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Truncation of Continuum Ambiguities in Phase-Shift Analysis

D. Atkinson[1,*] and I. S. Stefanescu[2]

1 CERN – Geneva 23, Switzerland
2 Institut für Theoretische Kernphysik, Postfach 6380, D-7500 Karlsruhe, Federal Republic of Germany

Abstract. The continuum ambiguity in the determination of phase shifts from scattering data consists of a family of amplitudes which have in general an infinite number of partial waves. In practical computations, however, the partial wave series is necessarily truncated. We discuss the relation of the resulting (truncated) amplitudes to those representing the true continuum ambiguity. In particular, we show that each of the latter is approximated increasingly well, as the cut-off tends to infinity, uniformly inside an ellipse in the $\cos\theta$ plane.

1. Introduction

It is well known that the determination of an elastic scattering amplitude from data on elastic scattering (differential cross-sections, polarizations, etc.), at a fixed energy over the whole angular range, generally suffers from a continuum ambiguity when the energy is such that inelastic channels are open. The family of amplitudes making up this continuum ambiguity can be explored by means of a generalization to function spaces of the implicit function theorem [1, 2]. However, in the practical implementation of this method for the determination of ambiguities [3], the partial wave series is necessarily truncated, whereas the true continuum ambiguity contains amplitudes with an infinite number of partial waves. In earlier work [3], it was generally assumed that the truncation error was unimportant: in this paper we discuss in a precise manner the relation between the truncated amplitudes produced by the computer and those belonging to the continuum ambiguity. There are two facets of this problem: i) on the one hand, one must show that, for any member of the continuum ambiguity, one can generate a sequence of increasingly better approximants by letting the truncation point recede to infinity; ii) if the finite algorithm of the computer generates a solution,

* Permanent address: Institute for Theoretical Physics, P.O.B. 800, Groningen, The Netherlands

one has to find a criterion to verify whether there corresponds to it a member of the true continuum ambiguity and, if so, estimate the error. The numerical work, which has been recently re-examined [4] with greatly improved accuracy, thus receives a detailed theoretical description.

In the case of spinless particles, the equations determining the absorptive part of the partial wave amplitudes from the differential cross-section are as follows:

$$A_\ell = A_\ell^2 + D_\ell^2 + I_\ell, \tag{1.1}$$

$$D_\ell = \frac{1}{2\pi i} \oint_{\partial E(z_0)} dz\, Q_\ell(z)\, [\sigma(z) - A^2(z)]^{1/2}, \tag{1.2}$$

$$A(z) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) A_\ell. \tag{1.3}$$

In (1.2), the integration in the $z = \cos\theta$ plane is taken around the circumference of the small Lehmann-Martin ellipse:

$$E(z_0) = \{z : |z + (z^2-1)^{1/2}| < z_0 + (z_0^2-1)^{1/2}\}, \tag{1.4}$$

and $\sigma(z)$, the normalized differential cross-section is a given function, holomorphic inside $E(z_0)$ and continuous in $\bar{E}(z_0)$. The inelasticities I_ℓ, limited to [0, 1/4], are taken to be given real numbers, such that

$$\|I\|_1 \equiv \sup_\ell I_\ell / Q_\ell(z_1) < \infty, \tag{1.5}$$

where z_1 lies in the interval $(z_0, 2z_0^2 - 1)$. The amplitude $F(z)$ is given by

$$F(z) = D(z) + iA(z). \tag{1.6}$$

A proof has been given [1] that the system (1.1)–(1.3) defines A_ℓ as an implicit function of I_ℓ for fixed $\sigma(z)$, and that the inelasticities can be varied in a continuous manner, thereby generating a continuum of amplitudes that correspond to precisely the same cross-section.

If the analytic continuation of $\sigma(z)$ to $E(z_0)$ has a finite number of zeros, then, if two solutions $F_1(z)$, $F_2(z)$, of the set of Eqs. (1.1)–(1.3) are sufficiently close to each other, they will differ in the physical region by a phase factor

$$F_1(z) = F_2(z) \exp[i\phi(z)], \tag{1.7}$$

with $\phi(z)$ holomorphic in $E(z_0)$. In general, every solution of Eqs. (1.1)–(1.3) has an infinite number of partial waves F_ℓ:

$$F_\ell = D_\ell + iA_\ell. \tag{1.8}$$

If, by chance, there exists a solution of (1.1)–(1.3) with a finite number of partial waves, F_ℓ, then one can easily show that all solutions of the system that are sufficiently close to it necessarily have an infinite number of non-vanishing partial waves.

In this paper, we analyze a system that is different from (1.1)–(1.3) in that only the first N partial waves are considered, although $\sigma(z)$ is kept unchanged, as is the case in practice.

In Sect. 2, we show that, if a solution $A(z)$ of the infinite system exists, then, if N is large enough, the truncated system has a solution, $A_N(z)$, which approximates $A(z)$ as well as one wishes, uniformly inside any ellipse $E(z')$, $z'<z_1$, where z_1 is given in (1.5).

In Sect. 3, a different point of view is taken: the truncated system yields the exact solution for an infinite system, where the cross-section has been slightly changed. It is shown that the difference between this effective cross-section and the original one tends to zero in norm as the cut-off point goes to infinity. Thus the results of the computer search for ambiguities show in fact the instability of polynomial amplitudes under small changes of the cross-section.

This leads to the following situation: assume we are given the results of a computer search for ambiguities; in general, as is shown in [5], to each approximate cross-section σ_N (corresponding to the result A_N), there exists a neighbourhood U of possible cross-sections σ, such that the complete system of Eqs. (1.1)–(1.3) has a solution $A=A(\sigma, I_N)$ (with the same inelasticities, I_ℓ, up to the N^{th} partial wave, and zero for $\ell \geqq N+1$) close to A_N. Essentially, the result, A_N, produced by the computer, corresponds to a true continuum ambiguity if the neighbourhood U extends up to the true σ. In fact, one should allow also the inelasticities with $\ell \geqq N+1$ to be free. In Sect. 4, we establish sufficient conditions under which, to each finite result, there corresponds a true continuum ambiguity.

