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On the Mass Spectrum of the 2+1 Gauge-Higgs Lattice Quantum Field Theory*

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Abstract. We investigate the mass spectrum of a 2+1 lattice gauge-Higgs quantum field theory with Wilson action $\beta A_p + \lambda A_H$, where $A_p(A_H)$ is the gauge (gauge-Higgs) interaction. We determine the complete spectrum exactly for all $\beta, \lambda > 0$ by an explicit diagonalization of the gauge invariant "transfer matrix" in the approximation that the interaction terms in the spatial directions are omitted; all gauge invariant eigenfunctions are generated directly. For fixed momentum the energy spectrum is pure point and disjoint simple planar loops and strings are energy eigenfunctions. However, depending on the gauge group and Higgs representations, there are bound state energy eigenfunctions not of this form. The approximate model has a rich particle spectrum with level crossings and we expect that it provides an intuitive picture of the number and location of bound states and resonances in the full model for small $\beta, \lambda > 0$. We determine the mass spectrum, obtaining convergent expansions for the first two groups of masses above the vacuum, for small β, λ and confirm our expectations.

1. Introduction

We continue our investigation of the energy-momentum spectrum of lattice gauge theories in the Euclidean formulation. For previous results see [1–7] and for an all statistical mechanics approach to particle spectrum see [8, 9]. For spectral results in the time-continuous Hamiltonian version of these models see [10]. For numerical results see [11, 12].

Here we consider a lattice gauge-Higgs theory with Wilson action A; the Boltzmann factor is formally given by

$$e^{-A} \equiv \exp\left\{\beta \sum_{p} \operatorname{Re}\chi(g_p) + \lambda \sum_{\langle x,y\rangle \in b} \operatorname{Re}\phi^+(\mathrm{x}) \mathrm{D}_H(g_{xy})\phi(y)\right\} \tag{1.1}$$

(see [13, 14] for notation) where $\beta \geqq 0$, $\lambda \geqq 0$. The sums occurring in (1.1) are over non-oriented plaquettes P and bonds b of the lattice. χ is the character of the

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irreducible representation D_G of the gauge group. The representations $D_G(D_H)$ of the gauge group in the pure gauge (gauge-Higgs) action may differ. For simplicity of analysis and presentation we restrict our attention to a $2+1$ lattice ($x=(x_0, x_1, x_2)=(x_0, \mathbf{x}) \in Z^3$) and the gauge group SU(2) with spin $\frac{1}{2}$ ($\frac{1}{2}$ and 1) for the gauge (Higgs) representations. In this case χ is real. Also we take the Higgs field to have unit length, i.e. $|\phi(x)|=1$. Similar methods apply in the general case.

The statistical mechanics and associated quantum field theory of the model are related by the Feynman-Kac formula,

$$\langle \bar{F}G(x_0, x)\rangle = (\hat{F}, e^{-H|x_0|}e^{i\mathbf{p}\cdot\mathbf{x}}\hat{G})_{\mathscr{H}}. \tag{1.2}$$

F, G are gauge invariant functions finitely supported in the $x_0=0$ hyperplane and $G(x_0, \mathbf{x})$ denotes the translation of G by $x=(x_0, \mathbf{x})$. The left side of (1.2) is the normalized ($\langle 1\rangle = 1$) infinite lattice expectation in the Gibbs ensemble with Boltzmann factor given by (1.1) and measure $d\mu(g)d\phi$, where $d\mu(g)$ is the product of Haar measures of the gauge group, one factor for each bond, and $d\phi$ is the product of invariant measures on $|\phi(x)|=1$, one for each site. The left side of (1.2) is used to define the Hilbert space $\mathscr{H}$ [with inner product $(\cdot,\cdot)_{\mathscr{H}}$], the vectors $\hat{F}$, $\hat{G} \in \mathscr{H}$ and energy-momentum operators H, $\mathbf{P}$ on the right side; for this connection see [2, 13–15]. Actually it is the positive self-adjoint semi-group $T^{|x_0|}$, $0 \leqq T \leqq 1$, that is well-defined, and if $T>0$ then T can be written $T=e^{-H}$, $H \geqq 0$. $\langle\cdot\rangle$ is well-defined for small β, λ, using the polymer expansion of [14], is translation invariant and independent of boundary conditions. We use the Schrödinger representation, i.e. F and G in (1.2) are supported on the zero time hyperplane.

An intuitive picture of the energy-momentum spectrum can be obtained by considering an approximate model obtained from (1.1) by dropping the interaction terms in the spatial (horizontal) directions, such as maintaining only terms with vertical plaquettes and bonds. In this approximation and for any dimension, gauge group and $\beta>0$, $\lambda>0$, the model is solved exactly in Sect. II. Our method produces the spectrum and all gauge invariant eigenfunctions of the "transfer matrix" T directly. For fixed momentum $\mathbf{p} \in (-\pi, \pi]^2$ the energy spectrum is pure point and the dispersion curves are flat. Disjoint planar simple loops and strings are gauge-invariant eigenfunctions. However, depending on the gauge group and Higgs representations, there are gauge invariant bound state eigenfunctions not of this form involving, for example, $3-j$ symbols [16]. In addition to mass spectrum occurring at n-particle thresholds there is other spectrum occurring above the two-particle threshold. It is well known [17, 18] that in a pure SU(2) gauge theory with action in the spin 1/2 representation, Wilson loops in the spin one representation have perimeter decay for small $\beta>0$. It is expected therefore that dynamical Higgs fields in the spin one representation are screened, and dressed single Higgs particles should appear in the spectrum. We verify this expectation explicitly in this paper.

In the full model for small β, λ we expect the approximate model to give the correct intuitive picture of the number and location of strongly-bound bound states and resonances. It is expected that as the horizontal interaction is turned on the mass spectrum above the two-particle threshold (and not corresponding with n-particle thresholds) disappears and corresponds with resonances. However, we do not understand, as yet, the precise spectral mechanism by which this may occur.

We point out that similar considerations can be applied to scalar and multi-component lattice spin models to give a unified picture of bound states and resonances.

For the full model with β, λ small in Sect. III we obtain, using the decoupling of hyperplane, Euclidean subtraction and expansion methods of [1–7], convergent expansions for the first two groups of masses above the vacuum and confirm the above expectations. In order to obtain additional masses below the two-particle threshold further complications arise. One has to make additional Euclidean subtractions which introduces new spurious poles in the complex energy plane whose relation with the spectrum becomes increasingly more complex.

In Sect. IV we obtain decay properties of correlation functions, convolution inverse and related functions which enter in the arguments of Sect. III. Section V provides the missing proofs of theorems in Sect. III.

In Sect. VI we make some concluding remarks. The mass expansions obtained here and in previous work result from finding a suitable implicitly defined function whose n^{th} derivative is given recursively (the classical implicit function theorem). We give in an appendix an explicit formula for the n^{th} derivative of the implicit function which may be useful in the numerical evaluation of masses employing the mass expansions given here and in [4–7].

II. Spectrum and Energy Eigenfunctions for Approximate Model

In this section we obtain the full energy-momentum spectrum of the model without horizontal plaquettes and bonds. For clarity we deduce the eigenvalues and gauge invariant eigenfunctions of the T operator ("transfer matrix") explicitly for the case of a $2+1$ SU(2) theory with the gauge (Higgs) field in the spin $\frac{1}{2}(1)$ representation. Pure temporal gauge models in the same approximation are considered in [13] with the purpose of testing specific ideas concerning the continuum limit. In [13] it is shown that $\prod_b D^{\sigma_b}_{i_b j_b}(g_b)$ are eigenfunctions $\left(\prod_b \text{ denoting the product over distinct bonds and } \sigma_b \text{ the representation}\right)$ of the temporal gauge transfer matrix (see [2, 14]). To obtain the eigenfunction expansion in the gauge invariant sector one applies the corresponding projection operator (which is an integration over all gauge transformations) to linear combinations of functions of the form above. We emphasize that in our diagonalization procedure all gauge invariant eigenfunctions are obtained directly.

In Figs. 1 and 2 for small β, λ we present graphs of the particle spectrum up to the two-particle threshold with the gauge (Higgs) in the spin $\frac{1}{2}(1)$ representation as well as the case with the gauge and Higgs both in the spin $\frac{1}{2}$ representation.

Recall (see [2]) that in the full model $T=E_0\,\mathrm{U}(1)\restriction\mathscr{H}$, where U(1) is the unitary time translation operator by one unit ($x_0=1$) in the Euclidean Hilbert space $\mathscr{E}$, i.e.

$$(\psi, \mathrm{U}(1)\phi)_{\mathscr{E}}=\langle\bar{\psi}\phi(x_0=1)\rangle\,,$$

and E_0 is the orthogonal projection on the time zero hyperplane (conditional expectation) defined by

$$E_0\psi(g,\phi)=\lim_{\Lambda\uparrow Z^3}\frac{\int \psi e^{-A_\Lambda}d\mu'(g)d\phi'}{\int e^{-A_\Lambda}d\mu'(g)d\phi'}\,,$$

where $'$ means omit the time zero variables and Λ denotes a finite hypercube. $E_0\psi$ only depends on the time zero variables. For $\phi \in \mathscr{H}$ and of finite support Λ_ϕ, then

$$T\phi(g, \phi) = \lim_{\Lambda \uparrow Z^3} \frac{\int \phi(x_0 = 1) e^{-A_\Lambda} d\mu'(g) d\phi'}{\int e^{-A_\Lambda} d\mu'(g) d\phi'}.$$

Now dropping the horizontal plaquettes and bonds we obtain partial cancellation of the numerator and denominator integrating over the horizontal bonds not in the support of ϕ and $\phi(x_0 = 1)$ to obtain

$$T\phi(g, \phi) = \frac{\int \phi(x_0 = 1) e^{-A_{01}} d\mu''(g) d\phi''}{\int e^{-A_{01}} d\mu''(g) d\phi''}, \tag{2.1}$$

where $''$ means integration over variables in the support of $\phi(x_0 = 1)$ and vertical bonds between $x_0 = 0$ and $x_0 = 1$ emanating from the support of $\phi(x_0 = 1)$, and

$$e^{-A_{01}} = \prod_{p \subset \Lambda_\phi} e^{\beta \chi(g_p)} \prod_{(x, y) \subset \Lambda_\phi} e^{\lambda(\phi_x, D^{(1)}(g_{xy})\phi_y)}, \tag{2.2}$$

with the products running through vertical plaquettes and bonds between the planes $x_0 = 0$ and $x_0 = 1$ whose bases belong to Λ_ϕ. Now we use the expansions

$$e^{\beta \chi(g)} = \sum_{n=0}^{\infty} c_n(\beta) \chi_{n/2}(g), \qquad c_n(\beta) = \int e^{\beta \chi_{1/2}(g)} \chi_{n/2}(g) d\mu(g),$$

where $\chi_{n/2}$ is the character of the spin $n/2$ representation of SU(2), and with $\mathbf{v} = D^{(1)}(g_{xy}) \mathbf{\Phi}_y$,

$$\begin{aligned} e^{\lambda(\phi_x, v)} &= \sum_{l=0}^{\infty} d_l(\lambda) \sum_{m=-l}^{l} \bar{Y}_{lm}(\phi_x) Y_{lm}(\mathbf{v}) \\ &= \sum_{l=0}^{\infty} d_l(\lambda) \sum_{m, m' = -l}^{l} \bar{Y}_{lm}(\phi_x) D^{(l)}_{mm'}(g_{xy}) Y_{lm'}(\phi_y) \\ &= \sum_{l=0}^{\infty} d_l(\lambda) (Y_l(\phi_x), D^{(l)}(g_{xy}) Y_l(\phi_y)), \end{aligned}$$

where $d_l(\lambda) = 2\pi \int_{-1}^{1} e^{\lambda x} P_l(x) dx$; $P_l(\cdot)$ is the l^{th} Legendre polynomial and Y_{lm} is the spherical harmonic. Substituting the expansions into (2.2) we obtain

$$\begin{aligned} e^{-A_{01}} = \sum_{\{n_p\}} \sum_{\{l_{xy}\}} &\Big(\prod_{p \subset \Lambda_\phi} c_{n_p} \chi_{n_p/2}(g_p) \Big) \\ &\cdot \Big(\prod_{(x, y) \subset \Lambda_\phi} d_{l_{xy}} (Y_{l_{xy}}(\phi_x), D^{(l_{xy})}(g_{xy}) Y_{l_{xy}}(\phi_y)) \Big). \end{aligned} \tag{2.3}$$

Now we insert (2.3) into (2.1) and perform the calculation of the numerator by first integrating over the vertical gauge bonds.

Each term of the sum in (2.3) has the form

$$\begin{aligned} &\Big(\prod_L c_{n_L} \Big) \Big(\prod_x d_{l_x} \Big) \Big(\sum_L D^{1/2 n_L}_{\alpha_1^L, \alpha_2^L}(g_L) D^{1/2 n_L}_{\alpha_2^L \alpha_2'^L}(g_L^{(2)}) D^{1/2 n_L}_{\alpha_2'^L \alpha_1'^L}(g_L'^{-1}) D^{1/2 n_L}_{\alpha_1'^L \alpha_1^L}(g_L^{(1)^{-1}}) \Big) \\ &\Big(\prod_{x \in \Lambda_\phi} (Y_{lx}(\phi_x), D^{(l_x)}(g_x) Y_{l_x}(\phi_x')) \Big), \end{aligned} \tag{2.4}$$

where vertical plaquettes and bonds are parametrized by their bases (a bond and a site, respectively). In general each vertical bond in Λ_ϕ will share five D's and the associated integral is

$$I = \int D^{1/2n^x_{L_1+}}_{\alpha_1^{L_1^x+}\alpha_1'^{L_1^x+}}(g^{-1}) D^{1/2n^x_{L_2+}}_{\alpha_1^{L_2^x+}\alpha_1'^{L_2^x+}}(g^{-1}) D^{1/2n^x_{L_1-}}_{\alpha_2^{L_1^x-}\alpha_2'^{L_1^x-}}(g) \cdot D^{1/2n^x_{L_2^x-}}_{\alpha_2^{L_2^x-}\alpha_2'^{L_2^x-}}(g) D^{(l_x)}_{m_x m'_x}(g)\, dg\,,$$

where $L^x_1{}_+$, etc. are the bonds containing x as an endpoint. Using unitarity and $D(g^{-1}) = D(g)^{-1}$, we can rewrite I as

$$I = \int D^{1/2\bar{n}^x_{L_1+}}_{\alpha_1^{L_1^x+}\alpha_1'^{L_1^x+}}(g) D^{1/2\bar{n}^x_{L_2+}}_{\alpha_1^{L_2^x+}\alpha_1'^{L_2^x+}}(g) D^{1/2n^x_{L_1-}}_{\alpha_2^{L_1^x-}\alpha_2'^{L_1^x-}}(g) \cdot D^{1/2n^x_{L_2-}}_{\alpha_2^{L_2^x-}\alpha_2'^{L_2^x-}}(g) D^{(l_x)}_{m_x m'_x}(g)\, dg\,,$$

where $D^{\bar{n}}$ denotes the complex conjugate of the representation $D^{(n)}$. The integral can be evaluated by reducing the tensor product

$$(\tfrac{1}{2}\bar{n}^x_{L_1+}) \otimes (\tfrac{1}{2}\bar{n}^x_{L_2+}) \otimes (\tfrac{1}{2}n^x_{L_1-}) \otimes (\tfrac{1}{2}n^x_{L_2-}) \otimes l_x$$

into irreducible components. Let

$$V^{(\frac{1}{2}\bar{n}^x_{L_1+})\otimes \ldots \otimes l_x}$$

be the unitary matrix implementing this reduction. Then,

$$I = \sum_{k=1}^{N_x} V^{(\frac{1}{2}\bar{n}^x_{L_1+})\otimes\ldots\otimes l_x}_{\alpha_1^{L_1^x+}\alpha_1^{L_2^x+}\alpha_2^{L_1^x-}\alpha_2^{L_2^x-}m_x;\,k} \cdot \bar{V}^{(\frac{1}{2}\bar{n}^x_{L_1-})\otimes\ldots\otimes l_x}_{\alpha_1'^{L_1^x+}\alpha_1'^{L_2^x+}\alpha_2'^{L_1^x-}\alpha_2'^{L_2^x-}m'_x;\,k}\,,$$

where $N_x[(\frac{1}{2}\bar{n}_{L_1^x+})\otimes\ldots\otimes l_x]$ is the number of times the identify representation is contained in the decomposition, and the index k is a convenient parametrization for some particular values of $(\alpha_1^{L_1^x+}, \ldots, m_x)$.

