2(r))$.

From (2.2) we obtain $h'(r) = 2\omega u(r)v(r) \geq 2\omega\delta > 0$.

This is a contradiction since h is periodic. Therefore $r_1 < \infty$. Assume now that $v(r_1) > 0$. Then $u(r_1) = 0$ and $u'(r_1) \leq 0$. From the first equation of (2.2) we get $g(v^2(r_1)) - (m - \omega) \leq 0$, hence $0 < v(r_1) \leq l$.

On applying (2.3) we obtain that $H(u(r_1), v(r_1)) < 0$, which is a contradiction. Hence $v(r_1) = 0$ and $u(r_1) > 0$. Therefore, on choosing r_0 close enough to r_1 we obtain (2.5).

We shall now study the system (2.1). We begin with:

Lemma 2.4. *Let $x \in \mathbb{R}$. Then there is $\tau > 0$ and there is $(u_x, v_x) \in C^1([0, \tau], \mathbb{R}^2)$, unique solution of (2.1) satisfying $u_x(0) = 0$, $v_x(0) = x$. In addition, (u_x, v_x) can be extended on a maximal interval $[0, R_x[$ with either $R_x = +\infty$ or $R_x < \infty$ and $\lim_{r \to R_x} |u_x| + |v_x| = +\infty$.*

Furthermore, (u_x, v_x) depend continuously on x, uniformly on $[0, R]$ for any $R < R_x$.

Proof. We just write (2.1) in the form:

$$u(r) = \frac{1}{r^\alpha} \int_0^r s^\alpha v(s)(g(v^2(s) - u^2(s)) - (m - \omega))ds$$

$$v(r) = x + \int_0^r u(s)(g(v^2(s) - u^2(s)) - (m + \omega))ds.$$

Since the right-hand side of (2.1) is a Lipschitz continuous function of (u, v) we may use a classical contraction mapping argument.

Next, we prove the following perturbation result:

Lemma 2.5. *Let $(u_0, v_0) \in \mathbb{R}^2$ and let (u, v) be the solution of (2.2) with initial data (u_0, v_0). Let (u_n^0, v_n^0) and ρ_n be such that*

$$\rho_n \underset{n \to \infty}{\longrightarrow} +\infty \quad and \quad (u_n^0, v_n^0) \underset{n \to \infty}{\longrightarrow} (u_0, v_0).$$

Let (u_n, v_n) be a solution of

$$u_n' + \frac{\alpha}{\rho_n + r} u_n = v_n(g(v_n^2 - u_n^2) - (m - \omega))$$

$$v_n' = u_n(g(v_n^2 - u_n^2) - (m + \omega))$$

such that $u_n(0) = u_n^0$, $v_n(0) = v_n^0$, and let $[0, \tau_n)$ be the maximal existence interval of (u_n, v_n). Then

(i) $\lim_{n \to \infty} \tau_n = +\infty$,

(ii) (u_n, v_n) *converges to (u, v) uniformly on bounded intervals.*

Proof. We have to prove that for any $R < \infty$ we have $\lim_{n \to \infty} \tau_n > R$ and (u_n, v_n) converges to (u, v) uniformly on $[0, R]$. Then, consider $R < \infty$. Since $\bigcup_{r \in [0, R]} \{(u(r), v(r))\}$ is a bounded set we obtain on applying a contraction argument

that there exist $\delta > 0$ and $\tau > 0$ with the following property: For any $\rho \geqq 1, r_0 \in [0, R]$ and $(w_0, z_0) \in \mathbb{R}^2$ such that $|w_0 - u(r_0)| + |z_0 - u(r_0)| \leqq \delta$, there is a solution $(w, z) \in C^1([0, \tau], \mathbb{R}^2)$ of:

$$w' + \frac{\alpha}{\rho + r} w = z(g(z^2 - w^2) - (m - \omega))$$

$$z' = w(g(z^2 - w^2) - (m - \omega)) \tag{2.6}$$

with $w(0) = w_0$, $z(0) = z_0$ and $\underset{r \in [0, \tau]}{\text{Sup}} |w(r)| + |z(r)| \leqq 2M$, where $M = \underset{r \in [0, R]}{\text{Sup}} |u(r)| + |v(r)|$.

