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Large Time Behaviour of Some N-Body Systems*

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Abstract. In this work we prove completeness for N-body systems that evolve asymptotically into either N free particles or a two cluster system with one of the clusters being a single particle. For the three body case our results imply completeness for a very general system with potentials decaying like $|x|^{-1-\varepsilon}$ at ∞.

Introduction

Completeness in many body scattering was first prove by Faddeev for three particles using a time-independent method in [1] and was followed up by many authors for the same case. All these methods are limited in that they make assumptions on the spectral properties of the subsystems. For an excellent review of these results see Ginibre [2]. On the other hand for the N-body short range systems Lavine [3] proved a completeness result when the potentials are repulsive, Iorio-O'Carroll [4] proved it when the potentials are weakly coupled and Sigal ([5] and the references given there) proved completeness for a class of generic short range potentials.

More recently Enss has continued his work on time-dependent scattering theory that he pioneered for the two body case to three body systems. In [6] he treated two cluster scattering of N-body systems. While in [7] he gave a rough sketch of the proof of three-body completeness, in [8] he made the proof clearer. See [8a] for a complete proof of the three-body long and short range cases. In [9] he has a slightly different approach to the proof and also some discussion of the general case. Almost simultaneously with [7, 8], Sinha et. al. [10] proved three-body completeness for pair potentials with $(2+\varepsilon)$ decay at ∞. They also incorporated some ideas of Enss in their work. See the work of Kitada for a different approach [11].

Mourre on the other hand has used his work on the spectral theory of many body operators [12] to determine some propagation properties of the N-body total evolution in some weighted spaces. He obtained L^2 estimates [13] in certain

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regions of the phase space for the three-body operator and obtained completeness in [14].

In [15] the work of Hagedorn and Perry [16] was used to prove completeness for a class of four-body systems, while Hagedorn and Perry themselves extended their work to the four-body case in [17], using in part the work of Hagedorn [18].

In an unpublished preprint [19], Mourre and Sigal combined some ideas from Mourre [13] with the phase-space analysis, refer to Deift and Simon [20] for a version of this, and proved completeness for a special class of N-body systems that they call admissible. It is one in which all the $(N-2)$ or lower-body subsystems do not have any negative eigenvalues. We prove in the present work a result similar to that of the Mourre–Sigal result but use the method of time-dependent scattering theory for the proof. We observe that for the three-body case this result means completeness for a wide class of short range potentials without any assumptions on the threshold eigenvalues and including a possibly infinite number of bound states of negative or zero energy for the subsystems.

Finally we note that a good review of the work done in completeness and related results for the N-body case can be found in the works of Reed and Simon [21], Sigal [5] and Hunziker and Sigal [22].

The model we consider for N-body scattering is the following. Corresponding to the partitions $D=\{D_1, D_2, \ldots, D_k\}$ of $\{1, \ldots, N\}$ into clusters, we have the associated cluster Hamiltonians

$$H(D) = H_0 + \sum_{\gamma \in i(D)} W_\gamma \equiv H_0 + W^D, \quad 1 \leqq \# D \leqq (N-1),$$

where

$$i(D) = \{\gamma = (ij) : i < j, i, j \in D_l, \quad \text{for some } 1 \leqq l \leqq k\},$$

and H_0 is the usual free Hamiltonian [23, 24] on the relative Hilbert space $L^2(\mathbb{R}^{(N-1)\nu})$, $\nu \geqq 3$. We scale the pair positions and momenta respectively by $x^\gamma \to (\mu_\gamma)^{1/2} x^\gamma$, $p^\gamma \to (\mu_\gamma)^{-1/2} p^\gamma$, where μ_γ is the reduced mass of the pair γ. Then we obtain H_0 in normal form, that is, $H_0 = -\frac{1}{2}\Delta$ in terms of the Laplacian Δ on $L^2(\mathbb{R}^{(N-1)\nu})$. As in [5] we write $L^2(\mathbb{R}^{(N-1)\nu})$ as $L^2(X^D) \otimes L^2(X_D)$, corresponding to the internal and external spaces of the clustering D. Then the cluster Hamiltonian splits into

$$H(D) = H^D + T_D, \quad H^D = T^D + W^D,$$

where $-2T^D$ and $-2T_D$ are the Laplacians on $L^2(X^D)$ and $L^2(X_D)$ respectively. We denote any x in $\mathbb{R}^{(N-1)\nu}$ by (x^D, x_D), corresponding to a clustering D, and x_D would further split as $x_D = (y_1, \ldots, y_{k-1})$ if D has k clusters. We denote by V_t, U_t, $V_t(D)$ and V_t^D the unitary groups generated by H, H_0, $H(D)$ and H^D respectively and the total Hamiltonian H is defind by $H = H(C)$ for $\# C = 1$. The cluster wave operators are given by

$$\Omega^{\pm}(D) = \operatorname*{slim}_{t \to \pm\infty} V_t^* V_t(D) E^D, \quad 1 < \# D \leqq N,$$

with E^D denoting the spectral projection corresponding to $\sigma_P(H^D)$ whenever $\# D \neq N$

and denoting the identity when $\#D = N$. The existence and mutual orthogonality of the ranges of the wave operators is well known [21, 23] when the pair potentials satisfy

(A1) $$W_\gamma(T^\gamma + 1)^{-1} \text{ is compact on } L^2(\mathbb{R}^\nu),$$

(A2) $$\rho_1(x^\gamma)^{-1} W_\gamma(T^\gamma + 1)^{-1} \text{ is bounded for}$$
$$\rho_1(x^\gamma) = (1 + (x^\gamma)^2)^{-\delta_1}, \quad \text{for some } \delta_1 > \tfrac{1}{2}.$$

We make two additional assumptions on the N-body system with one of them on the potentials, namely,

(A3) $$(T^\gamma + 1)^{-1}\{x^\gamma \nabla_\gamma W_\gamma(x^\gamma)\}(T^\gamma + 1)^{-1} \text{ is compact on } L^2(\mathbb{R}^\nu)$$

and

(A4) $$\sigma(H^D) = [0, \infty) \text{ for every clustering } D \text{ with } 3 \leqq \#D \leqq N.$$

We note that the assumption (A4) means that only a class of $(N-1)$-body subsystems are allowed to have negative eigenvalues. With these assumptions on the potentials our main theorem is the following.

Theorem I. *Let H be an N-body Hamiltonian with the pair potentials satisfying the conditions* (A1)–(A3) *and let the system satisfy the condition* (A4). *Then the scattering is asymptotically complete, that is,*

$$\bigoplus_{D:\#D \geqq 2} \operatorname{Range} \Omega^+(D) = \mathscr{H}_c(H) = \bigoplus_{D:\#D \geqq 2} \operatorname{Range} \Omega^-(D).$$

Remark. We observe that for more than three particles the condition (A4) might not be possible if we insist on the $(1+\varepsilon)$ decay for non-positive potentials at ∞ (Theorem X111.6, [24]). Nevertheless we retain Condition (A2) and are content with this remark.

1. Reduction of N-body Completeness

In this section we reduce N-body completeness to the verification of two conditions. Namely, the local decay and the lower energy decay of the total evolution of the scattering states (formulated as the conditions (LD) and (LED) later in the section). We make use of the theory of evolution of observables developed by Sinha and Muthuramalingam [25] and Enss [26]. We denote by $F(\cdot)$ the spectral projections and by K all absolute constants that occur in estimates. We assume throughout this section that the potentials satisfy the conditions (A1)–(A3).

To facilitate extracting the decay from the potentials, when they are locally singular, and to use it for estimates, we have a technical lemma whose proof is easy and is given in [15]. So we state it without proof. We set $\rho(\lambda) = (1 + \lambda^2)^{-\delta}$ for any $\delta > 0$. We define the generator A of dilations as in [10, 15], namely $A = \frac{1}{4}(x \cdot P + P \cdot x)$.

Proposition 1.1. *Let $\phi \in C_0^\infty(\mathbb{R})$ and let S be any of $\{H(D): 1 \leqq \#D \leqq N\}$, $2 \leqq N < \infty$ and $z \notin \sigma(S)$.*

(i) If $\psi(x)$ *is an S-bounded multiplication operator and* $\rho(x)^{-1}\psi(x)$ *is also S-bounded, then* $\psi(x)\,(S-z)^{-n}\rho(x)^{-1}$ *and* $\psi(x)\phi(S)\rho(x)^{-1}$ *are bounded operators for every positive integer n.*

(ii) *For any pairs* α, γ, $\rho(x^{\alpha})^{-1}\phi(T^{\gamma})\rho(x^{\alpha})$ *is bounded.*

(iii) $A^{\theta}\phi(S)(1+x^2)^{-\theta/2}$ *and* $A^{\theta}(S-z)^{-1}\,(1+x^2)^{-\theta/2}$ *are bounded for* $0 \leqq \theta \leqq 2$.

For any N-body Hamiltonian H, we set $\Sigma = \inf \sigma_{\mathrm{ess}}(H)$, $\sigma_{\mathrm{ess}}(H)$ being the essential spectrum (see [23, 24] for a definition) of H. Then we have the following decay results.

Proposition 1.2. *Consider* $\phi \in C_0^{\infty}(\mathbb{R})$ *with positive constants a, b and c. Then for every* $2 \leqq N < \infty$ *we have the following.*

(i) *Let* $\inf \operatorname{supp} \phi \geqq \frac{1}{2}b^2$. *If* $a + c < b$ *and* $|s| \leqq |t|$, *for arbitrary positive integer M,*

$$\|F(|x| \leqq a|t|)U_t\phi(H_0)F(|x| \leqq c|t|)\| \leqq K_M(1+|t|)^{-M}.$$

(ii) *There exists a positive constant* Λ *(depending upon N and the masses of the particles) such that if* $a > \Lambda^{-1}(b + c + \sqrt{-\Sigma})$, *then for all s, t with* $|s| + |t| \leqq |u|$, *we have*

$$\|F(|x| > a|u|)V_s\phi(H)U_t\phi(H_0)F(|x| \leqq c|u|)\| \leqq K(1+|u|)^{-\delta}$$

whenever $\phi \in C_0^{\infty}(\mathbb{R})$ *with* sup supp $\phi = \frac{1}{2}b^2$.

