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Conformal Regularization of the Kepler Problem

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Abstract. The manifold M of null rays through the origin of $\mathbb{R}^{2,n+1}$ is diffeomorphic to $S^1 \times S^n$, and it is a homogeneous space of $SO(2, n+1)$. This group therefore acts on T^*M, which we show to be the "generating manifold" of the extended phase space of the regularized Kepler Problem. A local canonical chart in T^*M is found such that the restriction to the subbundle of the null non-vanishing covectors is given by $p_0 + H(q, p) = 0$, where $H(q, p)$ is the Hamiltonian of the Kepler Problem. By means of this construction, we get some results that clarify and complete the previous approaches to the problem.

1. Introduction

By the Kepler Problem (KP) we mean the Hamiltonian system on the phase space $T^*(\mathbb{R}^n - 0)$, with the Hamiltonian

$$H = \tfrac{1}{2}p^2 - \frac{1}{q}, \tag{1.1}$$

q_k and p_k being canonical coordinates, $p^2 = \sum_k p_k^2$ and $q = (\sum_k q_k^2)^{1/2}$.

Let us recall some well known facts:

i) As every particle in a spherically symmetric field, the KP is completely integrable. Besides the obvious integrals of energy and angular momentum, one has conservation of the Lenz–Laplace vector

$$\left(\frac{\varepsilon}{2H}\right)^{1/2}\left(p^2 q_k - \frac{q_k}{q} - \langle q, p\rangle p_k\right), \tag{1.2}$$

where (E = numerical value of H):

$$\begin{aligned} &\varepsilon = \operatorname{sgn} E \quad (E \neq 0), \\ &\varepsilon = 0 \qquad\quad (E = 0). \end{aligned} \tag{1.3}$$

For $E = 0$ we define $(\varepsilon/2H)^{1/2}$ to be the modulus of the angular momentum. Under Poisson brackets, angular momentum and Lenz–Laplace vector yield the Lie algebra of $SO(n+1)$, $SO(n, 1)$ or $SO(n) \otimes_S \mathbb{R}^n$ (semidirect product) for negative, positive or null energy. These groups are the maximal invariance groups. Fock [1] in the quantum 3-dimensional case and Moser [2] in the classical n-dimensional case gave a geometrical picture of this dynamical symmetry (for $E < 0$) through the stereographic projection of the sphere S^n in the momentum space. In this way one obtains at the same time the regularization of the KP. In fact:

ii) The KP is not regular, i.e. the vector field generated by H in $T^*(\mathbb{R}^n - 0)$ is not complete since, in the collision orbits, the particle gets to the attractive center with infinite velocity in a finite time. Following Pham Mau Quan [3] we define "regularization" in this way: given a smooth manifold W and a non-complete vector field X on W, find a smooth manifold $\tilde{W}$, a complete vector field $\tilde{X}$ on $\tilde{W}$ and an embedding μ: $W \mapsto \tilde{W}$ such that $\mu(W)$ is a set open and dense in $\tilde{W}$ and the orbits of X are embedded in those of $\tilde{X}$. The regularization of the 3-dimensional KP may be also obtained by the Kustaanheimo–Stiefel-transformation [4][5]. Kummer [6] proved the equivalence of the two methods for $E \neq 0$. In both these regularization procedures, time is replaced by a new parameter α such that

$$\frac{dt}{d\alpha} = q. \tag{1.4}$$

It follows that:

iii) For any fixed value of $E < 0$ the phase space of the regularized KP is diffeomorphic to the unit T^*S^n. Notice that T^+S^n (i.e. T^*S^n with the zero section removed) is an orbit in the coadjoint representation of $\mathrm{SO}(2, n+1)$, which is locally isomorphic to the conformal group of $\mathbb{R}^{1,n}$ [7–9]. This dynamical group has been defined by Bacry [10] and Györgyi [11], who also introduced the so called Bacry-Györgyi variables, to be used alternatively to Fock variables (see [9] for the definitions).