Let K be the Lipschitz constant of the second line of (2.6) on the ball of radius $2M$ or $\mathbb{R}^2$. Then we get from (2.6), (2.2):

$$|w(r) - u(r + r_0)| + |z(r) - v(r + r_0)| \leqq |w_0 - u(r_0)| + |(z_0 - v(r_0)|$$

$$+ \int_0^r \frac{\alpha}{s + r_0} |w(s)| ds + \int_0^r K(|w(s) - u(s + r_0)| + |z(s) - v(r + r_0)|) ds.$$

Applying Gronwall's Lemma we obtain:

$$|w(r) - u(r + r_0)| + |(z(r) - v(r + r_0)|$$

$$\leqq \left(|w_0 - u(r_0)| + |(z_0 - v(r_0)| + \frac{2M\alpha\tau}{\rho} \right) \exp(K\tau) \quad \text{for} \quad 0 \leqq r \leqq \tau.$$

Applying this with $\rho = \rho_n$, $r_0 = 0$, $w_0 = u_n^0$, $z_0 = v_n^0$ we obtain that $\lim \tau_n \geqq \tau$ and (u_n, v_n) converges uniformly to (u, v) on $(0, \tau)$.

Iterating this argument n times with $n\tau \leqq R \leqq (n + 1)\tau$ we obtain Lemma 2.5.

Lemma 2.6. *Let $x \in \mathbb{R}$. Then for any $r \in [0, R_x)$ we have*

$$\frac{d}{dr} H(u_x(r), v_x(r)) = \frac{\alpha}{r} u_x^2(r)(g(v_x^2(r) - u_x^2(r)) - (m + \omega)). \tag{2.7}$$

Proof. Equation (2.7) is derived from elementary computations.

Lemma 2.7. *Let $x > 0$ be such that $m - \omega \leqq g(x^2) \leqq m + \omega$. Then $R_x = + \infty$ and $H(u_x(r), v_x(r)) < H(0, x)$ for $r > 0$.*

Proof. Let (u, v) be such that $g(v^2 - u^2) > m + \omega$. Then

$$H(u, v) = H(0, x) + (1/2) \int_{x^2}^{v^2 - u^2} (g(s) - m) ds + \omega(v^2 + u^2 - x^2) > H(0, x) + \omega u^2.$$

Therefore the sets $\{(u, v), H(u, v) \leqq H(0, x)\}$ and $\{(u, v), g(v^2 - u^2) - (m + \omega) > 0\}$ are disconnected.

Then the lemma follows from (2.7) and an obvious continuity argument.

Lemma 2.8. *Let $x > 0$ and $\rho \in (0, R_x)$ be such that $u_x(r) > 0$, $v_x(r) > 0$ for $0 < r < \rho$.*

Assume that there is $\theta \in (0, \rho)$ such that $u_x(\theta) - v_x(\theta) = \delta > 0$. Then $u_x(r) - v_x(r) \geqq \delta$ for $r \in [\theta, \rho]$.

Proof. Let x be as above and let $h(r) = v_x(r) - u_x(r)$ for $0 \leqq r \leqq \rho$. We have $h(0) > 0$

and $h(\theta) < 0$. Therefore there is $\beta \in (0, \theta)$ such that $h(\beta) = 0$ and $h'(\beta) \leq 0$.

From (2.1) we obtain immediately $\beta \geq \alpha/2\omega$, hence $\theta \geq \alpha/2\omega$. Let now $r \in [\theta, \rho]$. We have

$$h'(r) = (m - \omega - g(v^2 - u^2))h + \left(\frac{\alpha}{r} - 2\omega\right)u \leq (m - \omega - g(v^2 - u^2))h.$$

Since $(m - \omega - g(v^2 - u^2)) \geq m - \omega > 0$ when $h \leq 0$, the above inequality yields $h'(r) \leq 0$ for $r \in [\theta, \rho]$. Hence the lemma.

