Commun. Math. Phys. 102, 59–88 (1985)

Communications in Mathematical Physics

The Lipatov Argument for ϕ_3^4 Perturbation Theory

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Abstract. We extend to φ_3^4 the work of S. Breen on the leading behavior at large order of φ_2^4 perturbation theory. Using a phase space expansion to obtain new estimates on the high energy behavior of φ_3^4 Feynman graphs, and a rigorous semiclassical expansion, we prove that the radius of convergence of the Borel transform of the pertubative series for φ_3^4 Euclidean field theory is the one computed by the Lipatov method.

I. Introduction

The Lipatov method is a formal steepest descent method for finding the asymptotic behavior at large order of perturbation series in the Euclidean path integral formulation of quantum field theory. Following early work by Bender and Wu [1] and Lam [2], the first calculations by Lipatov [3] were restricted to massless φ^{2N} field theory in dimension $\frac{2N}{N-1}$. The method was extensively developed by Brézin, Le Guillou and Zinn-Justin [4] to compute the large order behavior of general bosonic theories. After arguments by Parisi and 't Hooft [5] it was realized that the result should hold only for superrenormalizable theories. Yet even there a general rigorous justification of the Lipatov method has not been given. Let us summarize the work done in this direction and the difficulties.

For simplicity we limit ourselves in this paper to the perturbative expansion for the pressure of the massive one-component φ^4 model in dimensions 1, 2, or 3, in which it is superrenormalizable. We rescale also the bare mass to be 1. Extensions to arbitrary mass, to N-component vector models and to general Schwinger functions are easy, once this simple case has been rigorously understood, and we do not discuss them here.

The partition function of the model in a volume Λ is defined by constructive field theory [6, 7] as:

$$Z_X(g) = \int e^{-gV(\varphi) + \text{counterterms}} d\mu_X(\Lambda), \tag{1.1}$$

in which $V(\varphi) = \int_\Lambda \varphi^4(x) d^d x$, $\Lambda = [-T/2,\ T/2]^d$ and $X = p$ (periodic) or D (Dirichlet) specifies the two possible types of boundary conditions that we will consider. The counterterms to make the theory finite depend on the dimension d and will be defined later. The mean zero Gaussian measure $d\mu_X(\Lambda)$ has covariance:

$$\int \varphi(f)\varphi(g)d\mu_X(\Lambda) = \langle f, (-\Delta_X+1)^{-1}g\rangle, \tag{1.2}$$

where $\langle\,,\,\rangle$ is the $L^2(\Lambda)$ inner product, and Δ_X is the Laplacian with X boundary conditions on Λ. The pressure may be defined at small g by a cluster expansion [6–8]:

$$p(g) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log Z_X(\Lambda), \tag{1.3}$$

and is independent of the choice of the boundary conditions. It has a renormalized perturbation series

$$p(g) \sim \sum_{n=0}^{\infty} (-1)^n a_n g^n, \tag{1.4}$$

which is known to be divergent [9, 10] and Borel summable [11] for $d=1,2,3$. This last result makes use of constructive field theory, but by "cheap" perturbative estimates one can prove the weaker result that the Borel transform $B(t)$ of (1.4), defined as

$$B(t) = \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{n!} t^n, \tag{1.5}$$

is analytic in a disk of non-zero radius [12].

From the results [6–12] one gets the impression of a unity of the superrenormalizable domain.

The Lipatov method applied to $\varphi^4_{1,2,3}$ gives always the same type of asymptotic formulae:

$$a_n \underset{n\to\infty}{\sim} n!\, a^n n^b c(1+0(1/n)), \tag{1.6}$$

where the coefficients a, b, and c depend on the dimension and have been explicitly computed [4]. Only c may depend on the renormalization scheme. Of particular interest is the coefficient a, which, if (1.6) is valid, is the inverse of the exact radius of convergence R of the power series (1.5).

Let us define the functional $S(\varphi)$ by:

$$\begin{aligned} S(\varphi) = 1/2 \int_{\mathbb{R}^d} [(\nabla\varphi)^2(x) + \varphi^2(x)]d^dx \\ - \log \int_{\mathbb{R}^d} \varphi^4(x)d^dx \end{aligned} \tag{1.7}$$

for φ in the Sobolev space $W^{1,2}(\mathbb{R}^d)$, which is the completion of $C_0^\infty(\mathbb{R}^d)$ in the norm

$$\|\varphi\|^2_{1,2} = \int_{\mathbb{R}^d} [(\nabla\varphi)^2(x) + \varphi^2(x)]\,d^dx\,. \tag{1.8}$$

The functional $S(\varphi)$ is bounded below and attains its infimum (see Lemma IV.5 below). The Lipatov prediction for a is:

$$a = R^{-1} = \limsup_{n\to\infty} \left[\frac{|a_n|}{n!}\right]^{1/n} = \exp[-\inf S(\varphi) + 2]\,. \tag{1.9}$$

To summarize rigorous results, let us call "full justification" (of the Lipatov results), a proof of (1.6) with the right values of a, b, and c, and "partial justification", a proof of (1.9). Then a full justification has only been obtained for φ_1^4 (anharmonic oscillator) [13], and a partial justification has been obtained for regularized lattice models in any dimension [14] [with S in (1.9) replaced by a lattice version of (1.7)], and for φ_2^4 in the continuum [15]. Here we prove (1.9) for φ_3^4, extending therefore the partial justification of the Lipatov method to basically all the superrenormalizable cases, where it is supposed to work. The extension to φ_3^4 has some interest in itself, since the three dimensional asymptotic formulae (1.6) were used to optimize numerical computations of critical exponents in our three dimensional world [16].

Let us sketch the difficulties met in proving a formula like (1.9) by a semi-classical expansion.

In any dimension $d=1,2,3$, one has to perform an infinite volume limit. This problem can be solved by periodic and Dirichlet bracketing inequalities, and lemmas relating the large order behavior of ordinary and connected functions [14, 15].

In dimensions 2 and 3, one has also to renormalize and to interchange large order and ultraviolet limits. This was accomplished in [15] for $d=2$ by introducing an order dependent ultraviolet cutoff. We follow the same strategy. However, there is a new difficulty. There exists in dimension 3 a mass counterterm which is not linear in g; it renormalizes the logarithmically divergent graph of Fig. 1, which we call the "blob":

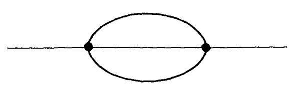


Fig. 1. The "blob" B

Therefore we cannot use directly, as in [14, 15] an integral representation for the n^{th} order of unconnected perturbation theory, b_n. In the case of φ_2^4, where the only renormalization is Wick ordering, one has:

$$b_n = \frac{1}{n!} \int [\int : \varphi^4(x) : d^2x]^n d\mu . \tag{1.10}$$

For interactions which are not linear in the coupling constant, one may use the following heuristic contour integral:

$$b_n = (-1)^n \frac{1}{2\pi i} \oint \int e^{-g\int \varphi^4 - \text{counterterms}} g^{-n-1} d\mu dg , \tag{1.11}$$

but it is difficult to justify. We prefer to establish some graphical estimates to bound the high energy behavior of perturbation theory and the effect of the blob counterterm, and use again an integral representation of type (1.10) with a cutoff measure, for which we can do a rigorous Laplace expansion.

These graphical estimates may have some interest in their own and are presented in Sect. III. In Sect. II notations and results are stated. Section IV depends heavily on [15]. Some technical results on lattice propagators are gathered in the Appendix.

We remark that for technical reasons we are unable to replace the lim sup in (1.9) by a simple limit, as expected, except for the class of subtraction schemes which subtract the blob at an energy sufficiently large compared to the bare mass. For this class of schemes we have more detailed information than simply (1.9) (see Sect. II).

Finally we believe that our graphical estimates, inspired by the phase space technique of [17], have analogues in just renormalizable theories. In particular one may hope to prove with such methods that the renormalization group improved perturbation series for $g\varphi_4^4$ with negative g, as defined in [18], should follow the large order behavior computed by Lipatov's method in 4 dimensions. This may be a key ingredient for a proof of existence of the first renormalon on the positive real axis in the Borel plane of φ_4^4.

II. Definition and Results

We have to introduce some definitions, most of them rather standard in perturbation theory.

In φ_3^4 the divergent graphs are those of Fig. 2:

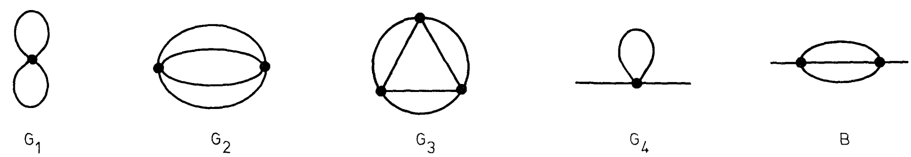


Fig. 2

We renormalize the theory by the usual Zimmermann scheme of subtraction at 0 external momenta [19], but Theorem II.1 below does not depend on this particular choice.

We use two different types of ultraviolet cutoffs; an exponential cutoff as in [17], convenient to derive graphical estimates, and a lattice cutoff, convenient for the infinite volume limit and for the Laplace expansion [14, 15]. These two cutoffs are related by the technical estimates of the Appendix.

To define the exponential cutoff, we pick an integer M, fixed in Sect. III to a large value. The propagator in x-space may be written, using a parametric representation [17]:

$$C(x,y)=\int \frac{d^3k}{(2\pi)^3}\frac{e^{ik(x-y)}}{1+k^2}=\frac{1}{(4\pi)^{3/2}}\int_0^\infty \frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x-y|^2}{4\alpha}}. \tag{2.1}$$

We define also:

$$C_0(x,y)=\frac{1}{(4\pi)^{3/2}}\int_1^\infty \frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x-y|^2}{4\alpha}}, \tag{2.2}$$

$$C_i(x,y)=\frac{1}{(4\pi)^{3/2}}\int_{M^{-2i}}^{M^{-2(i-1)}} \frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x-y|^2}{4\alpha}}, \quad (i\geqq 1), \tag{2.3}$$

$$C^j(x,y)=\sum_{i=0}^{j} C_i(x,y); \qquad C(x,y)=\sum_{i=0}^{\infty} C_i(x,y). \tag{2.4}$$

C^j can be thought of as a propagator with momentum cutoff of order M^j. Let us define the regularized propagator C_δ on the lattice $\delta\mathbb{Z}^3$ as:

$$C_\delta(x,y)=(-\Delta_\delta+1)^{-1}(x,y)$$
$$=\int_{-\pi/\delta}^{\pi/\delta}\frac{d^3k}{(2\pi)^3}\frac{e^{ik\cdot(x-y)}}{\left[1+2\delta^{-2}\sum_{\alpha=1}^{3}(1-\cos\delta k_\alpha)\right]}, \tag{2.5}$$

where Δ_δ is the usual discretization of the Laplacian. C_δ is a priori defined for $x, y\in\delta\mathbb{Z}^3$, but we may consider it also as a piecewise constant function; since it depends only on $x-y$, we evaluate it at the center of the cube of $\mathbb{R}^3$ of side δ, centered on $\delta\mathbb{Z}^3$, to which $x-y$ belongs.

The Gaussian measure with mean 0 and covariance C_δ is called $d\mu_\delta$. Similarly the lattice propagator $C_\delta^X(\Lambda)$ in a finite volume $\Lambda_\delta=\Lambda\cap\delta\mathbb{Z}^3$ with $X=p$ or $X=D$ boundary conditions is defined as usual, the corresponding Gaussian measures being called $d\mu_\delta^X(\Lambda)$. We may ensure in the rest of the paper that, as in [15], δ is such that the sides of Λ lie midway between lattice sites. We have the useful inequalities:

$$0\leqq C_\delta^D(\Lambda)(x,y)\leqq C_\delta(x,y), \tag{2.6}$$

$$0\leqq C_\delta^p(\Lambda)(x,y)\leqq\sum_{n=-\infty}^{+\infty}C_\delta(x-y+nT) \tag{2.7}$$

for $x, y\in\Lambda$. (Recall that $\Lambda=[-T/2, T/2]^3$.)

Other technical estimates on the behavior of C_δ and C^i are proved in the Appendix.

Perturbation theory expresses the coefficient a_n in (1.4) as a sum over all connected Feynman graphs G without any external legs and propagators C, of their renormalized Feynman amplitudes:

$$a_n=\sum_{\substack{G\\ n(G)=n}} I_G^R, \tag{2.8}$$

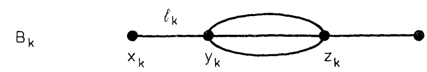
$$I_G^R=\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\int_\Lambda\dots\int_\Lambda\prod_v d^3x_v\quad\mathscr{R}\cdot\prod_\ell C(x_\ell,y_\ell). \tag{2.9}$$

In (2.9) the products are taken over all vertices v of G and over all internal lines ℓ, x_ℓ and y_ℓ being the positions of the ends of the line ℓ. The number of vertices and lines of G are called respectively $n(G)$ and $\ell(G)$. A "graph" means always an unlabeled graph. The renormalization operator acts in a way which can be summarized by the following rules:

- if $G=G_1$, G_2, or G_3, or if G contains a "tadpole" G_4, then $\mathscr{R}\cdot\prod_\ell C(x_\ell,y_\ell)\equiv 0$;
- in any other case $\mathscr{R}=\prod_k(1-t_{B_k})$, the product running over the blobs B_k in

G. We may define t_{B_k}, which subtracts the value of B_k at 0 external momenta, directly on the x-space integrand of (2.9). Let us associate to each B_k one of its two external lines, ℓ_k, in such a way that the lines associated to different B_k's are different. Call y_k the position of the vertex of B_k at the end of ℓ_k, x_k the position of the other end of ℓ_k, and z_k the position of the other vertex of B_k (see Fig. 3):

Fig. 3



Then t_{B_k} may be defined by

$$t_{B_k}\left[\prod_{\ell} C(x_\ell, y_\ell)\right] = C(x_k, z_k) \prod_{\ell \neq \ell_k} C(x_\ell, y_\ell). \tag{2.10}$$

Thanks to translation invariance, the integral (2.9) does not in fact depend on the choice of the line ℓ_k.