Thus, after performing the vertical integrations, (2.4) becomes

$$\left(\prod_{L\subset\Lambda_\phi} c_{nL}\right)\left(\prod_{x\subset\Lambda_\phi} d_{lx}\right)\left(\prod_{L\subset\Lambda_\phi} D^{1/2n_L}_{\alpha_1^L\alpha_2^L}(g_L)\bar{D}^{1/2n_L}_{\alpha_1'^L\alpha_2'^L}(g'_L)\right) \left(\prod_{x\in\Lambda_\phi} \bar{Y}_{l_x m_x}(\phi_x) Y_{l_x m'_x}(\phi'_x)\right) \sum_{\{k_x\}} \prod_{x\in\Lambda_\phi} V^{(\frac{1}{2}\bar{n}^x_{L_1+})\otimes\ldots\otimes l_x}_{\alpha_1^{L_1^x+}\alpha_1^{L_2^x+}\alpha_2^{L_1^x-}\alpha_2^{L_2^x-}m_x;\,k_x};$$

$$\bar{V}^{(\frac{1}{2}\bar{n}^x_{L_1+})\otimes\ldots\otimes l_x}_{\gamma_1'^{L_1^x-}\ldots m'_x;\,k_x} = \left(\prod_{L\subset\Lambda_\phi} c_{nL}\right)\left(\prod_{x\in\Lambda_\phi} d_{lx}\right) \sum_{\{k_x\}} \psi^{\{k_x\}}_{\{n_p\},\{l_{xy}\}}(g,\phi)\,\bar{\psi}^{\{k_x\}}_{\{n_p\},\{l_{xy}\}}(g',\phi')\,,$$

where we define

$$\psi^{\{k_x\}}_{\{n_p\},\{l_{xy}\}} \equiv \left(\prod_{L\subset\Lambda_\phi} D^{1/2n_L}_{\alpha_1^L\alpha_2^L}(g_L)\left(\prod_{x\in\Lambda_\phi}\bar{Y}_{l_x m_x}(\phi_x)\right)\right) \cdot \prod_{x\in\Lambda_\phi} V^{(\frac{1}{2}\bar{n}^x_{L_1+})\otimes\ldots\otimes l_x}_{\alpha_1^{L_1^x+}\alpha_1^{L_2^x+}\alpha_2^{L_1^x-}\alpha_2^{L_2^x-}m_x;\,k_x}. \tag{2.5}$$

Notice that the set of functions (2.5) is orthogonal, i.e. $(\psi^{\{k_x\}}_{\{n_p\}\{l_{xy}\}}, \psi^{\{k'_x\}}_{\{n'_p\},\{l'_{xy}\}}) = 0$ unless $\{n'_p\} = \{n_p\}$, $\{l'_{xy}\} = \{l_{xy}\}$ and $\{k'_x\} = \{k_x\}$. The orthogonality is obvious if $\{n'_p\} \neq \{n_p\}$ or $\{l'_{xy}\} \neq \{l_{xy}\}$. If $\{n'_p\} = \{n_p\}$ and $\{l'_{xy}\} = \{l_{xy}\}$, then

$$(\psi^{\{k_x\}}_{\{n_p\}\{l_{xy}\}}, \psi^{\{k'_x\}}_{\{n_p\}\{l_{xy}\}}) = \left(\prod_{L\subset\Lambda_\phi}\frac{1}{d_L}\right)\prod_{x\in\Lambda_\phi} V^{-1(\frac{1}{2}\bar{n}^x_{L_1+}\otimes\ldots\otimes l_x)}_{k_x;\,\alpha_1^{L_1^x+}\ldots m_x}$$

$$V^{(\frac{1}{2}\bar{n}^x_{L_1+})\otimes\ldots\otimes l_x}_{\alpha_1^{L_1^x+}\ldots m_x;\,k'_x} = \left(\prod_{L\subset\Lambda_\phi}\frac{1}{d_L}\right)\prod_{x\in\Lambda_\phi}\delta_{k_x,k'_x}\,,$$

verifying the assertion where d_L is the dimension of the representation $\frac{1}{2}n_L$. Furthermore, the ψ's are gauge invariant.

The numerator of (2.1) has the form

$$\sum_{\{n_p\}} \sum_{\{l_{xy}\}} \left(\prod_{p \subset \Lambda_\phi} c_{n_p} \right) \left(\prod_{xy \subset \Lambda_\phi} d_{l_{xy}} \right) \sum_{\{k_x\}} (\psi^{\{k_x\}}_{\{n_p\}\{l_{xy}\}}, \phi) \psi^{\{k_x\}}_{\{n_p\}\{l_{xy}\}}.$$

In the denominator we continue the integration over the horizontal bonds in the $x_0 = 1$ plane. In (2.3) only $c_{n_p} = 0$ can occur; integrating over the vertical gauge bond xy only $l_{xy} = 0$ can occur. Thus the denominator is just $\left(\prod_{p \subset \Lambda_\phi} c_0 \right) \left(\prod_{xy \subset \Lambda_\phi} d_0 \right)$.

In this way we obtain the spectral resolution given by

Theorem II.1. *$T\phi$ has the eigenfunction expansion*

$$T\phi = \sum_{\{n_p\}} \sum_{\{l_{xy}\}} \left| \left(\prod_{p \subset \Lambda_\phi} \frac{c_{n_p}}{c_0} \right) \left(\prod_{xy \subset \Lambda_\phi} \frac{d_{l_{xy}}}{d_0} \right) \left(\prod_{L \subset \Lambda_\phi} \frac{1}{d_L} \right) \right| \cdot \sum_{\{k_x\}} (\hat{\psi}^{\{k_x\}}_{\{n_p\}\{l_{xy}\}}, \phi)) \hat{\psi}^{\{k_x\}}_{\{n_p\}\{l_{xy}\}},$$

where $\hat{\psi}^{\{k_x\}}_{\{n_p\}\{l_{xy}\}} = \psi^{\{k_x\}}_{\{n_p\}\{l_{xy}\}} / |\psi^{\{k_x\}}_{\{n_p\}\{l_{xy}\}}|$ are a complete set of gauge invariant orthonormal energy eigenfunctions with eigenvalues

$$\left(\prod_p \frac{c_{n_p}}{c_0} \right) \left(\prod_{xy} \frac{d_{l_{xy}}}{d_0} \right) \left(\prod_L \frac{1}{d_L} \right).$$

Let $F_j(\lambda_j), j = 1, 2$ be the spectral family associated with the translation operator e^{iP_j}, then $d|F_j(\lambda_j)\hat{\psi}|^2 = d\lambda_j$, such as $\hat{\psi}$ has a uniform momentum distribution on $(-\pi, \pi]^2$.

Remarks. 1. The uniform momentum distribution follows by multiplying $(\psi, e^{i\mathbf{p}\cdot\mathbf{x}}\psi) = \langle \bar{\psi}\psi(\mathbf{x}) \rangle$ by $f(\mathbf{x})$, summing over $\mathbf{x}$ and noting that $\langle \bar{\psi}\psi(\mathbf{x}) \rangle = 0$ for $\mathbf{x} \neq 0$. In this way $\int f(\lambda) d|F(\lambda)\psi|^2 = \int f(\lambda) d\lambda \langle \bar{\psi}\psi \rangle = \int f(\lambda) d\lambda$ for all smooth f so that $d|F(\lambda)\psi|^2 = d\lambda$.

2. To see that disjoint simple planar loops and strings are eigenfunctions we refer to the claculations of [1, 2], where it is seen that the integral over a vertical bond where only two oppositely oriented bonds in the same representation overlap removes it and gives a factor of $1/r$, where r is the dimension of the representation.

3. A non-loop bound state eigenfunction occurs in the case of seven vertical plaquettes with the base of six forming an elementary rectangle and the base of the seventh dividing the rectangle two plaquettes. If D_G is the spin 1 representation (but not spin 1/2) then the integral over the vertical bond with 3 overlapping bonds is non-zero. The integral over the six vertical bonds gives rise to the eigenfunction with asymptotic eigenvalue $\sim -7 \ln \beta$.

4. A non-string particle occurs, for example, when the gauge (Higgs) is in the spin $\frac{1}{2}(1)$ representation. It arises from the integration of four vertical plaquettes whose base is a plaquette and a vertical bond along one edge. The eigenfunction has one Higgs fields, four gauge bonds and is given in Sect. III.

5. Related to Remark 3 by a direct calculation we obtain the inner products $(\chi_0, \chi_R) > 0$, $(\chi_0, \chi_B) > 0$, where $\chi_R(\|\chi_R\| = 1)$ is the elementary 2×1 rectangle eigenfunction, $\chi_B(\|\chi_B\| = 1)$ the bound state eigenfunction of Remark 3 and $\chi_0(\|\chi_0\| = 1)$ is the product function $\chi(g_p)\chi(g_{p'})$, where P and P' are plaquettes with one bond in common. Thus the spectrum, up to the two particle threshold $-\varepsilon$,

associated with the functions χ_0, $\chi(g_p)$ and their translates and rotates is the same as the spectrum of the full model, such as all correlation functions.

6. A candidate for a resonance comes from a 3×1, 8-sided rectangle with a ninth plaquette inside again with the spin 1 representation. The asymptotic eigenvaue is $-9\ln\beta$ for $\lambda=0$.

7. Although loop masses depend on β we see that the mass ratios are independent of β for $\lambda=0$.

We depict the mass spectrum in Figs. 1 and 2 for small β and λ. The asymptotic mass is plotted as a function of α^{-1} (or γ^{-1}), where $\lambda=\beta^{\alpha}(\beta=\lambda^{\gamma})$. The solid (dotted) lines correspond to the mass of the model with the gauge, Higgs representations $\frac{1}{2}$, $1(\frac{1}{2},\frac{1}{2})$. We have also indicated the two-particle thresholds although we emphasize that in the approximate model for fixed momentum there is no continuously varying energy typical of two asymptotically free particles. Some of the associated eigenfunctions are represented pictorially with a single line indicating a gauge bond in the spin $\frac{1}{2}$ representation and a double line indicating a gauge bond in the spin 1 representation; the degeneracy is determined by the number of distinct configurations (not related by translation) that can be produced by rotation about the x_0 axis. A dot indicates a Higgs field. The heavy line of the insets indicates the region of the β, λ plane displayed on the graph. The physical Hilbert space $\mathscr{H}$ can be written $\mathscr{H}=\mathscr{H}_o\oplus\mathscr{H}_e$, where $\mathscr{H}_e(\mathscr{H}_o)$ contains an even (odd number of Higgs fields, and we display the spectrum separately for $\mathscr{H}_o$ and $\mathscr{H}_e$.

Some of the interesting features of the graphs are the abundance of particles, level crossings and the fact that some particles are stable in one region but unstable in another. Furthermore we see the existence of dressed single Higgs fields as particles.

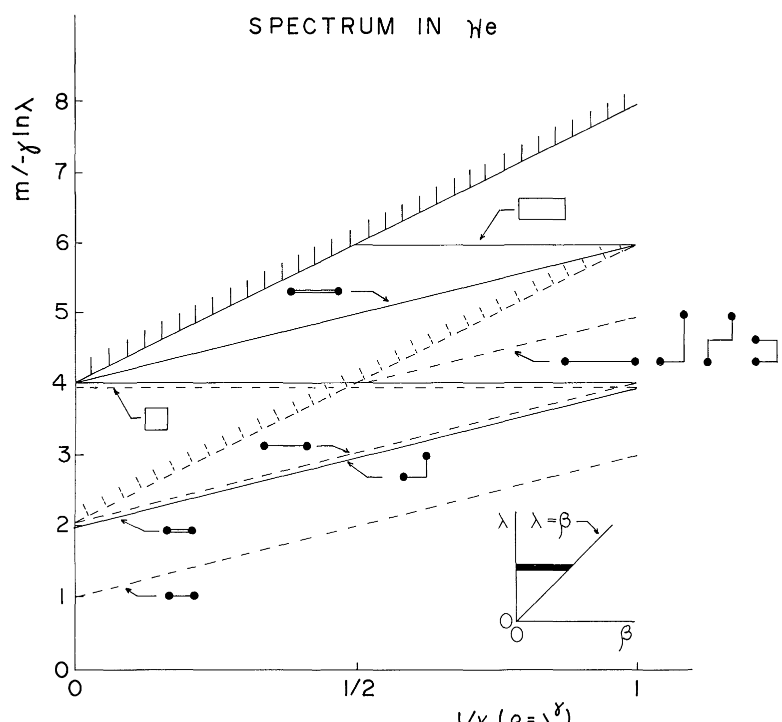

Fig. 1

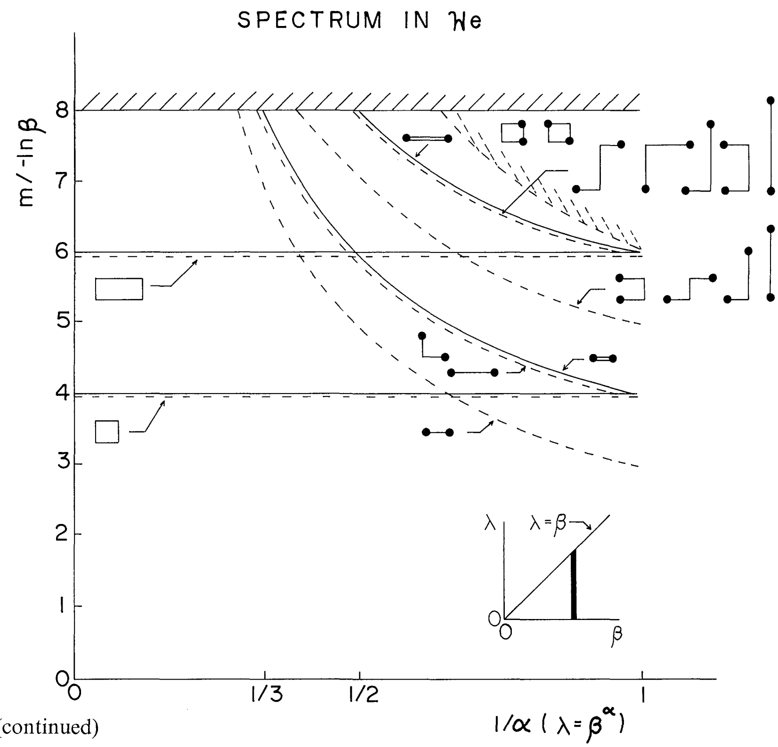

Fig. 1 (continued)