Finally, in Sect. 5, we extend these results to the case of spin $0-$spin $\frac{1}{2}$ scattering, in which both the polarization and the cross-section are given on the whole angular interval.

2. The Approximation of the Continuum Ambiguity

We rewrite the system (1.1)–(1.3) as the non-linear operator equation:

$$S(A, \sigma, I) = \sum_{\ell=0}^{\infty} (2\ell+1)\left[A_\ell - A_\ell^2 - D_\ell^2 - I_\ell\right] P_\ell(z) = 0\,, \tag{2.1}$$

with $D_\ell = D_\ell(\sigma, A)$ defined by (1.2). It is shown in ref. 5 that, if $\{I_\ell\}$ belongs to the space B_1 of sequences with the norm (1.5) (denoted by $\|\cdot\|_1$), and if $\sigma(z)$ is an element of B_0, the space of holomorphic functions in $E(z_0)$, continuous in $\bar{E}(z_0)$, with the norm $\|\sigma\|_0 = \sup_{\partial E(z_0)} |\sigma(z)|$, then S maps B_1 into itself. We call the ambiguity, $F(\sigma)$, associated with σ, the set of all solutions $\{A_\ell\}$ of (2.1) lying in B_1, obtained as $\{I_\ell\}$ sweeps over B_1, with $I_\ell \in [0, 1/4]$ and such that: i) the Fréchet derivative $S_A(A_1) \equiv (\partial S/\partial A)(A_1)$ has a bounded inverse; ii) the function $\bar{D}(z) = \{\sigma(z) - A^2(z)\}^{1/2}$ is holomorphic in $E(z_0)$; iii) the value $A(1)$ is fixed (since the forward amplitude can be unambiguously determined, thanks to the optical theorem); iv) we add the technical restriction that $\inf|\sigma(z) - A^2(z)|$ for $z \in \partial E(z_0)$ should be non-vanishing. In the following, we shall ignore conditions ii) and iii) in the definition of $F(\sigma)$: they may be incorporated by a finite-dimensional modification of the operator S in Eq. (2.1), as was shown in [1].

The approximate operator, $S_N(A_N, \sigma, I)$, mapping R^{N+1} into itself is defined as in (2.1), by restricting the summation to the first $N+1$ terms; $A_N(z)$ is defined by

(1.3), also restricted to the first $N+1$ terms. In this section, we prove the following theorem: let $A_0(z)$ belong to $F(\sigma)$. Then, for any $\varepsilon>0$, there exists an $N_0(\varepsilon, z', A_0)$, such that, for any $N>N_0$, i) the approximate equation

$$S_N(A_N, \sigma, I)=0 \tag{2.2}$$

has a solution $\bar{A}_N$, and ii) for any real $1<z'<z_1$,

$$\sup_{z\in\bar{E}(z')} |\bar{A}_N(z)-A_0(z)|<\varepsilon\,. \tag{2.3}$$

Consequently, any member of $F(\sigma)$ may be approximated as well as one wishes by solving the truncated system (2.2), for N large enough.

The proof uses many results of [5]. To describe it, we introduce first

$$\gamma_0(z)=\sum_{\ell=N+1}^{\infty}(2\ell+1)A_\ell^0 P_\ell(z)\,, \tag{2.4}$$

and define

$$S_{N,\gamma}(A_N, \sigma, I)=\sum_{\ell=0}^{N}(2\ell+1)\left[A_\ell^2+D_{\ell,\gamma}^2+I_\ell\right]P_\ell(z)\,, \tag{2.5}$$

which is a non-linear mapping from R^{N+1} to R^{N+1}, with

$$D_{\ell,\gamma}=\frac{1}{2\pi i}\oint_{\partial E(z_0)} dz\, Q_\ell(z)\{\sigma(z)-[A_N(z)+\gamma(z)]^2\}^{1/2}\,. \tag{2.6}$$

Clearly, if

$$A_0(z)=A_N^0(z)+\gamma_0(z)\,, \tag{2.7}$$

then

$$S_{N,\gamma_0}(A_N^0, \sigma, I)=0\,. \tag{2.8}$$

In fact, if $P_N x$ is the restriction of $x\in B_1$ to the subspace R^{N+1} spanned by its first $N+1$ components, then, by construction

$$P_N\partial S/\partial A(A_0)=\partial S_{N,\gamma_0}/\partial A_N(A_N^0)\,. \tag{2.9}$$

One should also notice that S_N is in fact $S_{N,\gamma}$ for $\gamma=0$.

The first step of the proof will be to show that $\partial S_{N,\gamma_0}/\partial A(A_N^0)$ has a bounded inverse if $\partial S/\partial A=S_A(A_0)$ has one, and N is large enough. We conclude then from the implicit function theorem that there exists a neighbourhood of γ_0, such that $S_{N,\gamma}(A_N, I)=0$ can be solved for $A_N=A_N(\gamma, I)$. In order to show that this neighbourhood contains $\gamma=0$, and to estimate the deviations of $\bar{A}_N=A_N(0, I)$ from A_0, we shall verify that the conditions of the more precise theorem of Kantorowich [6] are fulfilled, if N is sufficiently large. Application of the theorem will prove then our statement.