In this paper we proceed as follows. The manifold M of null unoriented rays through the origin of $\mathbb{R}^{2,n+1}$ is diffeomorphic to $S^1 \times S^n$, which is in turn a homogeneous space of $G = \mathrm{SO}(2, n+1)$. This group acts therefore on T^*M. As we shall see later, T^*M can be identified with the "generating manifold" of the extended phase space of the regularized KP with any energy. The action of G on T^*M is not transitive, so we restrict to consider the $(2n+1)$-dimensional subbundle N of T^*M given by null non-vanishing covectors: it results that $T^+S^n = N/S^1$. The main point is the following: it is possible to find three local canonical charts in T^*M (one for every value of ε) such that N is locally given by the equation

$$p_0 + H(q, p) = 0, \tag{1.5}$$

where $H(q, p)$ is the Hamiltonian (1.1) of the KP or the Hamiltonian with repulsive potential.

As is well known (see Theorem 2.6 below), by considering (1.5) as a constraint in $T^*(\mathbb{R}^n - 0) \times T^*\mathbb{R}$ one obtains the Hamilton equations

$$\frac{dq_k}{dq_0} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dq_0} = -\frac{\partial H}{\partial q_k}. \tag{1.6}$$

By means of this construction, we get some results that clarify and complete the previous approaches to the KP:

a) the introduction of the regularization parameter α in Eq. (1.4) is not postulated, but is a consequence of our approach;

b) the definition of Fock and Bacry–Györgyi variables is extended to the case $E=0$, and their relations are clarified;

c) the case of repulsive potential is automatically included;

d) the equivalence between Fock–Moser and KS regularization, which is basically due to the homomorphism $SO(2,4) \simeq SU(2,2)$, is here straightforwardly proved for any value of E;

e) as suggested in [12], the present construction can be generalized by considering the simple Lie groups whose maximal compact subgroup contains U(1) [13, Chap. VIII], and studying their action on the Bergman–Silov boundary;

f) as we shall show in a forthcoming paper, the geometric quantization (in the sense of Kostant and Souriau) of the Kepler manifold T^+S^n can be naturally obtained. Notice, for instance, that N is already the prequantum bundle.

As for the notation, the range of the indices is

$$\begin{aligned} A, B, C &= -1, 0 \ldots n+1, \\ \mu, \nu, \rho &= 0 \ldots n, \\ \alpha, \beta, \gamma &= 1 \ldots n+1, \\ i, j, k &= 1 \ldots n, \\ a, b, c &= 2 \ldots n. \end{aligned}$$

2. The "Generating Manifold"

Let $\eta_{AB} = \mathrm{diag}(- - + \cdots +)$ be the metric tensor of $\mathbb{R}^{2,n+1}$ and $m_{AB} = -m_{BA}$ a basis of the Lie algebra $\mathscr{G}$ of $G = SO(2, n+1)$. Then

$$[m_{AB}, m_{AC}] = \eta_{AA} m_{BC}, \tag{2.1}$$

or zero if all indices are different. It is convenient to introduce special symbols for the elements of the basis, namely:

$$\begin{aligned} m_{\mu\nu} &= J_{\mu\nu}, \\ m_{\mu n+1} &= A_\mu, \\ m_{-1\mu} &= B_\mu, \\ m_{-1n+1} &= D; \end{aligned} \tag{2.2a}$$

or alternatively:

$$\begin{aligned} P_\mu &= A_\mu + B_\mu, \\ C_\mu &= A_\mu - B_\mu. \end{aligned} \tag{2.2b}$$

We now recall some well known facts [14, 15]. Since the action of G on $\mathbb{R}^{2,n+1}$ is linear, it induces an action on the projective manifold of (unoriented) rays through the origin. Moreover G sends the null cone into itself and acts transitively on the manifold M of null rays. This manifold is diffeomorphic to $S^1 \times S^n$ and is endowed with a class of pseudoriemannian metrics g_γ obtained by restriction of the $SO(2, n+1)$ invariant metric η on any section γ of the null cone. The action of G on M is conformal; the metrics g_γ being conformaly flat, with signature $(- + \cdots +)$, the Lie algebra $\mathscr{G}$ of G is isomorphic to the Lie algebra of conformal vector fields on Minkowski space $\mathbb{R}^{1,n}$. So we can identify the generators in (2.2*ab*) as follows: $J_{\mu\nu}=$ Lorentz group, $D=$dilation, $P_\mu=$ translations, $C_\mu=$ conformal translations. Let H be the (closed) subgroup of G with Lie algebra $\mathscr{H}=\{J_{\mu\nu}, C_\mu, D\}$: it is the isotropy group of the origin in $\mathbb{R}^{1,n}$. Since $M=G/H$, we can identify M with the "conformal compactification" of $\mathbb{R}^{1,n}$. In other words, one can obtain M by adding to $\mathbb{R}^{1,n}$ a null cone at infinity.[1]