Lemma 2.9. *Let $x > 0$ and $\rho \in (0, R_x)$ be such that $u_x(r) > 0$ and $v_x(r) > 0$ for $r \in (0, \rho)$.*
Assume there is $\theta \in (0, \rho)$ such that $g(v_x^2(\theta) - u_x^2(\theta)) - (m + \omega) < 0$, and let $\tau = \mathrm{Sup}\{r \in [0, \rho], \quad g(v_x^2 - u_x^2) - (m + \omega) \geq 0 \quad on \quad [0, r]\}$. Then we have $\tau \geq (\alpha/2\omega)(u_x(\tau)/v_x(\tau))$ and $g(v_x^2(r) - u_x^2(r)) - (m + \omega) < 0$ for $r \in (\tau, \rho)$.

Proof. Suppose first that $g(x^2) - (m + \omega) \leq 0$. Applying Lemma 2.7 we obtain that $H(u_x(r), v_x(r)) < H(0, x)$ for $r > 0$.

As pointed out in the proof of Lemma 2.7, this implies that

$$g(v_x^2(r) - u_x^2(r)) - (m + \omega) < 0 \quad for \quad r > 0.$$

Therefore we have $\tau = 0$ and the conclusion of Lemma 2.9 holds.

Suppose now that $g(x^2) > (m + \omega)$. Then we have $\tau > 0$. Consider $k(r) = v_x^2(r) - u_x^2(r)$ for $r \in [0, \rho]$. From (2.1) we get:

$$k'(r) = 2\left(\frac{\alpha}{r}u_x(r) - 2\omega v_x(r)\right)u_x(r). \tag{2.8}$$

It follows from the definition of τ that $k'(\tau) \leq 0$, hence $\tau > (\alpha/2\omega)(u_x(\tau)/v_x(\tau))$. It remains to prove that $g(k(r)) - (m + \omega) < 0$ for $r \in (\tau, \rho)$. We argue by contradiction. If this is wrong we can find r_0 and r_1 such that $\tau < r_0 < r_1$, $g(k(r)) - (m + \omega) < 0$ for $r \in (r_0, r_1)$ and $g(k(r_0)) = g(k(r_1)) = m + \omega$.

However, we have $k'(r_0) \leq 0$ and $k'(r_1) \geq 0$. From (2.8) we obtain

$$\frac{u_x(r_0)}{v_x(r_0)} \leq \frac{u_x(r_1)}{v_x(r_1)}.$$

Since $g(k(r_0)) = g(k(r_1)) = m + \omega$, this implies that $v_x(r_1) \geq v_x(r_0)$.

On the other hand, considering (2.1) we see that $v_x' < 0$ for $r \in (r_0, r_1)$. Hence $v_x(r_1) < v_x(r_0)$ which is a contradiction. This ends the proof of Lemma 2.

Lemma 2.10. *Let $x > 0$ be such that $R_x = +\infty$ and*

$$0 < u_x(r) \leq v_x(r) \leq \mathrm{Cte} \ for \ r > 0.$$

Then there is C such that $v_x(r) \leq C \exp\left(-\frac{m - \omega}{2}r\right)$.

Proof. Let us denote (u_x, v_x) by (u, v). Let $\tau = \mathrm{Sup}\{r \geq 0, g(v^2 - u^2) - (m + \omega) \geq 0$ on $[0, r]\}$. Considering (2.1) we see that $v' \geq 0$ on $[0, \tau]$. Therefore $v(r) \geq x$ for $r \in [0, \tau]$.

We have then:

$$u' + \frac{\alpha}{r}u \geq 2\omega v \geq 2\omega x \quad \text{on} \quad [0, \tau].$$

Hence $u(\tau) \geq (2\omega x/(\alpha + 1))\tau$. Since u is bounded, τ is finite.

Applying Lemma 2.9 we have $g(v^2 - u^2) - (m + \omega) < 0$ on $(\tau, +\infty)$. Thus v is decreasing on $(\tau, +\infty)$. Then there is $\delta \geq 0$ such that $v(r) \underset{r \to +\infty}{\searrow} \delta$.

Assume for a moment that $\delta = 0$. Then $(u, v) \underset{r \to \infty}{\to} (0, 0)$ and $g(v^2 - u^2) \underset{r \to \infty}{\to} 0$.

Considering (2.1) we have for r large enough

$$u' \leq -\frac{\alpha}{r}u - \frac{m - \omega}{2}v$$

$$v' \leq -\frac{m + \omega}{2}u.$$

Hence: $(u + v)' + ((m - \omega)/2)(u + v) \leq 0$ for r large.