A sufficient condition for $\partial S_{N,\gamma_0}/\partial A(A_N^0)$ to have a bounded inverse is that a number $c>0$ exists, such that, for all $x\in P_N B_1\equiv R^{N+1}$,

$$\|\partial S_{N,\gamma_0}/\partial A(A_N^0)\,x\|_N\geqq c\,\|x\|_N\,, \tag{2.10}$$

where $\|\cdot\|_N$ is a norm on $P_N B_1$, defined by the restriction of (1.5) to the first $N+1$ terms. To verify (2.10), we write (as in [5]),

$$S_A(A_0) = 1 + K(A_0), \tag{2.11}$$

with $K(A_0)$ a compact operator [5] and conclude that, with $\|x\|_N = 1$,

$$\begin{aligned}\|\partial S_{N,\gamma_0}/\partial A(A_N^0)x\|_N &\geqq \inf_{\substack{x\in R^{N+1}\\ \|x\|_1\geqq 1}} \|S_A(A_0)x\|_1 - \sup_{\substack{x\in R^{N+1}\\ \|x\|_1\leqq 1}} \|R_N Kx\|_1 \\ &\geqq c_0 - f_N\,. \end{aligned}\tag{2.12}$$

In Eq. (2.12) we have used (2.9), the notation $R_N = 1 - P_N$, the fact that $\|x\|_N \geqq 1$ implies $\|x\|_1 \geqq 1$ [for the norm (1.5)] and that, by hypothesis, $S_A(A_0, I)$ has a bounded inverse, so that a $c_0 > 0$ exists such that (2.12) is true. It remains to show that $f_N \to 0$ as $N \to \infty$, using the explicit form of $K(A_0)$. This, however, follows from the estimate [5]

$$\begin{aligned}\sup_{\substack{x\in R^{N+1}\\ \|x\|_1=1}} \|R_N Kx\|_1 &= \sup_{\substack{x\in R^{N+1}\\ \|x\|_1=1}} \sup_{\ell>N} \left|\frac{1}{\pi}\oint_{\partial E(z_0)} dt \frac{Q_\ell(t)}{Q_\ell(z_1)}\left\{A_\ell - D_\ell \frac{A_0(t)}{[\sigma(t)-A_0^2(t)]^{1/2}}\right\}x(t)\right| \\ &\leqq \text{const} \sup_{l\geqq N+1} \frac{Q_\ell^2(z_0)}{Q_\ell(z_1)}\,. \end{aligned}\tag{2.13}$$

In Eq. (2.13) we have used the notation $x(t) = \sum_{\ell=0}^{N} (2l+1)x_\ell P_\ell(t)$, the majorizations $|D_\ell| < \text{const}\, Q_\ell(z_0)$ following from (1.2), $\inf_{t\in\partial E(z_0)} |\sigma(t) - A^2(t)|^{1/2} > n > 0$, following from condition iv) in the definition of $F(\sigma)$, as well as the inequality: $\sup_{\|x\|_N\leqq 1} \sup_{z\in\partial E(z_0)} |x(z)| < (z_1 - z_0)^{-1}$, following from (1.5) and the Heine formula. Since the last factor in (2.13) tends to zero for large N like $[z_1/(2z_0^2-1)]^N$, condition (2.10) is fulfilled for any $c = c_0 - \varepsilon$, $\varepsilon > 0$, as soon as $N > N_0(A_0, \varepsilon)$, for some N_0.

Now according to Kantorowich (Theorem 6, Chap. XVII, [6]), if i) the operator $S_N(A_N, I)$ has a continuous second derivative with respect to A_N in a certain domain $U_r: \|A_N - A_N^0\| < r$; ii) $(\partial S_N/\partial A_N)\,(A_N^0, I)$ has a bounded inverse; iii) the quantity

$$h \equiv \|[\partial S_N/\partial A_N(A_N^0)]^{-1}\|_N^2 \cdot \sup_{A_N\in U_r} \|\partial^2 S_N/\partial A_N^2\|_N \cdot \|S_N(A_N^0, I)\|_N \tag{2.14}$$

obeys $h < \frac{1}{2}$; iv) the quantity

$$r_0 \equiv \frac{1-(1-2h)^{1/2}}{h} \|[\partial S_N/\partial A_N(A_N^0)]^{-1}\|_N \cdot \|S_N(A_N^0, I)\|_N \tag{2.15}$$

obeys $r_0 < r$, then the equation $S_N(A_N) = 0$ has a solution in a neighbourhood $U_{r_0}: \|A_N - A_N^0\| < r_0$ of A_N^0, which may be constructed by the Newton-Kantorowich iteration (see [1, 6]).

We next sketch how one can verify that conditions i)–iii) are fulfilled if N is large enough and that, in fact, the bound r_0, Eq. (2.15), tends to zero as $N \to \infty$. To verify condition i), we notice first that, as a consequence of iv) in the definition of $F(\sigma)$, we may choose N so large that, for any $\varepsilon(=\eta/8$, say), $\inf_{t\in\partial E(z_0)} (\sigma(t)-(A_N^0)^2(t))>n-\varepsilon>0$. This follows from the fact that $\sup_{t\in\partial E(z_0)} |A_0(t)-A_N^0(t)|$ may be made as small as one wishes, by letting N increase, since $z_0<z_1$. Further, for all $A_N(t)$ obeying $\|A_N-A_N^0\|_N\leqq\varepsilon(z_1-z_0)/2\|A_N^0\|_0$, the Heine formula implies $\inf_{t\in\partial E(z_0)} |\sigma(t)-A_N^2(t)|>n-2\varepsilon$. With this, the majorizations needed to verify the boundedness of $\partial^2 S_N/\partial A_N^2$ in U_r, with $r=n(z_1-z_0)/4\|A_N^0\|_0$, are straightforward.