Let us now consider the symplectic action of G on $T^*(G/H)$. This action not being transitive, we decompose $T^*(G/H)$ in orbits of G. They are symplectic manifolds on which the group action is transitive, and so they may be identified (Kostani–Souriau Theorem, [16, p. 180] with (covering spaces of) orbits of G in $\mathscr{G}^*$. To get this identification, it is useful the following theorem due to Wolf [17]:

Theorem 2.1. *Let G be a Lie group with Lie algebra $\mathscr{G}$, $f\in\mathscr{G}^*$, G_f the isotropy subgroup of f* (i.e. $G_f\cdot f=f$) *and $\mathscr{G}_f$ the corresponding Lie algebra. Consider a closed subgroup $H\subset G$ with Lie algebra $\mathscr{H}$ such that*: a) $\dim\mathscr{H}=\frac{1}{2}(\dim\mathscr{G}+\dim\mathscr{G}_f)$; b) $\langle f,\mathscr{H}\rangle=0$; c) $\mathscr{G}_f\subset\mathscr{H}$. *Then $O_f=G\cdot f$ is equivariantly diffeomorphic to an open G-orbit in $T^*(G/H)$.*

If, as in the present case, G is semisimple, by means of the Cartan–Killing form B: $\mathscr{G}\times\mathscr{G}\mapsto\mathbb{R}$ we may identify $\mathscr{G}$ and $\mathscr{G}^*$. So, for $m\in\mathscr{G}$, we define $m^*\in\mathscr{G}^*$ by $\langle m^*,n\rangle=B(m,n)$, $\forall n\in\mathscr{G}$. Therefore $\mathscr{G}_{m^*}=\{n\in\mathscr{G}\colon [m,n]=0\}$. The basis (2.2*a*) is (pseudo)-orthonormal for B and so $B(P_\mu, C_\nu)=2\eta_{\mu\nu}$, $B(P,P)=B(C,C)=0$.

Proposition 2.2. *If $f_-=C_0^*$ and $f_+=C_1^*$, then $\mathcal{O}_{f_-}$ ($\mathcal{O}_{f_+}$) are the submanifolds of $T^*(G/H)$ given by timelike (spacelike) covectors.*

Proof. $\mathscr{G}_{f_-}=\{C_\mu, J_{hk}\}$ and $\mathscr{G}_{f_+}=\{C_\mu, J_{a0}, J_{ab}\}$, then $\mathscr{H}=\{J_{\mu\nu}, C_\mu, D\}$ satisfies the hypotheses of Theorem 2.1. Moreover G/H has tangent space $\mathscr{G}/\mathscr{H}$, hence cotangent space $\mathscr{H}^\perp=\{x\in\mathscr{G}^*\colon \langle x,\mathscr{H}\rangle=0\}$, and so $\mathscr{H}^\perp=\{C_\mu^*\}$. Remembering the signature of g_γ and that the action of G on G/H is conformal, the proposition is obtained. ■

Proposition 2.3. *The symplectic G invariant form induced in $\mathcal{O}_{f_\mp}$ by the canonical form of $T^*(G/H)$ through the equivariant diffeomorphism of Theorem 2.1, coincides with the Kirillov form.*

Proof. We remember that the Kirillov 2-form ω, (that makes every orbit of a group in the coadjoint representation a symplectic manifold) is defined as

$$\omega_f(u,v)=\langle f,[u,v]\rangle, \tag{2.3}$$

where $u,v\in$ to the Lie algebra of the group, and $f\in$ to the dual. The cotangent space to $\mathcal{O}_f$ in f is spanned by $\mathscr{G}/\mathscr{G}_f$: for, e.g., f_- we have $\mathscr{G}/\mathscr{G}_{f-}=\{P_\mu; D, J_{0k}\}$, where $\{P_\mu\}$ span the tangent space to G/H. A direct computation of (2.3) proves the proposition. Analogously for f_+. ■

The two orbits $\mathcal{O}_{f_\mp}$ are $2(n+1)$-dimensional. We now come to the $2n$-dimensional orbit considered by Onofri [8] and called "Kepler manifold."