We call G an ICW graph (irreducible connected Wick-ordered graph) if it is connected (C), does not contain any blobs (I), and does not contain any tadpoles (W). Under the same conditions except that G contains blobs, we call G a BCW graph (B for blobs). Similarly we define CW, C, and W graphs; for instance a W graph has no tadpoles, but may be disconnected and may or may not contain blobs. Unless otherwise specified, all the graphs considered have $n(G)=n$ fixed. Then we introduce the following objects:

$$a_n = a_n^B + a_n^I, \tag{2.11}$$

$$a_n^B = \sum_{G\,\mathrm{BCW}} I_G^R; \qquad a_n^I = \sum_{G\,\mathrm{ICW}} I_G^R. \tag{2.12}$$

Similarly we define $I_G^{W,i}$ and $I_G^{W,\delta}$ by (2.9) with the renormalization operator $\mathscr{R}$ suppressed and C replaced, respectively, by C^i and C_δ, and we write

$$a_n^{B,i} = \sum_{G\,\mathrm{BCW}} I_G^{W,i}; \qquad a_n^{I,i} = \sum_{G\,\mathrm{ICW}} I_G^{W,i}, \tag{2.13}$$

$$a_n^{W,i} = a_n^{B,i} + a_n^{I,i}; \qquad a_n^{W,\delta} = a_n^{B,\delta} + a_n^{I,\delta}, \tag{2.14}$$

$$a_n^{B,\delta} = \sum_{G\,\mathrm{BCW}} I_G^{W,\delta}; \qquad a_n^{I,\delta} = \sum_{G\,\mathrm{ICW}} I_G^{W,\delta}. \tag{2.15}$$

We introduce also at finite volume Λ:

$$a_n^{W,\delta,X}(\Lambda) = \sum_{G\,CW} I_G^{W,\delta,X}(\Lambda), \tag{2.16}$$

$$I_G^{W,\delta,X}(\Lambda) = \prod_v \left(\delta^3 \sum_{x_v \in \Lambda_\delta}\right) \prod_\ell C_\delta^X(\Lambda)(x_\ell, y_\ell), \tag{2.17}$$

$$b_n^{W,\delta,X}(\Lambda) = \sum_{G\,W} \prod_{G_k \subseteq G} I_{G_k}^{W,\delta,X}(\Lambda), \tag{2.18}$$

where in (2.18) the G_k's are the connected components of G. Moreover we define

$$W_\delta^X(\varphi) = \delta^3 \sum_{x \in \Lambda_\delta} :\varphi^4:(x), \tag{2.19}$$

$$V_\delta(\varphi) = \delta^3 \sum_{x \in \Lambda_\delta} \varphi^4(x), \tag{2.20}$$

the Wick ordering being with respect to $d\mu_\delta^X(\Lambda)$. One has also:

$$b_n^{W,\delta,X}(\Lambda) = \frac{1}{n!} \int [W_\delta^X(\varphi)]^n d\mu_\delta^X(\Lambda)(\varphi)$$
$$= \frac{1}{n!} n^{2n} \int [W_\delta^X(\varphi/\sqrt{n})]^n d\mu_\delta^X(\Lambda)(\varphi), \tag{2.21}$$

where in (2.21) it should be understood that the Wick ordering scales as in [15], (1.14); if C_δ is the Wick constant:

$$W_\delta^X(\varphi/\sqrt{n}) = \left(\frac{\varphi}{\sqrt{n}}\right)^4 - 6\left(\frac{\varphi}{\sqrt{n}}\right)^2 \frac{C_\delta}{n} + 3\left(\frac{C_\delta}{n}\right)^2. \tag{2.22}$$

Finally we define

$$b_n^{\delta,X}(\Lambda) = \frac{(-1)^n}{n!} n^{2n} \int [V_\delta(\varphi/\sqrt{n})]^n d\mu_\delta^X(\Lambda)(\varphi). \tag{2.23}$$

We have also to introduce the finite volume analogues of the function $S(\varphi)$ defined by (1.7). We consider:

$$S_{X,\Lambda}^\delta(\varphi) = \tfrac{1}{2}\langle \varphi, A_X^\delta \varphi\rangle - \log[V_\delta(\varphi)], \tag{2.24}$$

$$A_X^\delta \equiv -\Delta_X^\delta + 1. \tag{2.25}$$

$S_{X,\Lambda}^\delta$ is defined on piecewise constant functions φ in Λ_δ, with $X=p$ or $X=D$. Its continuum counterpart is:

$$S_{X,\Lambda}(\varphi) = \tfrac{1}{2}\int_\Lambda [(\nabla_X\varphi)^2(x) + \varphi^2(x)]d^3x - \log\int_\Lambda \varphi^4(x)d^3x \tag{2.26}$$

for $\varphi \in W_0^{1,2}(\Lambda)$ if $X=D$ and $\varphi \in W^{1,2}(\Lambda)$ if $X=p$; these spaces are respectively the completion of $C_0^\infty(\Lambda)$ and of periodic $\mathscr{C}_1$ functions for the norm $\int_\Lambda [(\nabla_X\varphi)^2(x) + \varphi^2(x)]d^3x$ [15]. By Lemma IV.5 below, the functionals S, $S_{X,\Lambda}$, and $S_{X,\Lambda}^\delta$ are bounded below and attain their infimum. Our main result is:

Theorem II.1. *Let R be the radius of convergence of the series* (1.5). *Then*

$$R^{-1} = \limsup_{n\to\infty} \left[\frac{|a_n|}{n!}\right]^{1/n} = \exp[-\inf S(\varphi) + 2]. \tag{2.27}$$

In our case (1 component φ_3^4 with mass 1) one can compute [4]:

$$\exp[-\inf S(\varphi)+2] \simeq \frac{4!}{\pi}\cdot(36.091\ldots)^{-1} \Rightarrow R \simeq 4.72\ldots. \quad \square \tag{2.28}$$

To prove the theorem we put $M^i = \pi\delta^{-1} \simeq n^\varepsilon$, where ε will be fixed later to a small value, and we relate a_n in (2.27) to $\inf S(\varphi)$ through 8 successive steps:

$$a_n \xrightarrow{①} a_n^{W,i} \xrightarrow{②} a_n^{W,\delta} \xrightarrow{③} a_n^{W,\delta,X}(\Lambda) \xrightarrow{④} b_n^{W,\delta,X}(\Lambda)$$
$$\xrightarrow{⑤} b_n^{\delta,X}(\Lambda) \xrightarrow{⑥} \inf S_{X,\Lambda}^\delta \xrightarrow{⑦} \inf S_{X,\Lambda} \xrightarrow{⑧} \inf S. \tag{2.29}$$

Section III (with the Appendix) corresponds to performing steps 1 to 2, and Sect. IV to steps 3 to 8, which are basically the steps performed in [15].

The reader may convince himself after reading Sects. III and IV that Theorem II.2 holds for any choice of the subtraction scheme. In fact the minus sign in the renormalized amplitude of the blob creates difficulties in finding lower bounds on a_n; to solve them we rely on the technique of [10]; the price to pay is that one cannot replace the lim sup in (2.27) by a lim as expected. However, if one is willing to use a scheme which subtracts the blob at large momentum μ, the negative part of its amplitude is pushed far into ultraviolet region and may be dominated by the positive low momentum part of its amplitude. One obtains the more detailed result:

Theorem II.2. *Let a_n^μ be the n^{th} order of perturbation with subtraction of the blob at momentum μ. If $\mu \gg 1$, a_n^μ has the sign of $(-1)^n$ for n large enough, and:*

$$a = R^{-1} = \lim_{n\to\infty} \left[\frac{a_n^\mu}{n!}\right]^{1/n} = \exp[-\inf S(\varphi)+2]. \tag{2.30}$$

Therefore the Borel transform $B(t)$ has a singularity at $t=-R$.

The proof of Theorem II.2 follows easily from the bounds derived in Sects. III and IV and is left to the reader. We remark that to remove the (purely technical) restriction μ large in our opinion requires a strict improvement of the bounds of Sect. III, in particular of the speed at which the limit in (2.27) is attained.

III. Graphical Estimates

In this section we will prove graphical estimates and use them to introduce a lattice cutoff of order n^ε and to remove the blob renormalization, hence to perform steps 1 and 2 in (2.29). We remark that by translation invariance, if G is ICW and V_0 is a fixed vertex of G:

$$I_G^R \equiv I_G = \sum_\mu I_{G,\mu}, \tag{3.1}$$

$$I_G^\mu = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \prod_{v \neq v_0} d^3x_v \prod_{\ell\in G} C_{i_\ell}(x_\ell, y_\ell)|_{x_{v_0}=0}, \tag{3.2}$$

where, as in [17], μ is an assignment of momenta $\{i_\ell\}$, $\ell \in G$, hence is in $\mathbb{N}^{\ell(G)}$. For G ICW, F any subgraph of G and any $i \geqq 1$, our main technical estimate will bound the amplitude $I_{G,F}^i$ of G, computed by (3.2) but with full propagators C if $\ell \in F$ and with propagators C^i if $\ell \notin F$ [see (2.4)], by the amplitude I_G^i computed by (3.2) with all propagators replaced by C^i, losing a multiplicative factor exponential in the size of F (but not in the size of G, which would be disastrous for our purpose).

More precisely let us consider the set $\mathscr{A}_F^i$ of assignments $\mu = \{i_\ell\}$ such that $i_\ell \leqq i$ if $\ell \notin F$, and $\mathscr{B}^i$ the set of all μ's such that $i_\ell \leqq i$ $\forall \ell$. Then $I_{G,F}^i = \sum_{\mu\in\mathscr{A}_F^i} I_{G,\mu}$; also $I_G^i = \sum_{\mu\in\mathscr{B}^i} I_{G,\mu}$, and we have:

Theorem III.1. *For M large enough, there exists a function $K(M)>0$ such that for any* ICW *graph G, any $F \subseteqq G$ and any $i \geqq 1$*

$$I_{G,F}^i \leqq [K(M)]^{\ell(F)} I_G^i. \quad \square \tag{3.3}$$

We will prove Theorem III.1 by applying inductively a more precise version in which the indices of the highest lines in an assignment μ are lowered down by one unit. Probably this is not the simplest proof of Theorem III.1, but it is the only one we found. For any $\mu=\{i_\ell\}$, let us define $i(\mu)=\max_{\ell\in G} i_\ell$ and $H(\mu)=\{\ell\in G,\ i_\ell=i(\mu)\}$.

We introduce also a decomposition on length scales by considering, for any $j\geqq 1$, $\mathbb{R}^3$ as the disjoint union of A_j^0, A_j^1, and A_j^2,

$$A_j^0=\{x,|x|\leqq 4M^{-j}(\log M)^{1/2}\}\,, \tag{3.4}$$

$$A_j^1=\{x,4M^{-j}(\log M)^{1/2}<|x|<3M^{-(j-1)}(\log M)^{1/2}\}\,, \tag{3.5}$$

$$A_j^2=\{x,3M^{-(j-1)}(\log M)^{1/2}\leqq|x|\}\,. \tag{3.6}$$

For any $j\geqq 1$ and $F\subseteqq G$ we may want to know whether $x_\ell-y_\ell$ belongs to A_j^0, A_j^1, or A_j^2. Therefore we call $\mathscr{P}(F)$ the set of "length assignments" π which to $\ell\in F$ associate a value $\pi(\ell)=0$, 1 or 2 and we write

$$I_{G,\mu}=\sum_{\pi\in\mathscr{P}(F)} I_{G,\mu}^{\pi,j}, \tag{3.7}$$

where $I_{G,\mu}^{\pi,j}$ is defined by the integral (3.2) with additional restrictions $x_\ell-y_\ell\in A_j^{\pi(\ell)}$ for any $\ell\in F$. We will call $F^k(\pi)=\{\ell\in F,\pi(\ell)=k\}$. Hence $F=F^0(\pi)\cup F^1(\pi)\cup F^2(\pi)$. Also we define $F^{\leqq 1}(\pi)=F^0(\pi)\cup F^1(\pi)$.

Theorem III.2. *For M large enough, there exist two positive functions $K_0(M)$ $K_1(M)$, with $\lim_{M\to\infty} K_0(M)=0$, such that, for any ICW graph G, any $F\subseteqq G$, any $i\geqq 1$, any $H\subseteqq F$, $H\neq\emptyset$, any $j>i$, and any $\pi\in\mathscr{P}(H)$:*

$$\sum_{\substack{\mu\in\mathscr{A}_F^i\\ i(\mu)=j\\ H(\mu)=H}} I_{G,\mu}^{\pi,j}\leqq[K_0(M)]^{\ell(H)}[K_1(M)]^{\ell(H^1(\pi))}\sum_{\substack{\mu'\in\mathscr{A}_F^i\\ i(\mu')=j-1\\ H\subseteqq H(\mu')}} I_{G,\mu'}^{\pi',j-1}, \tag{3.8}$$

where π' is a function of π, which belongs to $\mathscr{P}(H^{\leqq 1}(\pi))$ and is identically $0:\pi'(\ell)=0$ $\forall\ell\in H^{\leqq 1}(\pi)$. Clearly, there is no more assignment for the lines of H in A_j^2, and the lines of H in $A_j^0\cup A_j^1$ are in A_{j-1}^0 in the right hand side of (3.8). □

Comment. The left-hand side of (3.8) is in fact an integral similar to III.2 with propagators C^i on the lines of $G-F$, C^{j-1} on the lines of $F-H$, and C_j on the lines of H, and restrictions $x_\ell-y_\ell\in A_j^{\pi(\ell)}$ for $\ell\in H$. The sum in the right-hand side of (3.8) may be thought of as the same integral with propagators C^i on the lines of $G-F$, C^{j-1} on the lines of $F-H$, C_{j-1} on the lines of H, and restrictions $x_\ell-y_\ell\in A_{j-1}^0$ if $\ell\in H^{\leqq 1}(\pi)$. As announced, the lines of H have been lowered by one unit.