Condition ii) requires the existence of a bounded inverse of $\partial S_N/\partial A_N(A_N^0, I)$, provided $\|\gamma_0\|_0\equiv \sup_{z\in\partial E(z_0)} |\gamma_0(z)|$ is small enough. To do this, we show that the bilinear mapping $\partial^2 S_{N,\gamma}/\partial A_N\partial\gamma$ from $R^{N+1}\times B_0$ into R^{N+1} is bounded. Indeed, if the latter is the case, we may apply a generalized mean value theorem [7, p. 45] to conclude that:

$$\|[\partial S_{N,\gamma_0}/\partial A_N-\partial S_N/\partial A_N](A_N^0)\|_N\leqq \sup_{0<\lambda<1} \|\partial^2 S_{N,\lambda\gamma_0}/\partial A_N\partial\gamma(A_N^0)\|\cdot\|\gamma_0\|_0\,, \tag{2.16}$$

so that $\partial S_N/\partial A_N(A_N^0)$ may approach in norm arbitrarily closely to $\partial S_N, \gamma_0/\partial A_N(A_N^0)$, as $N\to\infty$. Then (2.16) implies:

$$\begin{aligned}&\|[\partial S_N/\partial A_N(A_N^0)]^{-1}\|_N\\ &\leqq\left\|\left\{1-\left[\frac{\partial S_{N,\gamma_0}}{\partial A_N}\right]^{-1}\left[\frac{\partial S_N}{\partial A_N}-\frac{\partial S_{N,\gamma_0}}{\partial A_N}\right](A_N^0)\right\}^{-1}\right\|_N\\ &\quad\cdot\|\{(\partial S_{N,\gamma_0}/\partial A_N)(A_N^0)\}^{-1}\|_N\leqq\frac{\text{const}}{1-\text{const}\,\|\gamma_0\|_0}<c_1\end{aligned} \tag{2.17}$$

for N large enough, which is condition ii). Thus we only have to verify the boundedness of $\|\partial^2 S_{N,\gamma}/\partial A_N\partial\gamma\|$ for small $\|\gamma\|_0$. As in condition i), this is ensured if $|\sigma(z)-[A_N^0(z)+\gamma(z)]^2|$ is non-vanishing on $\partial E(z_0)$, and this is in turn true if N is so large that $\sup_{z\in\partial E(z_0)} |A_N(z)-A_N^0(z)|<n/8\|A_0\|_0$ and $\|\gamma\|_0<n/8\|A_0\|_0$. This disposes of condition ii).

Using again the mean value theorem of (2.16), together with Eq. (2.8), we see that:

$$\|S_N(A_N^0, I)\|_N\leqq \sup_{0<\lambda<1} \|\partial S_N/\partial\gamma(A_0, \lambda\gamma_0)\|_N\cdot\|\gamma_0\|_0\leqq c_1\|\gamma_0\|_0 \tag{2.18}$$

if γ_0 is restricted to a neighbourhood of the origin of B_0, where the derivative $\partial S_N/\partial\gamma$ may be uniformly bounded. This is in turn possible if, as above, $\|\gamma\|_0$ is chosen so that $\inf_{z\in\partial E(z_0)} |\sigma(z)-(A_N^0+\gamma)^2|>n_1>0$, for some n_1.

From (2.18), (2.17) and the boundedness of $\partial^2 S_N/\partial A_N^2$ in U_r, we see that the quantity h of Eq. (2.14), in condition iii), may be made to obey the bound $h<\frac{1}{2}$ if we only choose $\|\gamma_0\|_0$ small enough, i.e. we let N increase correspondingly. In fact, we can make h as small as we wish, and thus also r_0 of Eq. (2.15), by allowing N to be large enough.

We conclude that all the conditions of the Kantorowich theorem are fulfilled. Consequently, the system $S_N(A_N, I)=0$ has a solution if N is large enough and its solution departs from A_N^0 in norm $\|\cdot\|_N$ by a quantity which can be as small as one wishes if N is increased correspondingly. This is equivalent to the statement of our theorem, since, for any $z'<z_1$,

$$\sup_{z\in\bar{E}(z')}|\bar{A}_N(z)-A_0(z)|\leqq \sup_{z\in\bar{E}(z')}\{|\bar{A}_N(z)-A_N^0(z)|+|A_N^0(z)-A_0(z)|\}$$
$$\leqq r_N/(z_1-z')+\sup_{z\in\bar{E}(\bar{z})}|A_N^0(z)-A_0(z)|, \tag{2.19}$$

and both quantities vanish as $N\to\infty$.

Thus, given any precision ε, we may find $N(\varepsilon, z', A_0)$, so that $A_0(I)$ is approximated better than ε by the solution of $S_N(A_N, I)=0$ [in the sense of (2.3)]. In fact, given A_0, there exists a neighbourhood $U(A_0)$ of it in B_1, in which the approximation will be uniform if the system (1.1)–(1.3) is truncated after $N(\varepsilon, z', A_0)$ waves. By extracting a finite cover from the infinite covering with $U(A_0)$ of a certain compact set K of amplitudes in $F(\sigma)$, we may obtain an order of truncation $N(\varepsilon, z', K)$ for which approximation within ε occurs uniformly over K. Of course, since $F(\sigma)$ is unknown, we do not know a priori how good our approximation is.

3. Approximation of the Cross-Section

Another consequence of the previous section is that the dispersive parts of the approximating amplitudes:

$$D_N(z)=\sum_{\ell=0}^{N}(2\ell+1)D_{\ell,N}P_\ell(z)$$
$$=\sum_{\ell=0}^{N}(2\ell+1)\frac{P_\ell(z)}{2\pi i}\oint_{\partial E(z_0)}dy[\sigma(y)-A_N^2(y)]^{1/2}Q_\ell(y) \tag{3.1}$$

tend to the true dispersive part uniformly in any ellipse, strictly contained in $E(z_0)$. Indeed, using the truncated Heine formula [8, p. 179], one may write

$$D_N(z)=\{\sigma(z)-A_N^2(z)\}^{1/2}-(N+1)\{P_{N+1}(z)R_N(z)-P_N(z)R_{N+1}(z)\}, \tag{3.2}$$

with

$$R_N(z)=\frac{1}{2\pi i}\oint_{\partial E(z_0)}\frac{dy}{y-z}\{\sigma(y)-A_N^2(y)\}^{1/2}Q_N(y). \tag{3.3}$$