Integrating the above equation we see that (u, v) has the prescribed exponential decay. Therefore it only remains to prove that $\delta = 0$. We argue by contradiction and we assume $\delta > 0$.

Let $r_n \to +\infty$ be such that $u(r_n) \to k$.

Let (w, z) be the solution of (2.2) with initial data (k, δ). It follows from Lemma 2.5 that $(u(r_n + \cdot), v(r_n + \cdot))$ converges to (w, z) uniformly on bounded intervals.

Since $v(r_n + r) \to \delta$ for any $r > 0$, we have $z(r) \equiv \delta$, hence from (2.4) we have $k = 0$ and $\delta = l$.

This argument shows that indeed $(u, v) \underset{r \to +\infty}{\to} (0, l)$. We now put $v = l + w$. The equations for (u, w) are:

$$u' + \frac{\alpha}{r}u = (l + w)(g(l^2 + w^2 + 2wl - u^2) - (m - \omega))$$

$$w' = u(g(l^2 + w^2 + 2wl - u^2) - (m + \omega)).$$

Let us put $a = g'(l^2) > 0$. Since $g(l^2) = (m - \omega)$, we have

$$g(l^2 + w^2 + 2wl - u^2) = (m - \omega) + 2law + o(u + w).$$

Therefore we have:

$$u' + \frac{\alpha}{r}u = 2l^2 aw + o(u + w)$$

$$w' = -2\omega u + o(u + w). \tag{2.9}$$

Hence: $(w - u)' = \omega(w - u) - (\omega - (\alpha/r))u - (\omega + 2l^2 a)w + o(u + w)$.

For r large we obtain $(w - u)' - \omega(w - u) \leq 0$. Hence $e^{-\omega r}(w - u)$ is non-increasing for r large. Since $e^{-\omega r}(w - u) \underset{r \to \infty}{\to} 0$ we must have $w - u \geq 0$ for r large.

Therefore we can replace $o(u + w)$ by $o(w)$.

From the first equation of (2.9) we obtain

$$u' + \frac{\alpha}{r}u \geq l^2 aw \text{ for } r \text{ large.}$$

Hence $u' \geq (l^2 a - \alpha/r)u$ for r large. This implies $u(r) \geq Ce^{(l^2 a/2)r}$ for r large which is a contradiction and Lemma 2.10 is proved.

3. The Main Result

We are now in a position to state and prove our main result. We assume that α, ω, m and g satisfy the same conditions as in Sect. 2. Then we have:

Theorem 3.1. *There is a solution* $(u,v) \in C^1([0, +\infty), \mathbb{R}^2)$ *of the system*

$$u' + \frac{\alpha}{r}u = v(g(v^2 - u^2) - (m - \omega))$$

$$v' = u(g(v^2 - u^2) - (m + \omega))$$

such that $u(0) = 0$, *and*

$$0 < u(r) \leq v(r) \leq \text{Cte} \exp\left(-\frac{m - \omega}{2}r\right) \quad \text{for} \quad r > 0.$$

Remark 3.2. Theorem 1.2 is a consequence of Theorem 3.1 above. Indeed, let F satisfy the hypothesis of Theorem 1.2, and consider g defined by:

$$g(x) = F(x) \qquad \text{for} \quad x \geq 0,$$
$$g(x) = -F(-x) \quad \text{for} \quad x \leq 0.$$

Then g satisfies the hypothesis of Theorem 3.1, and therefore Theorem 1.2 is proved.

Proof of Theorem 3.1. The proofs proceeds in several steps. We define a set I as follows:

$$I = \{x > l/\exists r_x \in (0, R_x), u_x > 0 \text{ and } v_x > 0 \text{ on } (0, r_x) \text{ and } u_x(r_x) = 0\}.$$

In Step 1 we shall prove that I is a non-empty open set. Then in Step 2 we shall establish estimates for solutions of (2.1) with initial data in I.

Finally, in Step 3 we shall prove that $(u_{\bar{x}}, v_{\bar{x}})$ with $\bar{x} = \text{Sup } I$ satisfies the conclusions of Theorem 3.1.

Step 1. I is a non-empty open set.