Proof of Theorem III.1, Assuming Theorem III.2. Let us consider G, F, and i as in Theorem III.1. We write:

$$I_{G,F}^i=\sum_{\mu\in\mathscr{A}_F^i} I_{G,\mu}=I_G^i+\sum_{j>i}\sum_{\substack{H_1\subseteqq F\\ H_1\neq\emptyset}}\sum_{\substack{\mu_1\in\mathscr{A}_F^i\\ i(\mu_1)=j\\ H(\mu_1)=H_1}}\sum_{\pi_1\in\mathscr{P}(H_1)} I_{G,\mu_1}^{\pi_1,j}. \tag{3.9}$$

Applying (3.8) we get:

$$I_{G,F}^i\leqq I_G^i+\sum_{j>i}\sum_{\substack{H_1\subseteqq F\\ H_1\neq\emptyset}}[K_0(M)]^{\ell(H_1)}\sum_{\pi_1\in\mathscr{P}(H_1)}[K_1(M)]^{\ell(H_1^1(\pi_1))}\sum_{\substack{\mu_1'\in\mathscr{A}_F^i\\ i(\mu_1')=j-1\\ H(\mu_1')\supseteqq H_1}} I_{G,\mu_1'}^{\pi_1',j-1}, \tag{3.10}$$

where $\pi_1'\in\mathscr{P}(H_1^{\leqq 1}(\pi_1))$ and is identically 0.

If $j-1>i$, we decompose again:

$$\sum_{\substack{\mu_1'\in\mathscr{A}_F^i\\ i(\mu_1')=j-1\\ H(\mu_1')\supseteqq H_1}} I_{G,\mu_1'}^{\pi_1',j-1} = \sum_{\substack{H_2\\ H_1\subsetneqq H_2\subsetneqq F}} \sum_{\substack{\pi_2\in\mathscr{P}(H_2)\\ H_1^{\leqq}(\pi_1)\subsetneqq H_2^0(\pi_2)}} \sum_{\substack{\mu_2\in\mathscr{A}_F^i\\ i(\mu_2)=j-1\\ H(\mu_2)=H_2}} I_{G,\mu_2}^{\pi_2,j-1}, \tag{3.11}$$

and we apply again Theorem III.2. We repeat this until we arrive at a large sum with end contributions of the form:

$$\sum_{\substack{\mu_{j-i}'\in\mathscr{A}_F^i\\ i(\mu_{j-i}')=i\\ H(\mu_{j-i}')\supseteqq H_{j-i}}} I_{G,\mu_{j-i}'}^{\pi_{j-i}',i},$$

which we can bound uniformly by I_G^i. Therefore we obtain:

$$I_{G,F}^i \leqq I_G^i \Big\{1+\sum_{j>i}\ \sum_{\emptyset\subset H_1\subsetneqq H_2\subsetneqq\ \subsetneqq H_{j-i}\subsetneqq F}\ \sum_{\substack{\pi_1,\ ,\pi_{j-i}\\ \pi_k\in\mathscr{P}(H_k)\\ H_{k-1}^{\leqq 1}(\pi_{k-1})\subsetneqq H_k^0(\pi_k)}} \cdots$$

$$\ldots[K_0(M)]^{\sum_{k=1}^{j-i}\ell(H_k)}[K_1(M)]^{\sum_{k=1}^{j-i}\ell(H_k^1(\pi_k))}\Big\}. \tag{3.12}$$

But since $H_{k-1}^{\leqq 1}(\pi_{k-1})\subsetneqq H_k^0(\pi_k)$, a line in $H_{k-1}^1(\pi_{k-1})$ is not in $H_k^1(\pi_k)$, and thus $\sum_{k=1}^{j-i}\ell(H_k^1(\pi_k))\leqq\ell(H_{j-i})\leqq\ell(F)$. Furthermore each sum over π_k is over at most $3^{\ell(H_k)}$ possibilities. We may choose M large enough, so that $K_0(M)\leqq\frac{1}{12}$. Then (3.12) becomes:

$$I_{G,F}^i\leqq[K_1(M)]^{\ell(F)}I_G^i\left[1+\sum_{j>i}\ \sum_{\emptyset\subset H_1\subsetneqq\ldots\subsetneqq H_{j-i}\subsetneqq F}\left(\tfrac{1}{4}\right)^{\sum_{k=1}^{j-i}\ell(H_k)}\right]. \tag{3.13}$$

But we may bound the choice of H_{j-i} by $2^{\ell(F)}$, the choice of H_{j-i-1} by $2^{\ell(H_{j-i})}$ etc...., and obtain, since $\ell(H_k)\geqq 1$ for $1\leqq k\leqq j-i$:

$$1+\sum_{j>i}\ \sum_{\emptyset\subset H_1\subsetneqq\ldots\subsetneqq H_{j-i}\subsetneqq F}\left(\tfrac{1}{4}\right)^{\sum_{k=1}^{j-i}\ell(H_k)}\leqq\ldots$$

$$\ldots 1+\sum_{j>i}2^{\ell(F)}\left(\tfrac{1}{2}\right)^{j-i}\leqq 1+2^{\ell(F)}, \tag{3.14}$$

which proves (3.3) with $K(M)=3K_1(M)$ if $F\neq\emptyset$ (if $F=\emptyset$, (3.3) is trivial).

Proof of Theorem III.2. We write

$$\sum_{\substack{\mu\in\mathscr{A}_F^i\\ i(\mu)=j\\ H(\mu)=H}} I_{G,\mu}^{\pi,j} = \int\ldots\int_{\substack{x_\ell-y_\ell\in A_j^{\pi(\ell)}\\ \ell\in H}} \prod_{v\neq v_0} d^3x_v \prod_{\ell\notin F} C^i(x_\ell,y_\ell)$$

$$\cdot\prod_{\substack{\ell\in F\\ \ell\notin H}} C^{j-1}(x_\ell,y_\ell)\prod_{\ell\in H} C_j(x_\ell,y_\ell)\big|_{x_{v_0}=0}\,. \tag{3.15}$$

i) If $\ell\in H$, $\pi(\ell)=2$, one has $|x_\ell-y_\ell|^2\geqq 9M^{-2(j-1)}\log M$, and by (2.3) and the rescaling $\alpha\to M^2\alpha$ (using $1=8/9+1/10+1/90$, $j\geqq 2$ since $j>i\geqq 1$, and assuming from now on that $M \geqq 10$):

$$C_j(x_\ell, y_\ell) \leqq \frac{1}{(4\pi)^{3/2}} \int_{M^{-2(j-1)}}^{M^{-2(j-2)}} \frac{d\alpha}{\alpha^{3/2}} M \cdot \exp\left\{-\frac{\alpha}{M^2} - \frac{|x_\ell - y_\ell|^2}{4\alpha} \cdot \frac{M^2}{90} - 2\log M - \frac{|x_\ell - y_\ell|^2 M^2}{40\alpha}\right\}$$

$$\leqq \left(\frac{e}{M}\right) C_{j-1}(x_\ell, y_\ell) \exp\left\{-\frac{|x_\ell - y_\ell|^2 M^{2(j-1)}}{40}\right\}. \tag{3.16}$$

ii) If $\ell \in H$, $\pi(\ell) = 1$, one has similarly:

$$C_j(x_\ell, y_\ell) = \frac{1}{(4\pi)^{3/2}} \int_{M^{-2(j-1)}}^{M^{-2(j-2)}} \frac{d\alpha}{\alpha^{3/2}} M \cdot \exp\left[-\frac{\alpha}{M^2} - \frac{|x_\ell - y_\ell|^2 M^2}{4\alpha}\right]$$

$$\leqq \frac{e}{M} \cdot M^2 C_{j-1}(x_\ell, y_\ell). \tag{3.17}$$

Applying inequalities (3.16) and (3.17) to the lines of $H^1(\pi) \cup H^2(\pi)$ we get:

$$\sum_{\substack{\mu \in \mathscr{A}_F^i \\ i(\mu) = j \\ H(\mu) = H}} I_{G,\mu}^{\pi,j} \leqq \left(\frac{e}{M}\right)^{\ell(H^1(\pi) \cup H^2(\pi))} [M^2]^{\ell(H^1(\pi))} \int_{\left\{\substack{x_\ell - y_\ell \in A_j^{\pi(\ell)} \\ \text{if } \ell \in H}\right.} \cdots \int \prod_{v \neq v_0} d^3 x_v \cdots \prod_{\ell \in H^2(\pi)} e^{-\frac{|x_\ell - y_\ell|^2 M^{2(j-1)}}{40}} \prod_{\ell \notin F} C^i(x_\ell, y_\ell) \prod_{\substack{\ell \in F \\ \ell \notin H}} C^{j-1}(x_\ell, y_\ell)$$

$$\cdot \prod_{\substack{\ell \in H \\ \ell \notin H^0(\pi)}} C_{j-1}(x_\ell, y_\ell) \prod_{\ell \in H^0(\pi)} C_j(x_\ell, y_\ell) \Big|_{x_{v_0} = 0}. \tag{3.18}$$

Let us call $H_\alpha \equiv H^0(\pi)$ and H_β, $\beta = 1, \ldots, r$ the connected components of H_α. Holding fixed the positions of the ends of the external lines of H_α which are not in H_α, we will bound the partial integrations over internal vertices of H_α in (3.18), using the constraint that all internal distances in H_α are short relative to scale j. Let us call $n_\alpha = n(H_\alpha)$, $\ell_\alpha = \ell(H_\alpha)$, E_α the set of external lines of H_α, and e_α the number of lines in E_α. We will bound:

$$J_\alpha = \int_{\substack{x_\ell - y_\ell \in A_j^{\pi(\ell)} \\ \text{if } \ell \in H_\alpha \cup [E_\alpha \cap H]}} \cdots \int \prod_{v \in H_\alpha,\, v \neq v_0} d^3 x_v \prod_{\ell \in H_\alpha} C_j(x_\ell, y_\ell) \prod_{\substack{\ell \in E_\alpha \\ \pi(\ell) = 2}} e^{-\frac{|x_\ell - y_\ell|^2 M^{2(j-1)}}{40}}$$

$$\cdots \prod_{\substack{\ell \in E_\alpha \\ \ell \notin F}} C^i(x_\ell, y_\ell) \prod_{\substack{\ell \in E_\alpha \cap F \\ \ell \notin H}} C^{j-1}(x_\ell, y_\ell) \prod_{\ell \in E_\alpha \cap H} C_{j-1}(x_\ell, y_\ell) \big|_{x_{v_0} = 0}. \tag{3.19}$$

Let us use an elementary lemma.

Lemma III.1. *For any connected graph H_β, there exists a subset V'_β of the set V_β of internal vertices of H_β, of n'_β elements, with $n'_\beta = \sup\{1, I(n_\beta/4)\}$, (I meaning "integral part of", and $n_\beta \equiv n(H_\beta)$), such that for any $v \in V_\beta$ there is a chain of at most 6 lines of H_β joining v to a vertex of V'_β. If $v_0 \in H_\beta$, one may further require $v_0 \in V'_\beta$.* □

Proof. Take a tree of H_β, draw it on the plane, number the vertices (starting from v_0 if $v_0 \in H_\beta$) by "turning around the tree in the plane," as in [17], and choose for V'_β the vertices with numbers 1, 8, 12, 16, etc.... It is tedious but easy to check that this is a convenient choice of V'_β.

We can split V_β into the disjoint union of n'_β subsets V^w_β, $w \in V'_\beta$ such that if $v \in V^w_\beta$ there is a chain of at most 6 lines in H_β joining v to w, and such that $w \in V^w_\beta$ for any $w \in V'_\beta$.

Now we use a one-to-one change of variables which is made of partial rescalings with local rescaling centers in V'_β, for $\beta = 1, \dots, r$. More precisely we define (assuming $M > 56(\log M)^{1/2}$):

$$x'_v = [M/56(\log M)^{1/2}]x_v - \{[M/56(\log M)^{1/2}] - 1\}x_w \quad \text{if } v \in V^w_\beta \text{ for some } \beta = 1, \dots, r \text{ and some } w \in V'_\beta;$$
$$x'_v = x_v \quad \text{otherwise}. \tag{3.20}$$

Remark. We use "local, partial rescalings" to control the short distance region. The reason is that with a single rescaling of H_β with one scaling center as in [17], holding the ends of the external lines of H_β fixed, a problem occurs for n_β large. Far from the scaling center, external lines of H_β, although of lower frequency, "feel" the scaling in the sense that the ratio between the external propagators before and after the scaling may be of order e^{n_β}. If there are many such legs ($\sim n_\beta$), this gives a disastrous factor $e^{n^2_\beta}$. Moving also the external lines of H_β seems untractable, and "breaking" them (as in [17] or Lemma III.2 below) seems to lead inevitably to the loss of some $K^{n(G)}$. It is this difficulty which is solved by organizing the scaling into local partial ones and losing a fraction of the power counting in a manner typical of superrenormalizable theories. Clearly this should be improved if we were to apply this technique to $-g\varphi^4_4$.