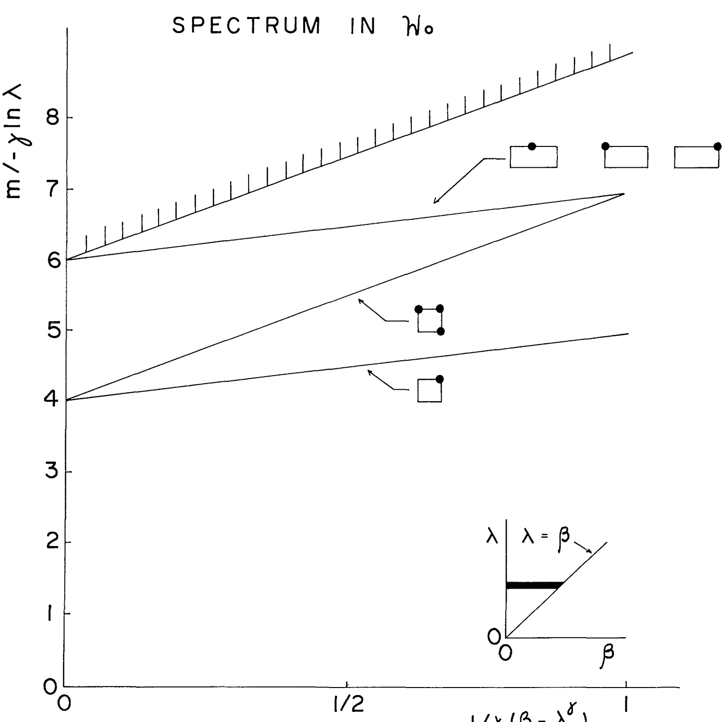

Fig. 2

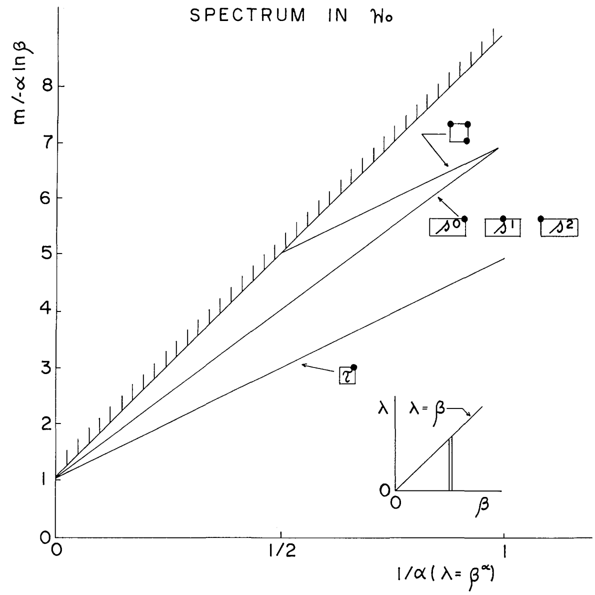

Fig. 2 (continued)

Fig. 1. Mass spectrum for small β, λ in the even Higgs subspace $\mathscr{H}_e$ for the gauge group SU(2) with no magnetic interaction. The solid (dotted) lines are for the pure gauge in the spin 1/2 representation and the Higgs in the spin 1 (spin 1/2) representation. The solid (dotted) cross hatch indicates the corresponding two-particle thresholds. The inset indicates the region depicted. Associated eigenfunctions are represented by a single (double) line for a spin 1/2 (1) gauge bond, a circle for a Higgs field

Fig. 2. Same as Fig. 1 but in the odd Higgs subspace $\mathscr{H}_0$

III. First Two Mass Groups of Full Model

In this section we state the theorems which give the particle structure and convergent expansions for the first two groups of masses above the vacuum for β, λ small in the full model. Correlation function (hereafter abbreviated cf) decay properties and the missing proofs of the theorems of this section are given in Sects. IV and V, respectively. The results are in agreement with the mass spectrum obtained in the previous section (displayed in Figs. 1 and 2) for the approximate model. Different regions of the β, λ plane are treated separately. We restrict our analysis to the case of the gauge (Higgs) interaction in the spin $\frac{1}{2}(1)$ representation; the case of spin $\frac{1}{2}(\frac{1}{2})$ can be handled similarly. We consider zero momentum states and work within a definite Z_4 (discrete angular momentum) sector as in [3] which serves to reduce degeneracy and simplify the analysis.

In part A we treat the region of Fig. 1 ($\mathscr{H}_e$, $\lambda < \beta^3$) and show that the first two groups of masses arise from the plaquette and elementary rectangle (window) cf's. a result similar to the pure gauge case treated in [3, 7]. We give detailed arguments here as our method is new, simplifies that of [3] and generalizes to the degenerate case of part B.

In part B we treat the region of Fig. 2 ($\mathscr{H}_0$, $\lambda < \beta^3$) and the identity sector of Z_4; the masses obtained are in agreement with those displayed in the figure. Here the spectral problem for the determination of the second mass group has a new aspect as this group is triply degenerate. However, a spatial coordinate reflection (parity) symmetry is used to further reduce the degeneracy. Other regions of the β, λ plane (β, λ small) and representations of Z_4 can be treated similarly (see Sect. VI).

We let $G_{\phi\psi}(x,\beta,\lambda)$ denote the truncated ϕ, ψ cf, $\hat{G}_{\phi\psi}(x_0,\beta,\lambda) = \sum_{\mathbf{x}} G_{\phi\psi}(x=(x_0,\mathbf{x}),\beta,\lambda)$ the zero space momentum cf and

$$G_{\phi\psi}(p,\beta,\lambda)=\sum_x e^{-ipx}G_{\phi\psi}(x,\beta,\lambda)\,,$$

$$p\cdot x=\sum_{i=0}^{2} p_i x_i(\tilde{G}_{\phi\psi}(p_0,\beta,\lambda)\equiv\tilde{G}_{\phi\psi}(p_0,\mathbf{p}=0,\beta,\lambda))\,,$$

the Fourier transform. We also consider $G_{\phi\psi}(x-y)\,(\hat{G}_{\phi\psi}(x_0-y_0))$ as kernels of convolution operators on $l_2(Z^3)\,(l_2(Z))$. The above cf's are analytic for β and λ small, and we suppress the β, λ dependence when no confusion can arise. In what follows positive constants will be denoted by c_i; c_{m_n}, d_{m_n} refer to coefficients in the Taylor expansion of various cf's and their values are given in Sect. IV. ε will denote a strictly positive constant which may depend on β, but $\varepsilon(\beta)\overset{\beta\to 0}{\to}0$. Throughout our results hold for all sufficiently small β, λ unless stated otherwise. $\tilde{G}_{\phi\psi}(p_0)$, $|\mathrm{Re}\,p_0|\leqq\pi$, admits the spectral representation formally given by

$$\tilde{G}_{\phi\psi}(p_0)=\int_0^\infty \frac{\sinh\lambda_0}{\cosh\lambda_0-\cos p_0}\int_{\boldsymbol{\lambda}\in(-\pi,\pi]^2}\delta(\boldsymbol{\lambda})d(\phi,E(\lambda_0)F(\boldsymbol{\lambda})\psi)_{\mathscr{H}}$$

(see [15] for a precise statement). Spectral results follow from analyticity properties of $\tilde{G}_{\phi\phi}(p_0)$. Note that from the spectral representation $\tilde{G}_{\phi\psi}(p_0)$ is analytic for $\mathrm{Re}\,p_0\neq 0$. Points of p_0 non-analyticity ($\mathrm{Im}\,p_0\geqq 0$, $\mathrm{Re}\,p_0=0$) are in the energy (or mass) spectrum. We use the same notation as [3] regarding the Z_4 group, R denotes rotation about the x_0 axis by $\pi/2$ and $\{P_i\}$ denote the projections on the irreducible representations.

A. Here we work in $\mathscr{H}_e$. As in [3] we denote by $\chi(\chi_i,\ 1\leqq i\leqq 4)$ the plaquette function (the horizontal 1×2 rectangle (window) function in the representation i of Z_4). We first consider the spectrum in the subspaces generated by χ_i, $i=3,4$. That there is no mass spectrum in $[0,\ -(1-\varepsilon)\ln\beta^8)$ follows from

Theorem III.1. $\tilde{G}_{\chi_i\chi_i}(p_0)$, $i=3,4$ *are analytic in* $0\leqq\mathrm{Im}\,p_0<-(1-\varepsilon)\ln\beta^8$.

Now we consider the subspace generated by χ_2. We let $-\hat{\Gamma}_{\chi_2\chi_2}(x_0)$ denote the convolution inverse of $\hat{G}_{\chi_2\chi_2}(x_0)$. We have

Theorem III.2. $\tilde{\Gamma}_{\chi_2\chi_2}(p_0)$ *is analytic on* $0\leqq\mathrm{Im}\,p_0<-(1-\varepsilon)\ln\beta^8$.

From Theorem III.2 and the spectral representation for $\tilde{G}_{\chi_2\chi_2}(p_0)$ we see that $\tilde{\Gamma}_{\chi_2\chi_2}(p_0)$ has at most one zero. $\tilde{\Gamma}_{\chi_2\chi_2}(p_0)$ does indeed have a zero at $p_0=im_2\cong -i6\ln\beta$ and a convergent expansion for m_2 is obtained by setting $\lambda=\alpha\beta^3$, $0<\alpha<1$, α fixed (a procedure which we use for the expansions in this section), and writing the β Taylor expansion of $\tilde{\Gamma}_{\chi_2\chi_2}$ as

$$\tilde{\Gamma}_{\chi_2\chi_2}=\tilde{\Gamma}^d_{\chi_2\chi_2}+\tilde{\Gamma}^r_{\chi_2\chi_2}\,,$$

where $\tilde{\Gamma}^d_{\chi_2\chi_2} = -2 + c_{66}\beta^6 e^{-ip_0}$. The zeroes of $\tilde{\Gamma}^d_{\chi_2\chi_2}$ determine the correct singular and constant term in the expansion for m_2. After making a non-linear transformation from the variables p_0, β to $w = -2 + c_{66}\beta^6 e^{-ip_0}$, β the analytic implicit function theorem [19] applies to find the zero of the function $H(w, \beta)$, where $H(w = -2 + c_{66}\beta^6 e^{-ip_0}, \beta) = \tilde{\Gamma}(p_0, \beta)$. In this way we obtain

Theorem III.3. $m_2 = -\ln\beta^6 + \ln\frac{2}{c_{66}} + r(\beta)$, *where* $r(\beta) = \ln(1 + w(\beta))$ *is analytic at* $\beta = 0$ *and* $r(0) = 0$.

Passing now to the subspace generated by χ we let $\hat{\Gamma}_{\chi\chi}(x_0)$ denote the convolution inverse of $-\hat{G}_{\chi\chi}(x_0)$. We have

Theorem III.4. $\tilde{\Gamma}_{\chi\chi}(p_0)$ *is analytic on* $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^6$.

By the same technique as used in arriving at Theorem III.3. we obtain a mass m_0 as the zero of $\tilde{\Gamma}_{\chi\chi}(p_0)$ given by

Theorem III.5. $m_0 = -\ln\beta^4 + \ln c_{44} + r_0(\beta)$, $r_0(\beta)$ *analytic at* $\beta = 0$ *and* $r_0(0) = 0$.

As in [3] to go up in the spectrum we look for additional zeroes of $\tilde{\Gamma}_{\chi\chi}$. Introduce

$$\tilde{F}_{\chi_1\chi_1} = \tilde{G}_{\chi_1\chi_1} + \tilde{G}_{\chi_1\chi}\tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi\chi_1}, \tag{3.1}$$

which is expected to subtract out the physical pole contributions to $\tilde{G}_{\chi_1\chi_1}$. That it does subtract out the m_0 pole contribution is seen in

Theorem III.6. $\tilde{F}_{\chi_1\chi_1}(p_0)$ *is analytic in* $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^6$.

We denote by $-\hat{\Phi}_{\chi_1\chi_1}(x_0)$ the convolution inverse of $\hat{F}_{\chi_1\chi_1}(x_0)$ and we have

Theorem III.7. $\tilde{\Phi}_{\chi_1\chi_1}(p_0)$ *is analytic in* $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^8$.

We show below that $\tilde{\Phi}_{\chi_1\chi_1}$ has precisely one zero at $p_0 = i\varrho \approx -i\ln\beta^6$, that this zero is a pole of $\tilde{\Gamma}_{\chi\chi}$ and this is the only singularity of $\tilde{\Gamma}_{\chi\chi}$ in $-\ln\beta^5 < \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^8$. We then obtain an equation for $\tilde{\Gamma}_{\chi\chi}$ and a zero of $\tilde{\Gamma}_{\chi\chi}$ close to ϱ which corresponds to a mass m.

To find the zero of $\tilde{\Phi}_{\chi_1\chi_1}$ write the β Taylor expansion as

$$\tilde{\Phi}_{\chi_1\chi_1} = \tilde{\Phi}^d_{\chi_1\chi_1} + \tilde{\Phi}^r_{\chi_1\chi_1}, \quad \text{where} \quad \tilde{\Phi}^d_{\chi_1\chi_1} = -2 + c_{66}\beta^6 e^{-ip_0},$$

the zero of which determines the correct singular and constant term of ϱ. Proceeding as before one obtains an expansion for ϱ. As we do not know of a spectral representation for $\tilde{F}_{\chi_1\chi_1}$ or $\tilde{\Phi}_{\chi_1\chi_1}$, a Rouche argument is used to show that ϱ is the only zero of $\tilde{\Phi}$. We have

Theorem III.8. $\tilde{\Phi}_{\chi_1\chi_1}(p_0)$ *has one zero in* $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^8$ *given by* $\varrho = -\ln\beta^6 + \ln\frac{2}{c_{66}} + r_\varrho(\beta)$, *where* $r_\varrho(\beta)$ *is analytic at* $\beta = 0$ *and* $r_\varrho(0) = 0$.

We now rewirte (3.1) to obtain an equation for $\tilde{\Gamma}_{\chi\chi}$ [formally obtained by multiplying (3.1) on the right by $-\tilde{\Phi}_{\chi_1\chi_1}$ and on the left by $\tilde{\Gamma}_{\chi\chi}$]. Define

$$\tilde{L}_{\chi\chi_1} = \tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi\chi_1}\tilde{\Phi}_{\chi_1\chi_1}, \qquad \tilde{L}_{\chi_1\chi} = \tilde{\Phi}_{\chi_1\chi_1}\tilde{G}_{\chi_1\chi}\tilde{\Gamma}_{\chi\chi},$$

and

$$\tilde{M} = -\tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi_1\chi_1}\tilde{\Phi}_{\chi_1\chi_1}.$$

We have

$$\tilde{M} = \tilde{\Gamma}_{\chi\chi} - \tilde{L}_{\chi\chi_1}\tilde{F}_{\chi_1\chi_1}\tilde{L}_{\chi_1\chi}.$$

That $\tilde{\Gamma}_{\chi\chi} = \tilde{L}_{\chi\chi_1}\tilde{F}_{\chi_1\chi_1}\tilde{L}_{\chi_1\chi} + \tilde{M}$ is analytic in $0 < \mathrm{Im}\, p_0 < -(1-\varepsilon)\ln\beta^8$ except at the zero of $\Phi_{\chi_1\chi_1}$ follows from a) and c) of

Theorem III.9. a) $\tilde{L}_{\chi\chi_1}$, $\tilde{L}_{\chi_1\chi}$ *are analytic on* $0 < \mathrm{Im}\, o_0 < -(1-\varepsilon)\ln\beta^8$,
b) $\tilde{L}_{\chi\chi_1}(i\varrho) = \tilde{L}_{\chi_1\chi}(i\varrho) \neq 0$,
c) $\tilde{M}$ *is analytic on* $0 \leqq \mathrm{Im}\, p_0 < -(1-\varepsilon)\ln\beta^8$,
d) $\tilde{M}(p_0) \neq 0$ *for* $-(1-\varepsilon)\ln\beta^5 < \mathrm{Im}\, p_0 < -(1-\varepsilon)\ln\beta^8$.