Now, using the theorem of the previous section, given a sufficiently small ε, we may find N_0, such that, for all $N>N_0$,

$$\sup_{z\in\partial E(z_0)}|A_N(z)-A_0(z)|<\varepsilon \tag{3.4}$$

and

$$\inf_{z\in\partial E(z_0)}|\sigma(z)-A_N^2(z)|>n_1>0 \tag{3.5}$$

for some n_1. Consequently, for $N > N_0$ and z strictly inside $E(z_0)$,

$$R_N(z) < k(z) Q_N(z_0) . \tag{3.6}$$

Then, for such N's,

$$\begin{aligned} |D_N(z) - D_0(z)| &< |\{\sigma(z) - A_N^2(z)\}^{1/2} - \{\sigma(z) - A_0^2(z)\}^{1/2}| + k Q_N(z_0) P_{N+1}(z) \\ &\leqq \varepsilon \|A_0\|_0 / n_1 + k Q_N(z_0)/Q_N(z) , \end{aligned} \tag{3.7}$$

which proves our statement. It follows easily that the differential cross-sections $\sigma_N(z)$ obtained from the truncated solutions $A_N(z)$ tend to the true differential cross-section, uniformly in any ellipse strictly contained in $E(z_0)$.

This is not surprising; however, it is not necessary to assume the whole strength of the theorem of Sect. 2, in order to achieve convergence of the differential cross-sections. Indeed, assume we have a sequence of truncated equations $S_N(A_N, I_N) = 0$, which have solutions, $A_N(z)$, such that there exists an A_0 of $F(\sigma)$, with the property that, uniformly in N,

$$\sup_{z \in \partial E(z_0)} |A_N(z) - A_0(z)| < R \tag{3.8}$$

for some R. Then it is still true that $\sigma_N(z)$ tends to $\sigma(z)$ uniformly inside any ellipse $E(z_2)$, $z_2 < z_0$. Indeed, the estimate (3.6) remains true and therefore

$$|D_N(z) - \{\sigma(z) - A_N^2(z)\}^{1/2}| < k(N+1) P_{N+1}(z) Q_N(z_0) . \tag{3.9}$$

Consequently,

$$\begin{aligned} |\sigma_N(z) - \sigma(z)| &\equiv |D_N^2(z) + A_N^2(z) - \sigma(z)| \\ &\leqq |D_N(z) + \{\sigma(z) - A_N^2(z)\}^{1/2} | k(N+1) P_{N+1}(z) Q_N(z_0) . \end{aligned} \tag{3.10}$$

Using (3.8) and (3.9), it follows that a constant k' exists, independent of N, so that, for $z \in E(z_2)$,

$$|\sigma_N(z) - \sigma(z)| \leqq k' \left[\frac{z + (z^2 - 1)^{1/2}}{z_0 + (z_0^1 - 1)^{1/2}} \right]^N , \tag{3.11}$$

and the upper bound tends to zero, as $N \to \infty$, for $z \leqq z_2 < z_0$.

This result is interesting insofar as we did not use anywhere in its derivation the fact that the inelasticities I_ℓ, $\ell = 0, \ldots, N$ of the truncated system reproduce the inelasticities of the exact infinite system. Therefore, $A_N(z)$ may simply tend, as $N \to \infty$, to a member of $F(\sigma)$ different from $A_0(z)$. This leaves one, however, with the problem of deciding whether or not this is really so. More, precisely, given a solution $A_N(z)$ corresponding to the truncated system and to a set of inelasticities I_k, $k = 0, \ldots, N$, the problem is whether there exists a solution $A_0(z)$ of the infinite system, corresponding to the $N+1$ inelasticities supplemented by some set $\bar{I}' = \{I_\ell\}_{\ell = N+1}^{\infty}$ and to the same $\sigma(z)$. If so, then $A_N(z)$ may be interpreted as an approximant of $A_0(z)$; if not, it is simply an instability generated in the search for continuum ambiguities.

From a practical point of view, this problem is of no relevance, since if we simply verify that the output of a certain run leads to a differential cross-section that departs from $\sigma(z)$ in the physical region [which is much weaker than (3.11)] by less than a number ε, smaller than the experimental errors, it is an admissible amplitude. Nevertheless, in the next section, we shall establish sufficient criteria for a positive answer to our question.

4. The Information Contained in the Truncated System

In this section we shall assume N is fixed, a solution $A_N(z)$ of the truncated system is available and the inverse of the N dimensional Fréchet derivative exists. The first $N+1$ inelasticities $I_0, \ldots, I_N$ – the set $\tilde{I}_N$ – are fixed. Further, $\inf_{z \in \partial E(z_0)} |\sigma(z) - A_N^2(z)| > n_1 > 0$. The question we ask is: does there exist a set $\tilde{I}'_N$ of inelasticities $I'_N \in B_1$, $0 < I_\ell < \frac{1}{4}$, so that the infinite dimensional system

$$S(A, \tilde{I}_N, \tilde{I}'_N) = 0 \tag{4.1}$$

has a solution $A(z)$, $F(\sigma)$, with $F(\sigma)$ defined in Sect. 2?