Rescaling α's into $M^2\alpha$'s for $\ell \in H_\alpha$, the integral (3.19) becomes (since the jacobian of (3.20) is $[56(\log M)^{1/2}/M]^{3\sum_{\beta=1}^{r}(n_\beta - n'_\beta)}$):

$$J_\alpha = M^{\ell_\alpha}[56(\log M)^{1/2}/M]^{3\sum_{\beta=1}^{r}(n_\beta - n'_\beta)} \underset{\substack{x_\ell - y_\ell \in A_j^{\pi(\ell)} \\ \text{if } \ell \in H_\alpha \cup [E_\alpha \cap H]}}{\int \dots \int} \dots$$
$$\dots \prod_{v \in H_\alpha, v \neq v_0} d^3x'_v \prod_{\ell \in H_\alpha} D_{j-1}(x_\ell, y_\ell) \dots$$
$$\dots \prod_{\substack{\ell \in E_\alpha \cap H \\ \pi(\ell) = 2}} e^{-\frac{|x_\ell - y_\ell|^2 M^{2(j-1)}}{40}} \prod_{\ell \in E_\alpha, \ell \notin F} C^i(x_\ell, y_\ell) \prod_{\ell \in E_\alpha \cap F, \ell \notin H}$$
$$\cdot C^{j-1}(x_\ell, y_\ell) \prod_{\ell \in E_\alpha \cap H} C_{j-1}(x_\ell, y_\ell) \Big|_{x_{v_0} = 0}, \tag{3.21}$$

where

$$D_{j-1}(x_\ell, y_\ell) \equiv \frac{1}{(4\pi)^{3/2}} \int_{M^{-2(j-1)}}^{M^{-2(j-2)}} \frac{d\alpha}{\alpha^{3/2}} e^{\left[-\alpha/M^2 - \frac{|x_\ell - y_\ell|^2 M^2}{4\alpha}\right]}.$$

In (III.21) x_ℓ and y_ℓ are functions of x'_ℓ, y'_ℓ by inversion of (3.20).

For $\ell \in H_\alpha$, let us call $x_\ell = x_{v_1}$, $y_\ell = x_{v_2}$, $v_1 \in V_\beta^{w_1}$, $v_2 \in V_\beta^{w_2}$. Since

$$x'_\ell - y'_\ell = [M/56(\log M)^{1/2}]\,(x_\ell - y_\ell) - [M/56(\log M)^{1/2} - 1]\,(x_{w_1} - x_{w_2}),$$

and $M > 56(\log M)^{1/2}$:

$$\begin{aligned} x'_\ell - y'_\ell &\leqq [M/56(\log M)^{1/2}]\,[x_\ell - y_\ell| + |x_{w_1} - x_{w_2}|] \\ &\leqq \frac{14 \cdot 4 \cdot M^{-j}(\log M)^{1/2} \cdot M}{56(\log M)^{1/2}} \leqq M^{-(j-1)}, \end{aligned} \tag{3.22}$$

where we used the constraints in (3.21), the fact that w_1 is linked to w_2 via v_1, ℓ and v_2 by at most $6+1+6=13$ lines, and the triangular inequality. (3.22) implies:

$$x'_\ell - y'_\ell \in A^0_{j-1}, \tag{3.23}$$

$$D_{j-1}(x_\ell, y_\ell) \leqq e^{5/4} C_{j-1}(x'_\ell, y'_\ell), \tag{3.24}$$

since

$$\begin{aligned} &\exp\{-\alpha/M^2 - |x_\ell - y_\ell|^2 M^2/4\alpha\} \\ &\leqq 1 \leqq e^{5/4} \exp\left[-\alpha - \frac{|x'_\ell - y'_\ell|^2}{4\alpha}\right] \end{aligned}$$

if $M^{-2(j-1)} \leqq \alpha \leqq 1$ and $|x'_\ell - y'_\ell|^2 \leqq M^{-2(j-1)}$.

Similarly if $\ell \in E_\alpha$, we have

$$\begin{aligned} &||x_\ell - y_\ell| - |x'_\ell - y'_\ell|| \\ &\leqq |x_\ell - x'_\ell| + |y_\ell - y'_\ell| \\ &\leqq 12 \cdot [4M^{-j}(\log M)^{1/2}]\,[(M/56(\log M)^{1/2}) - 1] \leqq M^{-(j-1)}. \end{aligned} \tag{3.25}$$

Now using Lemma A.1 in the Appendix, and (3.25):

a) If $\ell \in E_\alpha$ and $\ell \notin F$, since $i<j$,

$$C^i(x_\ell, y_\ell) \leqq K C^i(x'_\ell, y'_\ell). \tag{3.26}$$

b) If $\ell \in E_\alpha \cap F$ and $\ell \notin H$,

$$C^{j-1}(x_\ell, y_\ell) \leqq K C^{j-1}(x'_\ell, y'_\ell). \tag{3.27}$$

c) If $\ell \in E_\alpha \cap H$ and $\pi(\ell) = 1$, $|x_\ell - y_\ell| < 3M^{-(j-1)}(\log M)^{1/2}$, hence

$$\begin{aligned} &|x'_\ell - y'_\ell| \leqq 3M^{-(j-1)}(\log M)^{1/2} + M^{-(j-1)} \\ &\Rightarrow x'_\ell - y'_\ell \in A^0_{j-1}, \end{aligned} \tag{3.28}$$

and

$$C_{j-1}(x_\ell, y_\ell) \leqq M^4 C_{j-1}(x'_\ell, y'_\ell), \tag{3.29}$$

since $\exp\left[-\dfrac{|x_\ell - y_\ell|^2}{4\alpha}\right] \leqq 1 \leqq M^4 e^{-\frac{|x'_\ell - y'_\ell|^2}{4\alpha}}$ if $\alpha \geqq M^{-2(j-1)}$ and $x'_\ell - y'_\ell \in A^0_{j-1}$.

d) If $\ell \in E_\alpha \cap H$ and $\pi(\ell) = 2$, $|x_\ell - y_\ell|^2 \geqq |x'_\ell - y'_\ell|^2 - 2M^{-(j-1)}|x_\ell - y_\ell| - M^{-2(j-1)}$, and therefore

$$e^{-\frac{|x_\ell - y_\ell|^2}{4\alpha}} \leqq e^{-\frac{|x'_\ell - y'_\ell|^2}{4\alpha}} \cdot e^{\left[\frac{|x_\ell - y_\ell| M^{-(j-1)}}{2\alpha} + \frac{M^{-2(j-1)}}{4\alpha}\right]},$$

and for $\alpha \geqq M^{-2(j-1)}$, $e^{\left[\frac{|x_\ell - y_\ell| \cdot M^{-(j-1)}}{2\alpha} + \frac{M^{-2(j-1)}}{4\alpha}\right]} \leqq e^{1/4 + \frac{M^{j-1}|x_\ell - y_\ell|}{2}}$. But since $x_\ell - y_\ell \in A_j^2$, $|x_\ell - y_\ell|^2 \geqq 3M^{-(j-1)}(\log M)^{1/2}|x_\ell - y_\ell|$, and

$$e^{-\frac{|x_\ell - y_\ell|^2 M^{2(j-1)}}{40}} \cdot C_{j-1}(x_\ell, y_\ell) \leqq e^{1/4} C_{j-1}(x'_\ell, y'_\ell) \leqq K C_{j-1}(x'_\ell, y'_\ell), \tag{3.30}$$

provoded $M \geqq e^{\frac{400}{9}}$, $K \geqq e^{1/4}$, which we assume now.

Putting together (3.23), (3.24), and (3.26)–(3.30), we obtain

$$\begin{aligned} J_\alpha \leqq & [e^{5/4} \cdot M]^{\ell_\alpha} \left[\frac{56(\log M)^{1/2}}{M}\right]^{3\sum_{\beta=1}^{r}(n_\beta - n'_\beta)} \\ & \cdot K^{e_\alpha} (M^4)^{\ell(E_\alpha \cap H^1(\pi))} \int_{\substack{x_\ell - y_\ell \in A_{j-1}^0 \\ \text{if } \ell \in H_\alpha \cup [E_\alpha \cap H^1(\pi)]}} \dots \int \\ & \prod_{v \in H_\alpha, v \neq v_0} d^3 x'_v \prod_{\substack{\ell \in E_\alpha \\ \ell \notin F}} C^i(x'_\ell, y'_\ell) \prod_{\substack{\ell \in E_\alpha \cap F \\ \ell \notin H}} C^{j-1}(x'_\ell, y'_\ell) \\ & \prod_{\ell \in H_\alpha \cup (E_\alpha \cap H)} C_{j-1}(x'_\ell, y'_\ell)|_{x_{v_0} = 0} \,. \end{aligned} \tag{3.31}$$

Putting $\ell_\beta = \ell(H_\beta)$ one has $3(n_\beta - n'_\beta) = 3n_\beta - 3$ if $n_\beta \leqq 7$, $3(n_\beta - n'_\beta) \geqq \frac{9n_\beta}{4}$ if $n_\beta \geqq 8$. Therefore, $3(n_\beta - n'_\beta) - \ell_\beta \geqq \ell_\beta/8$ if $n_\beta \geqq 8$ (since $\ell_\beta \leqq 2n_\beta$), and $3(n_\beta - n'_\beta) - \ell_\beta = 3n_\beta - \ell_\beta - 3 \geqq 1 \geqq \ell_\beta/14$ if $n_\beta \leqq 7$, since $3n_\beta - \ell_\beta - 3$ is the convergence degree of H_β, always bigger than one since G is ICW. Therefore, since $\ell_\alpha = \sum_\beta \ell_\beta$ and $n_\beta - n'_\beta \leqq \ell_\beta$:

$$M^{\ell_\alpha} \left[\frac{56(\log M)^{1/2}}{M}\right]^{3\sum_{\beta=1}^{r}(n_\beta - n'_\beta)} \leqq \left[\frac{(56)^3(\log M)^{3/2}}{M^{1/14}}\right]^{\ell_\alpha}. \tag{3.32}$$

Putting together (3.31), (3.32), and (3.18) achieves the proof of Theorem III.2 with $K_1(M) = M^6$, and

$$K_0(M) = \frac{e^{5/4} \cdot K^6 \cdot (56)^3 (\log M)^{3/2}}{M^{1/14}}, \tag{3.33}$$

since $e_\alpha \leqq 6\ell_\alpha$, and $e/M \leqq K_0(M)$.

When a graph has many lines of high momenta we may use a cruder bound on its amplitude by "breaking" all but one of the external legs of its high momenta components:

Lemma III.2. *There exist constants $K_1 > 0$, $K_2 > 0$ and a function $K'_0(M)$ with $\lim_{M \to \infty} K'_0(M) = 0$, such that for any* ICW *graph G and any assignment $\mu = \{i_\ell\}, \ell \in G$, one has*

$$I_{G,\mu} \leqq K_1^n [K'_0(M)]^{\sum_{\ell \in G} i_\ell}, \tag{3.34}$$

$$I_G \geqq I_G^i \geqq I_G^0 \geqq K_2^n \quad \forall i. \quad \square \tag{3.35}$$

Proof. The upper bound is a straightforward consequence of [17]. The lower bound may be obtained as in [9].

For the rest of the paper M is now fixed to a large value such that Theorem III.1 holds, and such that $\sup(K_0(M), K_0'(M))$ is less than 1/12. $\varepsilon(n)$ will be the generic name for a positive function of n which tends to 0 as n goes to infinity (in fact the reader may verify that all the $\varepsilon(n)$ used in this paper tend to 0 at least as $\mathrm{const}(\sqrt{\log\log n})^{-1}$. Finally i is fixed to be the integer part of $\varepsilon \log n/\log M$, so that $M^i \simeq n^\varepsilon$.

Lemma III.3

$$a_n^{I,i} \leqq a_n^I \leqq a_n^{I,i}(1+\varepsilon(n))^n . \quad \square \tag{3.36}$$

Proof. Inequality (3.36) is a simple consequence of

$$\forall G \,\mathrm{ICW}, \quad \sum_{F \subseteq G} J_{G,F}^i \leqq I_G^i (1+\varepsilon(n))^n , \tag{3.37}$$

where $J_{G,F}^i = \sum_{\mu \in \mathscr{C}_F^i} I_{G,\mu}$, $\mathscr{C}_F^i = \{\mu, i_\ell > i \Leftrightarrow \ell \in F\}$. Obviously $J_{G,F}^i \leqq I_{G,F}^i$.

We say that F is small if $\ell(F) \leqq n/(\log n)^{1/2}$, and large if $\ell(F) > n/(\log n)^{1/2}$. Then using Theorem III.1 and Lemma III.2:

$$\begin{aligned} \sum_{F \text{ small}} J_{G,F}^i &\leqq \sum_{F \text{ small}} I_{G,F}^i \\ &\leqq K(M)^{n/(\log n)^{1/2}} \sum_{\ell=0}^{n/(\log n)^{1/2}} \binom{2n}{\ell} I_G^i \\ &\leqq I_G^i (1+\varepsilon(n))^n , \end{aligned} \tag{3.38}$$

$$\begin{aligned} \sum_{F \text{ large}} J_{G,F}^i &\leqq 2^\ell K_1^n (1/12)^{i \cdot n/(\log n)^{1/2}} \left[\sum_{k=0}^{\infty} (1/12)^k \right]^\ell \\ &\leqq \left[\frac{8K_1}{K_2} \right]^n (1/12)^{\frac{\varepsilon n (\log n)^{1/2}}{\log M}} I_G^i \\ &\leqq I_G^i [1+\varepsilon(n)]^n . \end{aligned} \tag{3.39}$$

Together, (3.38) and (3.39) prove (3.36).

Now we are going to bound the B graphs, which do contain blobs. When these graphs contain more than $n/(\log n)^{1/2}$ blobs, there are so few of them that one may just use the overwhelming number of irreducible graphs to bound them. But when these graphs have only a few blobs one should compare them to irreducible graphs with the same structure. More precisely, let us introduce as in [10] the "simplification" operator S, which associates to a graph G the graph obtained by reducing every maximal chain of blobs in G to a single line:

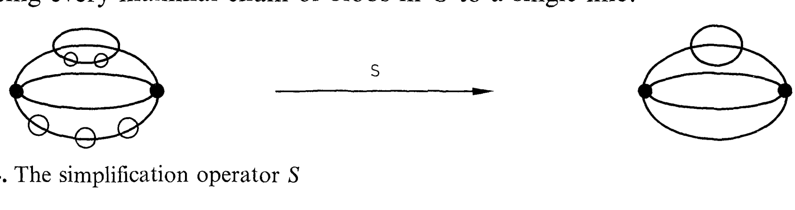


Fig. 4. The simplification operator S

Applying repeatedly S to a graph G one arrives after a finite number of steps at an irreducible graph called $S_\infty(G)$ as in [10]. S_∞ is a projection of the B graphs onto the I graphs.