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1 The relation of the conformal compactification with the regularization of the KP has been noted first by Kummer [6]

Proposition 2.4. *Let $f_0 = f_+ - f_-$: then $\mathcal{O}_{f_0}$ is symplectomorphic to T^+S^n, endowed with the canonical symplectic form.*

Proof. Let $G' = SO(1, n+1)$. As a subgroup of G, G' has the Lie algebra $\mathscr{G}' = \{J_{hk}, P_k, C_k, D\}$. Consider the subgroup H' of G' generated by $\mathscr{H}' = \{J_{hk}, C_k, D\}$. The manifold $M' = G'/H'$ can be identified with the projective manifold of the rays of the null cone in $\mathbb{R}^{1,n+1}$, and so $M' = S^n$. Being $f_+ = C_1^*$, then $\mathscr{G}'_{f_+} = \{C_k, J_{ab}\}$. From Theorem 2.1 we obtain: $\mathcal{O}'_{f_+} = T^+S^n$, where $\mathcal{O}'_{f_+} = G' \cdot f_+$. Let $\mathscr{P}$ be the orthogonal complement of $\mathscr{G}'$ in $\mathscr{G}$, i.e. $B(\mathscr{P}, \mathscr{G}') = 0$ and $\mathscr{G} = \mathscr{G}' \oplus \mathscr{P}$. Being $G' \cdot \mathscr{P}^* = \mathscr{P}^*$ and $f_+ \in \mathscr{G}'^*$, $f_- \in \mathscr{P}^*$, we have that $G' \cdot f_0$ is identifiable, through projection $\mathscr{G}^* \mapsto \mathscr{G}'^*$, with $\mathcal{O}'_{f_+}$. The orbit $G \cdot f_0$ contains $G' \cdot f_0$, but a dimension count shows that they must coincide: therefore we obtain the proposition. Computing $\mathscr{G}_{f_0}$ hence the cotangent space to $\mathcal{O}_{f_0}$, shows that the Kirillov form coincides with the canonical symplectic form of T^+S^n. ■

Analogously, we could prove Proposition 2.4 for $f_0 = f_+ + f_-$. Thus summarizing, we have

$$N/S^1 = \mathcal{O}_{f_0} \,(\mathrm{mod}\, \mathbb{Z}_2), \tag{2.4}$$

where N is the submanifold of null non-vanishing covectors in T^*M.

Proposition 2.5. *Identify $\mathscr{G}^*$ with $\wedge^2 \mathbb{R}^{2,n+1}$, then T^*M—(zero-section) is diffeomorphic to the manifold of the simple null non-vanishing bivectors,* i.e. *the bivectors of the type $Y \wedge X$, where $X, Y \in \mathbb{R}^{2,n+1}$ and $\eta(X,X) = 0$, $\eta(X,Y) = 0$.*

Proof. It is sufficient to verify that: a) f_- as simple bivector is generated by $X = (-10\ldots01)$ and $Y = (010\ldots0)$; b) f_+ by $X = \text{idem}$ and $Y = (0010\ldots0)$; c) f_0 by $X = \text{idem}$ and $Y = (0110\ldots0)$. ■

The following fact is crucial for our concern: the reduction $T^*M \mapsto T^+M'$ above described, may be interpreted as the reduction of the extended phase space of a mechanical system to the phase space. More exactly, we have the following classical theorem (see e.g. [18]):

Theorem 2.6. *Let $\mathscr{H}$: $T^*Q \mapsto \mathbb{R}$ be a "time" independent Hamiltonian, and (x_0, x_k, y_0, y_k) the canonical coordinates of T^*Q (Q is any differentiable manifold). Therefore $\mathscr{H}$ equals some constant h, and we may write, at least locally,*

$$y_0 + K(x_0, x_k, y_k) = 0. \tag{2.5}$$

Let us project the trajectories generated by $\mathscr{H}$ and belonging to the hypersurface (2.5) onto the hyperplane $y_0 = 0$: they are the solution of the hamiltonian system

$$\frac{dx_k}{dx_0} = \frac{\partial K}{\partial y_k}, \quad \frac{dy_k}{dx_0} = -\frac{\partial K}{\partial x_k}. \tag{2.6}$$

In our case $Q = M$ and $\mathscr{H} = g_y(y, y)$ with $h = 0$. Notice that g_y is conformally flat $\forall y$, so we can choose local coordinates (x_0, x_k) on M such that g_y is diagonal with x_0 timelike. Then x_0 is a local coordinate on the manifold of null rays in $\mathbb{R}^{2,1} \subset \mathbb{R}^{2,n+1}$. Apply Theorem 2.6: the reduced phase space is $T^+M' = \mathcal{O}_{f_0}$ and $K = \mp (g_{y'}(y', y'))^{1/2}$.