We rewrite $\tilde{\Gamma}_{\chi\chi}$ as

$$\tilde{\Gamma}_{\chi\chi} = \frac{\tilde{M}}{\tilde{\Phi}_{\chi_1\chi_1}}[\tilde{\Phi}_{\chi_1\chi_1} - \tilde{M}^{-1}\tilde{L}_{\chi\chi_1}\tilde{L}_{\chi_1\chi}]. \tag{3.2}$$

That $p_0 = i\varrho$ is indeed a simple pole of $\tilde{\Gamma}_{\chi\chi}$ follows from Theorem III.9b and c.

Next, we obtain a zero m_1 (near ϱ) of

$$\tilde{\Phi}_{\chi_1\chi_1} - \tilde{M}^{-1}\tilde{L}_{\chi\chi_1}\tilde{L}_{\chi_1\chi}, \tag{3.3}$$

the term in brackets in (3.2). That $p_0 = im_1$ is a zero of $\tilde{\Gamma}_{\chi\chi}(p_0)$ and hence in the spectrum follows from Theorem III.9d. The zero of (3.3) is obtained by the same method as used previously; here we separate out $\tilde{\Phi}^d_{\chi_1\chi_1}$, the zero of which gives the correct singular and constant term of m_1. The result is

Theorem III.10. $m_1 = -6\ln\beta + \ln\dfrac{2}{c_{66}} + r_1(\beta)$, *where* $r_1(\beta)$ *is analytic and* $r_1(0) = 0$.

From the spectral representation $\tilde{G}_{\chi\chi}$ is strictly monotonic in an interval of analyticity. Thus $\tilde{\Gamma}_{\chi\chi}$ can only have one zero in an interval of analyticity. Since $m_0 < \varrho$, we have $m_1 > \varrho$ and m_1 is the only zero of $\tilde{\Gamma}_{\chi\chi}$ in $\varrho < \mathrm{Im}\, p_0 < -(1-\varepsilon)\ln\beta^8$. Finally we point out that the spectral results obtained here extend to all $\mathscr{H}$ using the methods in [3].

B. Here we analyze the spectrum in $\mathscr{H}_0$ in the identity sector of Z_4. Referring to Fig. 2 in the approximate model of Sect. II, the first mass group is given by a single Higgs with four gluons and the second by a single Higgs with six gluons. In Fig. 2, we have labelled by τ one of the energy eigenfunctions of the first group and by ϱ^0, ϱ^1, ϱ^2 three of the eigenfunctions of the second group. These gauge invariant functions are obtained by integrating over appropriate vertical gauge bonds as explained in Sect. II. As these and related functions enter in the cf's to be analyzed we begin by giving their explicit form. We have (where we choose a counter-clockwise orientation for gauge bond loops)

$$\tau = \phi_{Am}D^{1/2}(g_{AB}g_{BC}g_{CD}g_{DA})_{ij}M^m_{ij};$$
$$\varrho^0 = \phi_{Am}D^{1/2}(g_{AB}g_{BC}g_{CD}g_{DE}g_{FG})_{ij}M^m_{ij}.$$

ϱ^1 and ϱ^2 are obtained in the obvious manner from ϱ^0 by a relabelling of the gauge bounds. Here M is the matrix which block diagonalizes the tensor product representation, i.e. $M(D^{1/2}(g)\otimes\bar{D}^{1/2}(g))M^+ = D^0(g)\otimes D^1(g)$ and is related to the $3-j$ symbols. In Lemma IV.14 in part B of Sect. IV it is shown that the effect of reversing the orientation of the gauge loop is to change the sign of the above functions.

Now we consider the effect of an x_1 coordinate reflection (parity) denoted by P. Specifically if the rotation R is $R(x_1, x_2)=(-x_2, x_1)$, then $P(x_1, x_2)=(-x_1, x_2)$. We find that $P_0P=PP_0$, $P_1P=PP_1$, but $P_2P=PP_3$ and $P_3P=PP_2$.

In particular as $P_0P=PP_0$ the reflection provides us with another selection rule in the identity representation of Z_4. We now show that this naturally breaks the triply asymptotically degenerate level into a non-degenerate and doubly degenerate level. With the same letter P denoting the action on functions $P_\pm = \frac{1}{2}(1\pm P)$ are the projections on functions with $\pm$ parity. Taking into account the change of sign of τ, ϱ^0, ϱ^1 under loop orientation reversal we have

$$\hat{G}_{\phi P\varrho^0} = -\hat{G}_{\phi\varrho^2}, \qquad \hat{G}_{\phi P\varrho^1} = -\hat{G}_{\phi\varrho^1}, \qquad \hat{G}_{\phi P\tau} = -\hat{G}_{\phi R\tau}.$$

We also obtain $\hat{G}_{\phi PR\tau} = -\hat{G}_{\phi\tau}$, $\hat{G}_{\phi PR^2\tau} = -\hat{G}_{\phi R^3\tau}$ and $\hat{G}_{\phi PR^3\tau} = G_{\phi R^2}$, $\hat{G}_{\phi PR_0\tau} = -\hat{G}_{\phi P_0\tau}$ which implies, letting $\tau_0 = P_0\tau$, $\hat{G}_{\phi P_+\tau_0}=0$, $\hat{G}_{\phi P_-\tau_0} = \hat{G}_{\phi P_-\tau_0}$. We let $\varrho_\pm \equiv \sqrt{2}P\pm\varrho^0 = \frac{1}{\sqrt{2}}(\varrho^0\mp\varrho^2)$, $\sigma_\pm \equiv \sqrt{2}P\pm\varrho^1$, $\varrho^0_\pm = P_0\varrho_\pm$ and $\sigma^0_\pm = P^0\sigma_\pm$. Thus as $\hat{G}_{\phi\sigma^0_+}=0$, we are led to consider, for positive parity, the cf, $\hat{G}_{\varrho^0_+\varrho^0_+}$, and for negative parity the cf's

$$\hat{G}_{\tau_0\tau_0}, \qquad \hat{G} \equiv \begin{pmatrix} \hat{G}_{\varrho^0_-\varrho^0_-} & \hat{G}_{\varrho^0_-\sigma^0_-} \\ \hat{G}_{\sigma^0_-\varrho^0_-} & \hat{G}_{\sigma^0_-\sigma^0_-} \end{pmatrix}.$$

The analysis of the singularities of $\hat{G}_{\varrho^0_+\varrho^0_+}$ is similar to that of $\tilde{G}_{\chi_2\chi_2}$ of part A. We have, letting $-\hat{\Gamma}_{\varrho^0_+\varrho^0_+}(x_0)$ denote the convolution inverse of $\hat{G}_{\varrho^0_+\varrho^0_+}(x_0)$,

Theorem III.10. *$\tilde{\Gamma}_{\varrho^0_+\varrho^0_+}(\varrho)$ is analytic on $0\leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^8\lambda$.*

$\tilde{\Gamma}_{\varrho^0_+\varrho^0_+}(p_0)$ has a zero at $p_0 = im$, $m\cong -\ln\beta^6\lambda$ and a convergent expansion for m is obtained as in Theorem III.3. We have

Theorem III.11. *$m = -\ln\beta^6\lambda + \ln\dfrac{8}{d_{66}} + r(\beta)$, where $r(\beta)$ is analytic at $\beta=0$ and $r(0)=0$.*

The determination of the spectrum in the negative parity sector is similar to that of the identity sector in part A except that the excited level of mass $\sim -\ln\beta^6\lambda$ is doubly degenerate and we use a matrix generalization of Eqs. (3.1–3). We let $-\hat{\Gamma}_{\tau_0\tau_0}(x_0)$ denote the convolution inverse of $\hat{G}_{\tau_0\tau_0}(x_0)$. We have

Theorem III.12. *$\tilde{\Gamma}_{\tau_0\tau_0}(p_0)$ is analytic on $0\leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln\beta^6\lambda$.*

$\tilde{\Gamma}_{\tau_0\tau_0}(p_0)$ has a zero at $p_0 = im_0$, $m_0\sim -\ln\beta^4\lambda$ and the mass is given by

Theorem III.13. *$m_0 = -\operatorname{In}\beta^4\lambda + \ln 8/d_{44} + r(\beta)$, where $r(\beta)$ is analytic at $\beta=0$ and $r(0)=0$.*

Define the 2×2 matrix-valued function

$$\tilde{F} = \tilde{G}(p) + \tilde{G}_{\tau_0}\tilde{\Gamma}_{\tau_0\tau_0}\tilde{G}^{\tau_0}, \tag{3.4}$$

where

$$\tilde{G}_{\tau_0} \equiv \begin{pmatrix} \tilde{G}_{\varrho^0 \tau_0} \\ \tilde{G}_{\sigma^0 \tau_0} \end{pmatrix}, \qquad \tilde{G}^{\tau_0} \equiv (\tilde{G}_{\tau_0 \varrho^0} \vdots \tilde{G}_{\tau^0 \sigma^0}).$$

We have

Theorem III.14. *$\tilde{F}(p_0)$ is analytic on $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^6 \lambda$.*

Let $-\tilde{\Phi}$ denote the 2×2 matrix-valued inverse of $\tilde{F}$. The analyticity properties of $\tilde{\Phi}$ are given by

Theorem III.15. *$\tilde{\Phi}(p_0)$ is analytic on $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^8 \lambda$.*

We show below that the only possible singularities of $\tilde{\Gamma}_{\tau_0\tau_0}$ on $-(1-\varepsilon)\ln \beta^5 \lambda < \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^8 \lambda$ are given by the zeros of $\det \tilde{\Phi}$. That there are at most two simple zeroes or one double zero is shown by separating out the term $\det \tilde{\Phi}^d = (-8 + d_{66}\beta^6 \lambda e^{-ip_0})^2$ from $\det \tilde{\Phi}$ and using a Rouche argument; a convergent expansion for the zero or zeroes is obtained by making a non-linear transformation on p_0, β and using the Weierstrass preparation theorem [20] to obtain $\det \tilde{\Phi}(p_0 = i\varrho_\pm) = 0$, where

Theorem III.16. *$\varrho_\pm = -\ln \beta^6 \lambda + \ln 8/d_{66} + r_\pm(\beta)$; $r_\pm(\beta)$ are analytic in $\beta^{1/2}$ or β and $r_\pm(0) = 0$.*

Remark. We are not claiming that $\varrho_\pm$ are distinct.

We now rewrite (3.4) to obtain an equation for $\tilde{\Gamma}_{\tau_0\tau_0}$. Define

$$\hat{L}^{\tau_0} = \hat{\Gamma}_{\tau_0\tau_0} \hat{G}^{\tau_0} \hat{\Phi}, \qquad \hat{L}_{\tau_0} = \hat{\Phi} \hat{G}_{\tau_0} \hat{\Gamma}_{\tau_0\tau_0},$$

and $\tilde{M}$ by

$$\tilde{M} = \tilde{\Gamma}_{\tau_0\tau_0} - \tilde{L}^{\tau_0} \tilde{F} \tilde{L}_{\tau_0}.$$

Analyticity properties of $\tilde{L}^{\tau_0}$, $\tilde{L}_{\tau_0}$ and $\tilde{M}$ are given in a) and c) of

Theorem III.17. a) *$\tilde{L}^{\tau_0}$, $\tilde{L}_{\tau_0}$ are analytic on $0 \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^8 \lambda$,*
b) *$\tilde{L}^{\tau_0}(i\varrho_\pm) \neq 0$ and $\tilde{L}_{\tau_0}(i\varrho_\pm) \neq 0$,*
c) *$\tilde{M}$ is analytic on $0 < \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^8 \lambda$ and $\tilde{M} \neq 0$ on $-(1-\varepsilon)\ln \beta^5 \lambda < \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^8 \lambda$.*

Writing $\tilde{F} = -\tilde{\Phi}^{-1} = \tilde{\Phi}'/\det \tilde{\Phi}$ we see from a) and c) of Theorems III.17 and III.16 that

$$\tilde{\Gamma}_{\tau_0\tau_0} = \tilde{L}^{\tau_0} \frac{\tilde{\Phi}'}{\det \tilde{\Phi}} \tilde{L}_{\tau_0} + \tilde{M} \tag{3.5}$$

is non-singular for $\varrho_0 \neq i\varrho_\pm$. Using the spectral representation for $\tilde{G}_{\tau_0\tau_0}(p_0)$, we see that $\tilde{\Gamma}_{\tau_0\tau_0}(p_0)$ can have at most two zeroes in $(\min\{\varrho_+, \varrho_-\} \leqq \operatorname{Im} p_0 < -(1-\varepsilon)\ln \beta^8 \lambda$. Rewriting (3.5) as

$$\tilde{\Gamma}_{\tau_0\tau_0} = \frac{\tilde{M}}{\det \tilde{\Phi}} [\det \tilde{\Phi} + \tilde{M}^{-1} \tilde{L}^{\tau_0} \tilde{\Phi}' \tilde{L}_{\tau_0}], \tag{3.6}$$

we see that the only possible zeroes of $\tilde{\Gamma}_{\tau_0\tau_0}$ are given by the zeroes of the term in brackets. By separating out $\det\tilde{\Phi}^d$ from $\det\tilde{\Phi}+\tilde{M}^{-1}\tilde{L}^{\tau_0}\tilde{\Phi}'\tilde{L}_{\tau_0}$, making a change of variables, and using the Weierstrass preparation theorem [20] we obtain

Theorem III.18. a) $\det\tilde{\Phi}+\tilde{M}^{-1}\tilde{L}^{\tau_0}\tilde{\Phi}'\tilde{L}_{\tau_0}(p_0=im_1^{\pm})=0$, *where* $m_1^{\pm}=-\ln\beta^6\lambda+\ln(8/d_{66})+r_1^{\pm}(\beta)$; $r_1^{\pm}(\beta)$ *are analytic in* $\beta^{1/2}$ *or* β *and* $r_1^{\pm}(0)=0$.

b) $m_1^{\pm}$ *are the only possible points in the spectrum in* $-(1-\varepsilon)\ln\beta^5\lambda<\operatorname{Im}p_0<-(1-\varepsilon)\ln\beta^8\lambda$.

Theorem III.18b follows by a Rouche argument.

The number of distinct zeroes and spectral points in Theorem III.18, which based on the results of Sect. II we expect to be two, can be determined by calculating the first non-vanishing terms in the differences of the remainders $r_1^{\pm}$ and $r^{\pm}$. This requires calculation of additional terms in the expansion of cf's which is considerally more involved than the previous ones because of continuum state contributions.

We point out that it is possible to establish the existence of precisely two nearly degenerate masses by making the following reasonable hypotheses:

1) the full spectrum is determined by the spectrum of the scalar quark two-point function, and

2) spectral points have multiplicity one.

IV. Decay Properties of Correlation and Related Functions and their Convolution Inverses

In this section we obtain estimates on the decay properties of cf's, related functions and their convolution inverses. These estimates imply analyticity properties of the Fourier transforms and are used in Sect. V where the missing proofs of the theorems of Sect. III are given. The estimates are obtained for a finite lattice Λ by a decoupling of hyperplane method (see, for example, [1–7]) and are uniform in Λ; they extend without change to the thermodynamic limit and are independent of boundary conditions by the polymer expansion of [14].