Lemma III.4. *There exist numerical constants K_3 and K_4 such that:*

a) *The number of* ICW *graphs with n vertices is at least n!*

b) *The number of graphs G with n vertices such that $S_\infty(G)=G'$ has $n'=n-2k$ vertices is bounded by $K_3^n n^{n-2k}$.*

c) *For a given* ICW *G' with n' vertices, the number of CW graphs G with $n=n'+2k$ vertices such that $S_\infty(G)=G'$ is bounded by $[K_3 n/k]^k$.*

d) *To a given* ICW *graph G' with n' vertices and to any subset W of the vertices of G' with $\# W \geqq n'/2$, we can associate at least $[K_4 n/k]^{2k}$* ICW *graphs G with $n=n'+2k$ vertices, $k \leqq n'/4$, obtained by inserting in G' 2k "bubbles" (see Fig. 5) which, when contracted, give reduction vertices which belong to W. The set of such graphs G is called $D(G', W, k)$, and we write*

$$D(G',k)=\bigcup_{W/\# W \geqq n'/2} D(G',W,k). \quad \square$$

Fig. 5. The bubble

Proof. a) follows from an easy adaptation of an argument in [9] and was used already in [10]. b) and c) can be considered as simple corollaries of the more complete combinatoric analysis in [20, Appendix C]. Finally d) is trivial: we choose $2k$ vertices in W $\left(\text{there are } \binom{\# W}{2k} \geqq \left[\frac{K_4 n}{k}\right]^{2k} \text{ possible choices}\right)$ and expand each of them into a bubble.

Lemma III.5. *Let G be a CW graph with n vertices and k' blobs; $G'=S_\infty(G)$ has $n'=n-2k$, $k' \leqq k$, vertices. There exists a constant K_5 such that:*

$$\sup\{|I_G^R|, I_G^{W,i}\} \leqq [K_5 \varepsilon \log n]^k I_{G'}, \tag{3.40}$$

$$|I_G^R| \leqq K_5^n. \tag{3.41}$$

Proof. Inequality (3.41) is an old result ([7, 12]). For (3.40), let us consider the graph G'' obtained from G by replacing every maximal chain of r blobs by a special line •—/—• which has a propagator $(\text{const})^r \cdot (p^2+1)^{-2+\varepsilon'}$ for a small positive ε' (this is an elementary bound on the behavior of the renormalized chain). Then obviously for some constant K_5 one has:

$$|I_G^R| \leqq K_5^{k'} I_{G''}. \tag{3.42}$$

Similarly since the blob is logarithmically divergent:

$$I_G^{W,i} \leqq K_5^{k'} (\varepsilon \log n)^{k'} I_{G''} \tag{3.43}$$

(recall that $M^i \simeq n^\varepsilon$). Now G' is obtained from G'' by reducing in G'' a convergent graph with $2(k'-k)$ vertices. In [17] such an object is bounded at any momentum by $(\text{const})^{k'-k}$. This achieves the proof of (3.40).

Lemma III.6. *There exists a numerical constant K_6 such that for any* ICW *graph G' with n' vertices and for any k less than $n'/4$, we have:*

$$I_{G'} \leqq [K_6 k/n]^{2k} \sum_{G \in D(G',k)} I_G . \quad \square \tag{3.44}$$

Proof. The "bubble" decays at large energy, hence its insertion may reduce the value of a graph. Therefore we decompose $I_{G'}$ as $\sum_{\mu'} I_{G',\mu'}$, and consider two cases.

i) If $\mu' = \{i'_{\ell'}\}$ is such that $\sum_{\ell' \in G'} i'_{\ell'} > \frac{3}{2} n' \log \left[\frac{2K_1}{K_2}\right]$, where K_1 and K_2 are as in Lemma III.2, we say that μ' is large, and we apply (III.34) to get:

$$\begin{aligned}
&\sum_{\mu' \text{ large}} I_{G',\mu'} \\
&\leqq K_1^{n'} [K_2/2K_1]^{3/2 n' \log 3} \sum_{\mu' \text{ large}} (1/4)^{\sum_{\ell' \in G'} i'_{\ell'}} \\
&\leqq K_2^n \cdot [K_2/K_1]^{[3/2n' - n]} (1/2)^{3/2 n'} (4/3)^{2n'} \leqq K_2^n ,
\end{aligned} \tag{3.45}$$

where we used $K'_0(M) \leqq 1/12 = 1/3 \cdot 1/4$, $\log 3 \geqq 1$, and assumed $K_1 \geqq 1$, $K_1 \geqq K_2$ (recall that $n' \leqq n = n' + 2k \leqq 3n'/2$).

For any $G \in D(G', k)$, using (3.35), Lemma III.4, d) and $k \leqq n'/4$, we get:

$$[K_4 n/k]^{2k} \sum_{\mu' \text{ large}} I_{G',\mu'} \leqq \sum_{G \in D(G',k)} I_G . \tag{3.46}$$

ii) If μ' is small $\left(\text{such that } \sum_{\ell' \in G'} i'_{\ell'} \leqq \frac{3}{2} n' \log \frac{2K_1}{K_2}\right)$, there exists at most $n'/4$ lines of G' with $i'_{\ell'} \geqq 6 \log \frac{2K_1}{K_2}$. Hence there are at least $n'/2$ vertices v of G' such that every line hooked to v has $i'_{\ell'} \leqq 6 \log \frac{2K_1}{K_2}$. Let $W(\mu')$ be the corresponding set of vertices. We will prove that for some constant K_7 and for any G in $D(G', W(\mu'), k)$:

$$I_{G',\mu'} \leqq K_7^k \sum_{\mu \text{ near } \mu'} I_{G,\mu} , \tag{3.47}$$

where "μ near μ'" means that $i_\ell = i'_{\ell'}$ if ℓ is a line of G', but not an external line of any bubble inserted in $W(\mu')$, that $i_\ell = i'_\ell$ or $i'_\ell + 1$ if ℓ is an external line of such a bubble, and that $i_\ell =$ anything if ℓ is an internal line of such a bubble. Indeed in momentum space it is trivial to verify that the amplitude of the bubble as a function of its 4 external momenta p_1, p_2, p_3, p_4, verifies (very sloppy bound...):

$$I_{bubble}(p_1, p_2, p_3, p_4) \geqq (\text{const}) \prod_{j=1}^{4} (p_j^2 + 1)^{-1} . \tag{3.48}$$

The left-hand side of (3.47) is computed with propagators $C_{i'}$ whose Fourier transform

$$\tilde{C}_{i'}(p) = (\text{const}) \cdot (p^2+1)^{-1} [e^{-M^{-2i'}(p^2+1)} - e^{-M^{-2(i'-1)}(p^2+1)}]$$

verifies $(p^2+1)^{-1}[\tilde{C}_{i'}(p) + \tilde{C}_{i'+1}(p)] \geqq K_8 \tilde{C}_{i'}(p)$. ($K_8$ does depend on M and on $j = 6 \log(2K_1/K_2)$). Therefore summing now on every small assignment μ', we get, using again Lemma III.4, d) (with $K_7 = K_8^4$),

$$(K_4 n/k)^{2k} \sum_{\substack{\mu' \text{ small} \\ W(\mu')=W}} I_{G',\mu'} \leqq K_7^k \sum_{\substack{G \in D(G',W,k) \\ \mu \text{ near } \mu'}} I_{G,\mu}. \tag{3.49}$$

Hence summing over all possible W's

$$(K_4 n/k)^{2k} \sum_{\mu' \text{ small}} I_{G',\mu'} \leqq (2^4 K_7)^k \sum_{G \in D(G',k)} I_G. \tag{3.50}$$

Indeed for a given $G \in D(G',k)$ and a given μ, the piece of amplitude $I_{G,\mu}$ can appear at most 2^{4k} times in the sum over W's of the right-hand side of (3.49), since there are at most 2^{4k} assignments μ' such that a given μ is near μ'.

Putting together (3.46) and (3.50) achieves the proof of (3.44).

Lemma III.7

$$\sup\{|a_n^B|, a_n^{B,i}\} \leqq a_n^{I,i}(1+\varepsilon(n))^n. \quad \square \tag{3.51}$$

Proof. Consider a BCW graph G, $G' = S_\infty(G)$, and $n(G') = n' = n - 2k$. We distinguish two cases:

i) k small ($k \leqq n/(\log n)^{1/2}$). Using Lemma III.4 c) and Lemma III.5 we have:

$$\sum_{G, S_\infty(G)=G'} \sup\{|I_G^R|, I_G^{W,i}\} \leqq \left[\frac{K_3 K_5 \varepsilon n \log n}{k}\right]^k I_{G'}. \tag{3.52}$$

Moreover, it is easy to verify that for a given ICW graph G with n vertices, the number of graphs G' such that $G \in D(G',k)$ is bounded by $(\text{const})^k \binom{n}{2k} \leqq K_9^k (n/k)^{2k}$ (hint: this bounds the choice of $2k$ bubbles in G). Therefore using Lemma III.6 and (3.36),

$$\sum_{\substack{G \text{ BCW} \\ k \text{ small}}} \sup\{|I_G^R|, I_G^{W,i}\} \leqq \sum_{k=1}^{n/(\log n)^{1/2}} \left(\frac{K_3 K_5 K_6^2 K_9 \varepsilon n \log n}{k}\right)^k a_n^I \leqq a_n^{I,i}(1+\varepsilon(n))^n. \tag{3.53}$$

ii) k large ($k > n/(\log n)^{1/2}$). Using Lemma III.4 b) and Lemma III.5:

$$\sum_{\substack{G \text{ BCW} \\ k \text{ large}}} \sup\{|I_G^R|, I_G^{W,i}\} \leqq \sum_{k=n/(\log n)^{1/2}}^{n} \left[\frac{K_5 \varepsilon \log n}{n^2}\right]^k (K_3 K_5 n)^n. \tag{3.54}$$

But from (3.35) and Lemma III.4 a), $a_n^{I,i} \geqq n! K_2^n$, hence the left-hand side of (3.54) is bounded by $a_n^{I,i}[\text{const} \log n]^n e^{-2n\sqrt{\log n}}$, hence by $a_n^{I,i}(1+\varepsilon(n))^n$. This achieves the proof of (3.51).

Lemma III.8. *For any* ICW *graph G:*

$$I_G^{W,\delta} \geqq K_{10}^n \tag{3.55}$$

$$\sup\{I_G^{W,\delta}, I_G^{W,i}\} \leqq (1+\varepsilon(n))^n \inf\{I_G^{W,\delta}, I_G^{W,i}\}. \quad \square \tag{3.56}$$

Proof. (3.55) is trivial (for instance using Lemma A.4 and (3.35). To prove (3.56), let us consider length assignments on the lines of G, as in the proof of Theorem III.1;

but now if $\pi(\ell)=1$, it means if $x=x_\ell-y_\ell$, that $\delta|\log\delta|\leqq\tilde{x}\leqq|x|\leqq\frac{1}{4}\log|\log\delta|$, where $\tilde{x}=\inf\{|x_1|, |x_2|, |x_3|\}$; $\pi(\ell)=0$ means that $\tilde{x}\leqq\delta|\log\delta|$ and $|x|\leqq\frac{1}{4}\log|\log\delta|$, and $\pi(\ell)=2$ means $|x|\geqq\frac{1}{4}\log|\log\delta|$. Let us introduce $r_0(\pi)=\#\ell$, $\pi(\ell)=0$, $r_2(\pi)=\#\ell$, $\pi(\ell)=2$. Then:

– If $r_2(\pi)\geqq n/\sqrt{\log\log n}$, let us say that π is "long." By Lemmas A.1 and A.4 and a standard analysis of type [17]:

$$\sup\left\{\sum_{\pi\text{ long}} I_G^{W,i,\pi}, \sum_{\pi\text{ long}} I_G^{W,\delta,\pi}\right\}\leqq[c']^n\left[e^{-\frac{c}{8}\log\log(n^\varepsilon)}\right]^{\frac{n}{(\log\log n)^{1/2}}}$$

$$\leqq c'^n e^{-cn(\log\log n)^{1/2}}. \tag{3.57}$$

Hence by (3.35) and (3.55)

$$\sup\left\{\sum_{\pi\text{ long}} I_G^{W,i,\pi}, \sum_{\pi\text{ long}} I_G^{W,\delta,\pi}\right\}$$

$$\leqq(1+\varepsilon(n))^n\inf\{I_G^{W,\delta}, I_G^{W,i}\}. \tag{3.58}$$

– If $r_0(\pi)\geqq\frac{n}{(\log n)^{1/2}}$ we say that π is "short." We can pick a tree of G with at least $\frac{n}{2(\log n)^{1/2}}$ "short" lines with $\pi(\ell)=0$. We bound the corresponding integral using Lemmas A.1 or A.4. Let us call $f(T)$ the product over the lines of this tree of the exponentially decreasing factors in (A.1) or (A.29), and g the rest of the integrand. We apply a standard Hölder inequality to bound $\int g\cdot f(T)$ by $[\int g^{1+\varepsilon'}]^{(1+\varepsilon')^{-1}}[f(T)^{\frac{1+\varepsilon'}{\varepsilon'}}]^{\frac{\varepsilon'}{1+\varepsilon'}}$ with ε' very small. For such an ε', by superrenormalisability and the fact that G is ICW we satisfy the conditions of "generalized convergence" of [17]; therefore, the first integral is bounded by c'^n, and it is easy to exhibit the "small volume" effect of the short lines in the second integral. Therefore one gets:

$$\sup\left\{\sum_{\pi\text{ short}} I_G^{W,i,\pi}, \sum_{\pi\text{ short}} I_G^{W,\delta,\pi}\right\}$$

$$\leqq(c'')^n\{\delta|\log\delta|\,[\tfrac{1}{4}\log|\log\delta|]^2\}^{\frac{\varepsilon' n}{2(\log n)^{1/2}}}. \tag{3.59}$$