We have $H(0, l) < 0$ and $H(0, x) \underset{x \to +\infty}{\longrightarrow} +\infty$ from Lemma 2.1 Thus there is $x_0 > l$ such that $H(0, x_0) = 0$. We put $x_1 = \text{Inf}(x_0, g^{-1}(m + \omega))$. Then we claim that $(l, x_1) \subset I$. Indeed, let $x \in (l, x_1)$.

From Lemma 2.7, we have $R_x = +\infty$ and $H(u_x(r), v_x(r)) < H(0, x)$, for $r > 0$. Since clearly $H(0, x) < 0$, we have

$$H(u_x(r), v_x(r)) < 0 \quad \text{for } r > 0.$$

On the other hand, a straightforward computation shows that $H(y, 0) \geq 0$ for $y \in \mathbb{R}$. As a consequence $v_x(r)$ can not vanish in a finite time. Therefore, $v_x(r) > 0$ for $r > 0$. As well, we can check easily that $H(u, v) \geq 0$ when $0 \leq |v| \leq |u|$. Therefore we have $|u_x(r)| \leq v_x(r)$ for $r > 0$.

On the other hand we have $(\alpha + 1)u'_x(0) = x(g(x^2) - (m - \omega)) > 0$. Hence $u_x(r) > 0$ for $r > 0$ and small.

Thus it only remains to prove that $u_x(r)$ vanishes for some $r > 0$. We argue by contradiction and we assume that $u_x(r) > 0$ for $r > 0$. Since $H(u_x, v_x)$ is bounded, (u_x, v_x) is bounded by Lemma 2.1 and we can apply Lemma 2.10, which asserts that $(u_x, v_x) \xrightarrow{r \to \infty} (0,0)$. This implies $\lim_{r \to \infty} H(u_x, v_x) = 0$, which is a contradiction since $H(u_x, v_x) \leqq H(0, x) < 0$. Therefore u_x has to vanish and $x \in I$.

The open character of I is an immediate consequence of the continuous dependence of (u_x, v_x) on x.

Step 2. Estimates for solutions of (2.1) with initial data in I. We begin with:

Lemma 3.3. *Let* $x \in I$. *Then*:
   (i) $0 < u_x(r) \leqq v_x(r)$ *for* $r \in (0, r_x)$
   (ii) *There is* $\tau_x \in [0, r_x)$ *such that* $v_x(r) \leqq v_x(\tau_x)$ *for* $r \in [0, r_x)$.

Proof.
   (i) We argue by contradiction and we assume that there is $r_0 \in (0, r_x)$ such that $u_x(r_0) - v_x(r_0) = \delta > 0$. Applying Lemma 2.8 we obtain that $u_x(r_x) - v_x(r_x) \geqq \delta$. This is impossible since this would imply $v_x(r_x) < 0$. Thus (i) is proved.
   (ii) Since $u_x(r_x) = 0$, we have $v_x(r_x) > 0$ because $(0,0)$ is an equilibrium point of (2.1) and can not be reached in a finite time. On the other hand, we have $u'_x(r_x) \leqq 0$, which implies $g(v_x^2(r_x)) - (m - \omega) \leqq 0$. Hence $g(v_x^2(r_x)) - (m + \omega) < 0$.

On applying Lemma 2.9, we obtain the existence of $\tau_x \in [0, r_x)$ such that $g(v_x^2 - u_x^2) - (m + \omega) \geqq 0$ on $[0, \tau_x]$ and $g(v_x^2 - u_x^2) - (m + \omega) < 0$ on (τ_x, r_x).

Therefore v_x is non-decreasing on $(0, \tau_x)$ and decreasing on (τ_x, r_x), thus (ii) holds.

Then we have the following estimate:

Lemma 3.4. *Let* $M = \underset{x \in I}{\text{Sup}}\, v_x(\tau_x)$. *Then* $M < +\infty$.

Proof. We use a contradiction argument and we assume that there is $x_n \in I$ such that $v_{x_n}(\tau_{x_n}) \xrightarrow[n \to \infty]{} \infty$.

For convenience we denote $u_{x_n}, v_{x_n}, \tau_{x_n}, r_{x_n}, R_{x_n}$ by $u_n, v_n, \tau_n, r_n, R_n$.