Three choices of γ are relevant for the KP, i.e. those obtained intersecting the null cone with:

i) a sphere with center in the vertex of the cone, thus γ is defined on $M = S^1 \times S^n$ and γ' on $M' = S^n$ with the usual metric induced by the immersion of S^n in $\mathbb{R}^{n+1}$ (more exactly, being x_0 a coordinate of the "time" type, we consider the universal covering $\tilde{G}$ instead of G, and so $\tilde{M} = \mathbb{R} \times S^n$ instead of M);

ii) a hyperboloid with same center; γ is defined on $M - \mathbb{Z}_2 \times C_\infty = H^1 \times H^n$, where C_∞ is the null cone at infinity in M and the H's are hyperboloids (the metric in H^n is induced by the immersion in $\mathbb{R}^{1,n}$); γ' is defined on M'—(two points at ∞) $= H^n$;

iii) a hyperplane parallel to a ray of the null cone; γ is defined on $M - C_\infty = P^{1,n}$ (hyperbolic paraboloid) and γ' on M'—(one point at ∞) $= P^n$; the two metric are flat.

The Hamiltonian K is the Hamiltonian of the unit geodesic flows on i) S^n, ii) H^n, iii) P^n and the invariance groups (i.e. the isometry groups of g_γ) are i) SO($n+1$), ii) SO(n, 1), iii) SO(n)$\otimes_S \mathbb{R}^n$.

The main point of the present work is the following

Theorem 2.7. *The extended phase space of the regularized* KP (*for negative, positive and null E*) *is symplectomorphic to the open submanifolds of* T^*M *given by the domain of the sections* γ *defined in* i) ii) iii) (*in this sense* T^*M *is the "generating manifold"*). *The Hamiltonian of the* KP *is a function of* K *and so has the same symmetry groups.*

3. Regularization of the KP

In this section we prove the theorem above and the points a), b) and c) of Sect. 1. To this end we construct the moment map $T^*M \mapsto \mathscr{G}^*$, in the three cases, using the following construction suggested by Proposition 2.5. Since $\gamma: M \mapsto \mathbb{R}^{2,n+1}$ is a section of the null cone, we can locally represent it by functions

$$X^A = \Gamma^A(x^\mu), \tag{3.1}$$

satisfying the null cone equation,

$$\eta(\Gamma, \Gamma) = 0. \tag{3.2}$$

The metric induced on the domain of γ by η is given by $g_{\gamma\mu\nu} = \psi^A_\mu \eta_{AB} \psi^B_\nu$, where $\psi^A_\mu = \partial \Gamma^A / \partial x^\mu$. Let Y_A and y_μ be the components of a covector respectively of $\mathbb{R}^{2,n+1}$ and M. Let $T^*\gamma$: $T^*M \mapsto T^*\mathbb{R}^{2,n+1}$ be the cotangent map, i.e. the map locally given by (3.1) and by $Y_A = \Pi_A(x^\mu, y_\nu)$, where

$$\Pi_A = \eta_{AB} \psi^B_\mu g^{\mu\nu}_\gamma y_\nu. \tag{3.3}$$

It is easy to check that

$$\Pi_A \Gamma^A = 0, \tag{3.4}$$

$$\Pi_A d\Gamma^A = y_\mu dx^\mu. \tag{3.5}$$

If f and g are differentiable mappings: $T^*\mathbb{R}^{2,n+1} \mapsto \mathbb{R}$, from (3.5) we have: $\{f, g\} \cdot T^*\gamma = \{f \cdot T^*\gamma, g \cdot T^*\gamma\}$, where $\{\cdot,\cdot\}$ are the Poisson brackets. If j: $T^*\mathbb{R}^{2,n+1} \mapsto \mathscr{G}^*$ is the moment map

$$m_{AB} = Y_A X_B - Y_B X_A, \tag{3.6}$$

then $J = j \cdot T^*\gamma$: $T^*M \mapsto \mathscr{G}^*$ is a moment map as well.