Again by (3.35) and (3.55) one concludes (since $\delta\simeq n^{-\varepsilon}$):

$$\sup\left\{\sum_{\pi\text{ short}} I_G^{W,i,\pi}, \sum_{\pi\text{ short}} I_G^{W,\delta,\pi}\right\}$$

$$\leqq(1+\varepsilon(n))^n\inf\{I_G^{W,i}, I_G^{W,\delta}\}. \tag{3.60}$$

If $r_2(\pi)\leqq n/\sqrt{\log\log n}$ and $r_0(\pi)\leqq n/(\log n)^{1/2}$ we say that π is "normal." Let us remark that in momentum space if $c\leqq 1$, $c'\geqq 1$,

$$\frac{1}{p^2+c}\leqq\frac{1}{c}\cdot\frac{1}{p^2+1};\qquad\frac{1}{p^2+1}\leqq\frac{c'}{p^2+c'}. \tag{3.61}$$

Therefore

$$\sum_{\pi\text{ normal}} I_G^{W,i,\pi}\leqq[c']^{2n/\sqrt{\log\log n}}\sum_{\pi\text{ normal}}\tilde{I}_G^{W,i,\pi}, \tag{3.62}$$

where $\tilde{I}_G^{W,i,\pi}$ has propagators $C_{c'}^i$ on lines ℓ with $\pi(\ell)=0,2$, and propagators C^i on lines ℓ with $\pi(\ell)=1$. Then, using Lemma A.2 and (A.35):

$$\sum_{\pi \text{ normal}} I_G^{W,i,\pi} \leqq (1+\varepsilon(n))^n [c'/c]^{2n/\sqrt{\log\log n}} \sum_{\pi \text{ normal}} I_G^{W,\delta,\pi} \leqq (1+\varepsilon(n))^n I_G^{W,\delta}. \tag{3.63}$$

Similarly, by Lemma A.2 and (A.35):

$$\sum_{\pi \text{ normal}} I_G^{W,\delta,\pi} \leqq (1+\varepsilon(n))^n [c']^{2n/\sqrt{\log\log n}} \sum_{\pi \text{ normal}} \tilde{\tilde{I}}_G^{W,i,\pi}, \tag{3.64}$$

where $\tilde{\tilde{I}}_G^{W,i,\pi}$ has propagators C_c^i on lines ℓ with $\pi(\ell)=0$ or 2 and propagators C^i on lines ℓ with $\pi(\ell)=1$. Going to momentum space and using (3.61) one gets again:

$$\sum_{\pi \text{ normal}} I_G^{W,\delta,\pi} \leqq (1+\varepsilon(n))^n [c'/c]^{2n/\sqrt{\log\log n}} \sum_{\pi \text{ normal}} I_G^{W,i,\pi} \leqq (1+\varepsilon(n))^n I_G^{W,i}. \tag{3.65}$$

Together, (3.58), (3.60), (3.63), and (3.65) prove (3.56).

Lemma III.9.

$$a_n^{W,i} \leqq (1+\varepsilon(n))^n a_n^{W,\delta}, \tag{3.66}$$

$$a_n^{W,\delta} \leqq (1+\varepsilon(n))^n a_n^{W,i}. \quad \square \tag{3.67}$$

Proof. (3.66) is a trivial consequence of Lemmas III.7 and III.8. Moreover, for the same reason that (3.40) is true, one may show that if G is BCW and $G' = S_\infty(G)$ (with k as in Lemma III.5):

$$I_G^{W,\delta} \leqq [K_5 \varepsilon \log n]^k I_{G'}^{W,\delta}. \tag{3.68}$$

By Lemma III.8, since G' is ICW,

$$I_G^{W,\delta} \leqq (K_5 \varepsilon \log n)^k (1+\varepsilon(n))^n I_{G'}^{W,i} \leqq (K_5 \varepsilon \log n)^k (1+\varepsilon(n))^n I_{G'}. \tag{3.69}$$

Therefore the proof of Lemma III.7 can be reproduced without any change, leading to the following analogue of (3.51):

$$a_n^{B,\delta} \leqq a_n^{I,i} \cdot (1+\varepsilon(n))^n. \tag{3.70}$$

By (3.70) and Lemma III.8:

$$a_n^{W,\delta} = a_n^{I,\delta} + a_n^{B,\delta} \leqq (1+\varepsilon(n))^n a_n^{I,i} \leqq (1+\varepsilon(n))^n a_n^{W,i}, \tag{3.71}$$

which achieves the proof of Lemma III.9.

The following result will be proved in the next section:

Theorem III.3. *Let* $a = \exp[-\inf S(\varphi) + 2]$ *as in (2.27). Then (recall that* $a_n^{W,\delta} > 0$*):*

$$a = \lim_{n\to\infty} [a_n^{W,\delta}/n!]^{1/n}. \quad \square \tag{3.72}$$

Assuming Theorem III.3, let us complete the proof of (2.27):

Proof of Theorem II.1.

A) *Upper Bound.* By Lemmas III.3, III.7, and (3.66), and since $a_n^{I,i} \leqq a_n^{W,i}$:

$$|a_n| \leqq a_n^I + |a_n^B| \leqq a_n^{I,i}(1+\varepsilon(n))^n \leqq a_n^{W,\delta}(1+\varepsilon(n))^n \tag{3.73}$$

(3.72) and (3.73) imply that $\limsup_{n\to\infty} [|a_n|/n!]^{1/n} \leqq a$.

B) *Lower Bound*. Let us suppose that $\limsup[|a_n|/n!]^{1/n} < a$. Then there exists n_0 and $b < a$ such that for $n \geqq n_0$, $|a_n| \leqq n!\, b^n$. Therefore for any λ such that $|\lambda| \leqq 1$, one has:

$$\left|\sum_{n=1}^{q} a_n \lambda^n\right| \leqq A_{n_0} + \sum_{n=1}^{q} n![b|\lambda|]^n$$
$$\leqq A_{n_0} + q![b|\lambda|]^q e^{\frac{1}{b|\lambda|}}$$

for some A_{n_0} independent of q. Now it is proved in [10] (first part of Eq. (31)] that, with the notations of this paper, for some constant M_4 and $\lambda_q = \frac{-M_4}{\sqrt{q}}$, one has:

$$\left|\sum_{n=1}^{q} a_n \lambda_q^n\right| \geqq \tfrac{1}{2} a_q^I |\lambda_q|^q . \tag{3.74}$$

But by (3.51), (3.67), and (3.72):

$$a_q^I \geqq a_q^{I,i} \geqq \frac{a_q^{B,i} + a_q^{I,i}}{(1+\varepsilon(q))^q} \leqq \frac{a_q^{W,i}}{(1+\varepsilon(q))^q}$$
$$\geqq \frac{a_q^{W,\delta}}{(1+\varepsilon(q))^q} \geqq \frac{a^q q!}{(1+\varepsilon(q))^q} .$$

This implies

$$1/2 \frac{a^q q!}{(1+\varepsilon(q))^q} |\lambda_q|^q \leqq A_{n_0} + q!\, b^q |\lambda_q|^q e^{\frac{1}{b|\lambda_q|}} ,$$

hence

$$a \leqq b \exp \frac{1}{bM_4\sqrt{q}} [1 + A_{n_0}/(\mathrm{const})^q \sqrt{q!}]^{1/q} [1 + \varepsilon(q)] ,$$

which obviously contradicts $a > b$ for q large enough.

For completeness, let us give a:

Scheme of the Proof of (3.74). In the "simplification" operation S, the reduction of a blob gives a factor $|\lambda|^2$ which, if $|\lambda| \sim \frac{1}{\sqrt{q}}$, allows us to decide in the reverse operation on which line this blob is inserted. In this way one can bound graphs of order n with blobs by graphs without blobs of lower order. Hence one obtains a lower bound only on partial sums of the perturbative expansion.

IV. The Semi-Classical Expansion

In this section we prove Theorem III.3, following Sects. 2 and 3 of [15] as closely as possible. We will indicate with parentheses the corresponding steps in (2.29) and references. We remark also that in most of this section φ is a discrete field on the lattice (which is called q in [15]).

Lemma IV.1 (Step 3; [14], Lemma 2, [15], Lemma 1.4). *For any n and* $\Lambda = [-T/2, T/2]$ ·

$$a_n^{W,\delta,D}(\Lambda) \leqq a_n^{W,\delta} \leqq a_n^{W,\delta,P}(\Lambda) . \quad \square \tag{4.1}$$

Proof. As in [14].

Lemma IV.2 (Step 4; [15, Corollary 1.3]).

$$[a_n^{W,\delta,X}(\Lambda)]^{1/n} = \left[\frac{b_n^{W,\delta,X}(\Lambda)}{|\Lambda|}\right]^{1/n}(1-\varepsilon(n)). \tag{4.2}$$

Proof. We can apply the proof of Corollary 1.3 in [15] without any change. With the notations of [15], we can check that $\lim_{k\to\infty}(b_k^*/b_k)^{1/k}=1$ follows also from Lemma IV.3 below and from the existence of $\lim_{n\to\infty, n \text{ even}} b_n^{1/n}$, which follows from Lemmas IV.4 and IV.6 below.

Let us introduce $p(n)$ and $q(n)$ which are equal to n if n is even and are respectively $n-1$ and $n+1$ if n is odd. Then:

Lemma IV.3.

$$(1+\varepsilon(n))^{-n} b_{p(n)}^{W,\delta,X}(\Lambda) \leqq b_n^{W,\delta,X}(\Lambda) \leqq b_{q(n)}^{W,\delta,X}(\Lambda)(1+\varepsilon(n))^n. \quad \square \tag{4.3}$$

Proof. We decompose $b_n^{W,\delta,X}(\Lambda)$ as the sum of the corresponding regularized Wick ordered amplitudes, which are all positive. The technique of "bubble insertion" of Sect. III (see Fig. 5) allows us to bound the amplitude of G by const $n^{8\varepsilon}$ times the amplitude of any graph obtained from G by inserting one bubble in any vertex of G; indeed the bubble decays at large energy, but since there is an ultraviolet cutoff of order n^ε, the sloppy bound (3.48) implies that to insert one bubble does not decrease the value of the graph by more than $(n^{2\varepsilon})^4$ (up to a constant). Then a trivial counting argument achieves the proof of (4.3)

Using Lemma IV.3 we may assume now that n is even, which is a source of considerable simplification in Sect. 2 of [15].

Lemma IV.4 (Step 5; [15, p. 188]).

$$\left|\left[\frac{b_n^{W,\delta,X}(\Lambda)}{n!}\right]^{1/n} - \left[\frac{b_n^{\delta,X}(\Lambda)}{n!}\right]^{1/n}\right| \leqq \varepsilon(n). \quad \square \tag{4.4}$$

Proof. Since n is even, we have

$$\left|\left[\frac{b_n^{W,\delta,X}(\Lambda)}{n!}\right]^{1/n} - \left[\frac{b_n^{\delta,X}(\Lambda)}{n!}\right]^{1/n}\right| = \frac{n^2}{(n!)^{2/n}}\left|\|W_\delta^X(\varphi/\sqrt{n})\|_n - \|V_\delta(\varphi/\sqrt{n})\|_n\right|. \tag{4.5}$$

By Stirling's formula, the hypercontractive estimate as in [15], (2.17), and since the tadpole is now linearly divergent in three dimensions one gets:

$$\begin{aligned} \left| \|W_\delta^X(\varphi/\sqrt{n})\|_n - \|V_\delta(\varphi/\sqrt{n})\|_n \right| &\leqq \|W_\delta^X(\varphi/\sqrt{n}) - V_\delta(\varphi/\sqrt{n})\|_n \\ &\leqq (n-1)\,\|W_\delta^X(\varphi/\sqrt{n}) - V_\delta(\varphi/\sqrt{n})\|_2 \\ &= \left(\frac{n-1}{n^2}\right)\|W_\delta^X(\varphi) - V_\delta(\varphi)\|_2 \\ &= O\left(\frac{n^{2\varepsilon}}{n}\right), \end{aligned} \tag{4.6}$$

which implies (4.4) if ε is fixed to a small value.

Lemma IV.5 [15, Lemma 2.1]. *The functionals $S_{X,\Lambda}$, $S_{X,\Lambda}^{\delta}$, and S all attain their infimums.* □

Proof. The proof in [15] uses a Sobolev inequality and the Rellich-Kondrachov theorem, still valid here in dimension 3.

Lemma IV.6 (Step 6; [15, pp. 185 and 189]).

$$\lim_{n \text{ even} \to \infty} [\int V_\delta^n(\varphi/\sqrt{n})\, d\mu_{\Lambda,\delta}^X]^{1/n} = \exp(-\inf S_{X,\Lambda}^{\delta}(\varphi)) . \quad \square \tag{4.7}$$

Proof. As in [15], we prove (4.7) by a lower and an upper bound. The lower bound uses Jensen's inequality exactly as in the first part of page 185 of [15]; it is even simpler since we have to use V_δ instead of W_δ^X in (4.7) ($V_{0,\delta}$ instead of V_δ with the notations of [15]). The upper bound is similar to the first part of [15] page 189, up to trivial changes: the dimension being three, the number of lattice points in Λ goes as $(T/\delta)^3 = \left[\frac{T}{\pi} n^{\varepsilon}\right]^3$, and the log appearing between Eq. (2.21) and (2.22) of [15] again has to be replaced by the "linearly divergent" factor n^{ε}. This does not affect of course the proof.