For n large enough we must have $g(x_n^2) > (m + \omega)$, otherwise v_n would be bounded by Lemmas 2.1 and 2.7. Therefore $\tau_n > 0$ and $g(v_n^2(\tau_n) - u_n^2(\tau_n)) = (m + \omega)$. Thus we have:

$$\underset{n \to \infty}{\text{Lim}} \frac{u_n(\tau_n)}{v_n(\tau_n)} = 1. \tag{3.1}$$

On applying Lemma 2.1 (ii) we obtain the existence of $k > 0$ such that $H(k, v) > 0$ for any $v \in \mathbb{R}$.

Since by (3.1) we have $u_n(\tau_n) \xrightarrow[n \to \infty]{} \infty$, and $u_n(r_n) = 0$, for n large enough there is $\rho_n \in (\tau_n, r_n)$ such that

$$u_n(r) > k \quad \text{for} \quad r \in (\tau_n, \rho_n), \quad u_n(\rho_n) = k.$$

From the definition of τ_n and from Lemma 3.3, (i) we have

$$0 \leq g(v_n^2(r) - u_n^2(r)) \leq m + \omega \quad \text{for} \quad r \in (\tau_n, r_n).$$

Thus

$$|g(v_n^2(r) - u_n^2(r)) - (m - \omega)| \leq 3m \quad \text{for} \quad r \in (\tau_n, r_n). \tag{3.2}$$

Furthermore, let $r \in (\tau_n, \rho_n)$. We have:

$$v_n^2(r) - u_n^2(r) \leq g^{-1}(m + \omega).$$

Thus:

$$v_n(r) \leq u_n(r) \left(1 + \frac{g^{-1}(m + \omega)}{u_n^2(r)} \right)^{1/2}$$

$$\leq u_n(r) \left(1 + \frac{g^{-1}(m + \omega)}{k^2} \right)^{1/2} \leq C u_n(r).$$

Considering (2.1) we obtain that

$$\left| u_n' + \frac{\alpha}{r} u_n \right| \leq C u_u \quad \text{on} \quad (\tau_n, \rho_n).$$

In particular we have

$$u_n' + \frac{\alpha}{r} u_n + C u_n \geq 0 \quad \text{on} \quad (\tau_n, \rho_n).$$

Consider

$$\phi_n(r) = \operatorname{Log} u_n(r) + Cr + \alpha \operatorname{Log} r.$$

Then $\phi_n' \geq 0$ on (τ_n, ρ_n). Therefore $\phi_n(\rho_n) \geq \phi_n(\tau_n)$. That is:

$$C\rho_n + \operatorname{Log} \rho_n^\alpha \geq -\operatorname{Log} k + \operatorname{Log} u_n(\tau_n) + C\tau_n + \operatorname{Log} \tau_n^\alpha.$$

From (3.1) and Lemma 2.9 we have $u_n(\tau_n) \to +\infty$ and $\lim_{n \to \infty} \tau_n \geq \alpha/2\omega$. It follows easily that:

$$\rho_n \underset{n \to +\infty}{\to} +\infty.$$

Let us set $(w_n, z_n) = (u_n(\rho_n + \cdot), v_n(\rho_n + \cdot))$. We have $w_n(0) = k$ and by (3.2) $z_n(0)$ is bounded. Therefore we can assume, possibly by extracting a subsequence that

$$z_n(0) \to \tilde{z} \geq k.$$

Let (w, z) be the solution of (2.8) with initial data $(k, \tilde{z})$. By definition of k, we have $H(k, \tilde{z}) > 0$.

On applying (2.5), we have the existence of $\gamma > 0$ such that $\underset{r \in [0, \gamma]}{\operatorname{Inf}} w(r) > 0$, $\underset{r \in [0, \gamma]}{\operatorname{Inf}} z(r) > 0$, and $z(\gamma) - w(\gamma) > 0$.

On the other hand, on applying Lemma 2.5, and since $\rho_n \underset{n \to \infty}{\to} \infty$, we obtain that (w_n, z_n) converges to (w, z) uniformly on $[0, \gamma]$. Therefore for n large enough we have:

$$\underset{r \in [0, \rho_n + \gamma]}{\operatorname{Inf}} u_n(r) \geq 0, \quad \underset{r \in [0, \rho_n + \gamma]}{\operatorname{Inf}} v_n(r) \geq 0$$

and $u_n(\rho_n + \gamma) > v_n(\rho_n + \gamma)$.