The proof of (i) in the above Proposition is standard [27, 30] and the result (ii) is a generalization of the result, of Enss [29], for the two-body case.

In the appendix a slightly more general version of Proposition 1.2 is proved which asserts that arbitrary power-law decay holds in (ii).

Now we are ready to formulate the conditions needed for completeness to follow. We recall that in the two-body case in the methods of Enss [26] or Sinha and Muthuramalingam [25] they conclude that whenever the scattering states evolve out of bounded regions in space (Local Decay), the asymptotic observables corresponding to the average velocity x/t and the momentum P are the same (in units where the reduced mass of the system is 1). We conclude the same result here. But since it is possible in the N-body case that asymptotically the particles might cluster so that some of them might stay in bounded regions of the cluster-internal-spaces, we have to rule out these states if we want the total momentum and the average velocities to be the same asymptotically.

Henceforth for any real valued continuous function ϕ, we set

$$\mathscr{E}\lim_{t\to\pm\infty}\phi(t) = \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t ds\,\phi(s)$$

whenever the limit exists. We also set,

$$\mathscr{H}^{\pm} = \mathscr{H}_c(H) \ominus \sum_{2 \leqq \# D \leqq N}^{\oplus} \operatorname{Range}(\Omega^{\pm}(D)),$$

where $\mathscr{H}_c(H)$ is the continuous spectral subspace of H. With this notation, the Local Decay condition is stated as, *Local Decay Condition* (*LD*): *Let* $f\in\mathscr{H}^{\pm}$. *Then for any function* $\phi\in C_0^\infty(\mathbb{R}^\nu)$ *and* any *pair* γ,

$$\mathscr{E}\lim_{t\to\pm\infty}\|\phi(x^\gamma)V_t f\|=0.$$

Given this condition the proof of the theorem on asymptotic observables is almost identical to the one for the two-body case [25, 26]. Hence we state it below in the form that we need, without proof. We take $Y_u=\exp(-iuA)$ for any real u in the following.

Theorem 1.3. *Whenever the condition* (LD) *is satisfied for a vector* $f\in\mathscr{H}^{\pm}$, *then*

(i) $\mathscr{E}\lim_{t\to\pm\infty}\|(V_t^* Y_{u/t}V_t-V_u)f\|=0,\quad \forall u\in\mathbb{R}.$

(ii) $\mathscr{E}\lim_{t\to\pm\infty}\left\|\left\{V_t^*\psi\left(\frac{A}{t}\right)V_t-\psi(H)\right\}f\right\|=0,\quad \forall\psi\in C_0^\infty(\mathbb{R}).$

(iii) $\mathscr{E}\lim_{t\to\pm\infty}\left\|\left\{V_t^*U_t\phi\left(\left|\frac{x}{t}\right|\right)U_t^*V_t-\phi(0)\right\}f\right\|=0,\quad \forall\phi\in C_b(\mathbb{R}^+).$

(iv) $\mathscr{E}\lim_{t\to\pm\infty}\left\|F(|x|>c|t|)U_t^*V_tf\right\|=0,\quad \forall c>0.$

Another consequence of the condition (LD) is,

Proposition 1.4. *Let* $\phi\in C_0^\infty(\mathbb{R})$ *and let* $f\in\mathscr{H}^{\pm}$ *satisfy the condition* (LD). *Then, for any clustering* D,

(i) $\mathscr{E}\lim_{t\to\pm\infty}\|\{\phi(H^D)-\phi(T^D)\}V_tf\|=0,$

(ii) $\mathscr{E}\lim_{t\to\pm\infty}\|\{\phi(H)-\phi(H(D))\}V_tf\|=0.$

Proof. By a standard argument using the Stone–Weierstrass theorem [31], we need to show only that for some real λ,

$$\mathscr{E}\lim_{t\to\pm\infty}\|\{(H^D-\lambda)^{-1}-(T^D-\lambda)^{-1}\}(T^D-\lambda)^{-M}V_tf\|=0$$

for every positive integer M. Using the second resolvent equation, the above term is dominated by

$$\sum_{\gamma\in i(D)}\mathscr{E}\lim_{t\to\pm\infty}\|(H^D-\lambda)^{-1}W_\gamma\rho_1(x^\gamma)^{-1}\|\cdot\|\rho_1(x^\gamma)(T^D-\lambda)^{-(M+1)}\rho_1(x^\gamma)^{-1}\|$$
$$\cdot\|\rho_1(x^\gamma)V_tf\|.$$

By Proposition 1.1 and the assumptions (A2), the first two factors in the above expression are bounded and it is dominated by

$$K\sum_{\gamma\in i(D)}\mathscr{E}\lim_{t\to\pm\infty}\|\rho_1(x^\gamma)V_tf\|,$$

which is zero by the condition (LD). The result (ii) is similarly proved. □

Now we give a very useful norm estimate connected to the wave operators corresponding to the case when all the particles are free asymptotically. We set $\Omega^{\pm}(0) \equiv \Omega^{\pm}(D)$ when $\#D = N$.

Theorem 1.5. *Let $\phi \in C_0^{\infty}((0,\infty))$ and let $2c < \sqrt{\text{inf supp } \phi}$. Then,*

$$\lim_{t\to\pm\infty} \|(\Omega^{\pm}(0)-1)\prod_{\substack{\gamma:\text{all}\\ \text{pairs}}} \phi(T^{\gamma})U_t F(|x| \leqq c|t|)\| = 0.$$

Proof. (+ case only). We have

$$(\Omega^{+}(0)-1)\prod_{\gamma} \phi(T^{\gamma})U_t F(|x| \leqq c|t|)$$

$$= \sum_{\alpha}\int_0^{\infty} ds i V_s^* W_\alpha U_s \prod_{\gamma} \phi(T^{\gamma})U_t F(|x| \leqq c|t|) = \sum_{\alpha}\int_0^{\infty} ds I(\alpha,s,t).$$

We shall prove that, for some $\mu_1 > 1$ and any α,

$$\|I(\alpha,s,t)\| \leqq K(1+|s|+|t|)^{-\mu_1},$$

thereby proving the result.

We have

$$\|I(\alpha,s,t)\| \leqq \|W_\alpha \prod_{\gamma\neq\alpha} \phi(T^{\gamma})(T^{\alpha}+1)^{-1}\rho_1(x^{\alpha})^{-1}\| \, \|\rho_1(x^{\alpha})(T^{\alpha}+1)\phi(T^{\alpha})U_{t+s}\cdot$$
$$\cdot F(|x| \leqq c|t|)\|.$$

Using Proposition 1.1, the first factor, in the above inequality, is finite. Then using Proposition 1.2(i), for the two-body case, the remaining part has the required estimate as in ([10], Lemma 6.2). □

Now we formulate the low energy decay condition.

Low Energy Decay Condition (LED). *There exists a set $\mathscr{D}^{\pm}$ dense in $\mathscr{H}^{\pm}$ such that for each $f \in \mathscr{D}^{\pm}$, there are constants $b^{\pm}(f)$ so that for each pair γ and $0 < b_\gamma \leqq b^{\pm}(f)$,*

$$\mathscr{E}\lim_{t\to\pm\infty} \|F(T^{\gamma} < \tfrac{1}{2}b_\gamma^2)V_t f\| = 0.$$

Then we have the N-body completeness result for which we set $F^{\pm}(D) \equiv \text{Range}(\Omega^{\pm}(D))$.

Theorem 1.6 *Let the N-body Hamiltonian H have potentials satisfying* (A1)–(A3) *and also let the system satisfy the conditions* (LD) *and* (LED). *Then,*

$$\sum_{2\leq \#D\leq N}^{\oplus} F^{+}(D) = \mathscr{H}_c(H) = \sum_{2\leq \#D\leq N}^{\oplus} F^{-}(D),$$

in particular the singular continuous spectrum for *H is absent.*

Proof. (+ case only). We show that the set $\mathscr{D}^{+}$ of the condition (LED) is $\{0\}$, thus proving the result. So we take an $f \in \mathscr{D}^{+}$, we also take $b > 0$ and a function $\phi \in C_0^{\infty}(\mathbb{R})$ with inf supp $\phi = \frac{1}{2}b^2$. Then, we have the following inequality.

$$\|f\|^2 \leqq \mathscr{E}\lim_{t\to\pm\infty}\left\{\sum_{\substack{\gamma:\text{all}\\ \text{pairs}}} \langle V_t f, (1-\phi(T^{\gamma}))V_t f\rangle + \langle V_t f, \prod_{\gamma}\phi(T^{\gamma})V_t f\rangle\right\}.$$

Clearly by the condition (LED) we can choose b so that the above inequality reduces to

$$\|f\|^2 \leqq \mathscr{E} \lim_{t\to\pm\infty} \langle V_t, \prod_\gamma \phi(T^\gamma) V_t f \rangle$$

$$\leqq \mathscr{E} \lim_{t\to\pm\infty} \Big\{ \langle V_t f, (-\Omega^+(0)+1) \prod_\gamma \phi(T^\gamma) U_t F(|x| \leqq c|t|) U_t^* V_t f \rangle$$

$$+ \langle V_t f, \Omega^+(0) \prod_\gamma \phi(T^\gamma) V_t f \rangle + \|F(|x| > c|t|) U_t^* V_t f\| \Big\}$$

for any positive number c. Now we choose the constant c to satisfy the conditions of Theorem 1.5 vis-a-vis b, so that the first term is zero in the above inequality by Theorem 1.5. The second term is zero since $(\Omega^+(0))^* f = 0$, and the last term is zero by Theorem 1.3 (iv). □

Remark 1.7. If the condition (LED) is altered as follows for each $f \in \mathscr{D}^\pm$ and each pair γ,

$$\lim_{\varepsilon \downarrow 0} \mathscr{E} \lim_{t\to\pm\infty} \|F(T^\gamma < \varepsilon) V_t f\| = 0,$$

even then Theorem 1.6 remains valid.