Lemma IV.7 (Step 7; [15, Lemma 2.3]).

$$\lim_{\delta \to 0} \inf S_{X,\Lambda}^{\delta} = \inf S_{X,\Lambda}, \quad \textit{for} \quad X = p, D . \quad \square \tag{4.8}$$

Proof. Let us call q_c, as in [15], a (lattice) field such that $S_{X,\Lambda}^{\delta}(q_c) = \inf S_{X,\Lambda}^{\delta}$. The proof of (4.8) is exactly similar to the proof of Lemma 2.3 in [15], except for the fact that (2.28) is no more valid, since the L^4 norm of C_δ^X is not uniformly bounded as $\delta \to 0$, since the graph G_2 in Fig. 2 is divergent. Hence we have to find another proof of the L^4 convergence of q_c towards ϕ (with the notations of [15]). Instead of proving, as in [15], that $\|q_c\|_\infty$ remains uniformly bounded as $\delta \to 0$ (and that $\|\phi\|_\infty$ is finite), we will prove that $\|q_c\|_{12}$ is uniformly bounded as $\delta \to 0$, (and that $\|\phi\|_{12}$ is finite) which is enough, by a standard Hölder estimate, to ensure the desired L^4 convergence.

By Hölder inequality one has:

$$\|C_\delta^X(x,y) q_c^3(y)\|_1 \leqq \|C_\delta^X(x,\cdot)\|_{5/2} \|q_c^3\|_{5/3} , \tag{4.9}$$

and

$$\begin{aligned} \|q_c^3\|_{5/3} &= [\int |q_c|^5]^{3/5} = [\int |q_c|^{3/2} |q_c|^{7/2}]^{3/5} \\ &\leqq [\int |q_c|^{12}]^{1/8 \cdot 3/5} [\int |q_c|^{7/2 \cdot 8/7}]^{7/8 \cdot 3/5} \\ &= \|q_c\|_{12}^{9/10} \cdot \|q_c\|_4^{21/10} . \end{aligned} \tag{4.10}$$

As in [15], q_c satisfies the classical equation:

$$q_c(x) = 4 \int_\Lambda C_\delta^X(x,y) q_c^3(y) d^3y / \|q_c\|_4^4 . \tag{4.11}$$

Therefore, we have the following analogue of [15], (2.28):

$$\|q_c\|_{12} \leqq 4 \cdot \left[\int_\Lambda \|C_\delta^X(x,\cdot)\|_{5/2}^{12}\right]^{1/12} \cdot \frac{\|q_c\|_{12}^{9/10}}{\|q_c\|_4^{19/10}} . \tag{4.12}$$

Hence

$$\|q_c\|_{12} \leqq 4^{10} \left[\int_{\Lambda} \|C_\delta^X(x,\cdot)\|_{5/2}^{12} \right]^{10/12} \|q_c\|_4^{-19}. \tag{4.13}$$

This proves that $\|q_c\|_{12}$ is uniformly bounded as $\delta \to 0$, since by power counting it is easy to verify that $\|C_\delta^X\|_{5/2}$ is uniformly bounded as $\delta \to 0$. Also as in [15] $\|q_c\|_4$ does not approach 0. The rest of the proof is as in [15]. Of course the numbers chosen are largely arbitrary....

Lemma IV.8 (Step 8; [15, Lemma 3.1]).

$$\lim_{\Lambda \to \infty} \inf S_{X,\Lambda}(\varphi) = \inf S(\varphi) \quad \text{for} \quad X = p, D. \quad \square \tag{4.14}$$

Proof. The proof in [15] remains valid up to dimension 3.

Together, Lemmas IV.1–IV.8 prove Theorem III.3.

Appendix

This Appendix is devoted to the proof of technical results on propagators with "exponential" or lattice cutoffs which are useful in particular for the proof of Lemma III.8.

Since the propagators considered are translation invariant, we will write systematically $C(x-y)$ instead of $C(x,y)$ to simplify notations. The letters c, c', K will be used for positive unimportant numerical constants, c for a small one (with respect to 1) and c' or K for a large one.

Lemma A.1. *There exist constants c, c', K such that:*

$$ce^{-|x|}/|x| \leqq C^i(x) \leqq c' \frac{e^{-|x|}}{|x|} \quad \text{if} \quad |x| \geqq M^{-i}, \tag{A.1}$$

$$cM^i \leqq C^i(x) \leqq c'M^i \quad \text{if} \quad |x| \leqq 3M^{-i}, \tag{A.2}$$

$$C^k(x) \leqq KC^k(y) \quad \text{if} \quad k \leqq j, \, ||x|-|y|| \leqq M^{-j}. \quad \square \tag{A.3}$$

Proof. By definition (2.4) one has

$$\forall i \; C^i(x) \leqq C(x) = \frac{e^{-|x|}}{|x|}. \tag{A.4}$$

Moreover

a) If $M^{-i} \leqq |x| \leqq 1$:

$$\begin{aligned} C^i(x) &\geqq c \int_{|x|^2}^{2|x|^2} \frac{d\alpha}{\alpha^{3/2}} e^{-2-\frac{|x|^2}{4\alpha}} \geqq c \int_{|x|^2}^{2|x|^2} e^{-9/4} \frac{d\alpha}{\alpha^{3/2}} \\ &\geqq c\left(\sqrt{\frac{1}{2|x|^2}} - \sqrt{\frac{1}{|x|^2}}\right) \geqq \frac{c}{|x|} \cdot \frac{2-\sqrt{2}}{2} \geqq \frac{ce^{-|x|}}{|x|}. \end{aligned} \tag{A.5}$$

b) If $|x| \geqq K'$, for some constant $K' > 2$ large enough,

$$C^i(x) \geqq \frac{1}{2|x|} e^{-|x|}. \tag{A.6}$$

Indeed

$$|C(x)-C^i(x)|=c\int_0^{M^{-2i}}\frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x|^2}{4\alpha}}$$
$$\leqq ce^{-M^{2i}\frac{|x|^2}{8}}\int_0^1\frac{d\alpha}{\alpha^{3/2}}e^{-\frac{|x|^2}{8\alpha}}$$
$$\leqq K''e^{-|x|^2/8},$$

since $|x|\geqq 1$; therefore choosing K' such that $2K''e^{-\frac{(K')^2}{8}}\leqq e^{-K'}$, one has for $|x|>K'$:

$$C^i(x)\geqq\frac{1}{|x|}e^{-|x|}-K''e^{-|x|^2/8}\geqq\frac{1}{2|x|}e^{-|x|},\tag{A.7}$$

which proves (A.6).

c) If $1\leqq|x|\leqq K'$:

$$C^i(x)\geqq c\int_1^2\frac{d\alpha}{\alpha^{3/2}}e^{-\frac{K'^2}{4\alpha}-\alpha}\geqq c_1\geqq c\frac{e^{-|x|}}{|x|}.\tag{A.8}$$

Together (A.4)–(A.6) and (A.8) imply (A.1). Moreover for $|x|\leqq 3M^{-i}$, $|x|^2\leqq 9M^{-2i}$, and therefore:

$$cM^i\leqq\int_{M^{-2i}}^{\infty}\frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-9/4}\leqq c''C^i(x)\leqq\int_{M^{-2i}}^{\infty}\frac{d\alpha}{\alpha^{3/2}}e^{-\alpha}\leqq c'M^i.\tag{A.9}$$

Let us finish the proof of Lemma A.1 using (A.1) and (A.2).

i) If $|y|\leqq 2M^{-k}|x|\leqq 2M^{-k}+M^{-j}\leqq 3M^{-k}$, and (A.2) proves (A.3).

ii) If $|y|\geqq 2M^{-k}$, $|x|\geqq 2M^{-k}-M^{-j}\geqq M^{-k}$, $\frac{|y|}{|x|}\leqq 2$, and by (A.1):

$$C^k(x)\leqq\frac{1}{|x|}e^{-|x|}\leqq e^{||x|-|y||}\frac{e^{-|y|}}{|y|}\cdot\frac{|y|}{|x|}\leqq 2\frac{e^{-|y|}}{|y|}\leqq K\cdot C^k(y).$$

This achieves the proof of (A.3).

The following lemmas control the behavior of lattice propagators. They are not rotation invariant. Let us suppose, by symmetry, that $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ is such that $0\leqq x_1\leqq x_2\leqq x_3$. If $x\in\delta\mathbb{Z}^3$, $x_1=\delta n_1$, $x_2=\delta n_2$, $x_3=\delta n_3$, $0\leqq n_1\leqq n_2\leqq n_3$.

Lemma A.2. *There exists a function $\varepsilon(\delta)$ which tends to 0 as δ tends to 0, such that, if $M^i=\pi/\delta$:*

$$|C_\delta(x)-C^i(x)|\leqq\varepsilon(\delta)C^i(x),$$
$$\text{if}\quad \delta|\log\delta|\leqq x_1\leqq|x|\leqq 1/4\log|\log\delta|.\quad\square\tag{A.10}$$

Proof. We will use the Fourier representation of C_δ and C^i:

$$C_\delta(x)=\frac{1}{(2\pi)^3}\int_{-\pi/\delta}^{\pi/\delta}\dots\int_{-\pi/\delta}^{\pi/\delta}\frac{d^3ke^{ik\cdot x}}{1+2\delta^{-2}\sum_\alpha(1-\cos\delta k_\alpha)}$$
$$\equiv\int\dots\int\Gamma_\delta(k)d^3k,\tag{A.11}$$

$$C^i(x)=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}\frac{d^3ke^{ikx-\delta^2k^2/\pi^2}}{1+k^2}$$
$$\equiv\int\dots\int\Gamma^i(k)d^3k.\tag{A.12}$$

We write $x=y+\zeta$, with $|\zeta|<\delta$ such that $C_\delta(x)=C_\delta(y)$ (recall that C_δ is piecewise constant). Using (2.3)–(2.4) one has:

$$C^i(y)-C^i(x)=\frac{1}{(4\pi)^{3/2}}\int_{\delta^2/\pi^2}^{\infty}\frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x|^2}{4\alpha}}\left[1-e^{\frac{|x|^2-|y|^2}{4\alpha}}\right]. \tag{A.13}$$

If $\alpha\geqq|x|\delta|\log\delta|^{1/2}$, since $||x|^2-|y|^2|\leqq 3|x|\,|y-x|\leqq 3\delta|x|$:

$$|1-e^{\frac{|x|^2-|y|^2}{4\alpha}}|\leqq c'/|\log\delta|^{1/2}=\varepsilon(\delta). \tag{A.14}$$

If $\alpha\leqq|x|\delta|\log\delta|^{1/2}$ since $|x|\geqq\delta|\log\delta|$ one has $\alpha\leqq|x|^2/|\log\delta|^{1/2}$; therefore:

$$\int_{\delta^2/\pi^2}^{|x|\delta|\log\delta|^{1/2}}\frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x|^2}{4\alpha}}\leqq e^{-\frac{|\log\delta|^{1/2}}{8}}\int_{\delta^2/\pi^2}^{|x|\delta|\log\delta|^{1/2}}\frac{d\alpha}{\alpha^{3/2}}e^{-\alpha-\frac{|x|^2}{8\alpha}}. \tag{A.15}$$

If $|x|\leqq 1$, using (A.1) we bound the right-hand side of (A.15) by:

$$e^{-\frac{|\log\delta|^{1/2}}{8}}C^i(x/\sqrt{2})\leqq\varepsilon(\delta)\frac{\sqrt{2}}{|x|}\leqq\varepsilon(\delta)C^i(x). \tag{A.16}$$

If $1\leqq|x|\leqq\dfrac{\log|\log\delta|}{4}$, we put $\alpha'=2\alpha$ and bound the right-hand side of (A.15) by

$$\begin{aligned}&\sqrt{2}e^{-\frac{|\log\delta|^{1/2}}{8}}e^{|x|\delta|\log\delta|^{1/2}}\int_{2\delta^2/\pi^2}^{2|x|\delta|\log\delta|^{1/2}}\frac{d\alpha'}{\alpha'^{3/2}}e^{-\alpha'-\frac{|x|^2}{4\alpha'}}\\&\leqq\varepsilon(\delta)C^i(x).\end{aligned} \tag{A.17}$$

Since $|x-y|\leqq\delta$, (A.15)–(A.17) remain true with x replaced by y. Also by (A.1), $C^i(y)\leqq c'C^i(x)$. Therefore, (A.14)–(A.17) together with this remark imply:

$$|C^i(x)-C^i(y)|\leqq\varepsilon(\delta)C^i(x). \tag{A.18}$$

Using (A.18), it is enough to prove (A.10) when $x\in\delta\mathbb{Z}^3$. Let $\gamma_\alpha(x)$ be such that:

$x_\alpha\cdot\gamma_\alpha(x)=0\bmod\pi$, and

$$1/2\gamma_\alpha(x)\leqq\frac{e^{|x|/4}|\log\delta|^{1/2}}{x_\alpha}\leqq\gamma_\alpha(x). \tag{A.19}$$

We consider the region $D_1:|k_\alpha|\leqq\gamma_\alpha(x)$, for any $\alpha=1,2,3$. In D_1 we can expand, since $\delta\gamma_\alpha(x)\ll 1$:

$$\begin{aligned}\cos\delta k_\alpha&=1-\frac{\delta^2k_\alpha^2}{2}+O(\delta^4k_\alpha^4);\\ e^{-\delta^2k^2/\pi^2}&=1+O(\delta^2k^2).\end{aligned} \tag{A.20}$$

Hence

$$\left|\frac{e^{-\delta^2k^2/\pi^2}}{1+k^2}-\frac{1}{1+2\delta^{-2}\sum_{\alpha=1}^{3}(1-\cos\delta k_\alpha)}\right|=\frac{O(\delta^2k^2)}{1+k^2}, \tag{A.21}$$

and

$$\int_{D_1}|\Gamma^i-\Gamma_\delta|d^3k\leqq c_1'\delta^2\prod_{\alpha=1}^{3}\gamma_\alpha(x). \tag{A.22}$$