We use a finite lattice approximation to the action, with complex coupling parameters $\{w_q\}$, z, $\{u_q\}$, v, given by

$$\begin{aligned}A_\Lambda=&\sum_q w_q\sum_{p\in P_q''}\chi(g_p)+z\sum_{p\in P^\perp}\chi(g_p)+\sum_q u_q\sum_{(i,j)\in Q_q''}(\phi(i),D^1(g_{ij})\phi(j))\\&+v\sum_{(i,j)\in Q^\perp}(\phi(i),D^1(g_{ij})\phi(j))\,.\end{aligned}\tag{4.1}$$

In (4.1) $P_q''(Q_q'')$ denote the plaquettes (Higgs bonds) parallel to the time (x_0) direction between the planes $x_0=q$ and $x_0=q+1$; $P^\perp(Q^\perp)$ denote the plaquettes (Higgs bonds) perpendicular to the time direction. For a function ϕ of the gauge fields $\{g_{ij}\}$ and Higgs fields $\{\phi(i)\}$ we define averages by

$$\langle\phi\rangle_\Lambda(\{w_q\},\{u_q\},z,v)=Z_\Lambda^{-1}\int\phi e^{A_\Lambda}dg_\Lambda d\mu_\Lambda\,,$$

where Z_Λ is such that $\langle 1\rangle_\Lambda=1$. From the polymer expansion [14] $\langle\phi\rangle_\Lambda$ is analytic in all variables if $\{|w_q|\}$, $\{|u_q|\}$, $|z|$, $|v|<\beta_0$, β_0 sufficiently small independent of Λ. Furthermore the thermodynamic limit exists, is translationally invariant and coincides with $\langle\phi\rangle$ when $\{w_q=\beta\}$ $\{u_q=\lambda\}$, $z=\beta$, $v=\lambda$. In addition, given ϕ and ψ of finite support, there exists an $m_0>0$, independent of $\{w_q\}, \ldots, v$ and Λ, such that

$$|\langle\phi(x)\psi(y)\rangle_\Lambda - \langle\phi(x)\rangle_\Lambda\langle\psi(y)\rangle_\Lambda| \leqq c_{\phi\psi}e^{-m_0|x-y|}. \tag{4.2}$$

In (4.2) $\psi(x)$ is ψ translated by $x\in Z^3$ and $c_{\phi\psi}$ is a constant depending only on ϕ, ψ.

In the sequel it is to be understood that our results hold for all sufficiently small values of β, λ, $\{w_q\}\ldots v$ and different constants may be denoted by the same letter. We also suppress the dependence on parameters when no confusion can arise. We define

$$G_{\phi\psi}(x,y;\Lambda) \equiv \langle\bar{\phi}(x)\psi(y)\rangle_\Lambda - \langle\bar{\phi}(x)\rangle_\Lambda\langle\psi(y)\rangle_\Lambda,$$

and adopt periodic conditions in the spatial direction letting

$$\hat{G}_{\phi\psi}(x_0,y_0;\Lambda) \equiv \sum_{\mathbf{y}} G_{\phi\psi}(x,y;\Lambda).$$

Due to (4.2)

$$|\hat{G}_{\phi\psi}(x_0,y_0;\Lambda)| \leqq c_{\phi\psi}e^{-m_0|x_0-y_0|}.$$

Decay properties follow more easily by writing $G_{\phi\psi}$ in terms of duplicate variables, i.e.

$$\begin{aligned} G_{\phi\psi}(x,y;\Lambda) &= \frac{1}{2Z_\Lambda^2}\int(\bar{\phi}(x)-\bar{\phi}'(x))\,(\psi(y)-\psi'(y))\,e^{A_\Lambda+A'_\Lambda}dg_\Lambda dg'_\Lambda d\mu_\Lambda d\mu'_\Lambda \\ &\equiv \frac{N_{\phi\psi}(x,y,\Lambda)}{2D(\Lambda)}, \end{aligned} \tag{4.3}$$

and typically are obtained by a double Taylor expansion in w_q, u_q. Although we consider here a double expansion many of the calculations are similar to those in [3] so that only the results will be stated. However, some estimates simplify as we only consider the gauge group SU(2) in the fundamental representation. Also as $G_{\phi\psi}(x,y,\Lambda)=0$ if $\phi(\psi)$ depends on an even (odd) number of Higgs fields, it is convenient to consider these cases separately. The even (odd) case being treated in part A(B). Each part is further divided into two subsections where in subsection 1) decay of $\hat{G}_{\phi\psi}$ is established for general $\phi\psi$ and in 2) properties of $\hat{G}_{\phi\psi}$ for specific, ϕ, ψ, related functions and their convolution inverses are established. From now on we only consider ϕ, ψ in $G_{\phi\psi}$ that are supported in $x_0=0$.

A.1) We begin by analyzing the Taylor series expansion of $G_{\phi\psi}$ in w_q, u_q. The simplification resulting from the spin 1/2 representation of SU(2) (even number of Higgs) are given in Theorem 4.1a(b) below. By considering the Taylor series expansion in w_q, u_q of the numerator and denominator of $G_{\phi\psi}$ for small w_q, u_q (dependent on Λ) we easily arrive at the structure of the Taylor series expansion for small w_q, u_q (independent of Λ). We have, writing

$$G_{\phi\psi} = \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} G_{\phi\psi}^{qkl}w_q^k u_q^l + R_{\phi\psi}^{qmn},$$

Theorem 4.1.

a) $G_{\phi\psi}^{qmn}(x,y,\Lambda)=0$, *m odd*,

b) $G_{\phi\psi}^{qmn}(x,y,\Lambda)=0$, *n odd*,

c) $G_{\phi\psi}(x,y,\Lambda)=G_{\phi\psi}^{q00}+G_{\phi\psi}^{q22}w_q^2u_q^2+R_{\phi\psi}^{q44}$,

d) $G_{\phi\psi}(x,y,\Lambda)=G_{\phi\psi}^{q40}w_q^4+G_{\phi\psi}^{q60}w_q^6+R_{\phi\psi}^{q80}$, $x_0\leqq q<y_0$.

Proof. a) As the pure gauge action has the spin 1/2 representation of SU(2) only an even number of plaquettes in P''_q can contribute. b) In the λ Taylor expansion of the Higgs exponential only an even number of Higgs can contribute. c) The absence of $G^{q20}_{\phi\psi}$ and $G^{q02}_{\phi\psi}$ follows from cancellation of volume dependent terms in the numerator and denominator, d) follows from using Peter–Weyl on the gauge bonds in P''_q.

Let $\hat{e}_0$ be the unit vector along the positive time direction. Recall that $\chi_h(\chi_v)$ denotes the elementary rectangular loop function with long axis along $x_1(x_2)$. Below $c_{44}>0$, $c_{66}>0$, c_{46} are combinatorial constants. The $G^{q40}_{\phi\psi}w_q^4$ and $G^{q60}_{\phi\psi}w_q^6$ terms in Theorem 4.1 *d* are calculated in [3] and we have

Theorem 4.2. *Let* $x_0 \leqq q < y_0$. *Then*

$$\begin{aligned} G_{\phi\psi}(x,y,\Lambda) = {} & c_{44} \sum_{t_0=q} G_{\phi\chi}(x,t,\Lambda) G_{\chi\psi}(t+e_0,y,\Lambda) w_q^4 \\ & + c_{46} \sum_{t_0=q} G_{\phi\chi}(x,t,\Lambda) G_{\chi\psi}(t+e_0,y,\Lambda) w_q^6 \\ & + c'_{66} \sum_{j=h,v} \sum_{t_0=q} G_{\phi\chi_j}(x,t,\Lambda) G_{\chi_j\psi}(t+\hat{e}_0,y,\Lambda) w_q^6 + R^{q82}_{\phi\psi}, \end{aligned}$$

where the G... on the right side are evaluated at $w_q = u_q = 0$.

For the partial Fourier transform we have

Theorem 4.3. *Let* $x_0 \leqq q < y_0$. *Then*

$$\begin{aligned} \hat{G}_{\phi\psi}(x_0,y_0,\Lambda) = {} & c_{44} \hat{G}_{\phi\chi}(x_0,q,\Lambda) \hat{G}_{\chi\psi}(q+1,y_0,\Lambda) w_q^4 \\ & + c_{46} \hat{G}_{\phi\chi}(x_0,q,\Lambda) \hat{G}_{\chi\psi}(q+1,y_0,\Lambda) w_q^6 \\ & + c'_{66} \sum_{j=h,v} \hat{G}_{\phi\chi_j}(x_0,q,\Lambda) \hat{G}_{\chi_j\psi}(q+1,y_0,\Lambda) w_q^6 + R^{q82}_{\phi\psi}, \end{aligned}$$

where the G... on the right side are evaluated at $w_q = u_q = 0$.

Taking into account the finite lattice Z_4 symmetry using the definitions $\chi_1 = P_1\chi_h$, $\chi_2 = P_2\chi_h$ and the fact that $P_3\chi_h = P_4\chi_h = 0$, $\chi_v = R\chi_h$ we have (suppressing the arguments in $\hat{G}$...)

$$\sum_{j=h,v} \hat{G}_{\phi\chi_j} \hat{G}_{\chi_j\psi} = \sum_{i,j=1}^{2} \hat{G}_{\phi P_i\chi_h} \hat{G}_{P_j\chi_h\psi} + \sum_{i,j=1}^{2} \hat{G}_{\phi P_i\chi_v} \hat{G}_{P_j\chi_v\psi}.$$

As $P_1\chi_v = P_1R\chi_h = P_1\chi_h$ and $P_2\chi_v = -\chi_2$, we have

$$\sum_{j=h,v} \hat{G}_{\phi\chi_j} \hat{G}_{\chi_j\psi} = 2\sum_{i=1}^{2} \hat{G}_{\phi\chi_i} \hat{G}_{\chi_i\psi}.$$

Therefore $\hat{G}_{\phi\psi}$ has the structure given in

Theorem 4.4. *Let* $x_0 \leqq q < y_0$.

a) *If* $\phi = P_i\phi$, $\psi = P_i\psi$, $i = 3, 4$, *then*

$$\hat{G}_{\phi\psi}(x_0,y_0,\Lambda) = R^{q82}_{\phi\psi}.$$

b) *If* $\phi = P_2\phi$, $\psi = P_2\psi$, *then*

$$\hat{G}_{\phi\psi}(x_0, y_0, \Lambda) = c_{66}\hat{G}_{\phi\chi_2}(x_0, q, \Lambda)\hat{G}_{\chi_2\psi}(q+1, y_0, \Lambda)w_q^6 + R_{\phi\psi}^{q82}.$$

c) *If* $\phi = P_1\phi$, $\psi = P_1\psi$, *then*

$$\begin{aligned}\hat{G}_{\phi\psi}(x_0, y_0, \Lambda) = {} & c_{44}\hat{G}_{\phi\chi}(x_0, q, \Lambda)\hat{G}_{\chi\psi}(q+1, y_0, \Lambda)w_q^4 \\ & + c_{46}\hat{G}_{\phi\chi}(x_0, q, \Lambda)\hat{G}_{\chi\psi}(q+1, y_0, \Lambda)w_q^6 \\ & + c_{66}\hat{G}_{\phi\chi_1}(x_0, q, \Lambda)G_{\chi_1\psi}(q+1, y_0, \Lambda)w_q^6 + R_{\phi\psi}^{q82},\end{aligned}$$

where the $\hat{G}\dots$ *on the right side above are evaluated at* $w_q = u_q = 0$.

In the rest of this part we assume $|\lambda| < |\beta|^3$, $\{|u_q| < |w_q|^3\}$ $|v| < |u|^3$. Iterating the above argument for all q, $x_0 \leqq q, < y_0$, and using a Cauchy estimate we have

Theorem 4.5. *Let* $x_0 \leqq q < y_0$.
a) *If* $\phi = P_i\phi$, $\psi = P_i\psi$, $i = 3, 4$, *then* $|\hat{G}_{\phi\psi}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{8|x_0 - y_0|}$.
b) *If* $\phi = P_2\phi$, $\psi = P_2\psi$, *then* $|\hat{G}_{\phi\psi}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{6|x_0 - y_0|}$.
c) *If* $\phi = P_1\phi$, $\psi = P_1\psi$, *then* $|\hat{G}_{\phi\psi}(x_0, y_0, \Lambda) \leqq c'|c\beta|^{4|x_0 - y|}$.

Proof. We bound $G_{\phi\psi}^{qrs} w_q^r u_q^s$ by the Cauchy estimate

$$|G_{\phi\psi}^{qrs} w_q^r u_q^s| = \left| \frac{1}{(2\pi i)^2} \oint_{|w'_q| = \beta_1, |u'_q| = \beta_1} dw'_q du'_q \frac{G(w'_q, u'_q)}{w'^{r+1}_q u'^{s+1}_q} w_q^r u_q^s \right|,$$

so that for $|w_q| = \beta$, $|u_q| = \lambda$,

$$|G_{\phi\psi}^{qrs} w_q^r u_q^s| \leqq c_{\phi\psi} \left(\frac{\beta}{\beta_1}\right)^r \left(\frac{\lambda}{\beta_1}\right)^s.$$

Assuming $\beta < \beta_1/2$ and $\lambda < \beta^3$, part (a) follows from the above estimate applied to each $R_{\phi\psi}^{q82}$ after noting that $G_{\phi\psi}^{q82} = 0$. Parts (b) and (c) are proven similarly.

2) Concerning $\hat{G}_{\chi_2\chi_2}(x_0, y_0, \Lambda)$ and its convolution inverse $-\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0, \Lambda)$ we have

Theorem IV.6.

a) $\hat{G}_{\chi_2\chi_2}(x_0, x_0, \Lambda) = \frac{1}{2} + O(\beta^2)$,

$$\hat{G}_{\chi_2\chi_2}(x_0, x_0 + 1, \Lambda) = \tfrac{1}{4}c_{66}\beta^6 + O(\beta^8),$$
$$|\hat{G}_{\chi_2\chi_2}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{6|x_0 - y_0|}.$$

b) $\hat{G}_{\chi_2\chi_2} = \hat{P}_{\chi_2\chi_2}[1 + \hat{P}_{\chi_2\chi_2}^{-1}(\hat{G}_{\chi_2\chi_2} - \hat{P}_{\chi_2\chi_2})]$,

$$\hat{P}_{\chi_2\chi_2}(x_0, y_0) = \hat{G}_{\chi_2\chi_2}(x_0, y_0)\delta_{x_0 y_0}.$$

c) $\hat{\Gamma}_{\chi_2\chi_2} = -\hat{G}_{\chi_2\chi_2}^{-1} = [1 + \hat{P}_2^{-1}(\hat{G}_{22} - \hat{P}_2)]^{-1}\hat{P}_2^{-1}$

$$= \sum_{n=0}^{\infty} (-1)^n [\hat{P}_2^{-1}(\hat{G}_{22} - \hat{P}_2)]^n \hat{P}_2^{-1}.$$

d) $\hat{\Gamma}_{\chi_2\chi_2}(x_0, x_0, \Lambda) = -2 + O(\beta^2)$,

$$\hat{\Gamma}_{\chi_2\chi_2}(x_0, x_0 + 1, \Lambda) = c_{66}\beta^6 + O(\beta^8),$$
$$|\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{7|x_0 - y_0|}, \quad |x_0 - y_0| > 1.$$

$\hat{\Gamma}_{\chi_2\chi_2}$ *is analytic in* β, λ.