2. Completeness for an N-body System

In this section we prove completeness for an N-body system that verifies conditions (A1)–(A3) on its potentials and also satisfies condition (A4). We briefly explain the contents of this section. We prove the Local Decay condition in Theorem 2.4 using a Theorem of Enss [7–9] which we state and prove in Proposition 2.2. (We note that this result appears in the work of Kitada [11] in a slightly different form.) We start with the RAGE Theorem [21] and Proposition 2.2 to conclude that whenever a scattering state leaves bounded regions in the internal spaces of clusters D with the same number of clusters, then for large times such a state also stays away from bounded regions in the internal spaces of any sub-clusterings C with $\#C = \#D + 1$ provided it is not asymptotically a cluster state corresponding to the clustering C. Thus we inductively conclude the Local Decay result for states in $\mathscr{H}^\pm$.

Having done this we note that if the total available energy for a state is positive (as it should be if it is in $\mathscr{H}^\pm$) then in view of the Local Decay result such a state can't stay in bounded regions along any pair directions which is possible only if all the pairs have strictly positive kinetic energies. This we conclude in Theorem 2.5. However for technical reasons and the limitations of our proof, we cannot prove this result directly but have to make use of the assumptions that (N-2) or lower-body subsystems do not have any negative eigenvalues and all the subsystems have complete scattering. Though we do not achieve it, it is our belief that completeness of the subsystems is not necessary in a general proof.

We start this section with an abstract lemma whose proof is found in the work of Enss [8].

Lemma 2.1. *Let H be any self-adjoint operator on $\mathscr{H}$ and B be any operator* *satisfying;*

$$\lim_{|T|\to\infty}\left\|\frac{1}{T}\int_0^T dt \exp(iHt)B^*B\exp(-iHt)P_c(H)\right\|=0.$$

Then for any $f\in\mathscr{H}$ *and any* $\varepsilon>0$, *there exists a* $T(\varepsilon)$ *(independent of* f*) such that for every* $|T|\geqq|T(\varepsilon)|$,

$$\sup_{\tau\in\mathbb{R}}\frac{1}{T}\int_\tau^{\tau+T} dt\,\|B\exp(-iHt)P_c(H)f\|\leqq\varepsilon\|f\|.$$

Using this abstract lemma we have the following proposition. We set $\rho(\lambda)=(1+\lambda^2)^{-\delta}$, $\delta>0$.

Proposition 2.2. *Let the potentials in the N-body Hamiltonian satisfy the condition* (A1). *Consider any* $f\in L^2(\mathbb{R}^{(N-1)\nu})$, $\|f\|=1$ *and* $\phi\varepsilon C_0^\infty(\mathbb{R})$ *with* $\operatorname{supp}\phi\subseteq\sigma_c(H^D)\backslash\sigma_p(H^D)$, *for every clustering D with k clusters in it. Then, for any* $\varepsilon>0$, *there is a* $T_1(\varepsilon)$ *and an* $R_0\equiv R_0(T_1(\varepsilon),\varepsilon)$ *such that for every* $R\geqq R_0$,

$$\frac{1}{T}\int_0^T dt\,\|\rho(x^D)\phi(H^D)\prod_{l=1}^{k-1}F(|y_l|>R)V_t(H+i)^{-1}P_c(H)f\|<\varepsilon$$

uniformly in $|T|\geqq|T_1(\varepsilon)|$.

Proof As in [8] it is enough to show that

$$\sup_{\tau\in\mathbb{R}}\frac{1}{T_1(\varepsilon)}\int_\tau^{\tau+T_1(\varepsilon)} dt\left\|\rho(x^D)\phi(H^D)\prod_{l=1}^{k-1}F(|y_l|>R)V_t(H+i)^{-1}P_c(H)f\right\|<\varepsilon$$

to conclude the result. By Lemma 2.1 this follows by showing that, for some $T_1(\varepsilon)$ and $R\geqq R_1(T_1(\varepsilon),\varepsilon)$

$$\left\|\frac{1}{T_1(\varepsilon)}\int_0^{T_1(\varepsilon)} dt(H+i)^{-1}V_t^*\phi(H^D)\left\{\rho(x^D)\prod_{l=1}^{k-1}F(|y_l|>R)\right\}^2\right.$$
$$\left.\cdot\phi(H^D)V_t(H+i)^{-1}\right\|\leqq K\varepsilon. \tag{2.1}$$

In view of the operator inequality, $(1-E^D)\cdot\phi(H^D)=\phi(H^D)$,

$$\frac{1}{T}\int_0^T dt\,V_t^*(D)\phi(H^D)\left\{\rho(x^D)\prod_{l=1}^{k-1}F(|y_l|>R)\right\}^2\phi(H^D)V_t(D)$$
$$\leqq\frac{1}{T}\int_0^T dt(V_t^D)^*\phi(H^D)\rho(x^D)^2\phi(H^D)V_t^D,$$

and the compactness of $\rho(x^D)\phi(H^D)$, there exists a $T_1(\varepsilon)$ such that,

$$\left\|\frac{1}{T_1(\varepsilon)}\int_0^{T_1(\varepsilon)} dt(H+i)^{-1}V_t^*(D)\phi(H^D)\left\{\rho(x^D)\prod_{l=1}^{k-1}F(|y_l|>R)\right\}^2\right.$$
$$\left.\cdot\phi(H^D)V_t(D)(H+i)^{-1}\right\|\leqq K\varepsilon/2. \tag{2.2}$$

On the other hand by an elementary estimate and Duhamel formula, the remaining term has the estimate,

$$\frac{2}{T_1(\varepsilon)}\int_0^{T_1(\varepsilon)} dt \left\| \prod_{l=1}^{k-1} F(|y_l|>R)\rho(x^D)\phi(H^D)\{V_t - V_t(D)\}(H+i)^{-1}\right\|$$

$$\leqq K \sum_{\alpha\in e(D)} T_1(\varepsilon) \sup_{0\leqq s\leqq T_1(\varepsilon)} \|F(|y_{l_\alpha}|>R)(T^D+i)^{-1}\rho(x^D)V_s^{D\cup\alpha}(D)W_\alpha(T^\alpha+i)^{-1}\|,$$

where y_{l_α} is the relative coordinate between the clusters connected by α, the clustering $D\cup\alpha$ is defined as in [32],

$$V_s^{D\cup\alpha}(D)\equiv \exp(-iH^{D\cup\alpha}(D)s), H^{D\cup\alpha}(D) = T^{D\cup\alpha}+W^D,$$

and $e(D)$ is defined by

$$e(D) = \{\gamma=(ij): i<j,\ i\in D_l,\ j\in D_{l'},\ l\neq l', 1\leqq l, l'\leqq k-1\}.$$

Now for each fixed s by assumption (A1), $(T^D+i)^{-1}\ \rho(x^D)V_s^{D\cup\alpha}(D)W_\alpha(T^\alpha+i)^{-1}$ is a compact operator on $L^2(X^{D\cup\alpha})$. This fact, the norm continuity of $((T^D+i)^{-1}\rho(x^D)V_s^{D\cup\alpha}(D)W_\alpha(T^\alpha+i)^{-1})$ in s uniformly in R and the inequality.

$$F(|y_{l_\alpha}|>R)\leqq F(|y_{l_\alpha}|>R_0),\quad R\geqq R_0$$

shows that there exists an $R_0\equiv R_0(T_1(\varepsilon),\varepsilon)$, so that for any $R\geqq R_0$,

$$\sup_{0\leqq s\leqq T_1(\varepsilon)} \|F(|y_{l_\alpha}|>R)(T^D+i)^{-1}\rho(x^D)V_s^{D\cup\alpha}(D)W_\alpha(T^\alpha+i)^{-1}\| < \frac{\varepsilon}{2T_1(\varepsilon)}.$$

Hence the result. □

We note that $\tilde{\Omega}^\pm(D) = \operatorname{slim}_{t\to\pm\infty} V_t^* V_t(D)$ are defined [23] when the potentials satisfy the conditions (A1) and (A2). We defined the thresholds $\mathrm{T}(H)$ as in [33] as $\mathrm{T}(H)=\{0\}\cup\left(\bigcup_{D\#D\geqq 2}\sigma_p(H^D)\right)$ and prove a compactness result related to the wave operators. We recall that it is possible to write the generator A of dilations as $A=A^D + A_D$ on $L^2(\mathbb{R}(N-1)\nu)$ corresponding to a clustering D. We set $\rho(\lambda)=(1+\lambda^2)^{-1}$ in the following lemma.

Lemma 2.3. *Let D be any clustering with $\#D=2$ and let $\psi\in C_0^\infty(\mathbb{R}\backslash\mathrm{T}(H))$. Then, there exists $b>0$ depending upon the support of ψ such that for any $\phi\in C_0^\infty(\mathbb{R})$ with* sup supp *$\phi=\frac{1}{2}b^2$,*

$$(\tilde{\Omega}^\pm(D)-1)\phi(H^D)\psi(H(D))\rho(x^D)F(A_D \gtrless 0)$$

is compact.