Since $|x|<1/4\log|\log\delta|$ and $x_1 \geqq \delta|\log\delta|$, we can bound the right-hand side of (A.22), using (A.1), by:

$$c_1' \prod_{\alpha=1}^{3} (2/x_\alpha) e^{3|x|/4} \delta^2 |\log\delta|^{3/2}$$
$$\leqq \frac{c'|\log\delta|^{3/16}}{|x|\,|\log\delta|^{1/2}} \leqq \frac{c'}{|x|} \frac{e^{-|x|}}{|\log\delta|^{1/16}} \leqq \varepsilon(\delta) C^i(x) . \tag{A.23}$$

It remains to bound the integrals of Γ_δ and Γ^i over the complement D_2 of D_1 (respectively in $[-\pi/\delta, \pi/\delta]^3$ and in $\mathbb{R}^3$). This can be done by repeated integration by parts, the surface terms being 0 by our choice of x and $\gamma_\alpha(x)$. In D_2, either $|k_3|$ is bigger than $\gamma_3(x)$, or $|k_3|$ is smaller than $\gamma_3(x)$ and $|k_\alpha|$ is bigger than $\gamma_\alpha(x)$, $\alpha=1$ or 2. Then integrating with respect to k_α, one gets in the first case (with $\Gamma=\Gamma_\delta$ or Γ^i):

$$\left|\int_{|k_3|\geqq\gamma_3(x)} \Gamma d^3k\right| \leqq \frac{1}{x_3^5} \int_{|k_3|\geqq\gamma_3(x)} \frac{d^5\Gamma}{(dk_3)^5} d^3k , \tag{A.24}$$

and in the second case:

$$\left|\int_{|k_\alpha|\geqq\gamma_\alpha(x);\,|k_3|\leqq\gamma_3(x)} \Gamma d^3k\right| \leqq \frac{1}{x_\alpha^5} \int_{|k_\alpha|\geqq\gamma_\alpha(x);\,|k_3|\leqq\gamma_3(x)} d^5\Gamma/(dk_\alpha)^5 d^3k . \tag{A.25}$$

Since for the lattice integrand $|k_\alpha| \leqq \pi/\delta$, we can bound in both cases $d^5/(dk_\alpha)^5 \cdot \Gamma(k)$ by $c'(1+k^2)^{-1-5/2}$. This bound when inserted in the right-hand side of (A.24)–(A.25) gives:

$$\left|\int_{D_2} \Gamma(k) d^3k\right| \leqq \frac{c'}{|x|} \frac{e^{-|x|}}{|\log\delta|^2} \leqq \varepsilon(\delta) C^i(x) , \tag{A.26}$$

which, together with (A.23), achieves the proof of (A.10).

Lemma A.3. *There exist constants c and c′ (c small, c′ large) such that:*

$$cC_\delta(0,0,x_1+x_2+x_3) \leqq C_\delta(x) \leqq c'C_\delta(0,0,x_3) . \quad \square \tag{A.27}$$

Proof. We use Symanzik's path representation of the lattice propagator:

$$C_\delta(x) = 1/\delta \sum_{\omega:0\to x} [1/(6+\delta^2)]^{|\omega|} , \tag{A.28}$$

where ω runs over all paths in $\delta\mathbb{Z}^3$ made of $|\omega|$ steps, starting at 0 and ending at x. Let us prove first the lower bound on C. Consider $e_1=(\delta,0,0)$, $e_2=(0,\delta,0)$, $e_3=(0,0,\delta)$. In (A.28) we may consider the paths from 0 to $(0,0,x_1+x_2+x_3)$ as an ordered list of p_1 steps e_1, p_1 steps $-e_1$, p_2 steps e_2, p_2 steps $-e_2$, $n_1+n_2+n_3+p_3$ steps e_3 and p_3 steps $-e_3$. Also the paths from 0 to x may be considered as ordered lists of n_1+p_1' steps e_1, p_1' steps $-e_1$, etc....

– If $p_1 \leqq p_2+n_1$ and $p_2 \leqq p_3+n_2$, we make a correspondence between paths of the first and of the second kind by changing first n_1+n_2 steps e_3 into steps e_2, then n_1 steps e_2 into steps e_1. The combinatoric factors are $\binom{n_1+n_2+n_3+p_3}{n_1+n_2}$, $\binom{n_1+n_2+p_2}{n_1}$ in the direct way and $\binom{n_1+p_1'}{n_1}$, $\binom{n_1+n_2+p_2'}{n_1+n_2}$ in the reverse way.

Since $p_1 = p'_1$, $p_2 = p'_2$ and $n_1 \leqq n_2 \leqq n_3$, one has:

$$\binom{n_1+p'_1}{n_1} \cdot \binom{n_1+n_2+p'_2}{n_1+n_2} \leqq \binom{n_1+n_2+n_3+p_3}{n_1+n_2} \cdot \binom{n_1+n_2+p_2}{n_1}.$$

– If $p_1 \geqq p_2 + n_1$ and $p_2 \leqq p_3 + n_2$, we consider a similar correspondence by changing first $n_1 + n_2$ steps e_3 into steps e_2, then n_1 steps $-e_1$ into steps $-e_2$. The direct and reverse combinatoric factors are respectively $\binom{n_1+n_2+n_3+p_3}{n_1+n_2}, \binom{p_1}{n_1}$ and $\binom{p'_2}{n_1}, \binom{n_2+p'_2}{n_1+n_2}$. Since $p'_2 = p_2 + n_1$, and $n_2 \leqq n_3$:

$$\binom{p'_2}{n_1}\binom{n_2+p'_2}{n_1+n_2} \leqq \binom{p_1}{n_1}\binom{n_1+n_2+n_3+p_3}{n_1+n_2}.$$

The two last cases ($p_2 \geqq p_3 + n_2$ and $p_1 \leqq n_1 + n_2 + p_3$, and $p_2 \geqq p_3 + n_2$ and $p_1 \geqq n_1 + n_2 + p_3$) are similar and left as an exercise. The lower bound in (A.27) is thus proved with $c = 1/4$ (since there are 4 cases). The upper bound is in the same spirit and is true with $c' = (6 + \delta^2)^2 \leqq 49$ (hint: change $I(n_1/2)$ steps e_1 into steps $-e_1$; if n_1 is odd, add one step $-e_1$. Do the same in the other direction e_2. The combinatoric factors are o.k..

Lemma A.4. *There exist constants c and c′ (c small, c′ large) such that*

$$\frac{ce^{-c'|x|}}{|x|} \leqq C_\delta(x) \leqq \frac{c'e^{-c|x|}}{|x|} \quad \textit{if} \quad |x| \geqq \delta, \tag{A.29}$$

$$c/\delta \leqq C_\delta(x) \leqq c'/\delta \quad \textit{if} \quad |x| \leqq \delta. \quad \square \tag{A.30}$$

Using Lemma A.3 it is sufficient to prove (A.29)–(A.30) when $x = (0, 0, \delta n_3)$, $n_3 \geqq 0$. If $n_3 = 0$, (A.30) follows obviously from representation (A.11). It remains to prove (A.29) for $x_3 = \delta n_3 > 0$. Shifting the k_3 integration contour in (A.11), one obtains by Cauchy's formula

$$C_\delta(0,0,x_3) = \frac{1}{(2\pi)^3}\,\pi \int_{-\pi/\delta}^{\pi/\delta} \int_{-\pi/\delta}^{\pi/\delta} dk_1\, dk_2\, \frac{\delta e^{-\tilde{k}_3 x_3}}{\sinh \delta \tilde{k}_3}, \tag{A.31}$$

where $\tilde{k}_3 > 0$ is defined by:

$$\cosh[\delta \tilde{k}_3(k_1,k_2)] = 1 + \frac{1 + 2\delta^{-2}[(1-\cos k_1\delta) + (1-\cos k_2\delta)]}{2\delta^{-2}}. \tag{A.32}$$

Since $|k_1|$ and $|k_2|$ are bounded by π/δ, one has

$$c(k_1^2 + k_2^2) \leqq \delta^{-2}[(1-\cos k_1\delta) + (1-\cos k_2\delta)] \leqq c'(k_1^2 + k_2^2). \tag{A.33}$$

Hence if $\vec{k} = (k_1, k_2)$:

$$c(1 + |\vec{k}|) \leqq \tilde{k}_3 \leqq c'(1 + |\vec{k}|). \tag{A.34}$$

Combining (A.34) with (A.31), it is easy to achieve the proof of (A.29).

One remarks that if C^i_m is the propagator with exponential cutoff M^i and bare mass m instead of 1, Lemma A.1 and A.4 imply that for some constants c and c' (c small, c' large):

$$\forall x \;\; cC^i_{c'}(x) \leqq C_\delta(x) \leqq c' C^i_c(x). \tag{A.35}$$

Acknowledgements. We thank Tom Spencer for useful comments and one of us (V.R.) thanks S. Breen for an interesting discussion which motivated this work.

References

1. Bender, C.M., Wu, T.T.: Anharmonic oscillator. II. A study of perturbation theory in large order. Phys. Rev. D**7**, 1620 (1973)
2. Lam, C.S.: Behavior of very high-order perturbation diagrams. Nuovo Cimento **55**A, 258 (1968)
3. Lipatov, L.N.: Calculation of the Gell-Mann-Low function in scalar theory with strong nonlinearity. Sov. Phys. JETP **44**, 1055 (1976) and Divergence of the perturbation-theory series and the quasi-classical theory. **45**, 216 (1977); Divergence of the perturbation-series and pseudoparticles. JETP Lett. **25**, 104 (1977)
4. Brézin, E., Le Guillou, J.C., Zinn-Justin, J.: Perturbation theory at large order. I. The ϕ^{2N} interaction, and II. Role of the vacuum instability. Phys. Rev. D**15**, 1544 and 1558 (1977); Perturbation theory of large orders for a potential with degenerate minima. Phys. Rev. D**16**, 408 (1977)
5. Parisi, G.: Asymptotic estimates in perturbation theory, Phys. Lett. **66**B, 167 (1977) and The Borel transform and the renormalization group. Phys. Rep. **49**, 215 (1979); 't Hooft, G.: Lectures given at Erice (1977)
6. Velo, G., Wightman, A. (eds.): Constructive quantum field theory. Lecture Notes in Physics, Vol. 25, Berlin, Heidelberg, New York: Springer 1973
7. Glimm, J., Jaffe, A.: Positivity of the ϕ_3^4 Hamiltonian. Fortschr. Phys. **21**, 327 (1973)
8. Feldman, J., Osterwalder, K.: The Wightman axioms and the mass gap for weakly coupled $(\phi^4)_3$ quantum field theories. Ann. Phys. **97**, 80 (1976)
   Magnen, J., Sénéor, R.: The infinite volume limit of the ϕ_3^4 model. Ann. Inst. Henri Poincaré **24**, 95 (1976)
9. Jaffe, A.: Divergence of perturbation theory for bosons. Commun. Math. Phys. **1**, 127 (1965)
10. Calan, de C., Rivasseau, V.: The perturbation series for Φ_4^4 field theory is divergent. Commun. Math. Phys. **83**, 77 (1982)
11. Graffi, S., Grecchi, V., Simon, B.: Borel summability: application to the anharmonic oscillator. Phys. Lett. **32**B, 631 (1970)
    Eckmann, J.-P., Magnen, J., Sénéor, R.. Decay properties and Borel summability for the Schwinger functions in $P(\Phi)_2$ theories. Commun. Math. Phys. **39**, 251 (1975)
    Magnen, J., Sénéor, R.: Phase space cell expansion and Borel summability for the Euclidean ϕ_3^4 theory. Commun. Math. Phys. **56**, 237 (1977)
12. Rivasseau, V., Speer, E.: The Borel transform in Euclidean ϕ_ν^4; local existence for $\mathrm{Re}\,\nu<4$. Commun. Math. Phys. **72**, 293 (1980)
13. Benassi, L., Grecchi, V., Harrell, E., Simon, B.: Bender-Wu formula and the Stark effect in hydrogen. Phys. Rev. Lett. **42**, 704 (1979)
    Harrell, E., Simon, B.: Duke Math. **47**, 845 (1980)
    Breen, S.: Mem. Am. Math. Soc. (to appear)
14. Spencer, T.: The Lipatov argument. Commun. Math. Phys. **74**, 273 (1980)
15. Breen, S.: Leading large order asymptotics for $(\Phi^4)_2$ perturbation theory. Commun. Math. Phys. **92**, 179 (1983)
16. Le Guillou, J.C., Zinn-Justin, J.: Critical exponents for the n-Vector model in three dimensions from field theory. Phys. Rev. Lett. **39**, 95 (1977) and Critical exponents from field theory. Phys. Rev. B**21**, 3976 (1980)
    Brézin, E., Parisi, G.: Critical exponents and large-order behavior of perturbation theory. J. Stat. Phys. **19**, 269 (1978)
17. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Bounds on completely convergent Euclidean Feynman Graphs. Commun. Math. Phys. **98**, 273 (1985)

18. Rivasseau, V.: Construction and Borel summability of planar 4-dimensional Euclidean field theory. Commun. Math. Phys. **95**, 445 (1984)
19. Zimmermann, W.: The power counting theorem for Minkowski metric, Commun. Math. Phys. **11**, 1 (1968) and Convergence of Bogoliubov's method for renormalization in momentum space, **15**, 208 (1969)
20. de Calan, C., Rivasseau, V.: Local existence of the Borel transform in Euclidean Φ_4^4. Commun. Math. Phys. **82**, 69 (1981)

Communicated by T. Spencer

Received February 8, 1985; in revised form April 18, 1985

Note added in proof. The simplification operator S introduced after Lemma III.3 might create tadpoles G_4. Therefore it should be replaced, throughout the paper, by $S \circ T$, where T in the simplification operator which transforms every maximal chain of tadpoles into a single line. The corresponding modifications are trivial. The change of S into $S \circ T$ should also be made in [10].