$\hat{\Gamma}_{\chi\chi}(x_0, y_0, \Lambda)$, the convolution inverse of $-\hat{G}_{\chi\chi}(x_0, y_0, \Lambda)$, is defined in a manner analogous to Theorem 4.5. We have

Theorem 4.7.
a) $\hat{G}_{\chi\chi}(x_0, x_0, \Lambda)=1+O(\beta^2)$, $\hat{G}_{\chi\chi}(x_0, x_0+1, \Lambda)=c_{44}\beta^4+O(\beta^8)$, $|\hat{G}_{\chi\chi}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{4|x_0-y_0|}$,
b) $\hat{\Gamma}_{\chi\chi}(x_0, x_0, \Lambda)=-1+O(\beta^2)$, $\hat{\Gamma}_{\chi\chi}(x_0, x_0+1, \Lambda)=c_{44}\beta^4+O(\beta^5)$,

$$|\hat{\Gamma}_{\chi\chi}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{5|x_0-y_0|}, \quad |x_0-y_0|>1 .$$

$\hat{\Gamma}_{\chi\chi}$ is analytic in β, λ.

$\hat{G}_{\chi_1\chi_1}$ and its convolution inverse $-\hat{\Gamma}_{\chi_1\chi_1}$ (defined as in Theorem 4.5) satisfy the same bounds as $\hat{G}_{\chi\chi}$ and $\tilde{\Gamma}_{\chi\chi}$, respectively. The behavior of $\hat{G}_{\chi\chi_1}$, $\hat{G}_{\chi_1\chi_1}$ and $\hat{\Gamma}_{\chi_1\chi_1}$ is given in

Lemma 4.8.
a) $\hat{G}_{\chi\chi_1}(x_0, x_0, \Lambda)=\beta+O(\beta^2)$,
b) $\hat{G}_{\chi_1\chi_1}(x_0, x_0, \Lambda)=\frac{1}{2}+O(\beta^2)$,
c) $\hat{\Gamma}_{\chi_1\chi_1}(x_0, x_0, \Lambda)=-2+O(\beta^2)$, *and $\hat{\Gamma}_{\chi_1\chi_1}$ is analytic in β, λ.*

We define $\hat{F}_{\chi_1\chi_1}=\hat{G}_{\chi_1\chi_1}+\hat{G}_{\chi_1\chi}\Gamma_{\chi\chi}\hat{G}_{\chi\chi_1}$. The properties of $\hat{F}_{\chi_1\chi_1}$ and its convolution inverse $-\hat{\Phi}_{\chi_1\chi_1}$ are given in

Theorem 4.9.
a) $\hat{F}_{\chi_1\chi_1}(x_0, x_0, \Lambda)=\frac{1}{2}+O(\beta)$, $\hat{F}_{\chi_1\chi_1}(x_0, x_0+1, \Lambda)=\frac{1}{4}c_{66}\beta^6+O(\beta^7)$,

$$|\hat{F}_{\chi_1\chi_1}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^7, \quad |x_0-y_0|>1 .$$

b) $\hat{\Phi}_{\chi_1\chi_1}=-\hat{F}^{-1}_{\chi_1\chi_1}=[1-\hat{\Gamma}_{\chi_1\chi_1}\hat{G}_{\chi_1\chi}\hat{\Gamma}_{\chi\chi}\hat{G}_{\chi\chi_1}]^{-1}\hat{\Gamma}_{\chi_1\chi_1}$.
c) $\hat{\Phi}_{\chi_1\chi_1}(x_0, x_0, \Lambda)=-2+O(\beta)$, $\hat{\Phi}_{\chi_1\chi_1}(x_0, x_0+1, \Lambda)=c_{66}\beta^6+O(\beta^7)$,

$$|\hat{\Phi}_{\chi_1\chi_1}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{7|x_0-y_0|}, \quad |x_0-y_0|>1 .$$

$\hat{F}_{\chi_1\chi_1}$ and $\hat{\Phi}_{\chi_1\chi_1}$ are analytic in β, λ.

We define $\hat{L}_{\chi\chi_1}=\hat{\Gamma}_{\chi\chi}\hat{G}_{\chi\chi_1}\hat{\Phi}_{\chi_1\chi_1}$, and $\hat{L}_{\chi_1\chi}=\hat{\Phi}_{\chi_1\chi_1}\hat{G}_{\chi_1\chi}\hat{\Gamma}_{\chi\chi}$. Their properties are given in

Theorem 4.10. *$\hat{L}_{\chi\chi_1}$ is analytic and*
a) $\hat{L}_{\chi\chi_1}(x_0, x_0, \Lambda)=2\beta+O(\beta^2)$, $\hat{L}_{\chi\chi_1}(x_0, x_0+1, \Lambda)=c\beta^8+O(\beta^9)$,
b) $|\hat{L}_{\chi\chi_1}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{7|x_0-y_0|}$, $|x_0-y_0|>1$; *the same for $\hat{L}_{\chi_1\chi}$.*

Multiplying the expression defining $\hat{F}_{\chi_1\chi_1}$ on the right by $\hat{\Phi}_{\chi_1\chi_1}$ and on the left by $-\hat{\Gamma}_{\chi\chi_1}$, we obtain $\hat{\Gamma}_{\chi\chi}=\hat{L}_{\chi\chi_1}\hat{F}_{\chi_1\chi_1}\hat{L}_{\chi_1\chi}-\hat{\Gamma}_{\chi\chi}\hat{G}_{\chi_1\chi_1}\hat{\Phi}_{\chi_1\chi_1}$. Let $\hat{M}=-\hat{\Gamma}_{\chi\chi}\hat{G}_{\chi_1\chi_1}\hat{\Phi}_{\chi_1\chi_1}$. $\hat{M}$ has the properties given in

Theorem 4.11. *$\hat{M}$ is analytic in β, λ.*
a) $\hat{M}(x_0, x_0, \Lambda)=-1+O(\beta)$, $\hat{M}(x_0, x_0+1, \Lambda)=c_{44}\beta^4+O(\beta^5)$.
b) $|\hat{M}(x_0, y_0, \Lambda)| \leqq c'|c\beta|^{8|x_0-y_0|}$, $|x_0-y_0|>1$.

B.1) Some properties of the Taylor expansion coefficients of $\hat{G}_{\phi\psi}$ in w_q, λ_q are given in

Theorem 4.12.
a) $\hat{G}^{qmn}_{\phi\psi}(x, y, \Lambda) = 0$, *m odd.*
b) $\hat{G}^{qmn}_{\phi\psi}(x_0, y_0, \Lambda) = 0$, *n even*, $x_0 \leqq q < y_0$.
c) $\hat{G}^{qon}_{\phi\psi}(x_0, y_0, \Lambda) = 0$, $x_0 \leqq q < y_0$.
d) $G^{q21}_{\phi\psi}(x_0, y_0, \Lambda) = 0$.

Proof. The proofs are based on an analysis of the expansion of the numerator and denominator of $\hat{G}_{\phi\psi}(x, y, \Lambda)$. a) There is an odd number of P''_q spin 1/2 bonds. b) Higgs bonds are uncoupled between $x_0 \leqq q$ and $y_0 > q$ and there is an odd number in $x_0 \leqq q$. c) Expanding in λ_q if there are an even number of spin 1 bonds then there are an odd number of Higgs in $x_0 \leqq q$; if an odd number then the integral over the bonds in Q''_q vanishes. d) If $x_0 \leqq q < y_0$, there is a free bond in P''_q or Q''_q which by $P - W$ gives zero. If $x_0 \leqq q$, $y_0 \leqq q$, there are an odd number of Higgs in $x_0 \leqq q$.

Thus $\hat{G}_{\phi\psi}$ has the structure given in

Corollary 4.13. *Let* $x_0 \leqq q < y_0$. *Then*

$$\hat{G}_{\phi\psi}(x_0, y_0, \Lambda) = \hat{G}^{q41}_{\phi\psi} w^4_q u_q + G^{q61}_{\phi\psi} w^6_q u_q + R^{q83}_{\phi\psi}.$$

We calculate the above derivatives $\hat{G}^{q41}_{\phi\psi}$ and $G^{q61}_{\phi\psi}$ by integrating over the gauge variables in $q \leqq x_0 < q+1$ to get, letting $\tau^0 = \tau$, $\tau^i = R^i \tau^0$, $0 \leqq i \leqq 3$ and $\varrho^{ij} = \mathrm{R}^j \varrho^i$, $0 \leqq i \leqq 2$, $0 \leqq j \leqq 3$,

Theorem 4.14. *Let* $x_0 \leqq q < y_0$. *Then*

$$\begin{aligned}\hat{G}_{\phi\psi}(x_0, y_0, \Lambda) = {} & d_{44}\left[\sum_{i=0}^{3} \hat{G}_{\phi\tau^i}(x_0, q, \Lambda)\hat{G}_{\tau^i\psi}(q+1, y_0, \Lambda)\right] w^4_q u_q \\ & + d_{46}\left[\sum_{i=0}^{3} G_{\phi\tau^i}(x_0, q, \Lambda)\hat{G}_{\tau^i\psi}(q+1, y_0, \Lambda)\right] w^6_q u_q \\ & + d_{66}\left[\sum_{i=0}^{2}\sum_{j=0}^{3} \hat{G}_{\phi\varrho^{ij}}(x_0, q, \Lambda)\hat{G}_{\varrho^{ij}\psi}(q+1, y_0, \Lambda)\right] w^6_q u_q + R^{q83}_{\phi\psi},\end{aligned}$$

where $d_{44} > 0$, $d_{66} > 0$ *and* d_{46} *are combinatorial constants.*

We now incorporate the Z_4 and x_1 coordinate reflection symmetries into the above expansion. But first we prove that the effect of loop orientation reversal in τ, ϱ^0, ϱ^1, ϱ^2 is a sign change, a fact used in Sect. III.B.

Lemma 4.15. τ, ϱ^0, ϱ^1, ϱ^2 *change sign under loop orientation reversal.*

Proof. It is enough to consider $\tau = (\phi_A)_m \bar{D}^{1/2}(g_{ABCD})_{ij} M^m_{ij}$, where we recall that

$$D^{1/2}(g)_{i_1 i_2} \bar{D}^{1/2}(g)_{l_4 l_3} = M_{i_1 l_4 : L m_1} U^L(g)_{m_1 m_2} (M^+)_{L m_2 : i_2 l_3},$$

where $U^1(g) \equiv U(g)$ is the real spin 1 representation and we set $M^m_{ij} \equiv M_{ij:1m}$. Thus multiplying the above by $U(g)_{m_1 m_2}$ and integrating over the group, we have

$$\int D^{1/2}(g)_{i_1 i_2} \bar{D}^{1/2}(g)_{l_4 l_4} U(g)_{m_1 m_2} dg = \tfrac{1}{3} M^{n_1}_{i_1 l_4} \bar{M}^{n_2}_{i_2 l_3} \delta_{n_1 m_1} \delta_{n_2 m_2}.$$

We denote by $\tau' = (\phi_A)_m \bar{D}^{1/2}(g_{ADCBA})_{ij} M^m_{ij}$ the function with the loop orientation reversed. Now $\tau' = (\phi_A)_m D^{1/2}(g_{ABCDA})_{ij} M^m_{ij}$ and from [16] $D^{1/2}$ and $\bar{D}^{1/2}$ are unitarily equivalent, i.e. $\bar{D}^{1/2}(g)=CD^{1/2}(g)C^{-1}$, where $C^T=-C$. Thus $\tau'=(\phi_A)_m \cdot \bar{D}^{1/2}(g_{ABCDA})_{kl}\bar{C}_{kj}M^m_{ij}C_{l_i}$. To determine the last matrix we compute

$$\bar{C}_{kj}C_{l_i}M^m_{ij}\bar{M}^n_{rs}=\bar{C}_{kj}C_{l_i}3\int D^{1/2}_{ir}\bar{D}_{js}U_{mn}dg\,.$$

Since $\bar{C}_{kj}\bar{D}^{1/2}_{js}=-D^{1/2}_{kj}\bar{C}_{sj}$, we have

$$\bar{C}_{kj}C_{li}M^m_{ij}\bar{M}^n_{rs}=-3C_{li}\bar{C}_{sj}\int D^{1/2}_{ir}D^{1/2}_{kj}U_{mn}dg=-C_{li}C_{sj}J^m_{ik}\bar{J}^h_{ri}\,,$$

where J^m_{ik} are proportional to the usual $3-j$ symbols in [16]. In matrix notation

$$(CM^mC^{-1})_{lk}\bar{M}^n_{rs}=-(CJ^m)_{lk}(\bar{J}^nC^{-1})_{rs}\,.$$

Thus

$$(CM^mC^{-1})_{lk}/(CJ^n)_{kl}=-(\bar{J}^nC^{-1})_{rs}/\bar{M}^n_{rs}=\lambda=\text{const}\,,$$

or $CMC^{-1}=\lambda CJ$. Therefore $M=\lambda JC$, $M^T=\lambda C^TJ=-\lambda CJ$, so that $CMC^{-1}=-M^T$, and it follows that

$$\tau'=-(\phi_A)_m\bar{D}^{1/2}(g_{ABCDA})_{kl}M^m_{kl}=-\tau\,.$$

Recall that $P_\pm=\frac{1}{2}(1\pm P)$ is the projection operator on functions with $\pm$ parity under x_1 coordinate reflection and P_i, $0\leqq i\leqq 3$, denote the projections on the representations of Z_4. Define

$$\varrho_+\equiv\sqrt{2}P_+\varrho^0=\frac{1}{\sqrt{2}}(\varrho^0-\varrho^2)\,,\qquad \varrho_-\equiv\sqrt{2}P_-\varrho^0=\frac{1}{\sqrt{2}}(\varrho^0+\varrho^2)\,,$$

so that $P_\pm\varrho_\pm=\varrho_\pm$, $P_\pm\varrho_\mp=0$. We have

Theorem 4.16. *Let $\phi=P_\pm P_0\phi$, $\psi=P_\pm P_0\psi$ and $x_0\leqq q<y_0$. Then*

$$\begin{aligned}\hat{G}_{\phi\psi}(x_0,y_0,\Lambda)=&\left(\frac{1\mp 1}{2}\right)d_{44}\hat{G}_{\phi\tau}(x_0,q)\hat{G}_{\tau\psi}(q+1,y_0)w_q^4\lambda_q\\&+\left(\frac{1\mp 1}{2}\right)d_{46}\hat{G}_{\phi\tau}(x_0,q)G_{\tau\psi}(q+1,y_0)w_q^6\lambda_q\\&+d_{6,6}\hat{G}_{\phi\varrho_\pm}(x_0,q)\hat{G}_{\varrho_\pm\psi}(q+1,y_0)w_q^6\lambda_q\\&+d_{6,6}\left(\frac{1\mp 1}{2}\right)\hat{G}_{\phi\varrho^1}(x_0,q)\hat{G}_{\varrho^1\psi}(q+1,y_0)w_q^6\lambda_q+R^{q83}_{\phi\psi}\,,\end{aligned}$$

where the $\hat{G}\ldots$ on the right side are evaluated at $w_q=\lambda_q=0$.