Proof. (+ case only). We have

$$(\tilde{\Omega}^+(D)-1)\phi(H^D)\psi(H(D))\rho(x^D)F(A_D>0)$$

$$= \sum_{\alpha\in e(D)} i\int_0^\infty ds\, V_s^* W_\alpha V_s(D)\phi(H^D)\psi(H(D))\rho(x^D)F(A_D>0)$$

$$\equiv \sum_{\alpha\in e(D)} \int_0^\infty ds\, I_\alpha(D,s). \tag{2.3}$$

Since there is a standard argument [10] using Theorem 2.6 of [32] for compactness, we show only that there exists $b > 0$ such that for some $\mu_1 > 1$,

$$\| I_\alpha(D,s) \| \leqq K(1+|s|)^{-\mu_1}$$

to conclude the result. Now for any positive number b, we consider $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$ satisfying,

$$\phi(H^D) = \{ \phi_1(H^D) + \phi_2(H^D) \} \phi(H^D)$$

with $\operatorname{supp} \phi_2 \subseteq (-\frac{1}{2}b^2, \frac{1}{2}b^2)$. Then we have,

$$\| I_\alpha(D,s) \| \leqq K\{ \| I_\alpha^1(D,s) \| + \| I_\alpha^2(D,s) \| \}, \tag{2.4}$$

where

$$I_\alpha^1(D,s) = \rho_1(x^\alpha) V_s(D) \phi_1(H^D) \psi(H(D))(H(D)+i) \rho(x^D) F(A_D > 0)$$

and

$$I_\alpha^2(D,s) = \rho_1(x^\alpha) V_s(D) \phi_2(H^D) \psi(H(D))(H(D)+i) \rho(x^D) F(A_D > 0).$$

Since the support properties of $\psi(H(D))$ and $\psi(H(D))(H(D)+i)$ are the same, we consider only $\psi(H(D))$ in the following estimates.

We have by a theorem of Froese and Herbst [33] and the hypothesis (A4) that the range of $\phi_1(H^D)$ is finite dimensional because $\sup \operatorname{supp} \phi_1 < 0$. Hence for estimating $\| I_\alpha^1(D,s) \|$ we take, without loss of generality, the range of $\phi_1(H^D)$ to correspond to the eigenvalue λ^D of H^D. Then moving $\rho(x^D)$ to the left using Proposition 1.1 (*i*), and using the inequality $\rho_1(x^\alpha)\rho(x^D) \leqq K(1+|x_D|)^{-\mu_1}$ for some $\mu_1 > 1$, we have, by a result of [34-36] and the support property of Ψ,

$$\| I_\alpha^1(D,s) \| \leqq K(1+|x_D|)^{-\mu_1} U_{s,D} \psi(\lambda^D + T_D) F(A_D > 0) \| \leqq K(1+|s|)^{-\mu_1},$$

where $U_{s,D} = \exp(-isT_D)$.

It is in the second term of (2.4) that we choose b depending upon $d = \inf\{\operatorname{supp}\psi \cap (0,\infty)\}$. We take $\psi_2 \in C_0^\infty((0,\infty))$ so that

$$\psi_2(T_D)\phi_2(H^D)\psi(H(D)) \equiv \phi_2(H^D)\psi(H(D)).$$

This is possible if b is small compared to d. We take $\frac{1}{2}b_1^2 = \inf \operatorname{supp} \psi_2$. Then by using Proposition 1.1 (i), and using a partition of the identity,

$$\begin{aligned} \| I_\alpha^2(D,s) \| \leqq K\{ &\| \rho_1(x^\alpha) \| \, \| F(|x^D| > a_1|s|) V_s^D \phi_2(H^D) \rho(x^D) \| \\ &+ \| \rho_1(x^\alpha) F(|x^D| \leqq a_1|s|, |x_D| > a_2|s|) \| \\ &+ \| \rho_1(x^\alpha) \| \, \| F(|x_D| \leqq a_2|s|) U_{s,D} \psi_2(T_D) F(A_D > 0) \| \, \| \rho(x^D) \| \} \end{aligned} \tag{2.5}$$

If for each $\alpha \in e(D)$,

$$x^\alpha = c(\alpha,D)x^D + d(\alpha,D)x_D, \tag{2.6}$$

then we set

$$\mu_1(D) = \min_{\alpha \in e(D)} |d(x,D)|, \mu_2(D) = \max_{\alpha \in e(D)} |c(\alpha,D)|,$$

and choose a_1, a_2, b, b_1 to satisfy

$$(b_1^2 + b^2) < d, \quad \mu_1(D)a_2 > \mu_2(D)a_1, \quad a_2 < b_1 \quad \text{and} \quad a_1 > \Lambda(D)^{-1}b, \qquad (2.7)$$

where we define $\Lambda(D) = \max\{1, \Lambda\}$, Λ being the constant appearing in Proposition 1.2, appropriate for the $(N-1)$ particle system given by H^D. We note that $\Lambda(D)$ is just a positive constant, and since the assumption (A4) guarantees that

$$\Sigma^D = \min_{\substack{C \subset D \\ \neq}} \inf \sigma(H^C) = 0$$

(i.e., the two cluster threshold, Hunziker limit for H^D is zero) if $\# D \geqq 2$, we can apply Proposition 1.2 (ii) to the first term of (2.5), using condition (2.7), to conclude that it is dominated by $K(1+|s|)^{-\mu_1}$. The last term has the same bound by a result of [36]. The estimate for the second term of (2.5) is obvious from condition (2.7). Thus we have the required estimate. □

We note as in [10] that whenever the k-particle system, $2 \leqq k \leqq (N-1)$, has complete scattering, it follows that for $\# D = 2$, Range $(\tilde{\Omega}^{\pm}(D))$ = Range $(\Omega^{\pm}(D)) \oplus$ Range $(\Omega^{\pm}(0))$ and for $3 \leqq \# C \leqq (N-1)$, Range $(\tilde{\Omega}^{\pm}(C))$ = Range $(\tilde{\Omega}(0))$ by assumption (A4). Then the Local Decay result follows in

Theorem 2.4. *The N-body system satisfying the condition* (A4) *with its potentials satisfying the conditions* (A1)–(A3) *satisfies also the Local Decay condition* (LD) *if every k particle system, $2 \leqq k \leqq (N-1)$, has complete scattering.*

Proof. (+ case only). We first claim the following.

Claim. We take ρ as in Lemma 2.3. Consider any $1 \leqq k \leqq (N-1)$ and suppose that for all clusterings C with $\# C < k$,

$$\mathscr{E} \lim_{t \to \infty} \|\rho(x^C) V_t f\| = 0$$

for some f. Then for the same f and for any clustering D with $\# D = k$ we have,

$$\mathscr{E} \lim_{t \to \infty} \|\rho(x^D) \phi(H^D) V_t f\| = 0$$

for all $\phi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \phi \subseteqq \sigma_c(H^D) \setminus \sigma_p(H^D)$. We have,

$$\begin{aligned}\frac{1}{T}\int_0^T dt \|\rho(x^D)\phi(H^D)V_t f\| &\leqq \frac{1}{T}\int_0^T dt \|\rho(x^D) \prod_{l=1}^{k-1} F(|y_l| > R)\phi(H^D)V_t f\| \\ &\quad + \sum_{l=1}^{k-1} \frac{1}{T}\int_0^T dt \|\rho(x^D)\phi(H^D)\rho(H^D)^{-1}\| \, \|\rho(x^D) \\ &\quad \cdot F(|y_l| \leqq R) V_t f\|.\end{aligned}$$

For any $\varepsilon > 0$, using Proposition 2.2 we choose $T_1(\varepsilon)$ and $R_1(T_1(\varepsilon), \varepsilon)$ so that the first term in the above inequality is smaller than $\frac{1}{2}\varepsilon$ when we take $R = R_1$. Given this R_1 we choose $T_2(R_1(T_1(\varepsilon), \varepsilon), \varepsilon)$ so that the second term is smaller than $\frac{1}{2}\varepsilon$ by the hypothesis of the claim, for any $|T| \geqq |T_2|$. Then we choose a $T_3 = \max(T_1, T_2)$ and exploit the uniformity statement of Proposition 2.2 to conclude that for any $|T| \geqq |T_3|$,

$$\frac{1}{T}\int_0^T dt \|\rho(x^D)\phi(H^D)V_t f\| < \varepsilon$$

which is the claim. Now we show that for any clustering C with $\#C = 2$,

$$\mathscr{E}\lim_{t\to\infty} \|\rho(x^C)V_t f\| = 0$$

for $f \in \mathscr{H}^{\pm}$; it is enough by density to take an f with $\psi(H)f = f$, $\psi \in C_0^\infty(\mathbb{R}\backslash T(H))$. By the RAGE Theorem [21] it is true that

$$\mathscr{E}\lim_{t\to\infty} \|\rho(x)V_t f\| = 0.$$

We choose ϕ as in Lemma 2.3., given the ψ. Then by *claim* we have, $\operatorname{supp}(1-\phi) \subseteqq \sigma_c(H^D)\backslash\sigma_p(H^D)$,

$$\mathscr{E}\lim_{t\to\infty} \|\rho(x^C)\{1-\phi(H^C)\}V_t f\| = 0.$$

Therefore by the Schwarz inequality we consider only

$$\begin{aligned}
&\mathscr{E}\lim_{t\to\infty} \|\rho(x^C)^{1/2}\phi(H^C)V_t f\|^2 \\
&\quad \leqq \mathscr{E}\lim_{t\to\infty} \{\|\rho(x^C)\{\psi(H) - \psi(H(C))\}V_t f\| \\
&\qquad + \langle V_t f, (-\tilde{\Omega}(C)+1)\psi(H(C))\phi(H^C)\rho(x^C)F(A_C > 0)V_t f\rangle \\
&\qquad + \langle V_t f, \tilde{\Omega}^+(C)\psi(H(C))\phi(H^C)\rho(x^C)F(A_C > 0)V_t f\rangle \\
&\qquad + \langle V_t f, (-\tilde{\Omega}^-(C)+1)\psi(H(C))\phi(H^C)\rho(x^C)F(A_C < 0)V_t f\rangle \\
&\qquad + \langle V_t f, \tilde{\Omega}^-(C)\psi(H(C))\phi(H^C)\rho(x^C)F(A_C < 0)V_t f\rangle\}.
\end{aligned}$$

Then the first two terms and the fourth term in the above inequality are zero by Lemma 2.3 and the RAGE theorem. The third term is zero since $\tilde{\Omega}^+(C)^* f = 0$, and the last term is zero because $\|F(A_C < 0)U_{t,C}f\| \to 0$ as $t \to \infty$ [36].