Explicitly we have the following three cases.

i) $T^*\gamma$ is given by

$$
\begin{aligned}
X^{-1} &= \cos x^0, \\
X^0 &= \sin x^0, \\
X^k &= \frac{2x^k}{x^2+1}, \\
X^{n+1} &= \frac{x^2-1}{x^2+1},
\end{aligned}
\tag{3.7a}
$$

and by

$$
\begin{aligned}
Y_{-1} &= -y_0 \sin x^0, \\
Y_0 &= y_0 \cos x^0, \\
Y_k &= \tfrac{1}{2}(x^2+1)y_k - \langle x, y\rangle x_k, \\
Y_{n+1} &= \langle x, y\rangle.
\end{aligned}
\tag{3.7b}
$$

Notice that x^0 do not parametrize S^1 but rather its covering space $\simeq \mathbb{R}$. The functions Γ^α are obtained through a stereographic projection of S^n onto $\mathbb{R}^n$. The metric g_γ is

$$\|y\|^2 = -y_0^2 + [\tfrac{1}{2}y(x^2+1)]^2, \tag{3.8}$$

where $y = (\sum y_k y_k)^{1/2}$ and $x^2 = \sum x^k x^k$.

ii) $T^*\gamma$ is given by

$$
\begin{aligned}
X^{-1} &= \frac{x^2+1}{x^2-1}, \\
X^0 &= \operatorname{Sinh} x^0, \\
X^k &= \frac{2x^k}{x^2-1}, \\
X^{n+1} &= \operatorname{Cosh} x^0,
\end{aligned}
\tag{3.9a}
$$

and by

$$
\begin{aligned}
Y_{-1} &= \langle x, y\rangle, \\
Y_0 &= y_0 \operatorname{Cosh} x^0, \\
Y_k &= \tfrac{1}{2}(x^2-1)y_k - \langle x, y\rangle x_k, \\
Y_{n+1} &= -y_0 \operatorname{Sinh} x^0.
\end{aligned}
\tag{3.9b}
$$

The functions Γ^{-1} and Γ^k are obtained through a stereographic projection of one sheet of H^n into $\mathbb{R}^n$ (i.e. onto the n-dimensional Poincaré disk) and a following inversion with respect to the origin, so that we have $x^2 > 1$. The metric g_γ is

$$\|y\|^2 = -y_0^2 + [\tfrac{1}{2}y(x^2-1)]^2. \tag{3.10}$$

iii) $T^*\gamma$ is given by

$$\begin{aligned} X^{-1} &= 1 + \frac{1}{x^2} - \frac{(x^0)^2}{4}, \\ X^0 &= x^0, \\ X^k &= \frac{2x^k}{x^2}, \\ X^{n+1} &= 1 - \frac{1}{x^2} + \frac{(x^0)^2}{4}, \end{aligned} \tag{3.11a}$$

and by

$$\begin{aligned} Y_{-1} &= \tfrac{1}{2}[\langle x, y\rangle - x^0 y_0], \\ Y_0 &= y_0, \\ Y_k &= \tfrac{1}{2}x^2 y_k - \langle x, y\rangle x_k, \\ Y_{n+1} &= \tfrac{1}{2}[\langle x, y\rangle - x^0 y_0]. \end{aligned} \tag{3.11b}$$

The mapping is obtained through a projection of $P^{1,n}$ onto $\mathbb{R}^{1,n}$ and a following inversion with respect to the origin in $\mathbb{R}^n$, so that we have $x^2 \neq 0$. The metric g_γ is given by

$$\|y\|^2 = -y_0^2 + [\tfrac{1}{2}yx^2]^2. \tag{3.12}$$

We stress the fact that, owing to stereographic projection in i) plus inversion in ii) and iii), we are missing one point in S^n, H^n and P^n: restoring this point corresponds just to regularization of the KP.

