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Comments

Symmetry Breaking in Landau Gauge
A comment to a paper by T. Kennedy and C. King

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Abstract. For the non-compact abelian lattice Higgs model in Landau gauge Kennedy and King (Princeton preprint, 1985) showed that the two point function $\langle\phi(x)\bar{\phi}(y)\rangle$ does not decay in the Higgs phase. We generalize their methods to show that for the same range of parameters there are states parametrized by an angle $\theta\in[0,2\pi)$ such that $\langle\phi(x)\rangle^{\theta}_{\text{Landau}}=e^{i\theta}\langle\phi(x)\rangle^{\theta=0}_{\text{Landau}}$ and $\langle\phi(x)\rangle^{\theta=0}_{\text{Landau}}>0$.

1. Introduction

In [1] Kennedy and King conjectured that the translation invariant pure phases of the lattice abelian Higgs model in three or more dimensions are parametrized by an angle $\theta\in[0,2\pi)$ such that

$$\langle\phi(x)\rangle_{\text{Landau}}=c\,e^{i\theta}$$

with $c>0$ in the Higgs region. Since in their paper they use boundary conditions which do not break the global gauge symmetry, they only could show that the two-point function doesn't decay. Using "Dirichlet" boundary conditions as explained below we generalize their methods to prove the following

Theorem 2. *In $d\geqq 3$, for any $\lambda>0$ there are states parametrized by an angle $\theta\in[0,2\pi)$, such that*

$$\langle\phi(x)\rangle^{\theta}_{\text{Landau}}=\langle\phi(x)\rangle^{\theta=0}_{\text{Landau}}e^{i\theta},$$

where $\langle\phi(x)\rangle^{\theta=0}_{\text{Landau}}>0$ is is uniformly bounded away from zero provided $e<e_0$ and $-m^2>R(\lambda)$ in the notation of Theorem (2.3) of [1].

Remark 1. This provides a *local* (in Landau gauge) order parameter for the phase transition established in [1].

Remark 2. We believe that our construction in fact yields $\langle\phi(x)\rangle=\langle\phi\rangle$ to be translation invariant, but in this comment we only prove it for the fixed length model.

2. The Boundary Conditions

We define the model on a d-dimensional rectangular lattice $\Lambda = \bigcup_{r=0}^{d} \Lambda^r$, which we take to be an *open* subcomplex of $\mathbb{Z}^d$. Starting from the set of sites Λ^0 this is defined recursively by the requirement [2] that an r-cell (i.e. bond for $r=1$, plaquette for $r=2$, etc.) belongs to Λ^r if and only if at least one $(r-1)$-cell in its boundary lies in Λ^{r-1} (in [3] this is called "∂^*-closed"). Visualizing Λ as a box with boundary $\partial\Lambda$ consisting of $(d-1)$-dimensional rectangles, this means that all the sites, links, plaquettes etc. contained in $\partial\Lambda$ *do not belong to* Λ, whereas they do, if Λ is closed as in [1].

With this difference in mind we write the electromagnetic part of the action as in [1]

$$S^{\Lambda}_{\mathrm{em}}(A) = \frac{1}{2}(dA, dA)_{\Lambda^2} + \frac{1}{2\alpha}(d^*A, d^*A)_{\Lambda^0},$$

where A is the real valued gauge field defined on Λ^1. $(dA)(p)$ is computed on plaquettes in the vicinity of $\partial\Lambda$ as usual by putting $A(b)=0$ for $b \notin \Lambda^1$. Note that an according specification is not needed for $(d^*A)(x)$, because by definition all the bonds emerging from $x \in \Lambda^0$ already belong to Λ^1. In fact although d^*A would also take on spurious values on sites one unit outside of Λ^0, these are not to be taken into account in order to have a proper gauge fixing function [4].

An important advantage of this choice of boundary conditions is, that now d^*d evaluated on 0-forms is nothing but the Laplacian $\Delta : \ell^2(\Lambda^0) \to \ell^2(\Lambda^0)$ with 0-Dirichlet boundary conditions on $\partial\Lambda^0$, which is clearly invertible. More generally we note as a standard fact [2, 5], that on an open subcomplex Λ of $\mathbb{Z}^d$ the Laplacian $\Delta = d^*d + dd^*$ is invertible on r-forms for all $0 \leqq r \leqq d-1$. In particular $D_\Lambda A e^{-S^{\Lambda}_{\mathrm{em}}(A)}$ with $D_\Lambda A = \prod_{b \in \Lambda^1} dA(b)$ is a well defined Gaussian without zero modes and hence integrable.

Finally we define the Higgs part of the action using the same notation as in [1]

$$S^{\Lambda}_{H}(A, \phi) = \frac{1}{2}\sum_{b \in \Lambda^1} |D\phi(b)|^2 + \sum_{x \in \Lambda^0 \cup \partial\Lambda^0} V(|\phi(x)|),$$

where the phase of the Higgs field $\phi(x)$ is constrained to be zero outside of Λ^0. This will correspond to $\theta = 0$ in Theorem 2. The general case is obtained by applying a global gauge transformation $\phi'(x) = e^{i\theta}\phi(x)$, $A'(b) = A(b)$.