Once we have this result for every clustering C with $\#C = 2$, the rest of the theorem follows inductively from *Claim*. $\square$.

We shall now prove the Low Energy Decay condition for the N-body evolution. For this we consider the sets

$$\mathscr{D}^{\pm} = \{f \in \mathscr{H}^{\pm} : \psi(H)f = f, \quad \text{for some } \psi \in C_0^\infty(\mathbb{R}\backslash\{0\})\}.$$

Clearly such $\mathscr{D}^{\pm}$ is dense in $\mathscr{H}^{\pm}$.

We take $\mathscr{D}^{\pm}$ in the above form since by Proposition 1.4 (ii), vectors in $\mathscr{D}^{\pm}$ cannot have negative energy and there are no positive energy thresholds by Froese and Herbst.

Theorem 2.5. *Let the N-body system satisfy the conditions* (A1)–(A4). *Then the system verifies the condition* (LED).

Proof. (+ case only) We take $\mathscr{D}^{\pm}$ defined above as the set stated in the condition

(LED) and consider $f \in \mathscr{D}^+$ with $\psi(H)f = f$. We set $d \equiv \inf\{\operatorname{supp}\psi \cap (0,\infty)\}$ and note that by Proposition 1.4 (which is valid in view of Theorem 2.4) for any $b < 2\sqrt{d}$,

$$\mathscr{E}\lim_{t\to\infty} \|F(H_0 < \tfrac{1}{2}b^2)V_t f\| = 0.$$

Then we prove the result by induction. We assume as the induction hypothesis that for every clustering C with $\#C \leqq k$, there exists some b, $\frac{1}{2}b^2 < d$ such that

$$\mathscr{E}\lim_{t\to\infty} \|F(T^C < \tfrac{1}{2}b^2)V_t f\| = 0,$$

and prove that for all clusters D with $\#D = k+1$, there exists a constant $b_1 > 0, b_1 < b$ such that

$$\mathscr{E}\lim_{t\to\infty} \|F(T^D < \tfrac{1}{2}b_1^2)V_t f\| = 0.$$

We set $T_l = \frac{1}{2}k_l^2$, where k_l is the momentum conjugate to y_l. Then for any $b_1 > 0$,

$$\begin{aligned}\mathscr{E}\lim_{t\to\infty} \|F(T^D < \tfrac{1}{2}b_1^2)V_t f\| \leqq \mathscr{E}\lim_{t\to\infty} \Bigg\{ &\|F(T^D < \tfrac{1}{2}b_1^2)\prod_{l=1}^{k} F(T_1 > \tfrac{1}{2}b_2^2)V_t f\| \\ &+ \sum_{l=1}^{k} \|F(T^D < \tfrac{1}{2}b_1^2)F(T_l < \tfrac{1}{2}b_2^2)V_t f\| \Bigg\}.\end{aligned} \tag{2.8}$$

Now we chose b_1, b_2 to satisfy

$$b_1^2 + b_2^2 < b^2. \tag{2.9}$$

Then all the terms except the first in (2.8) are zero by the induction hypothesis. Applying the Schwarz inequality to the remaining term of (2.8) we have

$$\mathscr{E}\lim_{t\to\infty} \|F(T^D < \tfrac{1}{2}b_1^2)V_t f\|^2 \leqq \mathscr{E}\lim_{t\to\infty} \left\langle V_t f, F(T^D < \tfrac{1}{2}b_1^2)\prod_{l=1}^{k} F(T_l > \tfrac{1}{2}b_2^2)V_t f \right\rangle. \tag{2.10}$$

We now take functions $\phi, \phi_1 \in C_0^\infty(\mathbb{R})$ with $\sup\operatorname{supp}\phi = \frac{1}{2}b_3^2$ and $\inf\operatorname{supp}\phi_1 = \frac{1}{2}b_4^2$, b_3 and b_4 satisfying,

$$b_3^2 + b_4^2 < b^2, \tag{2.11}$$

and pointwise,

$$F(T^D < \tfrac{1}{2}b_1^2)\prod_{l=1}^{k} F(T_l > \tfrac{1}{2}b_2^2) \leqq \phi(T^D)\prod_{l=1}^{k} \phi_1(T_l). \tag{2.12}$$

Then using the inequality (2.12) on the right-hand side of (2.10), replacing $V_t f$ by $U_t F\,(|x| \leqq c|t|)U_t^* V_t f$ (by means of Theorem 2.4 and Theorem 1.3 (iv)) and $\phi(T^D)$ by $\phi(H^D)$ (via Proposition 1.4) respectively we obtain,

$$\begin{aligned}\mathscr{E}\lim_{t\to\infty} \|F(T^D < \tfrac{1}{2}b_1^2)V_t f\|^2 \leqq \mathscr{E}\lim_{t\to\infty} \Bigg\{ &\Bigg\langle V_t f, (-\tilde{\Omega}^+(D) + 1)\phi(H^D)\prod_{l=1}^{k} \phi_1(T_l)U_t F(|x| \\ &\leqq c|t|)U_t^* V_t f \Bigg\rangle + K\|(\tilde{\Omega}^+(D))^* f\| \Bigg\}.\end{aligned}$$

The second term in the above inequality is zero by the choice of f. The first term is zero by choosing the constants c, b_3, b_4 as in the following lemma and applying the lemma. The result then follows inductively using Lemma 2.6. □

Lemma 2.6. *Let the N-body system satisfy the conditions* (A1)–(A4). *Let D be any clustering with k clusters* $1 < k < N$. *Then there exist positive constants* b_3, b_4 *and* c *such that for* ϕ, $\phi_1 \in C_0^\infty(\mathbb{R})$ *with* $\inf \operatorname{supp} \phi_1 = \frac{1}{2}b_4^2$ *and* $\sup \operatorname{supp} \phi = \frac{1}{2}b_3^2$, we have,

$$\lim_{t\to\infty} \|(\tilde{\Omega}^{\pm}(D) - 1)\phi(H^D) \prod_{l=1}^{k-1} \phi_1(T_l)U_t F(|x| \leqq c|t|)\| = 0.$$

Proof. (+ case only) Owing to the equality

$$(\tilde{\Omega}^{+}(D) - 1)\phi(T^D) \prod_{l=1}^{k-1} \phi_1(T_l)U_t F(|x| \leqq c|t|)$$

$$= \sum_{\alpha\in e(D)} i\int_0^\infty ds V_s^* W_\alpha V_s(D)\phi(H^D) \prod_{l=1}^{k-1} \phi_1(T_l)U_t F(|x| \leqq c|t|)$$

$$\equiv \sum_{\alpha\in e(D)} \int_0^\infty ds I_\alpha(D, s, t),$$

we prove only the estimate, for each $\alpha \in e(D)$,

$$\|I_\alpha(D, s, t)\| \leqq K(1 + |s| + |t|)^{-\mu_1}, \quad \mu_1 > 1$$

to conclude the result. Suppose y_{l_α} is the relative coordinate of the clusters connected by the pair α, then by the hypothesis (A2), Proposition 1.1 (i), (ii) we have,

$$\|I_\alpha(D, s, t)\| \leqq K\|\rho_1(x^\alpha)(H^D + i)\phi(H^D)V_s(D)(T_{l_\alpha} + i)\phi_1(T_{l_\alpha})U_t F(|x| \leqq c|t|)\|.$$

Since only the supports of ϕ and ϕ_1 play a role in the following estimate, we omit the factors $(H^D + i)$ and $(T_{l_\alpha} + i)$ from the above inequality in the following.

Then for some a_1, a_2, positive, the inequality just obtained becomes, using a partition of the identity,

$$\begin{aligned}\|I_\alpha(D, s, t)\| \leqq K\{ &\|\rho_1(x^\alpha)\| \, \|F(|x^D| > a_1|t+s|)V_\alpha^D \phi(H^D)U_t F(|x^D| \leqq c|t|)\| \\ &+ \|\rho_1(x^\alpha)F(|x^D| \leqq a_1|t+s|, |y_{l_\alpha}| > a_2|t+s|)\| \\ &+ \|\rho_1(x^\alpha)\| \, \|F(|y_{l_\alpha}| \leqq a_2|t+s|)\exp(-i(t+s)T_{l_\alpha})\phi(T_{l_\alpha}) \\ &\cdot F(|y_{l_\alpha}| \leqq c|t+s|)\|\}.\end{aligned} \tag{2.13}$$

We take the numbers $\mu_1(D)$, $\mu_2(D)$ defined in (2.6), an appropriate Λ from Proposition 1.2 (ii) and choose the constants, a_1, a_2, b_3, b_4 and c to satisfy,

$$\mu_1(D)a_2 > \mu_2(D)a_1, \quad \Lambda a_1 > b_3 + c \quad \text{and} \quad a_2 < b_4 + c, \tag{2.14}$$

from which the required estimate follows using Proposition 1.2 (i), (ii) and the relation (2.6) in the inequality (2.13) along with the assumption (A4). □

In view of Theorems 2.4 and 2.5 the theory of Sect. 1 applies to the N-body system satisfying the assumptions (A1)–(A4) whenever all the $(N-1)$ or lower-body systems satisfying the same hypotheses have complete scattering. Since completeness of the two particle scattering is well-known [30], Theorem I follows inductively.