From Theorem 2.6 we obtain that the Hamiltonian K is given in the three cases by

$$K = \tfrac{1}{2}y(x^2 - \varepsilon), \tag{3.13}$$

where ε is defined in (1.3). Let us reduce the three moment maps, i.e., in accordance with (2.4), put $\mathscr{K} = 0$ and $x^0 = 0$, and consider the unregularized problem. All the three moment maps $T^+\mathbb{R}^n \mapsto \mathscr{G}^*$ now become (see [19, p. 276] for the precise definition of $\hat{J}$, $\hat{A}$ etc.)

$$\begin{aligned} \hat{J}_{hk} &= y_h x_k - y_k x_h, \\ \hat{J}_{0k} &= -yx_k, \\ \hat{A}_0 &= -\tfrac{1}{2}y(x^2 - 1), \\ \hat{A}_k &= \tfrac{1}{2}(x^2 - 1)y_k - \langle x, y\rangle x_k, \\ \hat{B}_0 &= -\tfrac{1}{2}y(x^2 + 1), \\ \hat{B}_k &= \tfrac{1}{2}(x^2 + 1)y_k - \langle x, y\rangle x_k, \\ \hat{D} &= \langle x, y\rangle \end{aligned} \tag{3.14a}$$

and

$$\begin{aligned} \hat{P}_0 &= -yx^2, \\ \hat{P}_k &= x^2 y_k - 2\langle x, y\rangle x_k, \\ \hat{C}_0 &= y, \\ \hat{C}_k &= -y_k. \end{aligned} \tag{3.14b}$$

The Hamiltonian K equals in the three cases (modulo an uninteresting sign): i) $\hat{B}_0$, ii) $\hat{A}_0$, iii) $\hat{P}_0/2$ and thus have as symmetry groups the isotropy subgroups of these generators, i.e.: i) $SO(n+1)$, ii) $SO(1,n)$, iii) $\mathbb{R}^n \otimes_S SO(n)$.

Let us return to the moment map J and before reducing it, consider the canonical transformation $\mathscr{C}$

$$q_k = y_0 y_k \tag{3.15a}$$

$$p_k = -\frac{x_k}{y_0}, \tag{3.15b}$$

$$q_0 = \frac{y_0^3}{\varepsilon}\left[x_0 - \frac{\langle x, y\rangle}{y_0}\right], \tag{3.15c}$$

$$p_0 = \frac{\varepsilon}{-2y_0^2}. \tag{3.15d}$$

($\mathscr{C}$ is not defined for $\varepsilon = 0$. However, since ε enters in the formulas below only through the expression $(\varepsilon/2H)^{1/2}$, which we defined in the limit case also, we can safely take the limit $\varepsilon = 0$ in the final formulas.) $\mathscr{C}$ may be viewed as the composition of three canonical transformations: a) that given by exchanging coordinates and momenta; b) that given by (3.15*ab*), equivalent to an "energy rescaling"; c) that given by (3.15*d*). Note that (3.15*c*) is forced by requiring canonicity. Now $\mathscr{K} = 0$ reads as

$$p_0 + H(q, p) = 0, \tag{3.16}$$

where $H(q,p) = p^2/2 \mp q^{-1}$. Equation (3.15d) shows that H is a function of K, so it has the same symmetry groups. Note that x^0 is basically the regularization parameter: in fact

$$\frac{dq_0}{dx^0} = \{K, q_0\} + \frac{\partial q_0}{\partial x_0}, \tag{3.17}$$

and setting $\alpha = (\varepsilon/2H)^{1/2} x^0$, we get (1.4).

Let us consider the restriction to N of the moment maps $J \cdot \mathscr{C}^{-1}: T^*(\mathbb{R}^n - 0) \times T^*\mathbb{R} \mapsto \mathscr{G}^*$. We have

i) $\hat{B}_0 = (-2H)^{-1/2}$; $\hat{J}_{hk}$ = angular momentum; $\hat{A}_k$ = Lenz–Laplace vector; $\hat{B}_k$, $\hat{D}$ and $\hat{J}_{0k}$, $\hat{A}_0$ = Fock variables (for $x^0 = 0$) or Bacry–Györgyi variables (for $q_0 = 0$).

ii) $\hat{A}_0 = (2H)^{-1/2}$; $\hat{J}_{hk}$ = angular momentum; $\hat{B}_k$ = Lenz–Laplace vector; $\hat{D}$, $\hat{A}_k$ and $-\hat{B}_0$, $\hat{J}_{0k}$ = Fock variables (for $x^0 = 0$) or Bacry–Györgyi variables (for $q_0 = 0$).

iii) $\hat{P}_0/2 \mapsto (2H/\varepsilon)^{-1/2}$ (in the limit sense); $\hat{J}_{hk}$ = angular momentum; $\hat{P}_k$ = Lenz–Laplace vector; $\hat{C}_\mu$ and $\hat{D}$, $\hat{J}_{0k}$ = Fock variables (for $x^0 = 0$) or Bacry–Györgyi variables (for $q_0 = 0$).