We shall derive sufficient conditions for a positive answer from Kantorowich's theorem, already used in Sect. 2. We have thus the task to i) estimate $\|S(A_N, \tilde{I}_N, \tilde{I}'_N)\|_1$, and ii) find a bound on $\|(S_A(A_N))^{-1}\|_1$, where S is the operator of (2.1).

i) An estimate of $\|S(A_N, \tilde{I}_N, \tilde{I}'_N)\|_1$ is easy to obtain:

$$S(A_N, \tilde{I}_N, \tilde{I}'_N) = - \sum_{\ell = N+1}^{\infty} D_\ell^2(A_N) P_\ell(z)(2\ell + 1) - \tilde{I}'_N(z), \tag{4.2}$$

so that

$$\|S(A_N, \tilde{I}_N, \tilde{I}'_N)\|_1 \leqq \sup_{\ell \geqq N+1} \frac{Q_\ell^2(z_0)}{Q_\ell(z_1)} B_N + \|\tilde{I}'_N\|_1 = C_N + \|\tilde{I}'_N\|_1, \tag{4.3}$$

where the constants B_N, C_N depend on $\sigma(z)$ and $A_N(z)$.

ii) To estimate $\|S_A(A_N)^{-1}\|_1$ we compute, as in Sect. 2, a lower bound on $\|S_A(A_N)x\|_1$ for $\|x\|_1 \geqq 1$, using the lower bound

$$c_{0,N} \equiv \inf_{\|x\|_N = 1} \|\partial S_N / \partial A_N(A_N)x\|_N \tag{4.4}$$

which can in principle be evaluated numerically. To this end, we consider a number $k > 1$ and write

$$\begin{aligned} &\inf_{\|x\|_1 = 1} \|S_A(A_N)x\|_1 \\ &= \min \left\{ \inf_{\substack{\|P_N x\|_1 \geqq \frac{1}{k} \\ \|x\|_1 = 1}} \|S_A(A_N)x\|_1, \quad \inf_{\substack{\|P_N x\|_1 < \frac{1}{k} \\ \|x\|_1 = 1}} \|S_A(A_N)x\|_1 \right\} \equiv \min\{a, b\}. \end{aligned} \tag{4.5}$$

With the definition of Sect. 2, we write

$$a \geqq \inf_{\substack{\|P_N x\|_1 \geqq \frac{1}{k} \\ \|x\|_1 = 1}} \|x - K(A_N) P_N x\|_1 - \sup_{\|x\|_1 = 1} \|K(A_N) R_N x\|_1 \geqq \frac{c_{0,N}}{k} - d_N, \tag{4.6}$$

where we have used the fact that $\|x\|_1 \geqq \|x\|_N$. The quantity d_N may be estimated as in Sect. 2; it goes to zero as $N \to \infty$, and is thus expected to be a small number for the N used in calculations. On the other hand,

$$
\begin{aligned}
b \geqq & \inf_{\substack{\|P_N x\|_1 < \frac{1}{k} \\ \|x\|_1 = 1}} \|R_N x\|_1 - \sup_{\substack{\|P_N x\|_1 < \frac{1}{k} \\ \|x\|_1 = 1}} \|K R_N x\|_1 \\
& - \sup_{\substack{\|P_N x\|_1 < \frac{1}{k} \\ \|x\|_1 = 1}} \|P_N x - P_N K P_N x\|_1 - \sup_{\substack{\|P_N x\|_1 < \frac{1}{k} \\ \|x\|_1 = 1}} \|R_N K P_N x\|_1 \\
& = 1 - \frac{1}{k} - d_N - \frac{1}{k} \|1 - K(A_N)\|_N - \frac{1}{k} f_N \,,
\end{aligned}
\tag{4.7}
$$

where f_N is used also in Eq. (2.12). Letting $e_N \equiv \|1 - K(A_N)\|_N$, we conclude that

$$
\|S_A(A_N)^{-1}\|_1 < \frac{1}{\alpha_0}
\tag{4.8}
$$

with

$$
\alpha_0 = \min\left\{\frac{1}{k} c_{0,N} - d_N,\, 1 - \frac{1 + e_N + f_N}{k} - d_N\right\}
\tag{4.9}
$$

provided $\alpha_0 > 0$. In (4.9), d_N, f_N are expected to be small numbers, whereas e_N stays finite for N large. We can choose k so that α_0 is as large as possible. It is easy to see that this is achieved for $k = 1 + e_N + f_N + c_{0,N}$. The condition $\alpha_0 > 0$ is fulfilled if $c_{0,N} > d_N(1 + e_N + f_N + c_{0,N})$, which is true if d_N is small.

iii) By considerations strictly similar to those of Sect. 2, we may find a certain neighbourhood U_1 of A_N in B_1,

$$
\|A - A_N\|_1 < \Delta(n_1)\,,
\tag{4.10}
$$

such that, for $A \in U_1$, $\|\partial^2 S/\partial A^2(A_N)\|_1 < C_1$, for some C_1; $\Delta(n_1)$ can be estimated knowing n_1, determined by A_N.

We can now verify Kantorowich's conditions, which ensure that in the neighbourhood U_1, Eq. (4.10), there exists one solution of the infinite system. Comparing with (2.14), we see that they are

a) $$h < C_1(C_N + \|\tilde{I}'_N\|_1)/\alpha_0^2 < \tfrac{1}{2}\,, \tag{4.11}$$

b) $$\alpha_0 - \{\alpha_0^2 - 2C_1(C_N + \|\tilde{I}'_N\|_1)\}^{1/2} < \Delta(n_1) C_1\,. \tag{4.12}$$

According to Kantorowich, if we add the condition

c) $$\alpha_0 + \{\alpha_0^2 - 2C_1(C_N + \|\tilde{I}'_N\|_1)\}^{1/2} > \Delta(n_1) C_1\,, \tag{4.13}$$

then the ball U_1 contains a unique solution of the infinite system (4.1).

Now conditions (4.11) and (4.12) are likely to be satisfied if C_N, $\|\tilde{I}'_N\|$ are small. We see that, if (4.11) and (4.12) are fulfilled, then to each amplitude A_N obtained from the truncated system, there corresponds a whole set of amplitudes for different choices of $\tilde{I}'_N$, subject to (4.11) and (4.12).