As in arriving at Theorem 4.5 we have

Theorem 4.17. *Let $x_0\leqq q<y_0$.*

a) *If $\phi=P_+P_0\phi$, $\psi=P_+P_0\psi$, then $|\hat{G}_{\phi\psi}(x_0,y_0,\Lambda)|\leqq c'|c\beta^6\lambda|^{|x_0-y_0|}$.*

b) *If $\phi=P_-P_0\phi$, $\psi=P_-P_0\psi$, then $|\hat{G}_{\phi\psi}(x_0,y_0,\Lambda)|\leqq c'|c\beta^4\lambda|^{|x_0-y_0|}$.*

2) $\hat{\Gamma}_{\varrho+\varrho+}$, the convolution inverse of $-\hat{G}_{\varrho+\varrho+}$, is defined as in Theorem 4.6 and we have

Theorem 4.18.

a) $\hat{G}_{\varrho+\varrho+}(x_0, x_0, \Lambda) = 1/8 + O(\beta\lambda)$, $\hat{G}_{\varrho+\varrho+}(x_0, x_0+1, \Lambda) = \frac{1}{8\cdot 8} d_{66}\beta^6\lambda + O(\beta^7\lambda)$.

b) $\hat{\Gamma}_{\varrho+\varrho+}(x_0, x_0, \Lambda) = -8 + O(\beta\lambda)$, $\hat{\Gamma}_{\varrho+\varrho+}(x_0, x_0+1, \Lambda) = d_{66}\beta^6\lambda + O(\beta^7\lambda)$,

$$|\hat{\Gamma}_{\varrho+\varrho+}(x_0, y_0, \Lambda)| \leqq c'|c\beta^7\lambda)^{|x_0-y_0|}, \qquad |x_0-y_0|>1.$$

Proof. b) From Theorem 4.16 we have for $x_0 \leqq q < y_0$,

$$\hat{G}^{qn1}_{\varrho+\varrho+}(x_0, y_0, \Lambda) = 0, \qquad 0 \leqq n \leqq 5$$

and

$$\hat{G}^{q61}_{\varrho+\varrho+}(x_0, y_0, \Lambda) = d_{66}\hat{G}_{\varrho+\varrho+}(x_0, q_0, \Lambda)\hat{G}_{\varrho+\varrho+}(q+1, y_0, \Lambda)|_{w_q=u_q=0}.$$

From Leibniz's formula

$$\frac{\partial^{m+1}\hat{\Gamma}}{\partial w_q^m \partial\lambda_q} = \sum_{n=0}^{m-1} \binom{m}{n} \frac{\partial}{\partial\lambda_q}\left[\frac{\partial^n\hat{\Gamma}}{\partial w_q^n}\frac{\partial^{n-m}}{\partial w_q^{n-m}}\hat{G}\hat{\Gamma}\right],$$

and the above we obtain, for $x_0 \leqq q < y_0$,

$$\partial^{n+1}\hat{\Gamma}(x_0, y_0, \Lambda)/\partial w_q^n \partial\lambda_q|_{w_q=\lambda_q=0} \quad \text{for} \quad 0 \leqq m \leqq 5$$

and

$$\partial^7\hat{\Gamma}(x_0, y_0, \Lambda)/\partial w_q^6\partial\lambda_q|_{w_q=\lambda_q=0} = d_{66}\delta_{x_0,q}\delta_{q+1,y_0}.$$

By a Cauchy estimate the result follows. We now consider negative parity functions. Note that $P_-P_0\tau = P_0\tau$. Let $\tau_0 = P_0\tau$ and let $-\hat{\Gamma}_{\tau_0\tau_0}$ denote the convolution inverse of $\hat{G}_{\tau_0\tau_0}$. We have

Theorem 4.19.

a) $\hat{G}_{\tau_0\tau_0}(x_0, y_0, \Lambda) = \frac{1}{8} + O(\beta\lambda)$, $\hat{G}_{\tau_0\tau_0}(x_0, x_0+1, \Lambda) = \frac{d_{44}}{8\cdot 8}\beta^4\lambda + O(\beta^6\lambda)$.

b) $\hat{\Gamma}_{\tau_0\tau_0}(x_0, x_0, \Lambda) = -8 + O(\beta\lambda)$, $\hat{\Gamma}_{\tau_0\tau_0}(x_0, x_0+1, \Lambda) = d_{44}\beta^4\lambda + O(\beta^5\lambda)$,

$$|\hat{\Gamma}_{\tau_0\tau_0}(x_0, y_0, \Lambda)| \leqq c'|c\beta^5\lambda)^{|x_0-y_0|}.$$

Proof. b) Similar to Theorem 4.18. Here we use, for $x_0 \leqq q < y_0$,

$$\hat{G}^{qn1}_{\tau_0\tau_0}(x_0, y_0, \Lambda) = 0, \qquad 0 \leqq n \leqq 3$$

and

$$\hat{G}^{q41}_{\tau_0\tau_0}(x_0, y_0, \Lambda) = d_{44}\hat{G}_{\tau_0\tau_0}(x_0, q, \Lambda)\hat{G}_{\tau_0\tau_0}(q+1, y_0, \Lambda)|_{w_q=\lambda_q=0}.$$

We obtain, for $x_0 \leqq q < y_0$,

$$\hat{\Gamma}^{qn1}_{\tau_0\tau_0}(x_0, y_0, \Lambda) = 0, \qquad 0 \leqq n < 3,$$

and

$$\hat{\Gamma}^{q41}_{\tau_0\tau_0}(x_0, y_0, \Lambda) = d_{44}\delta(x_0, q)\delta(q+1, y_0).$$

We have, for

$$\hat{G}=\begin{pmatrix}\hat{G}_{\varrho\underset{-}{0}\varrho-} & \hat{G}_{\varrho-\sigma-}\\ \hat{G}_{\sigma-\varrho-} & \hat{G}_{\sigma-\sigma-}\end{pmatrix},\qquad \hat{G}^{\tau_0}=(\hat{G}_{\tau_0\varrho\bar{0}}\,\vdots\,\hat{G}_{\tau_0\varrho\overset{1}{0}}),\qquad \hat{G}_{\tau_0}=\begin{pmatrix}\hat{G}_{\varrho\bar{0}\tau_0}\\ \hat{G}_{\varrho\overset{1}{0}\tau_0}\end{pmatrix}.$$

Lemma 4.20.

a) $\hat{G}(x_0,x_0,\Lambda)=\frac{1}{8}+O(\beta\lambda)$,

b) $\hat{G}^{\tau_0}(x_0,x_0,\Lambda)=\left(\frac{1}{\sqrt{2}}\,\vdots\,1\right)\frac{\beta}{4}+O(\beta^2\lambda)$,

c) $\hat{G}_{\tau_0}(x_0,x_0,\Lambda)=\begin{pmatrix}1/\sqrt{2}\\ 1\end{pmatrix}\frac{\beta}{4}+O(\beta^2\lambda)$.

We define the 2×2 matrix function $\hat{F}=\hat{G}+\hat{G}_{\tau_0}\hat{\Gamma}_{\tau_0\tau_0}\hat{G}^{\tau_0}$, and let $-\hat{\Phi}$ denote its matrix convolution inverse. We have

Theorem 4.21.

a) $\hat{F}(x_0,x_0,\Lambda)=\frac{1}{8}+O(\beta\lambda)$, $\hat{F}(x_0,x_0+1,\Lambda)=d_{66}\frac{1}{8\cdot 8}\beta^6\lambda+O(\beta^7\lambda)$,

$$\|F(x_0,y_0,\Lambda)\|\leqq c'|c\beta^6\lambda|^{|x_0-y_0|},\qquad |x_0-y_0|>1,$$

b) $\hat{\Phi}(x_0,x_0,\Lambda)=-8+O(\beta\lambda)$, $\hat{\Phi}(x_0,x_0+1,\Lambda)=d_{66}\beta^6\lambda+O(\beta^7\lambda)$,

$$\|\hat{\Phi}(x_0,y_0,\Lambda)\|\leqq c'|c\beta^7\lambda)^{|x_0-y_0|},\qquad |x_0-y_0|>1,$$

and $\hat{\Phi}$ is analytic in β, λ.

Proof. a) We calculate w_q, u_q derivatives of $\hat{F}(x_0,y_0,\Lambda)$. First we have, for $x_0\leqq q<y_0$,

$$\begin{aligned}
\hat{G}^{q41}(x_0,y_0,\Lambda)&=d_{44}\hat{G}^{q00}_{\tau_0}(x_0,q,\Lambda)\hat{G}^{\tau_0q00}(q+1,y_0,\Lambda),\\
\hat{G}^{q61}(x_0,y_0,\Lambda)&=d_{46}\hat{G}^{q00}_{\tau_0}(x_0,q,\Lambda)\hat{G}^{\tau_0q00}(q+1,y_0,\Lambda)\\
&\quad+d_{66}\hat{G}^{q00}(x_0,q,\Lambda)\hat{G}^{q00}(q+1,y_0,\Lambda),\\
\hat{G}^{q41}_{\tau_0}(x_0,y_0,\Lambda)&=d_{44}\hat{G}^{q00}_{\tau_0}(x_0,q,\Lambda)\hat{G}^{q00}_{\tau_0\tau_0}(q+1,y_0,\Lambda);\\
\hat{G}^{q61}_{\tau_0}(x_0,y_0,\Lambda)&=d_{46}\hat{G}^{q00}_{\tau_0}(x_0,q,\Lambda)\hat{G}^{q00}_{\tau_0\tau_0}(q+1,y_0)\\
&\quad+d_{66}\hat{G}^{q00}(x_0,q,\Lambda)\cdot\hat{G}^{q00}_{\tau_0}(q+1,y_0,\Lambda),\\
\hat{G}^{\tau_0q41}(x_0,y_0,\Lambda)&=d_{44}\hat{G}^{q00}_{\tau_0\tau_0}(x_0,q,\Lambda)\cdot\hat{G}^{\tau_0q00}(q+1,y_0,\Lambda),
\end{aligned}$$

and

$$\begin{aligned}
\hat{G}^{\tau_0q61}(x_0,y_0,\Lambda)&=d_{46}\hat{G}^{q00}_{\tau_0\tau_q}(x_0,q,\Lambda)\hat{G}^{\tau_0q00}(q+1,y_0,\Lambda)\\
&\quad+d_{66}\hat{G}^{\tau_0q00}(x_0,q,\Lambda)\cdot\hat{G}^{q00}(q+1,y_0,\Lambda).
\end{aligned}$$

We need, in addition to the derivatives in the proof of Theorem 4.19b,

$$\begin{aligned}
\hat{\Gamma}^{q61}_{\tau_0\tau_0}(x_0,y_0,\Lambda)&=d_{46}\delta(x_0,q)\delta(q+1,y_0)\\
&\quad+d_{66}(\hat{\Gamma}^{q00}_{\tau_0\tau_0}\hat{G}^{\tau_0})(x_0,q,\Lambda)(\hat{G}^{q00}_{\tau_0}\hat{\Gamma}^{q00}_{\tau_0\tau_0})(q+1,y_0),
\end{aligned}$$

valid for $x_0 \leqq q < y_0$. Using the above results we find, for $x_0 \leqq q < y_0$, $\hat{F}^{qnm}(x_0, y_0, \Lambda) = 0$, $0 \leqq n \leqq 5$, $0 \leqq m \leqq 1$ from which the decay follows by a Cauchy estimate.

b) From Leibniz's rule and the properties of $\hat{F}$ in part a) we have, for $x_0 \leqq q < y_0$, $\hat{\Phi}^{qn1}(x_0, y_0, \Lambda) = 0$, $0 \leqq n \leqq 5$. For $x_0 \leqq q < y_0$ a lengthy computation gives

$$\hat{F}^{q61}(x_0, y_0, \Lambda) = d_{66} \hat{F}^{q00}(x_0, q, \Lambda) \hat{F}^{q00}(q+1, y_0, \Lambda)$$

and as

$$\hat{\Phi}^{q61}(x_0, y_0, \Lambda) = (\hat{\Phi}^{q00} \hat{F}^{q61} \hat{\Phi}^{q00})(x_0, y_0, \Lambda)$$

we have

$$\hat{\Phi}^{q61}(x_0, y_0, \Lambda) = d_{66} \delta(x_0, q) \delta(q+1, y_0)$$

from which the decay follows.

We define

$$\hat{L}^{\tau_0} = \hat{\Gamma}_{\tau_0\tau_0} \hat{G}^{\tau_0} \hat{\Phi} \quad \text{and} \quad \hat{L}_{\tau_0} = \hat{\Phi} \hat{G}_{\tau_0} \hat{\Gamma}_{\tau_0\tau_0}.$$

Theorem 4.22.

a) $\hat{L}^{\tau_0}(x_0, x_0, \Lambda) = 4\beta \left(\frac{1}{\sqrt{2}} \vdots 1 \right) + O(\beta^2)$.

b) $\hat{L}_{\tau_0}(x_0, x_0, \Lambda) = 4\beta \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} + O(\beta^2)$.

c) $\|\hat{L}^{\tau_0}(x_0, y_0, \Lambda)\| \leqq c' |c\beta^8 \lambda|^{|x_0 - y_0|}$, $|x_0 - y_0| > 1$ *and similarly for* $\hat{L}_{\tau_0}$.

Proof. c) Using the properties of the derivatives of $\hat{\Gamma}_{\tau_0\tau_0}$, $\hat{G}^{\tau_0}$ and $\hat{\Phi}$ obtained in the proceeding proofs we find, for $x_0 \leqq q < y_0$, $\hat{L}^{\tau_0 qmn}(x_0, y_0, \Lambda) = 0$ for $n = 1$, m odd and $0 \leqq m \leqq 5$, $0 \leqq n \leqq 1$. Also a lengthy computation gives

$$\hat{L}^{\tau_0 q61}(x_0, y_0, \Lambda) = d_{66} (\hat{\Gamma}_{\tau_0\tau_0} \hat{G}^{\tau_0})(x_0, q, \Lambda) [\hat{G}_{\tau_0} \hat{\Gamma}_{\tau_0\tau_0} \hat{G}^{\tau_0} \hat{\Phi} + \hat{G}\hat{\Phi} + I](q+1, y_0, \Lambda)|_{w_q = u_q = 0}.$$

But

$$\hat{G}_{\tau_0} \hat{\Gamma}_{\tau_0\tau_0} \hat{G}^{\tau_0} \hat{\Phi} + \hat{G}\hat{\Phi} + I = (\hat{G}_{\tau_0} \hat{\Gamma}_{\tau_0\tau_0} \hat{G}^{\tau_0} + \hat{G} - \hat{F}) \hat{\Phi} \equiv 0,$$

so that, for $x_0 \leqq q < y_0$, $\hat{L}^{\tau_0 q61}(x_0, y_0, \Lambda) = 0$ from which the decay follows.
Define $\hat{M} = \hat{\Gamma}_{\tau_0\tau_0} - \hat{L}^{\tau_0} \hat{F} \hat{L}_{\tau_0}$.

Theorem 4.23.

a) $\hat{M}(x_0, x_0, \Lambda) = -8 + O(\beta)$.

b) $\hat{M}(x_0, x_0+1, \Lambda) = d_{44}\beta^4\lambda + d_{46}\beta^6\lambda + O(\beta^7\lambda)$.

c) $|\hat{M}(x_0, y_0, \Lambda)| \leqq c' |c\beta^8 \lambda)^{|x_0 - y_0|}$, $|x_0 - y_0| > 1$.