Therefore Lemma 3.3(i) provides a contradiction and Lemma 3.4 holds.
3rd *Step.* Conclusion. From Lemmas 3.3 and 3.4 we have $I \subset \,]l, M[$. Therefore we can put $\bar{x} = \mathrm{Sup}\, I < \infty$. Clearly $\bar{x} \notin I$. Let $(\bar{u}, \bar{v}) = (u_{\bar{x}}, v_{\bar{x}})$ and $\bar{R} = R_{\bar{x}}$.

On the other hand, since $\bar{x} > l$ we have $\bar{u}'(0) > 0$, and thus $\bar{u} > 0$ and $\bar{v} > 0$ for $r > 0$ and small.

Since $\bar{x} \notin I$, $\bar{u}$ can not vanish before $\bar{v}$ does.

On the other hand, since $\bar{x} \in \bar{I}$, it follows from the continuous dependence of (u_x, v_x) with respect to x that $\bar{v}$ cannot vanish before $\bar{u}$. Since $(0, 0)$ is a rest point of (2.1), $\bar{u}$ and $\bar{v}$ cannot vanish simultaneously.

Therefore $\bar{u} > 0$ and $\bar{v} > 0$ for $r \in (0, \bar{R})$.

Let $x_n \in I$, $x_n \underset{n \to \infty}{\to} \bar{x}$. Let us denote $u_{x_n}, v_{x_n}, R_{x_n}, r_{x_n}$ by u_n, v_n, R_n, r_n. Let $\rho \in (0, \bar{R})$. Since $x_n \to \bar{x}$ we obtain that for n large enough we have $R_n > \rho$ and that (u_n, v_n) converges to $(\bar{u}, \bar{v})$ uniformly on $[0, \rho]$. Therefore we have $r_n > \rho$ for n large and applying Lemma 3.4 we have

$$\bar{u}(\rho) + \bar{v}(\rho) \leq 2M.$$

Therefore $\bar{u}$ and $\bar{v}$ are bounded and $\bar{R} = +\infty$.

A similar argument shows that $0 < \bar{u}(r) \leq \bar{v}(r)$ for $r > 0$. Then we can apply Lemma 2.10 and there is C such that

$$0 < \bar{u}(r) \leq \bar{v}(r) \leq C \exp\left(-\frac{m - \omega}{2} r\right).$$

The proof of Theorem 3.1 is thereby complete.

4. Further Remarks and Comments

Remark 4.1. The Dirac equation (1.1) is in great contrast with the Klein–Gordon equation, since it possesses stationary states without restrictions on the growth of the nonlinear interaction F. (Compare [2] for the Klein–Gordon equation.)

Remark 4.2. We do not know if the condition $0 < \omega < m$ is necessary in Theorem 1.1. On the other hand, this condition is almost necessary in Theorem 3.1. Indeed, it has been shown by L. Vazquez [19] that in the case $g(x) = x$, there is no solution of (2.1) such that the corresponding solution φ of (1.2) is square integrable, in the case $|\omega| > m$. His argument works as well in the more general framework of Theorem 3.1.

In addition, elementary computations show that if $-m \leq \omega \leq 0$, there is no solution of (2.1) going to 0 as $r \to \infty$. This had already been observed on numerical experiments (compare [16]).

Considering again Eq. (1.2) it seems that no elementary transformation of the solution given by Theorem 3.1 can give a solution of (1.2) for $\omega > 0$. For example, the charge conjugate of the solution given by Theorem 3.1 satisfies Eq. (1.2) with ω replaced by $-\omega$ but with $F(x)$ replaced by $F(-x)$.

Remark 4.3. Little seems to be known concerning the initial value problem (Cauchy problem) for Eq. (1.1). In particular we do not know if the stationary states are stable or not. For instance, some authors claim that the $(\bar{\psi}\psi)^2$ stationary states are unstable

[3, 21, 8] while others find regions of stable behaviour by using numerical computations [1]. Related to that question it would be of a great interest to solve directly Eq. (1.2) by a variational method, since better knowledge of the variational structure of the problem might give relevant information concerning the stability of stationary states. After the completion of this paper, the stability problem has been studied by Strauss and Vazquez [24].

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