Therefore we may state that the points of an ambiguity patch obtained from a numerical calculation may be separated into two classes: those for which conditions (4.11)–(4.13) are fulfilled and those for which they are not. We know that points of the first class correspond to true continuum ambiguities; for points of the second class, we may conservatively only state that they represent instabilities of the amplitude under perturbations of the cross-section. However, this refers only to the criterion (2.11)–(2.13), based on the Kantorowich theorem, and we certainly cannot exclude that other methods could nevertheless show that they also correspond to true continuum ambiguities.

We now indicate the modifications needed to include the analyticity of $D(z)$ in $E(z_0)$ and the constraint of the optical theorem. As shown in ref. 1, this is done by replacing the operator $S: B_1 \to B_1$ of Eq. (2.1) by an operator $\tilde{S}$ defined on $B_1 \times R^k \times R^{k+1}$, where the components of the first R^k are the coordinates of the k zeros of $D(z)$ inside $E(z_0)$ and those of R^{k+1} are $k+1$ inelasticities, denoted collectively by I_0 and taken to be independent variables. The operator $\tilde{S}$ has values in $B_1 \times R^k \times R^k \times R$ given by the following components:

$$\tilde{S}(A, I_0, z_i, I) = \begin{cases} S(A, I_0, I) \\ \sigma(z_i) - A^2(z_i)\,, \quad i = 1, 2, \ldots, k \\ d/dz[\sigma(z) - A^2(z)]\,(z_i)\,, \quad i = 1, 2, \ldots, k \\ \sum\limits_{\ell=0}^{\infty} (2\ell+1)A_\ell - A(1)\,. \end{cases} \tag{4.14}$$

A member of $F(\sigma)$ defined by conditions i)–iv) of Sect. 2 is the component in B_1 of a solution of $\tilde{S}(A, I_0, z_i, I) = 0$, given an element $I = \{I_{k+1}, I_{k+2}, \ldots\}$ of $R_k B_1$, for which the Fréchet derivative of $\tilde{S}$ has a bounded inverse. The Fréchet derivative of $\tilde{S}$ may again be written as an invertible operator plus a compact one and all the proofs of Sects. 2 to 4 may be carried through with sufficient, but not profound, care and so we shall skip them.

5. Spin 0 – Spin $\frac{1}{2}$ Scattering

In this section we sketch the generalization of the above results to the case that one of the particles has spin $\frac{1}{2}$ (e.g., pion-nucleon scattering). In the Barrelet formalism [9], the mapping $S(A, I)$, Eq. (2.1), is replaced by

$$S(A, I) = A(\zeta) - \sum_{\ell=-\infty}^{\infty} (j+\tfrac{1}{2})\mathbb{P}_\ell(\zeta)\,[A_\ell^2 + D_\ell^2 + I_\ell]\,, \tag{5.1}$$

$$D_\ell = \frac{1}{8\pi i} \oint_{\partial\Omega(\zeta_0)} d\zeta \left(1 - \frac{1}{\zeta^2}\right) \mathbb{Q}_\ell(\zeta)\hat{D}(\zeta) \tag{5.2}$$

$$\hat{D}(\zeta) = \tfrac{1}{2}[-\varrho(\zeta) + \Xi^{1/2}(\zeta)]/A(\zeta^{-1})\,, \tag{5.3}$$

$$\Xi(\zeta) = 4A(\zeta)A(\zeta^{-1})\,\{\sigma(\zeta) - A(\zeta)A(\zeta^{-1})\} + \varrho^2(\zeta)\,, \tag{5.4}$$

$$A(\zeta) = \sum_{\ell=-\infty}^{\infty} (j+\tfrac{1}{2})\mathbb{P}_\ell(\zeta)A_\ell(\zeta)\,, \tag{5.5}$$

with

$$\zeta = z + (z^2 - 1)^{1/2}, \tag{5.6}$$

and $j = \pm(\ell + \frac{1}{2})$ according to whether $\ell \geqq 0$ or $\ell < 0$, $\Omega(\zeta_0)$ is the annular image in the ζ plane, Eq. (5.6) of the small Lehmann ellipse, $\mathbb{P}_\ell$ and $\mathbb{Q}_\ell$ are the Barrelet generalizations of the Legendre functions of the first and second kind [see ref. 2]. The physical region is $|\zeta| = 1$, on which $\sigma(\zeta)$ and $-i\varrho(\zeta)/\sigma(\zeta)$ reduce to the differential cross-section and to $(+/-)$ the polarization. It is profitable to introduce $\tau(\zeta) \equiv \sigma(\zeta) + i\varrho(\zeta)$, which for $|\zeta| = 1$ obeys

$$\tau(\zeta = e^{i\theta}) = |D(\zeta) + iA(\zeta)|^2. \tag{5.7}$$

We look for solutions of (5.1)–(5.5) in the space B'_1 of functions $A(\zeta)$ holomorphic and real analytic in the annulus $\Omega(\zeta_1)$ – the image under (5.6) of the large Lehmann ellipse; $\zeta_1 < \zeta_0^2$ – with the norm $\sup_\ell |A_\ell|/\mathbb{Q}_\ell(\zeta_1)$, as the sequence $I_\ell \in [0, \frac{1}{4}]$ varies in B'_1. The set of solutions $F(\tau)$ obtained in this way is the continuum ambiguity. As in Sect. 2, we add to this definition the conditions: i) the Fréchet derivative of S has a bounded inverse at points of $F(\tau)$; ii) $D(\zeta)$ is holomorphic in $\Omega(\zeta_0)$; iii) $A(1)$ is given; iv) for simplicity we assume that, for $\zeta \in \partial\Omega(\zeta_0)$, $\Xi(\zeta) \neq 0$, $A(\zeta) \neq 0$.