3. The Order Parameter

Inspired by Eq. (2.5/6) of [1] we now define

$$G(x) = \phi(x)\, e^{-ie(A,h)} = r(x)\, e^{i(\theta, g) - ie(A,h)},$$

where h is given by

$$h = d\Delta^{-1} g$$

and $g = \delta_x$. As a crucial property like in [1] we note that $\|h\|^2_\infty \leqq \|h\|^2_2 = (g, \Delta^{-1}g) = (\Delta^{-1})_{xx}$ is uniformly bounded in x and $|\Lambda|$ for $d \geqq 3$. $G(x)$ is gauge invariant since $d^*h = g$ and reduces to $\phi(x)$ in Landau gauge since $(A, h) = (d^*A, \Delta^{-1}g)$.

Let us first focus on the fixed length model, i.e. the limit $\lambda \to \infty$ in

$$V(\phi) = \lambda(|\phi|^2 - a^2)^2$$

Theorem 1.

(i) *For $d \geqq 3$ and for any $0 < \gamma < 1$, there are constants $e(\gamma)$, $a(\gamma)$ such that for $e < e(\gamma)$ and $a > a(\gamma)$*

$$\langle G(x) \rangle \geqq \gamma \cdot a$$

uniform in the volume $|\Lambda|$.

(ii) *There is a constant $\delta > 0$ such that for $a < \delta$ and for all e*

$$\langle G(x) \rangle = 0$$

in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$.

Proof. First we note that in [1] the transformation of Balaban et al. [7] was implicitly used. This transformation is based on the identity

$$\sum_{\substack{\text{1-forms} \\ n: \Lambda^1 \to \mathbb{Z}}} = \sum_{\substack{v: \Lambda^2 \to \mathbb{Z} \\ dv = 0}} \sum_{s: \Lambda^0 \to \mathbb{Z}}$$

which also holds with our boundary conditions, with the minor simplification that now the 0-forms s are not constrained to be zero at a fixed point $x_0 \in \Lambda^0$, since $ds = 0$ already implies $s = 0$. Moreover Lemma (3.2) of [1] can be proven analogously, once we observe that in an open complex a maximal tree of Λ^1 has to be constructed in such a way that it touches $\partial\Lambda^0$ at only one point (which then is the base for this tree). From then one we can literally use the proof of Theorem (2.1i) of [1], since only the uniform boundedness of $\|h\|_2$ respectively $\|h\|_\infty$ is used and all correlation inequalities also apply for $G(x)$. In particular $\langle G(x) \rangle \geqq \langle \phi(x) \rangle_{XY\text{-model}}$ which also proves ii). □

Let furthermore $G_\infty := \lim_{|x-y| \to \infty} \langle G(x, y) \rangle$ the order parameter of King, Kennedy [1], then we also have the following generalization of their theorem (2.5).

Corollary. *In $d \geqq 3$ there are constants e_0 and μ_0 such that for $e < e_0$, $ae > \mu_0$ and all x*

i) $$G_\infty = |\langle G(x) \rangle|^2,$$

ii) $$\langle G(x); \bar{G}(y) \rangle = \frac{\sigma}{|x-y|^{d-2}} + O\left(\frac{1}{|x-y|^{d-1}}\right),$$

where $\langle \cdot ; \cdot \rangle$ denotes truncated expectation.

Proof.

i) This follows from Lemma (4.2) and Proposition (4.3) of [1], since also with our boundary conditions for any g of compact support $d_\Lambda \Delta_\Lambda^{-1} g$ converges in ℓ^2 to its infinite volume value in $d \geqq 3$. Hence

$$G_\infty = \langle G(x) \rangle^2 = a^2 e^{2F(h_0)},$$

where $F(h)$ is given by the translation invariant polymer expansion (4.1) of [1] and $h_0 = d\Delta^{-1}\delta_0$.

ii) Follows from the fact, that in the thermodynamic limit $\langle G(x,y)\rangle = \langle G(x)\bar{G}(y)\rangle$. $\square$

Finally we note, that the correlation inequality of [1], Appendix A, which relates expectations in the fixed length and variable length models, verbatim applies for $\langle G(x)\rangle$. Hence

$$\langle G(x)\rangle_{\text{var. length}} \geqq \langle G(x)\rangle_{\text{fixed length}}$$

for any λ, provided $-m^2$ is sufficiently large. This proves Theorem 2.

In conclusion we would like to mention that with the above boundary conditions the non-compact analogue of the factorization formula of [6] can be used to show[1]

$$\langle \phi(x)\rangle_\alpha = \langle \phi(x)\rangle_{\text{Landau}}\, e^{-\alpha/2(g,\Delta^{-2}g)}.$$

Since $(g,\Delta^{-2}g) = (\Delta^{-2})_{xx}$ diverges as $\Lambda \nearrow \mathbb{Z}^d$ in $d \leqq 4$, this proves absence of spontaneous symmetry breaking in $d \leqq 4$ for all $\alpha > 0$ in agreement with Theorem (2.4) of [1].

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1 Here $\langle\cdot\rangle_\alpha$ denotes expectation corresponding to the gauge fixing term $\frac{1}{2\alpha}(d^*A, d^*A)$ and $\langle\cdot\rangle_{\text{Landau}} \equiv \langle\cdot\rangle_{\alpha=0}$