Finally we observe that the assumption (A4) being a consequence of (A1)–(A3) for a three particle system, completeness follows for a three particle system with just the assumptions (A1)–(A3).

Appendix

In this appendix we prove Proposition 1.2 (ii). Enss proved a similar result for the two-body case, in a slightly different form, in [29] using the Gronwall inequality. We use instead a method suggested by the referee of [10]. We actually obtain a result stronger than Proposition 1.2 (ii) in that we deduce arbitrary time decay for the total evolution.

Throughout this appendix we fix the number of particles to be N_1 (a finite positive integer larger than one), assume the potentials of the N_1-particle system to satisfy the conditions (A1) and (A2) of the main paper, and assume the dimension of the configuration space of pairs of particles to be $\nu \geqq 1$. We also follow here the notation of the main paper except for one difference. For every k-cluster clustering D we denote by $\left\{y_l,\ l=1,\dots,n_D,\ n_D=\binom{k-1}{2}\right\}$ the relative coordinates between clusters in D.

Then we define the k-cluster threshold Σ_k of the N_1-particle system to be

$$\Sigma_k = \inf_{D \cdot \# D = k} \sigma(H^D),$$

and note that by the HVZ theorem [24] $\Sigma_2 \leqq \Sigma_3 \leqq \cdots \leqq \Sigma_{N_1} \equiv 0$, and Σ_2 is the Hunziker limit for the system.

Having outlined the notation we present a technical result on the localizing properties of functions of the Hamiltonians before we proceed to the main theorem.

Theorem 3.1. *Let $S = H(D)$, $1 \leqq \# D \leqq N_1$ and $2 \leqq N_1 < \infty$. Consider the bounded functions ψ, ψ_1 in $C^\infty(\mathbb{R})$ with disjoint supports and ϕ in $C_0^\infty(\mathbb{R})$. Then for any $r > 0$ and arbitrary integer $M > 0$,*

(i) $$\left\| \psi\left(\frac{|x|}{r}\right) \phi(S) \psi_1\left(\frac{|x|}{r}\right) \right\| \leqq K(1+r)^{-M}.$$

(ii) *For any $a > a_1$,*

$$\| F(|x| > ar) \phi(S) F(|x| \leqq a_1 r) \| \leqq K(1+r)^{-M}.$$

Proof. Clearly the result (ii) follows from (i) so we prove only (i). We choose a bounded function ψ_2 in $C^\infty(\mathbb{R}^{(N_1-1)\nu})$ with $\psi_2(x)\psi(|x|) = \psi(|x|)$ for each $x \in \mathbb{R}^{(N_1-1)\nu}$ and also such that the support of ψ_2 is disjoint from that of ψ_3 on $\mathbb{R}^{(N_1-1)\nu}$ defined by $\psi_3(x) = \psi_1(|x|)$, $x \in \mathbb{R}^{(N_1-1)\nu}$. Then we have,

$$\left\| \psi\left(\frac{|x|}{r}\right) \psi_2\left(\frac{x}{r}\right) \phi(S) \psi_1\left(\frac{|x|}{r}\right) \right\| = \left\| \psi\left(\frac{|x|}{r}\right) \left[\psi_2\left(\frac{x}{r}\right), \phi(S) \right] \psi_1\left(\frac{|x|}{r}\right) \right\| + 0.$$

Repeating this procedure, of transferring a ψ_2 to the right side, M times, we have

$$\left\|\psi\left(\frac{|x|}{r}\right)\phi(S)\psi_1\left(\frac{|x|}{r}\right)\right\| = \left\|\psi\left(\frac{|x|}{r}\right)\mathrm{Ad}^M_{\psi_2(x/r)}\{\phi(S)\}\psi_1\left(\frac{|x|}{r}\right)\right\|$$
$$\leqq K\,\|\mathrm{Ad}^M_{\psi_2(x/r)}\{\phi(S)\}\|.$$

The result now follows from the following Lemma. □

Lemma 3.2. *Consider a bounded function ψ in $C^\infty(\mathbb{R}^n)$ for any $n>0$ and $\phi\in C_0^\infty(\mathbb{R})$. Then for any $r>0$, S as in Theorem* 3.1,

$$\|\mathrm{Ad}^M_{\psi(x/r)}\{\phi(S)\}\| \leqq K(1+r)^{-M}$$

for every positive integer M.

Proof. We note that $-\infty<\mu<\sigma(S)$ exists. We then consider, for any $a<\mu$, the map $\chi(\lambda)=(\lambda-a)^{-1}$. χ is a bijection from (μ,∞) to $(0,(\mu-a)^{-1})$. Therefore it suffices to prove the result with $\phi_1((S-a)^{-1})$ replacing $\phi(S)$, where $\phi_1=\phi\circ\chi^{-1}$. Now by the self-adjointness of $(S-a)^{-1}$, setting $Z_t=\exp(-it(S-a)^{-1})$, we have

$$\mathrm{Ad}^M_{\psi(x/r)}\{\phi(S)\} = \int dt\,\hat{\phi}_1(t)\,\mathrm{Ad}^M_{\psi(x/r)}\{Z_t\}. \tag{3.1}$$

We claim that

$$\|\mathrm{Ad}^M_{\psi(x/r)}\{Z_t\}\| \leqq K(1+|t|)^{M+1}(1+r)^{-M}.$$

This follows by explicitly writing $\mathrm{Ad}^M_{\psi(x/r)}\{Z_t\}$ using Duhamel's formula and the following equality:

$$\begin{aligned} Z_t^*\psi\left(\frac{x}{r}\right)Z_t-\psi\left(\frac{x}{r}\right) &= -i\int_0^t ds Z_s^*(S-a)^{-1}\left[H_0,\psi\left(\frac{x}{r}\right)\right](S-a)^{-1}Z_s \\ &= -i\int_0^t ds Z_s^*(S-a)^{-1}\left\{\frac{1}{2r^2}(\Delta\psi)+\left(\frac{2}{r}(\nabla\psi)\left(\frac{x}{r}\right)P\right\}\right. \\ &\quad \cdot(S-a)^{-1}Z_s, \end{aligned} \tag{3.2}$$

where the expression for the commutator $[H_0,\psi(x/s)]$ can be a priori defined on $\mathscr{S}(\mathbb{R}^n)$ and then extended to $\mathscr{D}(S)$ without change by the boundedness of $\Delta\psi$, $\nabla\psi$ and the S boundedness of P. □

See for example [27] for the technique employed in the proof of the above lemma.

Lemma 3.3. *Consider the N_1-body Hamiltonian H with the pair potentials satisfying the conditions* (A1) *and* (A2). *Then for any $\phi\in C_0^\infty(\mathbb{R})$, $r>0$, and any clustering D, with $\delta\equiv 2\delta_1>1$,*

$$\left\|\prod_{\gamma\in e(D)}F(|x^\gamma|>r)\{\phi(H)-\phi(H(D))\}\right\| \leqq K(1+r)^{-\delta}.$$

The proof of the above lemma is easy since $\prod_{\gamma\in e(D)}F(|x^\gamma|>r)\{(H-a)^{-1}-(H(D)-a)^{-1}\}$ has the required decay for some real a. See for example [10].

We recall that $\Sigma_2, \Sigma_3, \ldots, \Sigma_{N_1}$ are the two, three, four,... N_1-cluster thresholds. Then our main theorem of the appendix is the following.

Theorem 3.4. *Consider a N_1-particle system with the pair potentials satisfying the conditions* (A1) *and* (A2). *Then for any positive numbers a, b, c, any positive integer M and any function $\phi \in C_0^\infty(\mathbb{R})$ with* $\sup\operatorname{supp}\phi = \frac{1}{2}b^2$, *there exists a positive constant Λ (depending upon M, N_1, and the masses of the particles) such that for all s, t, u satisfying* $|s| + |t| \leqq |u|$,

$$\|F(|x| > a|u|)V_s\phi(H)U_t\phi(H_0)F(|x| \leqq c|u|)\| \leqq K(1+|u|)^{-M\delta}$$

whenever $a > \Lambda^{-1}(b + \sqrt{-\Sigma_2} + c)$.

Before we proceed to the proof of the theorem we make a few comments on the result itself and also on the strategy of the proof. Firstly we stress that we give only a sufficient condition for separation of clusters. A necessary condition should read a "if the N_1-body evolution has $(1+|u|)^{-\delta}$ time decay, then it is necessary that $a > (2(E - \Sigma_2))^{1/2} + c$, where $E = \frac{1}{2}b^2$ is the maximum available total energy for the system." In other words $(2(E - \Sigma_2)^{1/2} + c)$ should give a lower bound on the speed for the forbidden region of propagation. Though it is intuitively expected, getting the necessary condition is not obvious with our method of proof especially for particle numbers higher than three. Secondly we take small positive energies $\frac{1}{2}b^2$ because in the application, we wanted to treat the possibility of an infinite number of $(N-1)$-body bound states at zero energy.