If condition iv) is fulfilled, then one may show that the first and second Fréchet derivatives of S [Eqs. (5.1)–(5.5)] exist at points of $F(\tau)$ as bounded operators from B'_1 (or $B'_1 \times B'_1$) into B'_1. The Theorem of Sect. 2 may then be formulated for this situation merely by replacing $E(z')$ by $\Omega(\zeta')$. Its proof follows certainly the same lines as in Sect. 2, with the use of the inequality,

$$|D_\ell| < C\mathbb{Q}_\ell(\zeta_0), \tag{5.8}$$

the generalization of the Heine formula,

$$\sum_{\ell=-\infty}^{\infty} (j + \tfrac{1}{2})\mathbb{P}_\ell(\xi)\mathbb{Q}_\ell(\zeta) = \frac{4\zeta^2}{\zeta^2 - 1}\frac{1}{\zeta - \xi}, \tag{5.9}$$

as well as the fact that for $\ell \to \infty$,

$$\mathbb{Q}_\ell^2(\zeta_0)/\mathbb{Q}_\ell(\zeta_1) \to 0. \tag{5.10}$$

These formulae are already listed or may be obtained from the properties of $\mathbb{P}_\ell$, $\mathbb{Q}_\ell$ derived in Appendix B of ref. 2.

A direct estimate of how close the differential cross-sections generated by the truncated system are to the true ones may be obtained by comparing the analytic continuation into $\Omega(\zeta_0)$ of Eq. (5.7), i.e.,

$$[D_N(\zeta) + iA_N(\zeta)]\,[D_N(\zeta^{-1}) - iA_N(\zeta^{-1})] \equiv \tau_N(\zeta), \tag{5.11}$$

with

$$[\hat{D}_N(\zeta) + iA_N(\zeta)]\,[\hat{D}_N(\zeta^{-1}) - iA_N(\zeta^{-1})] = \tau(\zeta), \tag{5.12}$$

where $\hat{D}_N(\zeta)$, the solution of Eq. (5.12), differs from $D_N(\zeta)$ of (5.11) by the addition of an infinite tail of partial waves:

$$\hat{D}_N(\zeta)-D_N(\zeta)=\frac{1}{8\pi i}\oint_{\partial\Omega(\zeta_0)} d\eta(1-\eta^{-2})D_N(\eta)\sum_{|\ell|>N}\mathbb{P}_\ell(\zeta)\mathbb{Q}_\ell(\eta)(j+\tfrac{1}{2}). \quad (5.13)$$

We show in the Appendix that if $|\zeta|<|\eta|=\zeta_0$, then

$$\sum_{|\ell|>N}\mathbb{P}_\ell(\zeta)\mathbb{Q}_\ell(\eta)(j+\tfrac{1}{2})<\text{const}\,|\zeta/\zeta_0|^N, \quad (5.14)$$

which implies that $\tau_N(\zeta)$ and $\tau(\zeta)$ differ by an amount of the same order.

The developments of Sect. 4 may also be repeated for this case where the definition of $F(\tau)$ differs somewhat from the scalar case. Apart from requiring that the zeros of $\Xi(\zeta)$ stay double, one may also be obliged to allow for the exceptional circumstance that for some ζ in $\Omega(\zeta_0)$, simultaneously a) $\varrho(\zeta)=0$, $A(\zeta)=0$ and $A(\zeta^{-1})=0$ or b) $\sigma(\zeta)=0$, $\varrho(\zeta)=0$ and $A(\zeta^{-1})=0$. This is handled by modifying the operator S of Eqs. (5.1)–(5.3) as shown in detail in [2].

Appendix. Evaluation of a Sum

We derive a formula for

$$H_L(\xi,\zeta)=\sum_{\ell=-L}^{L}(j+\tfrac{1}{2})\mathbb{P}_\ell(\xi)\mathbb{Q}_\ell(\zeta). \quad (A.1)$$

Using the definition of $\mathbb{P}_\ell(\zeta)$, $\mathbb{Q}_\ell(\zeta)$ in terms of Legendre functions [2], together with

$$\xi=x+(x^2-1)^{1/2}, \quad (A.2)$$

$$\zeta=z+(z^2-1)^{1/2}, \quad (A.3)$$

one verifies readily that

$$H_L(\xi,\zeta)=(z^2-1)^{-1/2}+\sum_{\ell=0}^{L}(2\ell+1)P_\ell(x)Q_\ell(z)-(x^2-1)^{1/2}(z^2-1)^{1/2}\cdot\sum_{\ell=1}^{L}\frac{2\ell+1}{\ell(\ell+1)}P'_\ell(x)Q'_\ell(z). \quad (A.4)$$

The first sum is the truncated Heine formula, which may be either looked up in ref. 8, or evaluated by multiplying the sum by $(x-z)$ and using the recurrence relations for $xP_\ell(x)$ or $zQ_\ell(z)$. One obtains [see also Eq. (3.2)]

$$\sum_{\ell=0}^{L}(2\ell+1)P_\ell(x)Q_\ell(z)=\frac{1}{z-x}-\frac{L+1}{z-x}\{Q_L(z)P_{L+1}(x)-Q_{L+1}(z)P_L(x)\}. \quad (A.5)$$

For the second sum, multiplication by $x-z$ and use of the recurrence relations for $xP_\ell^1(x)$ [or $zQ_\ell^1(z)$] leads to

$$\sum_{\ell=1}^{L} \frac{2\ell+1}{\ell(\ell+1)} P_\ell'(x) Q_\ell'(z) = \frac{1}{(x-z)(z^2-1)^{1/2}} + \frac{1}{(L+1)(x-z)} \{Q_L'(z) P_{L+1}(x) - Q_{L+1}'(z) P_L(x)\} . \quad \text{(A.6)}$$

Addition of (A.5) and (A.6) leads in the limit $L \to \infty$ to the Heine-type formula for $\mathbb{P}_\ell$, $\mathbb{Q}_\ell$ [Eq. (A.3) of ref. 2]. The rest of the series, cut off at a finite L, may be obtained immediately from (A.5) and (A.6). This leads to the majorization used in Sect. 5.

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