4. KS-transformation

As Kummer proved, the local isomorphism $SO(4,2) \simeq SU(2,2)$ yields the KS-transformation for $E \neq 0$. We first recall some of the Kummer's results. Let $\mathscr{E}$ be a matrix representation of the $U(2,2)$ invariant Hermitian form. We can choose a basis in $\mathbb{C}^{2,2}$ such that $\mathscr{E}$ has the form

$$\mathscr{E} = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \tag{4.1}$$

being σ_ν the Pauli matrices. Following Penrose we call twistors the elements of $\mathbb{C}^{2,2}$ on which U(2, 2) acts in the fundamental representation, and null twistor space T_0 the set of elements $\phi \in \mathbb{C}^{2,2}$ such that

$$\phi^\dagger \mathscr{E} \phi = 0. \tag{4.2}$$

Identifying the null twistors up a phase, i.e. $\phi \approx \phi \exp(i\theta)$, we get that the quotient $T_0/\approx$ is a real 6-dimensional manifold. Let $z \in \mathbb{C}^2 - 0$ and $w \in \mathbb{C}^2$ be such that $\psi = \begin{pmatrix} z \\ w \end{pmatrix} \in T_0$. It is easy to check that the matrices of the type

$$i\psi\psi^\dagger \mathscr{E} = i \begin{pmatrix} zw^\dagger & zz^\dagger \\ ww^\dagger & wz^\dagger \end{pmatrix} \tag{4.3}$$

describe a 6-dimensional orbit of SU(2, 2) in su*(2, 2). This orbit is equipped with the symplectic form $\omega = d\Theta$, where

$$\Theta = \frac{i}{2}(\psi^\dagger \mathscr{E} d\psi - d\psi^\dagger \mathscr{E} \psi). \tag{4.4}$$

On the basis of this construction, then Kummer proves the equivalence between Fock–Moser and KS regularization.

As an application of the present approach we prove the same result. As a byproduct we get the KS-transformation in a way which is independent of the sign of E and also covers the case $E = 0$. Let $\Xi = \vec{x} \cdot \vec{\sigma}$ and $2\Upsilon = y\sigma_0 + \vec{y} \cdot \vec{\sigma}$. Being $\det \Upsilon = 0$, we can define $\Upsilon^{1/2}$ as an element of $(\mathbb{C}^2 - 0)/\approx$ such that $\Upsilon^{1/2} \Upsilon^{\dagger 1/2} = \Upsilon$. Now

$$\psi = \begin{pmatrix} \Upsilon^{1/2} \\ i\Xi \Upsilon^{1/2} \end{pmatrix}, \tag{4.5}$$

provides a canonical system of coordinates for our orbit. In fact

$$\Theta = \langle y, dx \rangle. \tag{4.6}$$

The inverse of the bijective mapping (4.5) is the KS-transformation. To show it, immediately we have, from the mere definition, that

$$\Upsilon = zz^\dagger. \tag{4.7a}$$

Moreover multiply from the right both sides of $-iw = \Xi z$ by $z^\dagger (z^\dagger z)^{-1}$ and take the imaginary part. We obtain

$$\frac{1}{2} \frac{i}{z^\dagger z}(zw^\dagger - wz^\dagger) = \Xi + \sigma_0 \frac{\langle x, y \rangle}{y}. \tag{4.7b}$$

Equations (4.7*ab*) are easily seen to be equivalent to the KS-transformation as given by Kummer [6].

The relation with the KP is seen by composing (4.3) with (4.5), which gives

$$i\psi\psi^\dagger \mathscr{E} = i \begin{pmatrix} -i\Upsilon\Xi & \Upsilon \\ \Xi\Upsilon\Xi & i\Xi\Upsilon \end{pmatrix}, \tag{4.8}$$

and taking into account the isomorphism su*(2, 2) = so*(2, 4). In this way we obtain the moment map (3.14ab), which is valid for any value of E.

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