Now for an explanation of the proof. We observe that the system can cluster corresponding to a clustering D with the individual clusters being bound particles or particles propagating with speed small relative to the speeds of separation of the clusters. In the case when the clusters are bound particles with negative energy, the centers of mass of the clusters move with large relative speeds, having extracted energy from the bound states. So we decompose the velocity space into all such possible clusterings D, starting with clusterings of N_1 clusters and ending with the clusterings of two clusters. Having done this, we approximate the total evolution by the respective cluster evolutions in each of these regions. Therefore it is here that the minimal velocity of the forbidden region depends on the threshold energies of formation of the clusters. We handle the remaining tails using the Duhamel formula, where we exploit the decay of the potentials connecting the clusters so that each such step gives us a net $(u)^{1-\delta}$ time decay. Therefore we do this $(\delta - 1)^{-1}\delta$ times to obtain δ time decay. Finally we use a boot-strap method to obtain arbitrary decay starting with δ decay, but this is only technical.

With these comments we proceed to fix a constant (implicitly) related to the masses of the particles. Consider any clustering C. Then for any pair γ external to C, the position x^γ can be written in terms of the internal coordinate x^C and the relative coordinate y_l between the clusters joined by γ as, $x^\gamma = a(\gamma, C)x^C + b(\gamma, C)y_l$. Then we set.

$$\mu_1(C) = \min_{\gamma \in e(C)} |b(\gamma, C)|, \mu_2(C) = \max_{\gamma \in e(C)} |a(\gamma, C)|,$$

and

$$\mu = \max\{2, \max_{C:\#C>1} \mu_1(C)^{-1}\mu_2(C)\},$$

$$I_n = \sum_{\substack{C:\#C=k \\ n\leqq k\leqq N_1}}\sum \left\| F(|x^C| \leqq a_1(k_1)|u|) \prod_{l=1}^{n_C} F(|y_l| > a_2(k)|u|)V_s\phi(H)V_t\phi(H_0). \right.$$

$$\left. \cdot F(|x| \leqq c|u|) \right\|. \tag{3.1}$$

$I_{N_1+1} \equiv 0.$

Then,

Lemma 3.5. *Let* $\phi \in C_0^\infty(\mathbb{R})$ *with* sup supp $\phi = \frac{1}{2}b^2$. *Let* $1 < n \leqq N_1$ *and suppose that there exist positive constants* $\{a_1(k), a_2(k)\}_{n+1 \leqq k \leqq N_1}$ *satisfying* $a_1(k) = a_1(k+1) + a_2(k+1)$ such that

(H) $$\|I_{n+1}\| \leqq K_M(1+|u|)^{-M\delta}$$

is valid for arbitrary $M > 0$ *whenever* $|s| + |t| \leqq |u|$. *Then,*

$$\|I_n\| \leqq K_M(1+|u|)^{-M\delta}$$

holds if for all $n \leqq k \leqq N_1$,

$$a_2(k) > 2M(\mathrm{Max}\{b + c + \sqrt{-2\Sigma_k}, \quad \mu a_1(k)\}). \tag{3.2}$$

Proof. Clearly it suffices to prove, whenever $a_2(n)$ satisfies (3.2), that, for any clustering D with $\#D = n$,

$$\| F(|x^D| \leqq a_1(n)|u|) \prod_{l=1}^{n_D} F(|y_l| > a_2(n)|u|)V_s\phi(H)U_t\phi(H_0)F(|x| \leqq c|u|) \|$$

$$\leqq K_M(1+|u|)^{-M\delta}. \tag{3.3}$$

For proving the above estimate we choose an $\varepsilon > 0$ so that

$$a_2(n) > 2M(b + c + \sqrt{-2\Sigma_n} + \varepsilon), \tag{3.4}$$

and define the projections F_j^D, F_j^{nD}, $\tilde{F}_j^D$ and G_j^D as follows. Let $K^D = F(|x^D| \leqq a_1(n)|u|)$ and L be an integer $L > (\delta - 1)^{-1}\delta + 2$,

$$F_j^{nD} \equiv \prod_{l=1}^{n_D} F\left(|y_l| > \left(\frac{2M-1}{2M}\right)a_2(n)|u| + \left(\frac{1}{2M}a_2(n) - \frac{j\varepsilon}{L}\right)(|u| - s + s_j)\right.$$
$$\left. + \left(c + \frac{(L-j)\varepsilon}{L}\right)(s - s_j)\right),$$

$$\tilde{F}_j^{nD} \equiv \prod_{l=1}^{n_D} F\left(|y_l| > \left(\frac{2M-1}{2M}\right)a_2(n)|u| + \left(\frac{1}{2M}a_2(n) - \frac{(j+\frac{1}{2})\varepsilon}{L}\right)(|u| - s + s_j)\right.$$
$$\left. + \left(c + \frac{(L-(j+\frac{1}{2}))\varepsilon}{L}\right)\cdot(s - s_j)\right),$$

$$\tilde{F}_j^D \equiv \tilde{F}_j^{nD} K^D, \quad F_j^D \equiv F_j^{nD} K^D,$$

$$G_j^D \equiv \prod_{l=1}^{nD} \left(|y_l| > \frac{(2M-1)}{2M} a_2(n)|u| + \left(\frac{1}{2M} a_2(n) - \frac{j\varepsilon}{L} \right)(|u| - s) + \left(c + \frac{(L-j)\varepsilon}{L} \right) s \right),$$

and $G_{j+(1/2)}^D$ is defined by replacing j by $j + \frac{1}{2}$ in the above where $0 \leqq j \leqq L-2$ and $0 \leqq s_{L-2} \leqq \cdots \leqq s_1 \leqq s$. If the condition (3.4) is satisfied, then clearly there is a $b_1 > b$ such that

$$a_2(n) > 2M(b_1 + c + \sqrt{-2\Sigma_n} + \varepsilon). \tag{3.5}$$

We fix, then, a function $\tilde{\phi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\phi} \equiv 1$ on the support of ϕ and $\sup \operatorname{supp} \tilde{\phi} = \frac{1}{2} b_1^2$ and estimate the left side of (3.3) below by induction on M since (3.3) is valid trivially with $M = 0$. We assume, without loss of generality, that for any constants a_{M-1}^2 and a_{M-1}^1 satisfying

$$a_{M-1}^2 > (2M-1) \max \{ (b + c + \sqrt{-2\Sigma_n}), \mu a_{M-1}^1 \},$$

$$\| F(|x^D| \leqq a_{M-1}^1 |u|) \prod_{l=1}^{nD} F(|y_l| > a_{M-1}^2 |u|) V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \|$$

(H1)

$$\leqq K(1 + |u|)^{-(M-1)\delta}$$

holds for a given clustering D. We have then by Schwartz' inequality and the Duhamel formula,

$$\begin{aligned}
&\| F_0^D V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\quad \leqq \| F_0^D \{ \tilde{\phi}(H) - \tilde{\phi}(H(D)) \} \tilde{F}_0^D V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \| F_0^D \{ \tilde{\phi}(H) - \tilde{\phi}(H(D)) \} (1 - \tilde{F}_0^{nD}) V_s \phi(H) U_1 \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \| F_0^D \{ \tilde{\phi}(H) - \tilde{\phi}(H(D)) \tilde{F}_0^{nD} F(|x^D| > a_1(n)|u|) V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \| F_0^D \tilde{\phi}(H(D)) V_s(D)(1 - G_{1/2}^D) \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \| F_0^D \tilde{\phi}(H(D)) V_s(D) G_{1/2}^D \phi(H)(1 - G_1^D) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \| F_0^D \tilde{\phi}(H(D)) V_s(D) G_{1/2}^D \phi(H) G_1^D U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \int_0^s ds_1 \| F_0^D \tilde{\phi}(H(D)) V_{s-s_1}(D)(1 - F_1^{nD}) W_D V_{s_1} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \int_0^s ds_1 \| F_0^D \tilde{\phi}(H(D)) V_{s-s_1}(D) W_D F_1^{nD} F(|x^D| > a_1(n)|u|) \\
&\qquad\quad \cdot V_{s_1} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \| \\
&\qquad + \int_0^s ds_1 \| F_0^D \tilde{\phi}(H(D)) V_{s-s_1}(D) W_D F_1^D V_{s_1} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \|.
\end{aligned} \tag{3.6}$$

The second and the fifth terms in the above inequality are respectively dominated by

$$K\,\|F_0^D\{\tilde{\phi}(H)-\tilde{\phi}(H(D))\}(1-\tilde{F}_0^{nD})\| \quad \text{and} \quad \|G_{1/2}^D\,\phi(H)(1-G_1^D)\|.$$

Hence they are bounded by $K(1+|u|)^{-M\delta}$ by Theorem 3.1 (ii). The first term of (3.6) is dominated by

$$\|F_0^D\{\tilde{\phi}(H)-\tilde{\phi}(H(D))\}\|\,\|\tilde{F}_0^D V_s\phi(H)U_t\phi(H_0)F(|x|\leqq c|u|)\|. \tag{3.7}$$