Proof. b) and c) For $x_0 \leqq q < y_0$, $\hat{M}^{qmn}(x_0, y_0, \Lambda) = 0$ for $m = 0$ or $n = 0$ and $\hat{M}^{qm1}(x_0, y_0, \Lambda) = 0$ if $0 \leqq m \leqq 3$. Also, for

$$x_0 \leqq q < y_0, \quad \hat{M}^{q41}(x_0, y_0, \Lambda) = d_{44} \delta(x_0, q) \delta(q+1, y_0),$$

and we find

$$\begin{aligned}
&\hat{M}^{q61}(x_0, y_0, \Lambda)\\
&\quad = d_{46}\delta(x_0, q)\delta(q+1, y_0) + d_{66}(\hat{\Gamma}_{\tau_0\tau_0}\hat{G}^{\tau_0})(x_0, q, \Lambda)(\hat{G}_{\tau_0}\hat{\Gamma}_{\tau_0\tau_0})(q+1, y_0, \Lambda)|_{w_q=u_q=0}\\
&\qquad - \sum_{u_0, v_0} \hat{L}^{\tau_0}(x_0, u_0, \Lambda) d_{66}\hat{F}(u_0, q, \Lambda)\hat{F}(q+1, v_0, \Lambda)\hat{L}_{\tau_0}(v_0, y_0, \Lambda)|_{w_q=u_q=0}\,.
\end{aligned}$$

From $\hat{L}^{\tau_0} = \hat{\Gamma}_{\tau_0\tau_0}\hat{G}^{\tau_0}\hat{\Phi}$ and $\hat{L}_{\tau_0} = \hat{\Phi}\hat{G}_{\tau_0}\hat{\Gamma}_{\tau_0\tau_0}$ we have $\hat{L}^{\tau_0}\hat{F} = -\hat{\Gamma}_{\tau_0\tau_0}\hat{G}^{\tau_0}$ and $\hat{F}\hat{L}_{\tau_0} = -\hat{G}_{\tau_0}\hat{\Gamma}_{\tau_0\tau_0}$, which implies, for $x_0 \leqq q < y_0$, $\hat{M}^{q61}(x_0, y_0, \Lambda) = d_{46} \cdot \delta(x_0, q)\delta(q+1)$.

V. Missing Proofs of the Theorems in Sect. III

Here we give the missing proofs of Theorems in Sect. III. Analyticity results in p_0 of the Fourier transform of cf's and their convolution inverses follow from their corresponding decay properties given in Sect. IV. To find zeroes, corresponding to masses or to singularities of related functions, the technique used, in most cases, is that explained in arriving at Theorem III.3; the method also gives a convergent expansion. As this argument is used repeatedly throughout we give the proof of Theorem III.3. We also give the proof of Theorem III.10. which uses a variation of the argument, and a proof of Theorem III.16. which uses the Weierstrass preparation theorem [20]. Typically, after obtaining the zero, we want to show its uniqueness in the region of analyticity of the function. If the function has a spectral representation then the zero is simple and unique. This is the case except in Theorems III.8, 16, and 18; in the proof of Theorem III.8 below a Rouche argument is used to give uniqueness. The multiplicity part of Theorems III.16 and 18 is proved similarly using the estimates in Sect. IIIB. A Rouche argument is also used to prove Theorems III.9 and 17.

Proof of Theorem III.3. For notational simplicity drop the $\chi_2\chi_2$ indices from $\hat{\Gamma}$, write its $\beta=0$ Taylor series separating out terms up to and including order β^6, and take the Fourier transform to get

$$\begin{aligned}
\tilde{\Gamma}(p_0, \beta) = &-2 + \hat{\Gamma}_{R1}(x_0=0, \beta) + (e^{-ip_0} + e^{ip_0})c_{66}\beta^6\\
&+ \sum_{n=2}^{\infty} \hat{\Gamma}_{R7}(x_0=n, \beta)(e^{-ip_0 n} + e^{ip_0 n}),
\end{aligned}$$

where $\hat{\Gamma}_{Rm}(\hat{\Gamma}_n)$ denotes the Taylor series of $\hat{\Gamma}$ from m to ∞ (0 to n) and we have used Theorem 4.6. Making the non-linear transformation to the variables $w = -2 + c_{66}\beta^6 e^{-ip_0} = \tilde{\Gamma}^d$, β we obtain the function $H(w, \beta)$, where $H(w = -2 + c_{66}\beta^6 e^{-ip_0}, \beta) = \tilde{\Gamma}(p_0, \beta)$. Using the falloff of $\hat{\Gamma}$ given in Theorem 4.6d, $H(w, \beta)$ is jointly analytic for w, β small, $H(0,0)=0$ and $\dfrac{\partial H}{\partial w}(0,0)=1$. The analytic implicit function theorem [19] applies and gives a unique analytic $w(\beta)$, $w(0)=0$, for small β such that $H(w(\beta), \beta)=0$. Thus for $\beta > 0$, $w(\beta) = -2 + c_{66}\beta^6 e^{m(\beta)}$.

Proof of Theorem III.8. Dropping the $\chi_1\chi_1$ indices we show, using Rouche's theorem, that $\tilde{\Phi}$ has exactly one zero in the region $R = \{p_0 \,|\, |\mathrm{Re}\,p_0| < \pi,\ 0 < \mathrm{Im}\,p_0 < -(1-\varepsilon)7\ln\beta\}$. We write $\tilde{\Phi} = \tilde{\Phi}^d + \tilde{\Phi}_R$, $\tilde{\Phi}^d = -2 + c_{66}\beta^6 e^{-ip_0}$. It is easy to see that

$|\tilde{\Phi}^d(p_0 \in \partial R)| > 3/2$ for all $|\beta|$ sufficiently small and we now show that $|\tilde{\Phi}_R| < 3/2$ on ∂R. This follows directly for all terms except for

$$\sum_{n=1}^{\infty} \hat{\Phi}_{R7}(x_0 = n)\,(e^{-ip_0 n} + e^{ip_0 n}) \equiv \tilde{\Phi}'_R .$$

Using Theorem 4.9c we find $|\tilde{\Phi}'_R(p_0 \in \partial R)| \xrightarrow{\beta \to 0} 0$ uniformly on ∂R.

Proof of Theorem III.10. We find a zero of $\tilde{\Phi}_{\chi_1\chi_1} - \tilde{M}^{-1}\tilde{L}_{\chi\chi_1}\tilde{L}_{\chi_1\chi}$ near $p_0 = im$, $m \sim -6\ln\beta$. For notational simplicity drop the indices χ_1 and χ. We obtain the correct singular and constant term of m by transforming to the variables $w = -2 + c_{66}\beta^6 e^{-ip_0} \equiv \tilde{\Phi}^d$, β. We let $\tilde{M}'$, $\tilde{\Phi}'$, $\tilde{L}'$ denote $\tilde{M}$, $\tilde{\Phi}$, $\tilde{L}$, respectively written in terms of the w, β variables. Now, using Theorem 4.9 and 4.11, we can write

$$\tilde{\Phi}' = w + g(w, \beta), \qquad g(0,0) = 0, \qquad \frac{\partial g}{\partial w}(0,0) = 0,$$

$$\beta^2 \tilde{M}' = \frac{2c_{44}}{c_{66}} + h(w, \beta), \qquad h(0,0) = 0 .$$

Using Theorem 4.10 in addition, we see that $\tilde{L}'$ is jointly analytic as are g and h and that $(\beta^2\tilde{M}')^{-1}$ has a Taylor expansion beginning with a constant. Let

$$F(w, \beta) \equiv \tilde{\Phi}' - \beta^2\tilde{L}'^2(\beta^2\tilde{M}')^{-1} = w + g(w, \beta) - \beta^2\tilde{L}'^2(\beta^2\tilde{M}')^{-1} .$$

As $F(w, \beta)$ is jointly analytic, $F(0,0) = 0$, $\frac{\partial F}{\partial w}(0,0) = 1$ the analytic implicit function theorem gives a unique analytic $w(\beta)$, $F(w(\beta), \beta) = 0$ and $w(0) = 0$. Thus for $\beta >$ $w(\beta) = -2 + c_{66}\beta^6 e^{m(\beta)}$.

Proof of Theorem III.16. The $\beta = 0$ Taylor series of $\tilde{\Phi}_{ij}$, the i, j matrix element of $\tilde{\Phi}$, is, using Theorem 4.21b,

$$\tilde{\Phi}_{ij} = (-8 + d_{66}\beta^6\lambda e^{-ip_0})\delta_{ij} + d_{66}\beta^6\lambda e^{ip_0}\delta_{ij} + \hat{\Phi}_{ijR1}(x_0 = 0)$$
$$+ \sum_{n=1}^{\infty} \hat{\Phi}_{ijR7}(x_0 = n)\,(e^{-ip_0 n} + e^{ip_0 n}) .$$

Make a transformation to the variables $w = -8 + d_{66}\beta^6\lambda e^{-ip_0}$, β and introduce a 2×2 matrix function $H(w, \beta)$ such that

$$H_{ij}(w = -8 + d_{66}\beta^6\lambda e^{-ip_0}) = \tilde{\Phi}_{ij}(p_0, \beta) .$$

Using Theorem 4.21b we find, letting $F(w, \beta) \equiv \det H(w, \beta)$, $F(w, \beta) = w^2 + g(w, \beta)$, where $g(0,0) = 0$ and $\frac{\partial g}{\partial w}(0,0) = 0$. As $F(0,0) = 0$, $\frac{\partial F}{\partial w}(0,0) = 0$, but $\frac{\partial F}{\partial w}(0,0) = 2$ the Weierstrass preparation theorem [20] applies to give $w_{\pm}(\beta)$, analytic in $\beta^{1/2}$ or β, $w_{\pm}(0) = 0$ and $F(w_{\pm}(\beta), \beta) = 0$. Thus for $\beta > 0$, $w_{\pm}(\beta) = -8 + d_{66}\beta^6\lambda e^{\varrho_{\pm}}$.

VI. Concluding Remarks

In this paper we have explicitly analyzed the first two groups of masses in the region $\lambda < \beta^3$. The same techniques apply to other open regions, taking into account the degeneracy of the masses as given in the approximate model. The regions where there are intersection of lines and the degeneracy of masses is increased (see for example Fig. 1) can be treated by including this degeneracy in a suitable matrix of correlation functions. The problem of how to analyze mass groups beyond the second is currently being investigated.

In the increased region of convergence of the polymer expansion determined in [13], i.e. arbitrary β with λ large, the question as to the complexity of the mass spectrum also arises and has bearing on continuous models. In the case of the Z_2 gauge group we can show by a direct analysis, using methods similar to [15] or by duality, that the particle spectrum has the same complexity in this region as in the small β, λ region. Similar results are expected for continuous groups.

Some interesting open problems are the inclusion of Fermions and the study of resonances. Also for the Hamiltonian version of these models it would be interesting to develop a convergent perturbation theory in the magnetic coupling parameter along the lines of [21]. We point out that for the Euclidean model the results of this paper imply a convergent perturbation theory in the magnetic coupling parameters.

Appendix

Here we deduce an explicit formula for $f_n = \frac{1}{n!}\frac{d^n f(0)}{dz^n}$, the n^{th} Taylor coefficient of the analytic function $f(z)$, implicitly defined by $F(f(z),z)=0$, where $F(w,z)$ is jointly analytic, $F(0,0)=0$ and $\frac{\partial F}{\partial w}(0,0)\neq 0$. We point out that from the theory of Lie series [22] the non-autonomous ordinary differential equation obeyed by $f(z)$,

$$\frac{df(z)}{dz} = -\frac{F^{0,1}(f(z),z)}{F^{1,0}(f(z),z)}, \qquad F^{m,n}(w,z) \equiv \frac{\partial^{m+n}F(w,z)}{\partial w^m \partial z^n},$$

has a convergent series solution for small $|z|$ given by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}(D^n w)_{w=z=0}, \qquad D^0 = 1, \qquad D = \frac{\partial}{\partial z} - \frac{F^{0,1}(w,z)}{F^{1,0}(w,z)}\frac{\partial}{\partial w},$$

so that

$$f_n = \frac{1}{n!}(D^n w)_{w=z=0} = \frac{1}{n!}[D^1, \dots [D^n, w]\dots]_{w=z=0}.$$

Here we obtain a formula for f_n starting from the integral representation [19],

$$f(z) = \frac{1}{2\pi i}\int \frac{w\partial F(w,z)/\partial w}{F(w,z)} dw, \qquad \int \equiv \oint_{|w|=\varepsilon}. \tag{A.1}$$

We need the following two elementary power series lemmas.

Lemma A.1. *If* $h(z) = \sum_{n=0}^{\infty} h_n z^n$, $h_0 \neq 0$, *then* $h(z)^{-1} = \sum_{k=0}^{\infty} d_k z^k$, *where*

$$d_k = \sum_{m=0}^{k} (-1)^n \sum_{\{m_i\}\ \sum_{i}^{k} m_i = m,\ \sum_{l}^{k} l m_l = k} \binom{m}{m_1 \dots m_k} h_1^{m_1} \dots h_k^{m_k} h_0^{-(m+1)}.$$

Lemma A.2. *If* $F(w,0)/w=\sum_{i=0}^{\infty} a_i w^i$, $a_i=F^{i+1,0}(0,0)/(i+1)!$, *then* $(F(w,0)/w)^{-(l+1)}=\sum_{k=0}^{\infty} b_k w^k$, *where*

$$b_k=\sum_{m=0}^{k}(-1)^m \sum_{\{m_i\},\ \sum_{i=1}^{k} m_i=m,\ \sum_{j=1}^{k} jm_j=k} \begin{pmatrix} m \\ m_1\dots m_k \end{pmatrix} A_1^{m_1}\dots A_k^{m_k} A_0^{-(m+1)}$$

with

$$A_0=F^{1,0}(0,0)^{l+1}\equiv a_0^{l+1},$$

$$A_s=\sum_{\{l_r\},\ \sum_{r=0}^{s} l_r-l+1,\ \sum_{r=1}^{s} rl_r=s} \begin{pmatrix} l+1 \\ l_0 l_1\dots l_s \end{pmatrix} a_0^{l_0}a_1^{l_1}\dots a_s^{l_s}.$$

The formula for f_n is given by

Theorem A.1.

$$f_n=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{l=0}^{k}\frac{1}{(l-1)!}\sum_{l_1,l_2,l_3\ \sum_{u=1}^{3} l_u=l}\begin{pmatrix} l-1 \\ l_1 l_2 l_3 \end{pmatrix}$$

$$\cdot F^{l_1+1,n-k}(0,0)\,\frac{d^{l_2}P_l^k}{dw^{l_2}}(w,0)\bigg|_{w=0}\frac{d^{l_3}}{dw^{l_3}}\left(\frac{F(w,0)}{w}\right)^{-(l+1)}\bigg|_{w=0},$$

where

$$P_l^k(w,z)=(-1)^l\sum_{\substack{\{p_i\}\\ \sum_{i=1}^{k} p_i=l,\ \sum_{j=1}^{k} jp_j=k}}\frac{k!\,l!}{p_1!\dots p_k!}\left(\frac{F^{0,1}(w,z)}{1!}\right)^{p_1}$$

$$\cdot\left(\frac{(F^{0,2}(w,z))}{2!}\right)^{p_2}\dots\left(\frac{F^{0,k}(w,z)}{k!}\right)^{p_k},$$

$\frac{d^{l_2}P_l^k}{dw^{l_2}}(w,0)\Big|_{w=0}$ *is given by Leibniz rule and the last factor by Lemma A.2.*

Proof. Applying Leibniz' rule to Eq. (A.1), we obtain

$$\begin{aligned}\frac{d^n f(z)}{dz^n}&=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{1}{2\pi i}\int F^{1,n-k}(w,z)\left[\frac{d^k}{dz^k}F(w,z)^{-1}\right]w\,dw\\&=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{l=0}^{k}\frac{1}{2\pi i}\int F^{1,n-k}(w,z)P_l^k(w,z)F(w,z)^{-(l+1)}w\,dw,\end{aligned}$$

using Lemma A.1. Write $F(w,0)^{-(l+1)}w=(F(w,0)/w)^{-(l+1)}w^{-l}$, and use $(2\pi i)^{-1}\int K(w)w^{-(m+1)}dw=K^m(0)/m!$ with $m=l-1$ to obtain

$$f_n=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{l=0}^{k}\frac{1}{(l-1)!}\frac{d^{l-1}}{dw^{l-1}}\{F^{1,n-k}(w,0)P_l^k(w,0)\,(F(w,0)/w)^{-(l+1)}\}|_{w=0}.$$

Using Leibniz' rule again gives the result.

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