Here $a_2(n)>2\mu a_1(n)$ and $\tilde{F}_0^D \leqq \prod_{\gamma\in e(D)} F(|x^\gamma|>\sigma|u|), \tilde{F}_0^D \leqq \prod_{l=1}^{nD} F(|y_l|>\frac{1}{2}a_2(n)|u|)$ $F(|x^D|\leqq a_1(n)|u|)$, for some positive σ. Hence by Lemma 3.3 the first factor of (3.7) is bounded by $K(1+|u|)^{-\delta}$ while the second factor has $K(1+|u|)^{-(M-1)\delta}$ bound by hypothesis (H_1) since

$$\tilde{F}_0^D \leqq F(|x^D|\leqq a_1(n)|u|)\prod_{l=1}^{nD} F\left(|y_l|>\frac{(2M-2)}{2M}a_2(n)|u|\right)$$

and the constants $a_{M-1}^1 \equiv a_1(n)$ and $a_{M-1}^2 \equiv ((2M-1)/2M)a_2(n)$ satisfy the conditions of the hypothesis (H_1). Therefore the first term of (3.6) has the required decay. The fourth and the seventh terms of (3.6) are respectively dominated by

$$K\,\|F_0^D\tilde{\phi}(H(D))V_s(D)(1-G_{1/2}^D)\| \quad \text{and} \quad \int_0^s ds_1\,\|F_0^D\tilde{\phi}(H(D))V_{s-s_1}(D)(1-F_1^{nD})\|$$

and have similar estimates. We set

$$F_l \equiv F\left(|y_l|\leqq\frac{(2M-1)}{2M}a_2(n)|u|+\left(\frac{1}{2M}a_2(n)-\frac{\varepsilon}{L}\right)(|u|-s+s_1)\right.$$
$$\left.+\left(c+\frac{(L-1)\varepsilon}{L}\right)(s-s_1)\right)$$

and estimate only

$$\|F_0^D\tilde{\phi}(H(D))V_{s-s_1}(D)F_l\|$$

for all l. This expression is dominated by

$$\|F(|y_l|>a_2(n)|u|)\exp\left(-\frac{i}{2}(s-s_1)k_l^2\right)\tilde{\phi}(H(D))F_l\|, \tag{3.8}$$

where k_l is the momentum conjugate to y_l. In (3.8) the support property of $\tilde{\phi}$ restricts $H(D)$ to $H(D)\leqq\frac{1}{2}b_1^2$, which in turn restricts $\frac{1}{2}k_l^2\leqq\frac{1}{2}b_1^2-H^D$ by the positivity of $(T_D-\frac{1}{2}k_l^2)$. Since $H^D\geqq\Sigma_n$, we have, $\frac{1}{2}k_l^2\leqq(\frac{1}{2}b_1^2-\Sigma_n)$ which follows if $k_l^2\leqq(b_1+\sqrt{-2\Sigma_n})^2$. Therefore if we consider $\tilde{\phi}(H(D))$ as an operator of $\frac{1}{2}k_l^2$ fibred in $(H(D)-\frac{1}{2}k_l^2)$ and make use of condition (3.5) and Proposition 1.2 (i) we will have the required estimate because,

$$a_2(n)|u|-\frac{(2M-1)}{2M}a_2(n)|u|-\left(\frac{1}{2M}a_2(n)-\frac{\varepsilon}{L}\right)(|u|-s+s_1)$$
$$-(b_1+\sqrt{-2\Sigma_n}+\varepsilon+c)(s-s_1)\geqq\frac{\varepsilon}{L}|u|.$$

The sixth term being dominated by

$$K\| G_1^D U_t \phi(H_0) F(|x| \leqq c|u|)\|$$

is similarly estimated using Proposition 1.2 (i). Now the eighth term is dominated by

$$\int_0^s ds_1 \| \tilde{\phi}(H(D)) W_D F_1^{nD} F(|x^D| > a_1(n)|u|) V_{s_1} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|)\|$$

$$\leqq \int_0^s ds_1 \sum_{\substack{C.C \subseteq D \\ \#C=k \\ (n+1) \leqq k \leqq N_1}}\sum \| \tilde{\phi}(H(D)) W_D F_1^{nD} F(|x^D| > a_1(n)|u|) F(|x^C| \leqq a_1(k)|u|).$$

$$\cdot \prod_{l'=1}^{nC} F(|y_{l'}| > a_2(k)|u|) V_{s_1} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|)\|.$$

Now we note that if $C \subseteq D$, then a pair γ is external to C whenever it is external to D. This fact and the condition (3.2) imply that

$$\left\| \tilde{\phi}(H(D)) W_D F(|x^C| \leqq a_1(k)|u|) \prod_{l'=1}^{nC} F(|y_{l'}| > a_2(k)|u|) \right\| \leqq K(1+|u|)^{-\delta}$$

for all $(n+1) \leqq k \leqq N_1$. Hence the right-hand side of (3.9) is dominated by

$$\int_0^s ds_1 (1+|u|)^{-\delta} \sum_{\substack{C:\#C=k \\ (n+1) \leqq k \leqq N_1}}\sum \| F(|x^C| \leqq a_1(k)|u|) \prod_{l=1}^{nC} F(|y_l| > a_2(k)|u|).$$

$$\cdot V_{s_1} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|)\| \leqq K_M (1+|u|)^{-M\delta}$$

by the hypothesis (H) and the observation that for each l

$$F\left((|y_l| > \frac{(2M-1)}{2M} a_1(n)|u| + \left(\frac{1}{2M} a_2(n) - \frac{\varepsilon}{L} \right)(|u| - s + s_1) \right.$$

$$\left. + \left(c + \frac{(L-1)\varepsilon}{L} \right)(s - s_1) \right)$$

$$\leqq F\left(|y_l| > \frac{(2M-1)}{2M} a_2(n)|u| \right) \leqq F(|y_l| > a_2(k)|u| \Big) \tag{3.10}$$

for any $(n+1) \leqq k \leqq N_1$, since $a_2(n) > \mu a_1(n) \geqq 2(a_1(k) + a_2(k)) > 2a_2(k)$ for all $(n+1) \leqq k \leqq N_1$. Finally, the third term has the $K(1+|u|)^{-M\delta}$ bound by a similar argument. Thus collecting these estimates we have,

$$\| F_0^D V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|)\| \leqq O((1+|u|)^{-M\delta})$$

$$+ \int_0^s ds_1 \| F_0^D \tilde{\phi}(H(D)) V_{s-s_1}(D) W_D F_1^D V_{s_1} \phi(H) U_t \phi(H_0). F(|x| \leqq c|u|)\|,$$

which is the one similar to the term we started with. So we can repeat the splitting as in (3.6) and continue doing this $(L-2)$ times using the facts that for all

$1 \leqq j \leqq (L-2)$

$$\tilde{F}_j^D \leqq F(|x^D| \leqq a_1(n)|u|) \prod_{l=1}^{nD} F\left(|y_l| > \frac{(2M-1)}{2M}|u|\right),$$

$$F_j^D \leqq \prod_{\gamma \in e(D)} F(|x^\gamma| > \sigma|u|), \quad \sigma > 0,$$

and the hypothesis (H_1) to obtain the following estimates

$$\| F_0^D V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \|$$

$$\leqq 0((1+|u|)^{-M\delta}) + \underset{0 \leqq s_{L-2} \leqq \cdots \leqq s}{\int \cdots \int} ds_1 \cdots ds_{L-2} \| F_0^D \tilde{\phi}(H(D)) V_{s-s_1}(D) W_D F_1^D \cdots$$

$$\cdots W_D F_{L-2}^D V_{s_{L-2}} \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \|$$

$$\leqq 0((1+|u|)^{-M\delta}) + K_M (1+|u|)^{-(\delta-1)(L-2)} (1+|u|)^{-(M-1)\delta} \leqq K(1+|u|)^{-M\delta}.$$

Hence the result. □

Proof of Theorem 3.4. We have, for any positive numbers, a, b, c, the following inequality.

$$\| F(|x| > a|u|) V_s \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \|$$

$$\leqq \sum_{\substack{C: \# C = k \\ 1 \leqq k \leqq N_1}} \sum \| F(|x| > a|u|) F(|x^C| \leqq a_1(k)|u|) \prod_{l=1}^{nC} F(|y| > a_2(k)|u|) V_s$$

$$\cdot \phi(H) U_t \phi(H_0) F(|x| \leqq c|u|) \|. \tag{3.11}$$

We now recall the definition of μ and set

$$\mu_1 = (2M\mu + 1), \mu_2 = (2M\mu + 2) \text{ and } \Lambda^{-1} = 2M \max\left\{ (\mu_1)^{N_1 - 2} \sum_{k=0}^{N_1 - 3} \sqrt{2}(\mu_2)^k \right\},$$

and choose the constants a, b, c to satisfy $a > \Lambda^{-1}(b + c + \sqrt{-\Sigma_2})$. Then clearly there is an $\varepsilon > 0$, sufficiently small, so that we can choose the constants $\{a_1(k), a_2(k)\}$, $1 \leqq k \leqq N_1$ as follows.

$$a_1(1) = 2M(\mu_1)^{N_1 - 2}(b+c) + 2M \sum_{k=0}^{N_1 - 3} \sqrt{2}(\mu_1)^k (\sqrt{-\Sigma_{k+2}}) + (\mu_2)^{N_1 - 2}\varepsilon$$

$$a_1(n) = 2M \left\{ (\mu_1)^{(N_1 - 1) - n}(b+c) \right.$$
$$\left. + \sum_{k=0}^{(N_1 - 1) - (n+1)} \sqrt{2}(\mu_1)^k (\sqrt{-\Sigma_{n+k+1}}) + (\mu_2)^{(N_1 - 2) - n}\varepsilon \right\}$$

$$a_2(n) = 2M \left\{ 2M\mu \left[(\mu_1)^{(N_1 - 1) - n}(b+c) \right. \right.$$
$$\left. \left. + \sum_{k=1}^{(N_1 - 1) - n} \sqrt{2}(\mu_1)^k (\sqrt{-\Sigma_{n+k}}) \right] + \sqrt{-2\Sigma_n} + \mu_1 (\mu_2)^{(N_1 - 1) - n}\varepsilon \right\}$$

for all $2 \leqq n \leqq (N_1 - 1)$ and $a_2(N_1) = 2M(b + c + \varepsilon)$, $a_1(N_1) = 0$. With this choice we note that $F(|x| > a|u|)F(|x| \leqq a_1(1)|u|) \equiv 0$. Then we apply Lemma 3.5 inductively to the terms on the right-hand side of (3.11) starting with the term with $\#C = N_1$ and obtain the stated estimate. □

Now Proposition 1.4 (i) follows as a corollary of Theorem 3.4 when we take $M = 1$.

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